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ENDOMORPHISMS AND SUBALGEBRAS OF TARSKI ALGEBRAS

A b s t r a c t. In this note we prove that a Tarski algebra is determined by the monoid of its endomorphisms as well as by the lattice of its subalgebras.

1. Introduction

It is known that if **A** and **B** are Boolean algebras then

$$Sub(\mathbf{A}) \cong Sub(\mathbf{B}) \text{ iff } \mathbf{A} \cong \mathbf{B} \text{ iff } End(\mathbf{A}) \cong End(\mathbf{B}).$$

See [11] for the first equivalence and [8] or [10] for the second one. Implication algebras, also called Tarski algebras or semi-Boolean algebras (see [2, 9] for the basics of this kind of algebras) are the algebraic counterpart

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of the implication fragment of classical propositional logic. Implication algebras form a variety which corresponds the $\{\to\}$-subreducts of Boolean algebras. It is the purpose of this paper to prove that the above result is also true for Tarski algebras.

Following [4], two objects A, B in a category $\mathcal{K}$ are *equimorphic* if $End(A) \cong End(B)$ and, for a cardinal α, the category is said to be α-determined if every set of non-isomorphic equimorphic objects of $\mathcal{K}$ has a cardinality smaller than α. Using these definitions our second result assert that the category of Tarski algebras is 2- determined. In [5] we prove this result using a topological duality established in [3]. In this paper we prove the same result using the algebraic side. We think that this provide a more transparent definition of the endomorphisms involved, simplifies many of the proofs and gives the results a structure similar to the one in paper [6] were bounded distributive lattices are proved to be 3-determined.

Recall that a *Tarski algebra* is an algebraic structure $\mathbf{A} := \langle A, \to \rangle$ of type (2) that satisfies the following identities:

$$(x \to y) \to x \approx x, \tag{1}$$
$$(x \to y) \to y \approx (y \to x) \to x, \tag{2}$$
$$x \to (y \to z) \approx y \to (x \to z). \tag{3}$$

The identity $x \to x \approx y \to y$ is validated by $\mathbf{A}$. We denote by 1 the element of $\mathbf{A}$ such that $1 = a \to a$ for all $a \in A$. The binary relation $\leq$ defined on A by the prescription $x \leq y$ if and only if $x \to y = 1$ establishes a partial order on A with greatest element 1. We have that for all $a, b \in A$, $a, b \leq (a \to b) \to b = (b \to a) \to a$. We set $a \vee b := (a \to b) \to b$ and clearly, we have that $a \vee b = b \vee a$ and this element is the least upper bound of a and b in the order defined above. Indeed, $\langle A, \vee \rangle$ is a join semilattice whose principal filters are Boolean algebras.

2. Endomorphisms

Let $\mathbf{A}$ be a Tarski algebra. Recall that a filter of $\mathbf{A}$ is a non-empty subset F of A such that (i) $1 \in F$ and (ii) $a, a \to b \in F$ imply that $b \in F$. A maximal filter P is a proper filter that is not contained properly in another proper filter. In [3] (Theorem 2.4), it is proved that if P is a maximal filter then

$$z \to y \notin P \text{ iff } z \in P \text{ and } y \notin P, \tag{4}$$

In [1] maximal filters are characterized as follows:

$$\text{A proper filter } P \text{ is maximal iff } \forall a \notin P, \forall x \in A,\ a \to x \in P. \tag{5}$$

Maximal filters separate non-comparable elements, that is, if $a, b \in A$,

$$a \not\leq b \Longrightarrow \exists P, \text{maximal filter, such that } a \in P \text{ and } b \notin P \tag{6}$$

For the proof of this result see [3], Theorem 2.4.

It is easy to show that if $g \in End(\mathbf{A})$ and F is a filter of $\mathbf{A}$ then $g^{-1}(F)$ is a filter of $\mathbf{A}$.

Proposition 2.1. *If Q is a maximal filter of a Tarski algebra $\mathbf{A}$ and $g \in End(\mathbf{A})$ is such that $g^{-1}(Q) \neq A$ then $g^{-1}(Q)$ is a maximal filter of $\mathbf{A}$.*

Proof. Suppose that $g^{-1}(Q)$ is not a maximal filter of $\mathbf{A}$. Then, by property (5), there exists $a \notin g^{-1}(Q)$ such that $a \to b \notin g^{-1}(Q)$ for some $b \in A$. So, $g(a) \notin Q$ is such that $g(a) \to g(b) \notin Q$ and this is against the hypothesis that Q is a maximal filter, see (5). □

For each maximal filter Q of $\mathbf{A}$ and each $x \in A, x \neq 1$, define the function $f_{Q,x} : A \longrightarrow A$ by the formula

$$f_{Q,x}(t) = \begin{cases} 1, & \text{if } t \in Q\ ; \\ x, & \text{if } t \notin Q. \end{cases}$$

One can check that $f_{Q,x} \in End(\mathbf{A})$. The constant function from A into itself with constant value 1 is clearly an endomorphism. We denote this constant function by 1_A. For maximal filters P and Q of $\mathbf{A}$ and $g \in End(\mathbf{A})$, the following equalities are easily checked:

$$f_{P,x} \circ f_{Q,y} = \begin{cases} f_{Q,x}, & \text{if } y \notin P\ ; \\ 1_A, & \text{if } y \in P. \end{cases} \tag{7}$$

$$g \circ f_{P,x} = f_{P,g(x)}. \tag{8}$$

$$f_{P,x} \circ g = \begin{cases} f_{g^{-1}(P),x} & \text{if } g^{-1}(P) \neq A \text{ ;} \\ 1_A, & \text{if } g^{-1}(P) = A. \end{cases} \tag{9}$$

In what follows, $\mathbf{A}_1$ and $\mathbf{A}_2$ are Tarski algebras and

$$\Phi : End(\mathbf{A}_1) \longrightarrow End(\mathbf{A}_2)$$

is a monoid isomorphism.

Proposition 2.2. $\Phi(1_{A_1}) = 1_{A_2}$.

Proof. Assume, on the contrary that $\Phi(1_{A_1}) \neq 1_{A_2}$. Then, there exists $y \in A_2$, $y \neq 1$, such that $\Phi(1_{A_1})(x) = y$ for some $x \in A_2$. Clearly, $x \neq 1$. As $1_{A_1} \circ f = f \circ 1_{A_1} = 1_{A_1}$ for all $f \in End(\mathbf{A}_1)$ then

$$\Phi(1_{A_1}) \circ g = g \circ \Phi(1_{A_1}) = \Phi(1_{A_1}) \ \forall \ g \in End(\mathbf{A}_2). \tag{10}$$

Due to property (6) we can choose a maximal filter P such that $y \notin P$. From $\Phi(1_{A_1})(x) = y$ and equality (8) it follows that $\Phi(1_{A_1}) \circ f_{P,x} = f_{P,y}$. By (10), the left-hand side of this equality is $\Phi(1_{A_1})$, so $\Phi(1_{A_1}) = f_{P,y}$; but this is a contradiction because, using (8), it follows at once that $f_{P,y}$ does not enjoy property (10). □

Our next proposition asserts that endomorphisms of Tarski algebras of the form $f_{P,x}$, where P is a maximal filter, are preserved under monoid isomorphisms.

Proposition 2.3. *Let P be a maximal filter of $\mathbf{A}_1$ and $x \in A_1 \setminus \{1\}$. Then $\Phi(f_{P,x}) = f_{R,y}$ for some maximal filter R of $\mathbf{A}_2$ and $y \in A_2 \setminus \{1\}$.*

Proof. Let $u \in A_2 \setminus \{1\}$ and pick a maximal filter Q of $\mathbf{A}_2$. Set $\Psi := \Phi^{-1}$ and $\rho_u := \Psi(f_{Q,u})$. By Proposition 2.2, we can choose $t_u, y_u \in A_1, y_u \neq 1$ such that $\rho_u(t_u) = y_u$. Set $g_u := \Phi(f_{P,t_u})$ and $h_u := \Phi(f_{P,y_u})$. By (8), $\rho_u \circ f_{P,t_u} = f_{P,y_u}$. Applying Φ to both sides of this equality we receive $f_{Q,u} \circ g_u = \Phi(f_{P,y_u}) = h_u$ and by (8) we obtain

$$\Phi(f_{P,y_u}) = f_{g_u^{-1}(Q),u}.$$

Set $R := g_u^{-1}(Q)$. Since clearly f_{P,y_u} is not the constant endomorphism of value 1, due to Proposition 2.2 and Proposition 2.1, R is a maximal filter of $\mathbf{A}_2$.

Next we assert that there exists $u \in A_2$ such that $y_u \notin P$. For otherwise, due to (7), we have $f_{P,x} \circ f_{P,y_u} = 1$ and consequently $\Phi(f_{P,x}) \circ f_{R,u} = 1$ for all $u \in A_2$ and this means that $\Phi(f_{P,x})(u) = 1$ for all $u \in A_2$, a contradiction.

Let $y_u \notin P$. Then, by (7), $f_{P,x} \circ f_{P,y_u} = f_{P,x}$. Consequently $\Phi(f_{P,x}) = \Phi(f_{P,x}) \circ f_{R,u}$ and by (8) we have that

$$\Phi(f_{P,x}) = f_{R,v} \quad \text{where} \quad v := \Phi(f_{P,x})(u).$$

$\square$

The assertion in the following two lemmas are in the context of Proposition 2.3. There, P and P' are maximal filters of $\mathbf{A}_1$, Q and Q' are maximal filters of $\mathbf{A}_2$, $x, x' \in A_1 \smallsetminus \{1\}$ and $y, y' \in A_2$.

Lemma 2.4. *$\Phi(f_{P,x}) = f_{Q,y}$ and $\Phi(f_{P,x'}) = f_{Q',y'}$ imply $Q = Q'$.*

Proof. Pick $x'' \notin P$. Due to Proposition 2.3, $\Phi(f_{P,x''}) = f_{Q'',y''}$, for some maximal filter Q'' of $\mathbf{A}_2$ and $y'' \in A_2$. By (7), $f_{P,x} \circ f_{P,x''} = f_{P,x}$ and by applying Φ on both sides of this equality we get $f_{Q'',y} = f_{Q,y} \circ f_{Q'',y''} = f_{Q,y}$, the first equality being due to (7) (observe that $x'' \notin P$ implies $y'' \notin Q''$). Hence, $Q = Q''$. Similarly we get $Q' = Q''$. So, $Q = Q'$. $\square$

Lemma 2.5. *Let P and P' be maximal filters of $\mathbf{A}_1$ and $x \notin P, P'$. Set $\Phi(f_{P,x}) = f_{Q,y}$ and $\Phi(f_{P',x}) = f_{Q',y'}$. Then $y = y'$.*

Proof. By (7), $f_{P,x} \circ f_{P,'x} = f_{P,'x}$ and since Φ is and isomorphism we have that $f_{Q,y} \circ f_{Q',y'} = f_{Q',y'}$ which implies that $y' \notin Q$ ($y' \in Q$ implies $f_{Q,y} \circ f_{Q',y'} = 1_{A_2} \neq f_{Q',y'}$). As $f_{Q,y} \circ f_{Q',y'} = f_{Q',y}$ as well, $f_{Q',y'} = f_{Q',y}$; so, $y = y'$. $\square$

Define the function $\phi : A_1 \longrightarrow A_2$ by the following prescription:

$$\phi(1) = 1$$

and, for $x \neq 1$,

$$\phi(x) = y \text{ if } \Phi(f_{P,x}) = f_{Q,y}$$

where P is a maximal filter of $\mathbf{A}_1$ such that $x \notin P$ and Q is the maximal filter of $\mathbf{A}_2$ given by Proposition 2.3. By Lemma 2.5, this function is well defined and by Lemma 2.4, it is one to one. Indeed, by symmetry, the assignment $\psi : A_2 \longrightarrow A_1$, defined similarly with $\Psi = \Phi^{-1}$ instead of Φ, is such that $\phi^{-1} = \psi$. So, ϕ is a bijective function.

In what follows, we will prove that ϕ is an isomorphism. With this purpose, set

$$\phi(x \to x') = z, \quad \phi(x) = u \quad \text{and} \quad \phi(x') = u'. \tag{11}$$

We want to show that $u \to u' = z$. If $x = 1$ or $x' = 1$ or $x \leq x'$ (i.e., $x \to x' = 1$) the equality is easy to prove. So we assume that $x, x', x \to x' \neq 1$. First we assert that $u \to u' \leq z$; for otherwise, we can choose a maximal filter Q of $\mathbf{A}_2$ such that $u \to u' \in Q$ but $z \notin Q$. By definition of ϕ, there exists a maximal filter P of $\mathbf{A}_1$ such that $\Psi(f_{Q,z}) = f_{P,x\to x'}$ or, equivalently, $\Phi(f_{P,x\to x'}) = f_{Q,z}$. Pick now a maximal filter K of $\mathbf{A}_1$ such that $x \notin K$. Then, again by the definition of ϕ we may set

$$\Phi(f_{K,x}) = f_{U,u} \quad \text{and} \quad \Phi(f_{P,x'}) = f_{U',u'}, \tag{12}$$

where U and U' are maximal filters of $\mathbf{A}_2$. Observe here that since $z \notin Q$ then $x \to x' \notin P$ and consequently, $x \in P$ and $x' \notin P$. So, $f_{P,x\to x'} \circ f_{P,x'} = f_{P,x\to x'}$ and consequently, $f_{Q,z} \circ f_{U',u'} = f_{Q,z}$. It follows from this that $u' \notin Q$. Also, as $x \in P$, $f_{P,x\to x'} \circ f_{K,x} = 1$ and therefore, $f_{Q,z} \circ f_{U,u} = 1$ and this means that $u \in Q$; so, by (4), $u \to u' \notin Q$, a contradiction. This proves that $u \to u' \leq z$.

To complete the proof we need to show that $z \leq u \to u'$. We assume the contrary, i.e., $z \not\leq u \to u'$ and look for a contradiction. By (6), there exists a maximal filter T of $\mathbf{A}_2$ such that $z \in T$ but $u \to u' \notin T$. By (4), $u \in T$ and $u' \notin T$. Setting $\Psi(f_{T,u'}) = f_{P'',x''}$ and knowing that $\Psi(f_{U',u'}) = f_{P,x'}$ we have, by Lemma 2.5, that $x'' = x'$. Thus, $\Psi(f_{T,u'}) = f_{P'',x'}$. Observe that, as $u' \notin T$, $x' \notin P''$. Now, by Lemma 2.4, $\Psi(f_{T,u}) = f_{P'',t}$ for some $t \in A_1$ and now, since $\Phi(f_{K,x}) = f_{U,u}$, by Lemma 2.5 we have that $t = x$. (Observe that, as $u \in T$ then $x \in P''$ and since $x' \notin P''$ then $x \to x' \notin P''$). Summarising we have:

$$\Phi(f_{P'',x'}) = f_{T,u'}; \quad \Phi(f_{P'',x}) = f_{T,u} \text{ and } \quad \Phi(f_{P,x\to x'}) = f_{Q,z}. \tag{13}$$

It follows now from Lemma 2.5 and Lemma 2.4 that $\Phi(f_{P'',x\to x'}) = f_{T,z}$; but this is a contradiction because $x \to x' \notin P''$ implies $z \notin T$. We have proved this way the following result:

Theorem 2.6. *$End(\mathbf{A}_1) \cong End(\mathbf{A}_2)$ iff $\mathbf{A}_1 \cong \mathbf{A}_2$.*

3. Subalgebras

Through this section, if $\mathbf{A}$ is a Tarski algebra, $Sub(\mathbf{A})$ will denote the lattice of subalgebras of $\mathbf{A}$. For $S \subseteq A$ we denote by $\langle S \rangle$ the subalgebra of $\mathbf{A}$ generated by S. Clearly, $\{1, a\}$ is a subuniverse of $\mathbf{A}$ for each $a \in A$. In fact, the atoms of the lattice $Sub(\mathbf{A})$ are all the subalgebras $\langle a \rangle$ with universe $\{1, a\}$ for each $a \in A \setminus \{1\}$ and the trivial Tarski algebra $\langle 1 \rangle$ with universe $\{1\}$ is the least element of this lattice. So we have the following easy result:

Proposition 3.1. *If* $Sub(\mathbf{A}_1) \cong Sub(\mathbf{A}_2)$ *then* $|A_1| = |A_2|$.

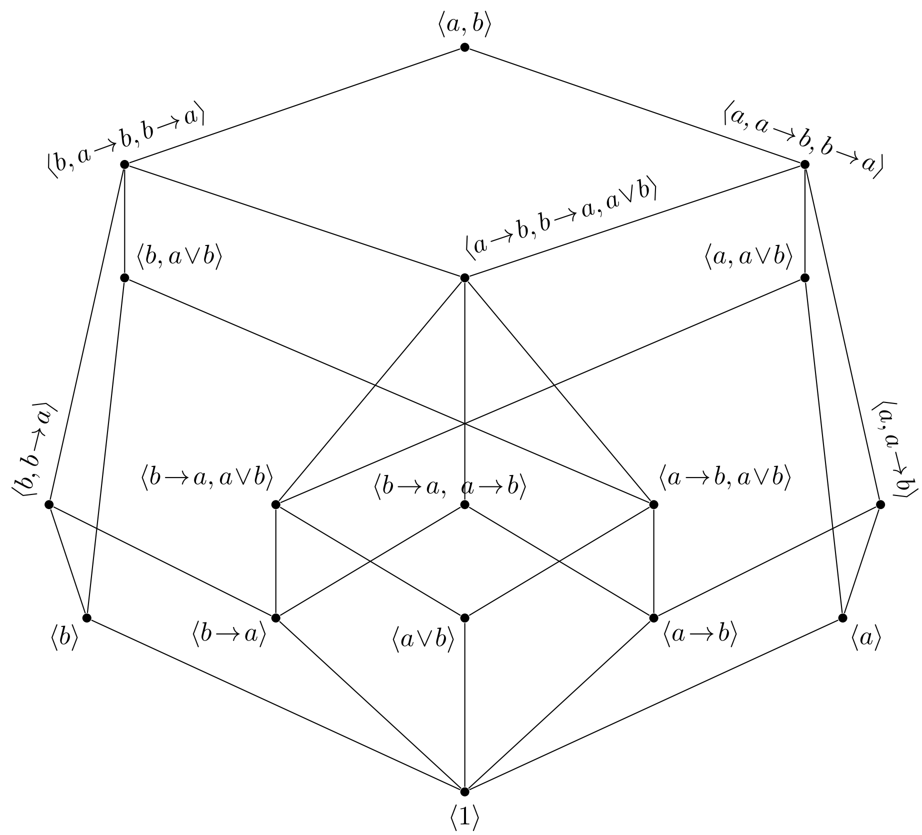
If $\Phi : Sub(\mathbf{A}_1) \longrightarrow Sub(\mathbf{A}_2)$ is a lattice isomorphism, define $\phi : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ by the prescription

$$\phi(x) = y \ \text{ iff } \ \Phi(\langle x \rangle) = \langle y \rangle. \tag{14}$$

We intend to prove that ϕ is an isomorphism of Tarski algebras. Certainly, ϕ is well defined and it is a bijective function. For $a, b \in A_1, a \neq b$, $\langle a, b \rangle = \langle a \rangle \vee \langle b \rangle$ and since Φ is a lattice isomorphism we have that $\Phi(\langle a, b \rangle) = \Phi(\langle a \rangle) \vee \Phi(\langle b \rangle) = \langle \phi(a) \rangle \vee \langle \phi(b) \rangle = \langle \phi(a), \phi(b) \rangle$; so,

$$\Phi \restriction_{Sub(\langle a,b \rangle)} : Sub(\langle a, b \rangle) \longrightarrow Sub(\langle \phi(a), \phi(b) \rangle)$$

is a lattice isomorphism. There are, up to isomorphisms, three Tarski algebras with two generators namely: (i) $\{1, a, b\}$ where $a \to b = b$, $b \to a = a$. (ii) $\{1, a, b, b \to a\}$ where $a \to b = 1$ and (iii) the free algebra with two genertors (it is described in [2] p.179).



The lattice $Sub(\langle a,b\rangle)$ in the case $\langle a,b\rangle$ is the free algebra with two generators is depicted above. We see that in this lattice, the subalgebra $\langle a, a\to b\rangle$ is the join of two atoms; more precisely,

$$\langle a, a\to b\rangle = \langle a\rangle \vee \langle a\to b\rangle.$$

In these equalities '$\vee$' obviously stands for the join operation in the lattice. Now, since Φ is a lattice isomorphism, in $Sub(\langle \phi(a), \phi(b)\rangle)$ we have that

$$\Phi(\langle a, a\to b\rangle) = \Phi(\langle a\rangle) \vee \Phi(\langle a\to b\rangle) = \langle \phi(a), \phi(a\to b)\rangle.$$

As $\mathrm{Sub}(\langle a,b\rangle) \cong \mathrm{Sub}(\langle \phi(a), \phi(b)\rangle)$ and $\langle a,b\rangle$ is the free algebra with two generators, so is $\langle \phi(a), \phi(b)\rangle$. In this algebra, the element which is the join of two atoms, one of them being $\langle \phi(a)\rangle$, is $\langle \phi(a), \phi(a)\to\phi(b)\rangle$; therefore we have that

$$\langle \phi(a), \phi(a)\to\phi(b)\rangle = \Phi(\{1, a, a\to b\}) = \langle \phi(a), \phi(a\to b)\rangle$$

from which it follows that $\phi(a \to b) = \phi(a) \to \phi(b)$. The other two cases of two-generated Tarski algebras are treated similarly arriving in both of them to the same conclusion, namely that $\phi(a \to b) = \phi(a) \to \phi(b)$. So we have proved the following result.

Theorem 3.2. *$Sub(\mathbf{A}_1) \cong Sub(\mathbf{A}_2)$ iff $\mathbf{A}_1 \cong \mathbf{A}_2$.*

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