REPORTS ON MATHEMATICAL LOGIC
53 (2018), 43–65
doi: 10.4467/20842589RM.18.003.8836

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A SEMANTICAL ANALYSIS OF CUT-FREE CALCULI FOR MODAL LOGICS

A b s t r a c t. We analyze semantically the logical inference rules in cut-free sequent calculi for the modal logics which are obtained from the least normal logic **K** by adding axioms from **T**, **4**, **5**, **D** and **B**. This implies Kripke completeness, as well as the cut-elimination property or the subformula property of the calculi. By slightly modifying the arguments, the finite model property of the logics also follows.

The purpose of this paper is to analyze semantically the logical inference rules in cut-free sequent calculi for modal logics, aiming at a cut-free or analytic version of Maehara [2], in which sequent calculi with cut are concerned with. This constitutes another proof of Kripke completeness as well as the cut-elimination property or the subformula property of the calculi. By modifying the arguments a bit, the finite model property also follows.

Received 9 June 2017
Keywords and phrases: Sequent calculus, Kripke completeness, finite model property, cut-elimination, subformula property.
AMS subject classification: 03B45, 03F05.

We consider all the propositional modal logics which are obtained from the least normal logic **K** by adding axiom schemata among the following:

T : $\Box A \supset A$. **4** : $\Box A \supset \Box\Box A$. **5** : $\neg\Box A \supset \Box\neg\Box A$.
D : $\Box A \supset \neg\Box\neg A$. **B** : $\neg A \supset \Box\neg\Box A$.

There are 15 mutually distinct logics of them, which we divide into the following four classes:

Class 1: **K**, **KT**, **KD**, **K4**, **K4D**, **S4**(=**KT4**).
Class 2: **K45**, **K45D**.
Class 3: **KB**, **KTB**, **KDB**, **K4B**, **S5**(=**KT4B**=**KT5**).
Class 4: **K5**, **K5D**.

Characterization of these logics by Kripke frame semantics is known (Goré [1], for example).

It is a classical result that each logic in Class 1 has a sequent calculus with the cut-elimination property (and so the subformula property). It is proved both syntactically and semantically in Shvarts [3] that those logics in Class 2, too, have sequent calculi with the cut-elimination property as well as the subformula property. The logics in Class 3 have sequent calculi with the subformula property but without the cut-elimination property (Takano [4]). Lastly, those in Class 4 have sequent calculi with a modified form of the subformula property but without the subformula property in the original sense (Takano [5]).

After common preliminaries in Section 1, sequent calculi for the logics in Classes 1–4 are dealt with in Sections 2–5, respectively.

The author hopes that our course of semantical analysis of inference rules as well as extension of the notion of subformula is refined and applied to other logics by the interested readers.

1. Preliminaries

In this paper, only $\neg$ (negation), $\supset$ (implication) and $\Box$ (necessity) are used as the logical symbols, and others are considered as abbreviations, for simplicity. Propositional letters and formulas are denoted by $p, q, r, \ldots$ and $A, B, C, \ldots$, respectively. A $\Box$-*formula* is a formula whose outermost logical symbol is the necessity symbol $\Box$. A *sequent* is an expression of the form $\Gamma \to \Theta$, where Γ and Θ are finite sequences of formulas.

Every sequent calculus which is taken up in this paper enjoys the following stipulation.

Stipulation 1. *The sequent calculus has $A \to A$ as an initial sequent for every A, and contains the following structural rules as inference rules.*

$$(\textbf{weakening}) \quad \frac{\Gamma \to \Theta}{A, \Gamma \to \Theta}\,, \qquad \frac{\Gamma \to \Theta}{\Gamma \to \Theta, A}\,.$$

$$(\textbf{exchange}) \quad \frac{\Delta, B, A, \Gamma \to \Theta}{\Delta, A, B, \Gamma \to \Theta}\,, \quad \frac{\Gamma \to \Theta, B, A, \Lambda}{\Gamma \to \Theta, A, B, \Lambda}\,.$$

$$(\textbf{contraction}) \quad \frac{A, A, \Gamma \to \Theta}{A, \Gamma \to \Theta}\,, \qquad \frac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}\,.$$

So, for convenience, the antecedent Γ and the succedent Θ of the sequent $\Gamma \to \Theta$ are recognized as sets also. Finite sequences (as well as finite sets) of formulas are denoted by $\Gamma, \Theta, \Delta, \Lambda, \dots$. We mean by $\mathrm{Sf}(\Gamma)$ the set of all the subformulas of some formulas in Γ, while by $\Box\Gamma$ the set $\{\Box A \mid A \in \Gamma\}$.

It must be noticed that the inference rule (**cut**), which is described below, is only admitted in some of our calculi with appropriate restriction:

$$(\textbf{cut}) \quad \frac{\Gamma \to \Theta, C \quad C, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\,.$$

Definition 1.1. Let GL be a sequent calculus. A sequent $\Gamma \to \Theta$ is *analytically saturated* in GL, iff the following properties hold.

(1.1-a) $\Gamma \to \Theta$ is unprovable in GL.

(1.1-b) Suppose $A \in \mathrm{Sf}(\Gamma \cup \Theta)$. If $A, \Gamma \to \Theta$ is unprovable in GL, then $A \in \Gamma$; while if $\Gamma \to \Theta, A$ is unprovable in GL, then $A \in \Theta$.

The set of all the analytically saturated sequents are denoted by W_{GL}. We denote analytically saturated sequents by $u, v, w, \dots$; besides, $\mathrm{a}(u)$ and $\mathrm{s}(u)$ denote the antecedent and succedent of u, respectively.

The following proposition will be used tacitly.

Proposition 1.2. *For a sequent calculus GL, $\mathrm{a}(u) \cap \mathrm{s}(u) = \emptyset$ for every analytically saturated sequent u.*

Lemma 1.3. *For a sequent calculus GL, if a sequent* $\Gamma \to \Theta$ *is unprovable in GL, then there is an analytically saturated sequent* u *with the following properties*:

(1.3-a) $\Gamma \subseteq \mathrm{a}(u)$ *and* $\Theta \subseteq \mathrm{s}(u)$.

(1.3-b) $\mathrm{a}(u) \cup \mathrm{s}(u) \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$.

(1.3-c) *Let* v *be an analytically saturated sequent such that* $\Gamma \subseteq \mathrm{a}(v)$, $\Theta \subseteq \mathrm{s}(v)$, $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u))$, *and every* $\Box$*-formula in* $\mathrm{a}(u)$ *and* $\mathrm{s}(u)$ *is also in* $\mathrm{a}(v)$ *and* $\mathrm{s}(v)$, *respectively. Then, every* $\Box$*-formula in* $\mathrm{a}(v)$ *and* $\mathrm{s}(v)$ *is also in* $\mathrm{a}(u)$ *and* $\mathrm{s}(u)$, *respectively.*

Proof. Let $A_1, A_2, \ldots, A_m, A_{m+1}, \ldots, A_n$ $(0 \leq m \leq n)$ be an enumeration of all the formulas in $\mathrm{Sf}(\Gamma \cup \Theta)$ such that $A_1, A_2, \ldots, A_m$ are $\Box$-formulas, while others are not. Put $\Gamma_1 = \Gamma$ and $\Theta_1 = \Theta$. Suppose that Γ_k and Θ_k have been defined $(1 \leq k \leq n)$. If $\Gamma_k \to \Theta_k, A_k$ is unprovable, then put $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k \cup \{A_k\}$; if $\Gamma_k \to \Theta_k, A_k$ is provable, but $A_k, \Gamma_k \to \Theta_k$ is unprovable, then put $\Gamma_{k+1} = \Gamma_k \cup \{A_k\}$ and $\Theta_{k+1} = \Theta_k$; otherwise, put $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k$.

We will show that $\Gamma_{n+1} \to \Theta_{n+1}$ is the desired sequent u.

Evidently, both properties (1.3-a) and (1.3-b) hold for $\Gamma_{n+1} \to \Theta_{n+1}$, namely, $\Gamma \subseteq \Gamma_{n+1}$, $\Theta \subseteq \Theta_{n+1}$ and $\Gamma_{n+1} \cup \Theta_{n+1} \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$.

Now, it will be shown that $\Gamma_{n+1} \to \Theta_{n+1}$ is analytically saturated. It is clearly unprovable. Suppose $A \in \mathrm{Sf}(\Gamma_{n+1} \cup \Theta_{n+1})$. It follows from (1.3-b) that $A \in \mathrm{Sf}(\Gamma_{n+1} \cup \Theta_{n+1}) \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$, and so A is A_k for some k $(1 \leq k \leq n)$. Suppose also that $A, \Gamma_{n+1} \to \Theta_{n+1}$ is unprovable. If $\Gamma_k \to \Theta_k, A$ were unprovable, then $A \in \Theta_{k+1} \subseteq \Theta_{n+1}$, and so $A, \Gamma_{n+1} \to \Theta_{n+1}$ would be provable, which contradicts our assumption. Hence $\Gamma_k \to \Theta_k, A$ is provable. But $A, \Gamma_k \to \Theta_k$ is unprovable, since $\Gamma_k \subseteq \Gamma_{n+1}$ and $\Theta_k \subseteq \Theta_{n+1}$. So $A \in \Gamma_{k+1} \subseteq \Gamma_{n+1}$. Similarly, if $\Gamma_{n+1} \to \Theta_{n+1}, A$ is unprovable, $A \in \Theta_{n+1}$. Thus, $\Gamma_{n+1} \to \Theta_{n+1}$ is analytically saturated.

It is left to check the property (1.3-c). So, let v be an analytically saturated sequent such that $\Gamma \subseteq \mathrm{a}(v)$, $\Theta \subseteq \mathrm{s}(v)$, $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\Gamma_{n+1} \cup \Theta_{n+1})$, and every $\Box$-formula $\Box B$ in Γ_{n+1} and Θ_{n+1} is also in $\mathrm{a}(v)$ and $\mathrm{s}(v)$, respectively. Suppose $\Box B \in \mathrm{a}(v)$. Since $\Box B \in \mathrm{Sf}(\Gamma_{n+1} \cup \Theta_{n+1}) \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$, $\Box B$ is A_k for some k $(1 \leq k \leq m)$. If $\Gamma_k \to \Theta_k, \Box B$ were unprovable, then $\Box B \in \Theta_{k+1} \subseteq \Theta_{n+1}$, and so $\Box B$ would be in $\mathrm{s}(v)$, which contradicts our assumption. Hence $\Gamma_k \to \Theta_k, \Box B$ is provable. But $\Box B, \Gamma_k \to \Theta_k$ is unprovable; for, since $1 \leq k \leq m$, it follows $\Gamma_k \subseteq \Gamma \cup \{\Box C | \Box C \in \Gamma_{n+1}\} \subseteq \mathrm{a}(v)$ and $\Theta_k \subseteq \Theta \cup \{\Box C | \Box C \in \Theta_{n+1}\} \subseteq \mathrm{s}(v)$. So $\Box B \in \Gamma_{k+1} \subseteq \Gamma_{n+1}$. Similarly, if $\Box B \in \mathrm{s}(v)$, then $\Box B \in \Theta_{n+1}$. This ends the proof that (1.3-c) holds for $\Gamma_{n+1} \to \Theta_{n+1}$. □

It is Sections 3, 4 and 5 that the property (1.3-b) is utilized, while solely in Section 3 for (1.3-c).

Lemma 1.4. *Let GL be a sequent calculus. Suppose that* $\langle W, R\rangle$ *is a Kripke frame with* $W \subseteq W_{GL}$*, and the following properties hold for every* A*,* B *and every* $u \in W$ *:*

($\neg$-**a**) $\neg A \in \mathrm{a}(u)$ *implies* $A \in \mathrm{s}(u)$*.*

($\neg$-**s**) $\neg A \in \mathrm{s}(u)$ *implies* $A \in \mathrm{a}(u)$*.*

($\supset$-**a**) $A \supset B \in \mathrm{a}(u)$ *implies* $A \in \mathrm{s}(u)$ *or* $B \in \mathrm{a}(u)$*.*

($\supset$-**s**) $A \supset B \in \mathrm{s}(u)$ *implies* $A \in \mathrm{a}(u)$ *and* $B \in \mathrm{s}(u)$*.*

($\Box$-**a**) $\Box A \in \mathrm{a}(u)$ *implies* $A \in \mathrm{a}(v)$ *for every* $v \in W$ *such that* uRv*.*

($\Box$-**s**) $\Box A \in \mathrm{s}(u)$ *implies* $A \in \mathrm{s}(v)$ *for some* $v \in W$ *such that* uRv*.*

Let $\models$ *be the satisfaction relation on* $\langle W, R\rangle$ *such that* $u \models p$ *iff* $p \in \mathrm{a}(u)$ *for every* $u \in W$ *and every* p*. Then,* $C \in \mathrm{a}(u)$ *implies* $u \models C$ *while* $C \in \mathrm{s}(u)$ *implies* $u \not\models C$*, for every* C *and every* $u \in W$*.*

Proof. By simultaneous induction on the construction of C.

Case 1: C is a propositional letter p. If $p \in \mathrm{a}(u)$, then $u \models p$ clearly. If $p \in \mathrm{s}(u)$, then $p \notin a(u)$, so $u \not\models p$.

Case 2: C is $\neg A$. Recall that $u \models \neg A$ iff $u \not\models A$. If $\neg A \in \mathrm{a}(u)$, then $A \in \mathrm{s}(u)$ by ($\neg$-**a**), so $u \not\models A$ by the hypothesis of induction, so $u \models \neg A$. The remainder is similar.

Case 3: C is $A \supset B$. Similar to Case 2.

Case 4: C is $\Box A$. Recall that $u \models \Box A$ iff $v \models A$ for every $v \in W$ such that uRv. Suppose first that $\Box A \in \mathrm{a}(u)$. For every $v \in W$ such that uRv, it follows $A \in \mathrm{a}(v)$ by ($\Box$-**a**), so $v \models A$ by the hypothesis of induction. So $u \models \Box A$. Suppose next that $\Box A \in \mathrm{s}(u)$. By ($\Box$-**s**), it follows $A \in \mathrm{s}(v)$ for some $v \in W$ such that uRv. By the hypothesis of induction, $v \not\models A$. So $u \not\models \Box A$. □

Definition 1.5. An inference is *admissible* in a sequent calculus GL, iff either some of the upper sequents of the inference is unprovable in GL, or the lower one is provable in GL.

Think of the following inference rules:

$$(\neg \rightarrow) \quad \frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta} . \qquad (\rightarrow \neg) \quad \frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A} .$$

$$(\supset \rightarrow) \quad \frac{\Gamma \rightarrow \Theta, A \quad B, \Gamma \rightarrow \Theta}{A \supset B, \Gamma \rightarrow \Theta} . \quad (\rightarrow \supset) \quad \frac{A, \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \supset B} .$$

Proposition 1.6. *For a sequent calculus GL, the following equivalences hold for every A and B.*

(1) *The inference $(\neg \rightarrow)$ is admissible in GL for every Γ and Θ, iff* ($\neg$-**a**) *holds for every u.*

(2) *The inference $(\rightarrow \neg)$ is admissible in GL for every Γ and Θ, iff* ($\neg$-**s**) *holds for every u.*

(3) *The inference $(\supset \rightarrow)$ is admissible in GL for every Γ and Θ, iff* ($\supset$-**a**) *holds for every u.*

(4) *The inference $(\rightarrow \supset)$ is admissible in GL for every Γ and Θ, iff* ($\supset$-**s**) *holds for every u.*

Proof. (1) *The 'if' part*: Suppose that $\neg A, \Gamma \rightarrow \Theta$ is unprovable. Then, by Lemma 1.3, $\neg A \in \mathrm{a}(u)$, $\Gamma \subseteq \mathrm{a}(u)$ and $\Theta \subseteq \mathrm{s}(u)$ for some u. It follows $A \in \mathrm{s}(u)$ by ($\neg$-**a**), and so $\Gamma \rightarrow \Theta, A$ is unprovable, since $\Gamma \subseteq \mathrm{a}(u)$ and $\Theta \cup \{A\} \subseteq \mathrm{s}(u)$. *The 'only if' part*: Suppose $\neg A \in \mathrm{a}(u)$. Since $\neg A, \mathrm{a}(u) \rightarrow \mathrm{s}(u)$ is unprovable, neither is $\mathrm{a}(u) \rightarrow \mathrm{s}(u), A$ by the assumption. Moreover $A \in \mathrm{Sf}(\mathrm{a}(u)) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u))$. So $A \in \mathrm{s}(u)$.

(2)–(4) Similar to (1). □

We put the second stipulation on a sequent calculus. So by the above proposition, ($\neg$-**a**), ($\neg$-**s**), ($\supset$-**a**) and ($\supset$-**s**) always hold for any sequent calculus.

Stipulation 2. *The sequent calculus contains $(\neg \rightarrow)$, $(\rightarrow \neg)$, $(\supset \rightarrow)$ and $(\rightarrow \supset)$ as inference rules.*

Let GL be a sequent calculus for a logic L. Suppose that $\langle W, R \rangle$ is a Kripke frame with $W \subseteq W_{GL}$ such that

(a) if $\Gamma \to \Theta$ is unprovable in GL, then $\Gamma \subseteq \mathrm{a}(u)$ and $\Theta \subseteq \mathrm{s}(u)$ for some $u \in W$,

(b) the Kripke frame $\langle W, R\rangle$ enjoys the properties ($\Box$-**a**) and ($\Box$-**s**), and

(c) the accessibility relation R meets the condition of the Kripke frames for the logic L,

and $\models$ is the satisfaction relation on $\langle W, R\rangle$ defined as in Lemma 1.4.

Then, if $\Gamma \to \Theta$ is unprovable in GL, then $\Gamma \subseteq \mathrm{a}(u)$ and $\Theta \subseteq \mathrm{s}(u)$ for some $u \in W$ by (a), and so by Lemma 1.4 and (b), u rejects $\Gamma \to \Theta$, that is, $C \in \Gamma$ implies $u \models C$ while $C \in \Theta$ implies $u \not\models C$. This together with (c) implies that, GL is complete with respect to the Kripke frame semantics for L, and $\langle W, R, \models\rangle$ forms a universal Kripke model for L. Note that, when $W = W_{GL}$, the condition (a) holds by Lemma 1.3.

2. The logics K, KT, KD, K4, K4D and S4

This section concerns the logics in Class 1, namely the logics **K**, **KT**, **KD**, **K4**, **K4D** and **S4**. By the Kripke frames made of the analytically saturated sequents, the inference rules that are added to the sequent calculi for these logics are analyzed semantically. As a result, completeness as well as the finite model property of the calculi and logics follow. The sequent calculi are cut-free and have the subformula property naturally.

Consider the following inference rules:

$$(K)\quad \frac{\Gamma \to A}{\Box\Gamma \to \Box A}\,. \qquad (D)\quad \frac{\Gamma \to}{\Box\Gamma \to}\,. \qquad (4)\quad \frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}\,.$$

$$(4D)\quad \frac{\Gamma, \Box\Gamma \to}{\Box\Gamma \to}\,. \qquad (S4)\quad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\,. \qquad (T)\quad \frac{A, \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}\,.$$

The additional inference rules, besides those in Stipulations 1 and 2, of the sequent calculus GL as well as the condition on the accessibility relations of the Kripke frames for the logic L are described in the following table, where $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$.

Logic	Additional rules	Condition on relations
K	(K)	none
KT	$(K), (T)$	reflexive
KD	$(K), (D)$	serial
K4	(4)	transitive
K4D	$(4), (4D)$	transitive and serial
S4	$(S4), (T)$	reflexive and transitive

Remember that a binary relation R on a set W is *serial*, iff for every $u \in W$, uRv for some $v \in W$.

Definition 2.1. For a sequent calculus GL, the binary relations $R_{\mathbf{K}}$, $R_{\mathbf{K4}}$ and $R_{\mathbf{S4}}$ on W_{GL} are defined as follows.

(1) $uR_{\mathbf{K}}v$, iff $\Box B \in \mathrm{a}(u)$ implies $B \in \mathrm{a}(v)$ for every B.

(2) $uR_{\mathbf{K4}}v$, iff $\Box B \in \mathrm{a}(u)$ implies $B, \Box B \in \mathrm{a}(v)$ for every B.

(3) $uR_{\mathbf{S4}}v$, iff $\Box B \in \mathrm{a}(u)$ implies $\Box B \in \mathrm{a}(v)$ for every B.

Proposition 2.2. *For a sequent calculus GL, the following equivalences hold for every A.*

(1) *The inference (K) is admissible in GL for every Γ, iff for every u, $\Box A \in \mathrm{s}(u)$ implies $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{K}}v$.*

(2) *The inference (D) is admissible in GL for every Γ, iff $R_{\mathbf{K}}$ is serial.*

(3) *The inference (4) is admissible in GL for every Γ, iff for every u, $\Box A \in \mathrm{s}(u)$ implies $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{K4}}v$.*

(4) *The inference $(4D)$ is admissible in GL for every Γ, iff $R_{\mathbf{K4}}$ is serial.*

(5) *The inference $(S4)$ is admissible in GL for every Γ, iff for every u, $\Box A \in \mathrm{s}(u)$ implies $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{S4}}v$.*

(6) *The inference (T) is admissible in GL for every Γ and Θ, iff $\Box A \in \mathrm{a}(u)$ implies $A \in \mathrm{a}(u)$ for every u.*

Proof. (1) *The 'if' part*: Suppose that $\Box\Gamma \to \Box A$ is unprovable. Then, by Lemma 1.3, $\Box\Gamma \subseteq \mathrm{a}(u)$ and $\Box A \in \mathrm{s}(u)$ for some u. So $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{K}}v$ by the assumption. If $B \in \Gamma$, then $\Box B \in \Box\Gamma \subseteq \mathrm{a}(u)$, so $B \in \mathrm{a}(v)$ by $uR_{\mathbf{K}}v$; hence $\Gamma \subseteq \mathrm{a}(v)$, and so $\Gamma \to A$ is unprovable. *The 'only if' part*: Suppose $\Box A \in \mathrm{s}(u)$. Put $\Gamma = \{B | \Box B \in \mathrm{a}(u)\}$. Since $\Box\Gamma \subseteq \mathrm{a}(u)$ and $\Box A \in \mathrm{s}(u)$, it follows that $\Box\Gamma \to \Box A$ is unprovable; hence, neither is $\Gamma \to A$ by the assumption. So $\Gamma \subseteq \mathrm{a}(v)$ and $A \in \mathrm{s}(v)$ for some v by Lemma 1.3. It follows $uR_{\mathbf{K}}v$ from $\Gamma \subseteq \mathrm{a}(v)$.

(2)–(5) Similar to (1).

(6) Similar to Proposition 1.6 (1). □

Let $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$. Then, GL is complete with respect to the Kripke frame semantics for L by the following proposition. Moreover, for an unprovable sequent $\Gamma \to \Theta$ in GL, even if we limit the analytically saturated sequents to those u's such that $\mathrm{a}(u) \cup \mathrm{s}(u) \subseteq \mathrm{Sf}(\Gamma \cup \Theta)$, the following argument remains valid; hence, the finite model property for L also follows (Corollary 2.4).

Proposition 2.3. *Suppose* $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$ *and consider the Kripke frame* $\langle W_{GL}, R_L \rangle$, *where* $R_{\mathbf{KT}} = R_{\mathbf{KD}} = R_{\mathbf{K}}$ *and* $R_{\mathbf{K4D}} = R_{\mathbf{K4}}$.

(1) *The Kripke frame* $\langle W_{GL}, R_L \rangle$ *enjoys the properties* ($\Box$-**a**) *and* ($\Box$-**s**).

(2) *The accessibility relation* R_L *on* W_{GL} *meets the condition of the Kripke frames for* L.

Proof. *The case where* $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}\}$: (1) Immediate from the definition of $R_{\mathbf{K}}$ and Proposition 2.2 (1). (2) By Proposition 2.2 (6) and (2).

The case where $L \in \{\mathbf{K4}, \mathbf{K4D}\}$: (1) Immediate from the definition of $R_{\mathbf{K4}}$ and Proposition 2.2 (3). (2) By Proposition 2.2 (4).

The case where $L = \mathbf{S4}$: (1) If $\Box A \in \mathrm{a}(u)$ and $uR_{\mathbf{S4}}v$, then $\Box A \in \mathrm{a}(v)$, and so $A \in \mathrm{a}(v)$ by Proposition 2.2 (6); hence ($\Box$-**a**) holds. The property ($\Box$-**s**) follows from Proposistion 2.2 (5). (2) Immediate from the definition of $R_{\mathbf{S4}}$. □

Corollary 2.4. *Suppose* $L \in \{\mathbf{K}, \mathbf{KT}, \mathbf{KD}, \mathbf{K4}, \mathbf{K4D}, \mathbf{S4}\}$. *The sequent calculus* GL *and so the logic* L *are complete with respect to the Kripke frame* *semantics, and have the finite model property. The calculus GL is cut-free and has the subformula property.*

Remark 2.5. A sequent calculus does not necessarily has the subformula property, even if it is cut-free. For example, the sequent calculus that is obtained from $G\mathbf{KT}$ by adding the inference rule

$$\frac{\Box\Box A, \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}$$

is cut-free and is complete with respect to the Kripke frame semantics for **S4**, but does not have the subformula property.

3. The logics K45 and K45D

This section concerns the logics in Class 2, namely the logics **K45** and **K45D**. By the Kripke frames made of the analytically saturated sequents in a rather complicated manner, the inference rules that are added to the sequent calculi for these logics are analyzed semantically. Similar properties of the sequent calculi and the logics as the previous section follow. The sequent calculi are cut-free and have the subformula property naturally.

Let (45) and (45D) be the following inference rules:

$$(45) \quad \frac{\Gamma, \Box\Gamma \to \Box\Theta, A}{\Box\Gamma \to \Box\Theta, \Box A} . \quad (45D) \quad \frac{\Gamma, \Box\Gamma \to \Box\Theta}{\Box\Gamma \to \Box\Theta} .$$

Let $L \in \{\mathbf{K45}, \mathbf{K45D}\}$. The additional inference rules, besides those in Stipulations 1 and 2, of the sequent calculus GL as well as the condition on the accessibility relations of the Kripke frames for the logic L are described in the following table.

Logic	Additional rules	Condition on relations
K45	(45)	transitive and euclidean
K45D	(45), (45D)	transitive, euclidean and serial

Remember that a binary relation R on a set W is *euclidean*, iff uRv and uRw imply vRw for every $u, v, w \in W$.

To deal with the logics **K45** and **K45D**, the notion of maximality of analytically saturated sequents is needed.

Definition 3.1. For a sequent calculus GL, the binary relations $R'_{\mathbf{S4}}$ and $R_{\mathbf{S5}}$ are defined as follows.

(1) $uR'_{\mathbf{S4}}v$, iff $\Box B \in \mathrm{s}(u)$ implies $\Box B \in \mathrm{s}(v)$ for every B.

(2) $uR_{\mathbf{S5}}v$, iff $uR_{\mathbf{S4}}v$, $uR'_{\mathbf{S4}}v$, $vR_{\mathbf{S4}}u$ and $vR'_{\mathbf{S4}}u$.

Definition 3.2. An analytically saturated sequent u is called *maximal*, iff $uR_{\mathbf{S4}}v$, $uR'_{\mathbf{S4}}v$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\{\Box B | \Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)\})$ imply $uR_{\mathbf{S5}}v$, for every v.

It is easy to see that, if u is maximal and $uR_{\mathbf{S5}}v$, then v is also maximal. The property (1.3-c) is used to obtain the following lemma, which is an immediate corollary to Lemma 1.3.

Lemma 3.3. *Let GL be a sequent calculus. For every analytically saturated sequent u, there is an analytically saturarted sequent u^* such that $uR_{\mathbf{S4}}u^*$, $uR'_{\mathbf{S4}}u^*$, $\mathrm{a}(u^*) \cup \mathrm{s}(u^*) \subseteq \mathrm{Sf}(\{\Box B | \Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)\})$ and u^* is maximal.*

Proof. Given u, apply Lemma 1.3 to the sequent $\{\Box B | \Box B \in \mathrm{a}(u)\} \rightarrow \{\Box B | \Box B \in \mathrm{s}(u)\}$. □

For each u, we fix u^* described in the above lemma, in the rest of this section. If u is maximal, $uR_{\mathbf{S5}}u^*$.

Definition 3.4. For a sequent calculus GL, the binary relation $R_{\mathbf{K45}}$ on W_{GL} is defined by: $uR_{\mathbf{K45}}v$, iff $u^*R_{\mathbf{S5}}v$ and $vR_{\mathbf{K}}v$.

Proposition 3.5. *For a sequent calculus GL, the following equivalences hold for every A.*

(1) *The inference* (45) *is admissible in GL for every Γ and Θ, iff for every u, $\Box A \in \mathrm{s}(u)$ implies $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{K45}}v$.*

(2) *The inference* (45D) *is admissible in GL for every Γ and Θ, iff $R_{\mathbf{K45}}$ is serial.*

Proof. Since the proof of (2) is similar to that of (1), we will confine ourselves to the proof of (1).

The 'if' part: Suppose that $\Box\Gamma \rightarrow \Box\Theta, \Box A$ is unprovable. By Lemma 1.3, $\Box\Gamma \subseteq \mathrm{a}(u)$, $\Box\Theta \subseteq \mathrm{s}(u)$ and $\Box A \in \mathrm{s}(u)$ for some u. So $A \in \mathrm{s}(v)$ for some

v such that $uR_{\mathbf{K45}}v$ by the assumption. If $B \in \Gamma$, then $\Box B \in \Box\Gamma \subseteq \mathrm{a}(u)$, so $\Box B \in \mathrm{a}(v)$ by $uR_{\mathbf{S4}}u^*$ and $u^*R_{\mathbf{S4}}v$, and so $B \in \mathrm{a}(v)$ by $vR_{\mathbf{K}}v$; hence $\Gamma \cup \Box\Gamma \subseteq \mathrm{a}(v)$. If $B \in \Theta$, on the other hand, $\Box B \in \Box\Theta \subseteq \mathrm{s}(u)$, so $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}u^*$ and $u^*R'_{\mathbf{S4}}v$; hence $\Box\Theta \subseteq \mathrm{s}(v)$. So $\Gamma, \Box\Gamma \to \Box\Theta, A$ is unprovable.

The 'only if' part: Suppose $\Box A \in \mathrm{s}(u)$. Then $\Box A \in \mathrm{s}(u^*)$ by $uR'_{\mathbf{S4}}u^*$. Put $\Gamma = \{B | \Box B \in \mathrm{a}(u^*)\}$ and $\Theta = \{B | \Box B \in \mathrm{s}(u^*)\}$. Since $\Box\Gamma \to \Box\Theta, \Box A$ is unprovable, neither is $\Gamma, \Box\Gamma \to \Box\Theta, A$ by the assumption. So, $\Gamma \cup \Box\Gamma \subseteq \mathrm{a}(v)$, $\Box\Theta \subseteq \mathrm{s}(v)$, $A \in \mathrm{s}(v)$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\Gamma \cup \Box\Gamma \cup \Box\Theta \cup \{A\})$ for some v by Lemma 1.3. We will prove $uR_{\mathbf{K45}}v$ by showing $u^*R_{\mathbf{S5}}v$ and $vR_{\mathbf{K}}v$ in order. First, if $\Box B \in \mathrm{a}(u^*)$, then $B \in \Gamma$, so $\Box B \in \Box\Gamma \subseteq \mathrm{a}(v)$; while, if $\Box B \in \mathrm{s}(u^*)$, then $B \in \Theta$, so $\Box B \in \Box\Theta \subseteq \mathrm{s}(v)$; hence $u^*R_{\mathbf{S4}}v$ and $u^*R'_{\mathbf{S4}}v$. Moreover,

$$\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\Gamma \cup \Box\Gamma \cup \Box\Theta \cup \{A\}) = \mathrm{Sf}(\{\Box B | \Box B \in \mathrm{a}(u^*) \cup \mathrm{s}(u^*)\}).$$

Since u^* is maximal, it follows $u^*R_{\mathbf{S5}}v$. Next, to show $vR_{\mathbf{K}}v$, suppose $\Box B \in \mathrm{a}(v)$. Then $\Box B \in \mathrm{a}(u^*)$, since $u^*R_{\mathbf{S5}}v$ implies $vR_{\mathbf{S4}}u^*$. Hence $B \in \Gamma \subseteq \mathrm{a}(v)$. So $vR_{\mathbf{K}}v$ also holds. □

By the way, the analytically saturated sequent v introduced in the proof of the 'only if' part is maximal.

Proposition 3.6. *Suppose $L \in \{\mathbf{K45}, \mathbf{K45D}\}$ and consider the Kripke frame $\langle W_{GL}, R_{\mathbf{K45}}\rangle$.*

(1) *The Kripke frame $\langle W_{GL}, R_{\mathbf{K45}}\rangle$ enjoys the properties* (□-**a**) *and* (□-**s**).

(2) *The accessibility relation $R_{\mathbf{K45}}$ on W_{GL} meets the condition of the Kripke frames for L.*

Proof. (1) If $\Box A \in \mathrm{a}(u)$ and $uR_{\mathbf{K45}}v$, then $\Box A \in \mathrm{a}(v)$ by $uR_{\mathbf{S4}}u^*$ and $u^*R_{\mathbf{S4}}v$, so $A \in \mathrm{a}(v)$ by $vR_{\mathbf{K}}v$; hence (□-**a**) holds. The property (□-**s**) follows from Proposition 3.5 (1).

(2) To show first that $R_{\mathbf{K45}}$ is transitive, suppose that $uR_{\mathbf{K45}}v$ and $vR_{\mathbf{K45}}w$. Since u^* is maximal and $u^*R_{\mathbf{S5}}v$, it follows that v is also maximal, and so $vR_{\mathbf{S5}}v^*$. This together with $u^*R_{\mathbf{S5}}v$ and $v^*R_{\mathbf{S5}}w$ implies $u^*R_{\mathbf{S5}}w$. In addtion, $wR_{\mathbf{K}}w$ by $vR_{\mathbf{K45}}w$. So $uR_{\mathbf{K45}}w$. To show next that $R_{\mathbf{K45}}$ is euclidean, suppose that $uR_{\mathbf{K45}}v$ and $uR_{\mathbf{K45}}w$. We obtain $vR_{\mathbf{S5}}v^*$ as above. This together with $u^*R_{\mathbf{S5}}v$ and $u^*R_{\mathbf{S5}}w$ implies $v^*R_{\mathbf{S5}}w$. In

addition, $wR_{\mathbf{K}}w$ by $uR_{\mathbf{K45}}w$. So $vR_{\mathbf{K45}}w$. When $L = \mathbf{K45D}$, the fact that the relation $R_{\mathbf{K45}}$ is serial follows from Proposition 3.5 (2). □

Corollary 3.7. *Suppose $L \in \{\mathbf{K45}, \mathbf{K45D}\}$. The sequent calculus GL and so the logic L are complete with respect to the Kripke frame semantics, and have the finite model property. The calculus GL is cut-free and has the subformula property.*

4. The logics KB, KTB, KDB, K4B and S5

This section concerns the logics in Class 3, namely the logics **KB**, **KTB**, **KDB**, **K4B** and **S5**. Similarly to Section 2, the inference rules that are added to the sequent calculi for these logics are analyzed semantically. As a result, completeness as well as the finite model property of the calculi and logics follow. The sequent calculi are not cut-free but have the subformula property naturally.

Consider the following inference rules:

$$(\mathbf{cut})^{\mathrm{a}} \quad \frac{\Gamma \to \Theta, C \quad C, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}, \ \textit{where } C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda).$$

$$(B)^{\mathrm{a}} \quad \frac{\Gamma \to \Box\Omega, A}{\Box\Gamma \to \Omega, \Box A}, \ \textit{where } \Box\Omega \subseteq \mathrm{Sf}(\Gamma \cup \{A\}).$$

$$(BD)^{\mathrm{a}} \quad \frac{\Gamma \to \Box\Omega}{\Box\Gamma \to \Omega}, \ \textit{where } \Box\Omega \subseteq \mathrm{Sf}(\Gamma).$$

$$(B45)^{\mathrm{a}} \quad \frac{\Gamma, \Box\Gamma \to \Box\Theta, \Box\Omega, A}{\Box\Gamma \to \Box\Theta, \Omega, \Box A}, \ \textit{where } \Box\Omega \subseteq \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\}).$$

$$(S5) \quad \frac{\Box\Gamma \to \Box\Theta, A}{\Box\Gamma \to \Box\Theta, \Box A}.$$

The additional inference rules, besides those in Stipulations 1 and 2, of the sequent calculus GL as well as the condition on the accessibility relations of the Kripke frames for the logic L are described in the following table, where $L \in \{\mathbf{KB}, \mathbf{KTB}, \mathbf{KDB}, \mathbf{K4B}, \mathbf{S5}\}$.

Logic	Additional rules	Condition on relations
KB	$(\mathbf{cut})^{\mathrm{a}}, (B)^{\mathrm{a}}$	symmetric
KTB	$(\mathbf{cut})^{\mathrm{a}}, (B)^{\mathrm{a}}, (T)$	symmetric and reflexive
KDB	$(\mathbf{cut})^{\mathrm{a}}, (B)^{\mathrm{a}}, (BD)^{\mathrm{a}}$	symmetric and serial
K4B	$(\mathbf{cut})^{\mathrm{a}}, (B45)^{\mathrm{a}}$	symmetric and transitive
S5	$(\mathbf{cut})^{\mathrm{a}}, (S5), (T)$	equivalence relation

Proposition 4.1. *For a sequent calculus GL, the inference* $(\mathbf{cut})^{\mathrm{a}}$ *is admissible for every* $\Gamma, \Theta, \Delta, \Lambda$ *and* C *with the restriction that* $C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$*, iff* $\mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$ *for every* u*.*

Proof. *The 'if' part*: Suppose that $\Gamma, \Delta \to \Theta, \Lambda$ is unprovable, and $C \in \mathrm{Sf}(\Gamma \cup \Theta \cup \Delta \cup \Lambda)$. By Lemma 1.3, $\Gamma \cup \Delta \subseteq \mathrm{a}(u)$ and $\Theta \cup \Lambda \subseteq \mathrm{s}(u)$ for some u. Since $C \in \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u))$, it follows $C \in \mathrm{a}(u) \cup \mathrm{s}(u)$ by the assumption. So, either $C, \Gamma \to \Theta$ or $\Delta \to \Lambda, C$ is unprovable, according to whether $C \in \mathrm{a}(u)$ or $C \in \mathrm{s}(u)$.

The 'only if' part: Suppose $C \in \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u))$. Then either $\mathrm{a}(u) \to \mathrm{s}(u), C$ or $C, \mathrm{a}(u) \to \mathrm{s}(u)$ is unprovable; for, if both were provable, $\mathrm{a}(u) \to \mathrm{s}(u)$ would be also provable by the assumption, which is a contradiction. Hence, either $C \in \mathrm{s}(u)$ or $C \in \mathrm{a}(u)$. □

Definition 4.2. For a sequent calculus GL, the binary relations $R_{\mathbf{KB}}$ and $R_{\mathbf{K4B}}$ on W_{GL} are defined as follows.

(1) $uR_{\mathbf{KB}}v$, iff $uR_{\mathbf{K}}v$ and $vR_{\mathbf{K}}u$.

(2) $uR_{\mathbf{K4B}}v$, iff $uR_{\mathbf{K4}}v$, $uR'_{\mathbf{S4}}v$, $vR_{\mathbf{K4}}u$ and $vR'_{\mathbf{S4}}u$.

Proposition 4.3. *For a sequent calculus GL with the inference rule* $(\mathbf{cut})^{\mathrm{a}}$*, the following equivalences hold for every* A*.*

(1) *The inference* $(B)^{\mathrm{a}}$ *is admissible in GL for every* Γ *and* Ω *with the restriction that* $\Box\Omega \subseteq \mathrm{Sf}(\Gamma \cup \{A\})$*, iff for every* u*,* $\Box A \in \mathrm{s}(u)$ *implies* $A \in \mathrm{s}(v)$ *for some* v *such that* $uR_{\mathbf{KB}}v$*.*

(2) *The inference* $(BD)^{\mathrm{a}}$ *is admissible in GL for every* Γ *and* Ω *with the restriction that* $\Box\Omega \subseteq \mathrm{Sf}(\Gamma)$*, iff* $R_{\mathbf{KB}}$ *is serial.*

(3) *The inference* $(B45)^{\mathrm{a}}$ *is admissible in GL for every* Γ*,* Θ *and* Ω *with the restriction that* $\Box\Omega \subseteq \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\})$*, iff for every* u*,* $\Box A \in \mathrm{s}(u)$ *implies* $A \in \mathrm{s}(v)$ *for some* v *such that* $uR_{\mathbf{K4B}}v$*.*

(4) *The inference* ($S5$) *is admissible in GL for every* Γ *and* Θ*, iff for every* u*,* $\Box A \in \mathrm{s}(u)$ *implies* $A \in \mathrm{s}(v)$ *for some* v *such that* $uR_{\mathbf{S5}}v$*.*

Proof. *Proof of* (1). *The 'if' part*: Suppose that $\Box\Gamma \to \Omega, \Box A$ is unprovable, and $\Box\Omega \subseteq \mathrm{Sf}(\Gamma \cup \{A\})$. By Lemma 1.3, $\Box\Gamma \subseteq \mathrm{a}(u)$, $\Omega \subseteq \mathrm{s}(u)$ and $\Box A \in \mathrm{s}(u)$ for some u. So $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{KB}}v$ by the assumption. If $B \in \Gamma$, then $\Box B \in \Box\Gamma \subseteq \mathrm{a}(u)$, so $B \in \mathrm{a}(v)$ by $uR_{\mathbf{K}}v$; hence $\Gamma \subseteq \mathrm{a}(v)$. So, it suffices to show $\Box\Omega \subseteq \mathrm{s}(v)$; for, $\Gamma \subseteq \mathrm{a}(v)$, $\Box\Omega \subseteq \mathrm{s}(v)$ and $A \in \mathrm{s}(v)$ imply that $\Gamma \to \Box\Omega, A$ is unprovable. Suppose $B \in \Omega$, and we will show $\Box B \in \mathrm{s}(v)$. By Proposition 4.1, $\Box B \in \Box\Omega \subseteq \mathrm{Sf}(\Gamma \cup \{A\}) \subseteq \mathrm{Sf}(\mathrm{a}(v) \cup \mathrm{s}(v)) \subseteq \mathrm{a}(v) \cup \mathrm{s}(v)$. If $\Box B$ were in $\mathrm{a}(v)$, then $B \in \mathrm{a}(u)$ would follow by $vR_{\mathbf{K}}u$, while $B \in \Omega \subseteq \mathrm{s}(u)$, which is a contradiction; hence $\Box B \notin \mathrm{a}(v)$. So $\Box B \in \mathrm{s}(v)$.

The 'only if' part: Suppose $\Box A \in \mathrm{s}(u)$. Put $\Gamma = \{B | \Box B \in \mathrm{a}(u)\}$ and $\Omega = \{B \in \mathrm{s}(u) | \Box B \in \mathrm{Sf}(\Gamma \cup \{A\})\}$. Since $\Box\Omega \subseteq \mathrm{Sf}(\Gamma \cup \{A\})$ but $\Box\Gamma \to \Omega, \Box A$ is unprovable, neither is $\Gamma \to \Box\Omega, A$ by the assumption. So, $\Gamma \subseteq \mathrm{a}(v)$, $\Box\Omega \subseteq \mathrm{s}(v)$, $A \in \mathrm{s}(v)$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\Gamma \cup \Box\Omega \cup \{A\})$ for some v by Lemma 1.3. It follows $uR_{\mathbf{K}}v$ from $\Gamma \subseteq \mathrm{a}(v)$. So, it suffices to show $vR_{\mathbf{K}}u$; for, then $uR_{\mathbf{KB}}v$. Suppose $\Box B \in \mathrm{a}(v)$. We will show $B \in \mathrm{a}(u)$. It follows $\Box B \in \mathrm{Sf}(\Gamma \cup \Box\Omega \cup \{A\}) = \mathrm{Sf}(\Gamma \cup \{A\})$. So $B \in \mathrm{Sf}(\Gamma \cup \{A\}) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$ by Proposition 4.1. If B were in $\mathrm{s}(u)$, then $B \in \Omega$ and so $\Box B \in \mathrm{s}(v)$ would follow, which is a contradiction; hence $B \notin \mathrm{s}(u)$. So $B \in \mathrm{a}(u)$.

Proof of (2). Similar to (1).

Proof of (3). *The 'if' part*: Suppose that $\Box\Gamma \to \Box\Theta, \Omega, \Box A$ is unprovable, and $\Box\Omega \subseteq \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\})$. By Lemma 1.3, $\Box\Gamma \subseteq \mathrm{a}(u)$, $\Box\Theta \cup \Omega \subseteq \mathrm{s}(u)$ and $\Box A \in \mathrm{s}(u)$ for some u. So $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{K4B}}v$ by the assumption. If $B \in \Gamma$, then $\Box B \in \Box\Gamma \subseteq \mathrm{a}(u)$, so $B, \Box B \in \mathrm{a}(v)$ by $uR_{\mathbf{K4}}v$; hence $\Gamma \cup \Box\Gamma \subseteq \mathrm{a}(v)$. If $B \in \Theta$, then $\Box B \in \Box\Theta \subseteq \mathrm{s}(u)$, so $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$; hence $\Box\Theta \subseteq \mathrm{s}(v)$. So, it is left to show $\Box\Omega \subseteq \mathrm{s}(v)$; for, $\Gamma \cup \Box\Gamma \subseteq \mathrm{a}(v)$, $\Box\Theta \cup \Box\Omega \subseteq \mathrm{s}(v)$ and $A \in \mathrm{s}(v)$ imply that $\Gamma, \Box\Gamma \to \Box\Theta, \Box\Omega, A$ is unprovable. Suppose $B \in \Omega$. We will show $\Box B \in \mathrm{s}(v)$. It follows

$$\Box B \in \Box\Omega \subseteq \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\}) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$$

by Proposition 4.1. Now, suppose temporarily that $\Box B \in \mathrm{a}(u)$. It follows $\Box B \in \mathrm{a}(v)$ by $uR_{\mathbf{K4}}v$, so $B \in \mathrm{a}(u)$ by $vR_{\mathbf{K4}}u$. But $B \in \Omega \subseteq \mathrm{s}(u)$, which is a contradiction. Hence $\Box B \notin \mathrm{a}(u)$, so $\Box B \in \mathrm{s}(u)$, and so $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$.

The 'only if' part: Suppose $\Box A \in \mathrm{s}(u)$. Put $\Gamma = \{B|\Box B \in \mathrm{a}(u)\}$, $\Theta = \{B|\Box B \in \mathrm{s}(u)\}$ and $\Omega = \{B \in \mathrm{s}(u)|\Box B \in \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\})\}$. Since $\Box\Omega \subseteq \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\})$ but $\Box\Gamma \to \Box\Theta, \Omega, \Box A$ is unprovable, neither is $\Gamma, \Box\Gamma \to \Box\Theta, \Box\Omega, A$ by the assumption. So, $\Gamma \cup \Box\Gamma \subseteq \mathrm{a}(v)$, $\Box\Theta \cup \Box\Omega \subseteq \mathrm{s}(v)$, $A \in \mathrm{s}(v)$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\Gamma \cup \Box\Gamma \cup \Box\Theta \cup \Box\Omega \cup \{A\})$ for some v by Lemma 1.3. It remains to show $uR_{\mathbf{K4B}}v$, which can be obtained by showing $uR_{\mathbf{K4}}v$, $uR'_{\mathbf{S4}}v$, $vR_{\mathbf{K4}}u$ and $vR'_{\mathbf{S4}}u$ successively. If $\Box B \in \mathrm{a}(u)$, then $B \in \Gamma$, so $B, \Box B \in \Gamma \cup \Box\Gamma \subseteq \mathrm{a}(v)$; hence $uR_{\mathbf{K4}}v$. If $\Box B \in \mathrm{s}(u)$, then $B \in \Theta$, so $\Box B \in \Box\Theta \subseteq \mathrm{s}(v)$; hence $uR'_{\mathbf{S4}}v$. To show $vR_{\mathbf{K4}}u$ next, suppose $\Box B \in \mathrm{a}(v)$. We will show $B, \Box B \in \mathrm{a}(u)$. It is remarked that $B \notin \Theta$; for, if B were in Θ, then $\Box B \in \Box\Theta \subseteq \mathrm{s}(v)$ would follow, which is a contradiction. Since $\Box B \in \mathrm{Sf}(\Gamma \cup \Box\Gamma \cup \Box\Theta \cup \Box\Omega \cup \{A\}) = \mathrm{Sf}(\Box\Gamma \cup \Box\Theta \cup \{A\})$, it follows $\Box B \in \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\})$. So,

$$B, \Box B \in \mathrm{Sf}(\Box\Gamma \cup \Theta \cup \{A\}) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$$

by Proposition 4.1. If B were in $\mathrm{s}(u)$, then $B \in \Omega$ and so $\Box B \in \Box\Omega \subseteq \mathrm{s}(v)$ would follow, which is a contradiction; hence $B \notin \mathrm{s}(u)$. Moreover, $\Box B \notin \mathrm{s}(u)$, since $B \notin \Theta$. Hence $B, \Box B \in \mathrm{a}(u)$. To show $vR'_{\mathbf{S4}}u$ last of all, suppose $\Box B \in \mathrm{s}(v)$. We will show $\Box B \in \mathrm{s}(u)$. Similarly to the above,

$$\Box B \in \mathrm{Sf}(\Gamma \cup \Box\Gamma \cup \Box\Theta \cup \Box\Omega \cup \{A\}) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u).$$

If $\Box B$ were in $\mathrm{a}(u)$, then $B \in \Gamma$ and so $\Box B \in \Box\Gamma \subseteq \mathrm{a}(v)$ would follow, which is a contradiction; hence $\Box B \notin \mathrm{a}(u)$. So $\Box B \in \mathrm{s}(u)$.

Proof of (4). *The 'if' part*: Suppose that $\Box\Gamma \to \Box\Theta, \Box A$ is unprovable. By Lemma 1.3, $\Box\Gamma \subseteq \mathrm{a}(u)$, $\Box\Theta \subseteq \mathrm{s}(u)$ and $\Box A \in \mathrm{s}(u)$ for some u. So $A \in \mathrm{s}(v)$ for some v such that $uR_{\mathbf{S5}}v$ by the assumption. If $B \in \Gamma$, then $\Box B \in \Box\Gamma \subseteq \mathrm{a}(u)$, so $\Box B \in \mathrm{a}(v)$ by $uR_{\mathbf{S4}}v$; hence $\Box\Gamma \subseteq \mathrm{a}(v)$. If $B \in \Theta$, then $\Box B \in \Box\Theta \subseteq \mathrm{s}(u)$, so $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$; hence $\Box\Theta \subseteq \mathrm{s}(v)$. So $\Box\Gamma \to \Box\Theta, A$ is unprovable.

The 'only if' part: Suppose $\Box A \in \mathrm{s}(u)$. Put $\Gamma = \{B|\Box B \in \mathrm{a}(u)\}$ and $\Theta = \{B|\Box B \in \mathrm{s}(u)\}$. Since $\Box\Gamma \to \Box\Theta, \Box A$ is unprovable, neither is $\Box\Gamma \to \Box\Theta, A$ by the assumption. So, $\Box\Gamma \subseteq \mathrm{a}(v)$, $\Box\Theta \subseteq \mathrm{s}(v)$, $A \in \mathrm{s}(v)$ and $\mathrm{a}(v) \cup \mathrm{s}(v) \subseteq \mathrm{Sf}(\Box\Gamma \cup \Box\Theta \cup \{A\})$ for some v by Lemma 1.3. We will prove $uR_{\mathbf{S5}}v$ by showing $uR_{\mathbf{S4}}v$, $uR'_{\mathbf{S4}}v$, $vR_{\mathbf{S4}}u$ and $vR'_{\mathbf{S4}}u$ successively. If

$\Box B \in \mathrm{a}(u)$, then $B \in \Gamma$, so $\Box B \in \Box\Gamma \subseteq \mathrm{a}(v)$; hence $uR_{\mathbf{S4}}v$. If $\Box B \in \mathrm{s}(u)$, then $B \in \Theta$, so $\Box B \in \Box\Theta \subseteq \mathrm{s}(v)$; hence $uR'_{\mathbf{S4}}v$. If $\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v)$, then

$$\Box B \in \mathrm{Sf}(\Box\Gamma \cup \Box\Theta \cup \{A\}) \subseteq \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$$

by Proposition 4.1. So, if $\Box B \in \mathrm{a}(v)$, then $\Box B \in \mathrm{a}(u)$; for, otherwise, $\Box B \in \mathrm{s}(u)$ and so $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$, which is a contradiction; hence $vR_{\mathbf{S4}}u$. If $\Box B \in \mathrm{s}(v)$ then $\Box B \in \mathrm{s}(u)$; for, otherwise, $\Box B \in \mathrm{a}(u)$ and so $\Box B \in \mathrm{a}(v)$ by $uR_{\mathbf{S4}}v$, which is a contradiction; hence $vR'_{\mathbf{S4}}u$. □

Proposition 4.4. *Suppose* $L \in \{\mathbf{KB}, \mathbf{KTB}, \mathbf{KDB}, \mathbf{K4B}, \mathbf{S5}\}$ *and consider the Kripke frame* $\langle W_{GL}, R_L\rangle$, *where* $R_{\mathbf{KTB}} = R_{\mathbf{KDB}} = R_{\mathbf{KB}}$.

(1) *The Kripke frame* $\langle W_{GL}, R_L\rangle$ *enjoys the properties* ($\Box$-**a**) *and* ($\Box$-**s**).

(2) *The accessibility relation* R_L *on* W_{GL} *meets the condition of the Kripke frames for* L.

Proof. *The case where* $L \in \{\mathbf{KB}, \mathbf{KTB}, \mathbf{KDB}\}$: By the definition of $R_{\mathbf{KB}}$, Propositions 4.3 (1), 2.2 (6) and 4.3 (2). *The case where* $L = \mathbf{K4B}$: By the definition of $R_{\mathbf{K4B}}$ and Proposition 4.3 (3). *The case where* $L = \mathbf{S5}$: By the definition of $R_{\mathbf{S5}}$, Propositions 2.2 (6) and 4.3 (4). □

Corollary 4.5. *Suppose* $L \in \{\mathbf{KB}, \mathbf{KTB}, \mathbf{KDB}, \mathbf{K4B}, \mathbf{S5}\}$. *The sequent calculus GL and so the logic L are complete with respect to the Kripke frame semantics, and have the finite model property. The calculus GL is not cut-free but has the subformula property.*

5. The logics K5 and K5D

This section concerns the logics in Class 4, namely the logics **K5** and **K5D**. By the Kripke frames made of specific analytically saturated sequents, the inference rules that are added to the sequent calculi for these logics are analyzed semantically. As a result, completeness as well as the finite model property of the calculi and logics follow. The sequent calculi are not cut-free and lack the subformula property in the original sense, but enjoy an extended subformula property.

Since the sequent calculi $G\mathbf{K5}$ and $G\mathbf{K5D}$ lack the subformula property in the original sense, we extend the notion of subformula (cf. Takano [5]).

Definition 5.1. (1) An *internal subformula* of A is a subformula of some formula C such that $\Box C$ is a subformula of A.

(2) A **K5**-*subformula* of A is either a subformula of A or the formula of the form $\Box\neg\Box B$ or $\neg\Box B$, where $\Box B$ is an internal subformula of A.

The sets of all the internal subformulas and **K5**-subformulas of some formulas in Γ are denoted by $\mathrm{InSf}(\Gamma)$ and $\mathrm{Sf}_{\mathbf{K5}}(\Gamma)$, respectively.

If $\Box A$ is an internal subformula of B, and B is a **K5**-subformula of C, then $\Box A$ is an internal subformula of C. If A is a **K5**-subformula of B, and B is a **K5**-subformula of C, then A is a **K5**-subformula of C.

Let $(\mathbf{cut})^5$, (5) and $(5D)$ be the following inference rules:

$$(\mathbf{cut})^5 \quad \frac{\Gamma \to \Theta, C \quad C, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\ , \ \textit{where}\ C \in \mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta \cup \Delta \cup \Lambda).$$

$$(5) \quad \frac{\Gamma \to \Box\Theta, A}{\Box\Gamma \to \Box\Theta, \Box A}\ . \quad (5D) \quad \frac{\Gamma \to \Box\Theta}{\Box\Gamma \to \Box\Theta}\ .$$

The additional inference rules, besides those in Stipulations 1 and 2, of the sequent calculus GL as well as the condition on the accessibility relations of the Kripke frames for the logic L are described in the following table, where $L \in \{\mathbf{K5}, \mathbf{K5D}\}$.

Logic	Additional rules	Condition on relations
K5	$(\mathbf{cut})^5$, (5)	euclidean
K5D	$(\mathbf{cut})^5$, (5), $(5D)$	euclidean and serial

The inference $(\mathbf{cut})^5$ can be characterized quite similarly to Proposition 4.1, provided that Definition 1.1 of analytical saturation is modified so as to concern the K5-subformulas, yet we don't involve ourselves in the modified one here. Then, it is to be remarked that, even if GL contains $(\mathbf{cut})^5$, and if u is an analytically saturated sequent in GL (in the original sense of Definition 1.1), it is not always the case that $\mathrm{Sf}_{\mathbf{K5}}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$. For, suppose $A \in \mathrm{Sf}_{\mathbf{K5}}(\mathrm{a}(u) \cup \mathrm{s}(u))$. Then, either $\mathrm{a}(u) \to \mathrm{s}(u), A$ or $A, \mathrm{a}(u) \to \mathrm{s}(u)$ is unprovable by $(\mathbf{cut})^5$. But, to apply the property (1.1-b), A's being in $\mathrm{Sf}_{\mathbf{K5}}(\mathrm{a}(u) \cup \mathrm{s}(u))$ is insufficient, but it is necessary that $A \in \mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u))$. It is always the case that $\mathrm{Sf}(\mathrm{a}(u) \cup \mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$ instead (cf. Proposition 4.1).

Definition 5.2. For a sequent calculus GL, $W^{\star}_{GL}$ is defined to be the set of all the analytically saturated sequents u's in GL that satisfy the following property:

(5.2-a) For every B, if $\Box B \in \mathrm{InSf}(\mathrm{a}(u) \cup \mathrm{s}(u))$ then either $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$.

Lemma 5.3. *For a sequent calculus GL with the inference rule* $(\mathbf{cut})^5$, *if* $\Gamma \to \Theta$ *is unprovable in GL, then* $\Gamma \subseteq \mathrm{a}(u)$ *and* $\Theta \subseteq \mathrm{s}(u)$ *for some* $u \in W^{\star}_{GL}$.

Proof. Let $A_1, A_2, \ldots, A_n$ be an enumeration of all the formulas in $\mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta)$. Put $\Gamma_1 = \Gamma$ and $\Theta_1 = \Theta$. Suppose that Γ_k and Θ_k have been defined so that $\Gamma \subseteq \Gamma_k,\ \Theta \subseteq \Theta_k$ but $\Gamma_k \to \Theta_k$ is unprovable $(1 \leq k \leq n)$. Then, either $\Gamma_k \to \Theta_k, A_k$ or $A_k, \Gamma_k \to \Theta_k$ is unprovable; for, if both were provable, since $A_k \in \mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta) \subseteq \mathrm{Sf}_{\mathbf{K5}}(\Gamma_k \cup \Theta_k)$, it would follow that $\Gamma_k \to \Theta_k$ is provable by $(\mathbf{cut})^5$, which contradicts our assumption. Hence, put $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k \cup \{A_k\}$, or $\Gamma_{k+1} = \Gamma_k \cup \{A_k\}$ and $\Theta_{k+1} = \Theta_k$ so that $\Gamma_{k+1} \to \Theta_{k+1}$ is also unprovable.

We claim that $\Gamma_{n+1} \to \Theta_{n+1}$ is the required $u \in W^{\star}_{GL}$. Evidently $\Gamma \subseteq \Gamma_{n+1}$, $\Theta \subseteq \Theta_{n+1}$, and $\Gamma_{n+1} \to \Theta_{n+1}$ is unprovable; moreover, $\Gamma_{n+1} \cup \Theta_{n+1} = \mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta)$. It is left to check the properties (1.1-b) and (5.2-a) for $\Gamma_{n+1} \to \Theta_{n+1}$. Let's show (5.2-a) first. So, suppose $\Box B \in \mathrm{InSf}(\Gamma_{n+1} \cup \Theta_{n+1})$. Since $\Box\neg\Box B \in \mathrm{Sf}_{\mathbf{K5}}(\Gamma_{n+1} \cup \Theta_{n+1})$ and

$$\mathrm{Sf}_{\mathbf{K5}}(\Gamma_{n+1} \cup \Theta_{n+1}) = \mathrm{Sf}_{\mathbf{K5}}(\mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta)) \subseteq \mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta) = \Gamma_{n+1} \cup \Theta_{n+1},$$

it follows $\Box\neg\Box B \in \Gamma_{n+1} \cup \Theta_{n+1}$. So (5.2-a) has been shown. To show (1.1-b) next, suppose $A \in \mathrm{Sf}(\Gamma_{n+1} \cup \Theta_{n+1})$. Then, since $\mathrm{Sf}(\Gamma_{n+1} \cup \Theta_{n+1}) \subseteq \mathrm{Sf}_{\mathbf{K5}}(\Gamma_{n+1} \cup \Theta_{n+1}) \subseteq \Gamma_{n+1} \cup \Theta_{n+1}$, it follows $A \in \Gamma_{n+1} \cup \Theta_{n+1}$. So, if $A, \Gamma_{n+1} \to \Theta_{n+1}$ is unprovable, then $A \notin \Theta_{n+1}$, and so $A \in \Gamma_{n+1}$. Similarly, if $\Gamma_{n+1} \to \Theta_{n+1}, A$ is unprovable, then $A \in \Theta_{n+1}$. Thus, (1.1-b) for $\Gamma_{n+1} \to \Theta_{n+1}$ has been also shown. □

Definition 5.4. For a sequent calculus GL, the binary relation $R_{\mathbf{K5}}$ on $W^{\star}_{GL}$ is defined as follows, by first introducing an auxiliary relation S.

(1) uSv, iff $\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v)$ implies $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$ for every B.

(2) $uR_{\mathbf{K5}}v$, iff $uR_{\mathbf{K}}v$, $uR'_{\mathbf{S4}}v$ and uSv.

Proposition 5.5. *For a sequent calculus GL with the inference rule* $(\mathbf{cut})^5$*, the following equivalences hold for every* A*.*

(1) *The inference* (5) *is admissible in GL for every* Γ *and* Θ*, iff for every* $u \in W^\star_{GL}$*,* $\Box A \in \mathrm{s}(u)$ *implies* $A \in \mathrm{s}(v)$ *for some* $v \in W^\star_{GL}$ *such that* $uR_{\mathbf{K5}}v$*.*

(2) *The inference* $(5D)$ *is admissible in GL for every* Γ *and* Θ*, iff* $R_{\mathbf{K5}}$ *is serial on* $W^\star_{GL}$*.*

Proof. Since the proof of (2) is similar to that of (1), we will confine ourselves to the proof of (1).

The 'if' part: Suppose that $\Box\Gamma \to \Box\Theta, \Box A$ is unprovable. By Lemma 5.3, $\Box\Gamma \subseteq \mathrm{a}(u)$, $\Box\Theta \subseteq \mathrm{s}(u)$ and $\Box A \in \mathrm{s}(u)$ for some $u \in W^\star_{GL}$. So $A \in \mathrm{s}(v)$ for some $v \in W^\star_{GL}$ such that $uR_{\mathbf{K5}}v$ by the assumption. If $B \in \Gamma$, then $\Box B \in \Box\Gamma \subseteq \mathrm{a}(u)$, so $B \in \mathrm{a}(v)$ by $uR_{\mathbf{K}}v$; hence $\Gamma \subseteq \mathrm{a}(v)$. If $B \in \Theta$, then $\Box B \in \Box\Theta \subseteq \mathrm{s}(u)$, so $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$; hence $\Box\Theta \subseteq \mathrm{s}(v)$. So, $\Gamma \to \Box\Theta, A$ is unprovable.

The 'only if' part: Suppose $\Box A \in \mathrm{s}(u)$, where $u \in W^\star_{GL}$. Put $\Gamma = \{B|\Box B \in \mathrm{a}(u)\}$ and $\Theta = \{B|\Box B \in \mathrm{s}(u)\}$. Since $\Box\Gamma \to \Box\Theta, \Box A$ is unprovable, neither is $\Gamma \to \Box\Theta, A$ by the assumption. So, $\Gamma \subseteq \mathrm{a}(v)$, $\Box\Theta \subseteq \mathrm{s}(v)$, $A \in \mathrm{s}(v)$ and $\mathrm{a}(v)\cup\mathrm{s}(v) \subseteq \mathrm{Sf}(\Gamma\cup\Box\Theta\cup\{A\})$ for some analytically saturated sequent v by Lemma 1.3. Since $uR_{\mathbf{K}}v$ and $uR'_{\mathbf{S4}}v$ follow from $\Gamma \subseteq \mathrm{a}(v)$ and $\Box\Theta \subseteq \mathrm{s}(v)$ respectively, it is left to check that the property (5.2-a) for v and the relation uSv hold. Let's show uSv first. Since

$$\begin{aligned}\mathrm{a}(v) \cup \mathrm{s}(v) &\subseteq \mathrm{Sf}(\Gamma \cup \Box\Theta \cup \{A\}) = \Box\Theta \cup \mathrm{Sf}(\Gamma \cup \Theta \cup \{A\}) \\ &\subseteq \Box\Theta \cup \mathrm{InSf}(\Box\Gamma \cup \Box\Theta \cup \{\Box A\}) \subseteq \mathrm{s}(u) \cup \mathrm{InSf}(\mathrm{a}(u) \cup \mathrm{s}(u)),\end{aligned}$$

if $\Box B \in \mathrm{a}(v) \cup \mathrm{s}(v)$, then either $\Box B \in \mathrm{s}(u)$ or $\Box B \in \mathrm{InSf}(\mathrm{a}(u) \cup \mathrm{s}(u))$; in the latter case, by (5.2-a) for u, either $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u) \cup \mathrm{s}(u)$. Thus uSv has been shown. Lastly, let us show (5.2-a) for v. So, suppose $\Box B \in \mathrm{InSf}(\mathrm{a}(v)\cup\mathrm{s}(v))$. Since $\mathrm{InSf}(\mathrm{a}(v)\cup\mathrm{s}(v)) \subseteq \mathrm{Sf}(\mathrm{a}(v)\cup\mathrm{s}(v)) \subseteq \mathrm{a}(v)\cup\mathrm{s}(v)$ by Proposition 4.1, it follows from uSv just shown that either $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u)$ or $\Box\neg\Box B \in \mathrm{s}(u)$. In the first (second, respectively) case, $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$ (by $uR_{\mathbf{K}}v$ and Proposition 1.6 (1), respectively). In the last case, $\Box\neg\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$. So, (5.2-a) for v has been also shown. $\Box$

By the following proposition together with Lemma 5.3, GL is complete with respect to the Kripke frame semantics for L, where $L \in \{\mathbf{K5}, \mathbf{K5D}\}$. So we have obtained another proof of the modified subformula property for GL, that is, if $\Gamma \to \Theta$ is valid in the sense of the Kripke frame semantics for L, it has a proof in GL such that every formula occurring in it belongs to $\mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta)$, which was first proved in Takano [5]. Moreover, for an unprovable sequent $\Gamma \to \Theta$ in GL, by limiting the analytically saturated sequents to those u's such that $\mathrm{a}(u) \cup \mathrm{s}(u) \subseteq \mathrm{Sf}_{\mathbf{K5}}(\Gamma \cup \Theta)$, the finite model property for L also follows (Corollary 5.7).

The following remark is useful in proving the next proposition: *If* $L \in \{\mathbf{K5}, \mathbf{K5D}\}$ *and* u *is analytically saturated in* GL*, then* $\Box\neg\Box B \in \mathrm{s}(u)$ *implies* $\Box B \in \mathrm{a}(u)$. For, since the sequent $\to \Box B, \Box\neg\Box B$ is provable in GL, it is not the case that both $\Box B$ and $\Box\neg\Box B$ are in $\mathrm{s}(u)$; moreover, $\Box\neg\Box B \in \mathrm{s}(u)$ implies $\Box B \in \mathrm{Sf}(\mathrm{s}(u)) \subseteq \mathrm{a}(u) \cup \mathrm{s}(u)$ by Proposition 4.1.

Proposition 5.6. *Suppose* $L \in \{\mathbf{K5}, \mathbf{K5D}\}$ *and consider the Kripke frame* $\langle W^\star_{GL}, R_{\mathbf{K5}}\rangle$.

(1) *The Kripke frame* $\langle W^\star_{GL}, R_{\mathbf{K5}}\rangle$ *enjoys the properties* ($\Box$–**a**) *and* ($\Box$–**s**).

(2) *The accessibility relation* $R_{\mathbf{K5}}$ *on* $W^\star_{GL}$ *meets the condition of the Kripke frames for* L.

Proof. (1) If $\Box A \in \mathrm{a}(u)$ and $uR_{\mathbf{K5}}v$, then $A \in \mathrm{a}(v)$ by $uR_{\mathbf{K}}v$; hence ($\Box$–**a**) holds. The property ($\Box$–**s**) follows from Proposition 5.5 (1).

(2) We will show that the relation $R_{\mathbf{K5}}$ is euclidean by deriving $vR_{\mathbf{K5}}w$ from $uR_{\mathbf{K5}}v$ and $uR_{\mathbf{K5}}w$. First, to show $vR_{\mathbf{K}}w$, suppose $\Box B \in \mathrm{a}(v)$. Since uSv, it follows that either $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u)$ or $\Box\neg\Box B \in \mathrm{s}(u)$ holds. In the first (second, respectively) case, $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$ (by $uR_{\mathbf{K}}v$ and Proposition 1.6 (1), respectively), and this contradicts to $\Box B \in \mathrm{a}(v)$. In the last case, it follows $\Box B \in \mathrm{a}(u)$, and so $B \in \mathrm{a}(w)$ by $uR_{\mathbf{K}}w$. Next, to show $vR'_{\mathbf{S4}}w$, suppose $\Box B \in \mathrm{s}(v)$. Similarly to the above, it follows that either $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u)$ or $\Box\neg\Box B \in \mathrm{s}(u)$ holds. In the first (second, respectively) case, $\Box B \in \mathrm{s}(w)$ by $uR'_{\mathbf{S4}}w$ (by $uR_{\mathbf{K}}w$ and Proposition 1.6 (1), respectively). In the last case, $\Box\neg\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$, and so $\Box B \in \mathrm{a}(v)$, which contradicts to $\Box B \in \mathrm{s}(v)$. It is left to

show vSw. So, suppose $\Box B \in \mathrm{a}(w) \cup \mathrm{s}(w)$. Since uSw, it follows that either $\Box B \in \mathrm{s}(u)$ or $\Box\neg\Box B \in \mathrm{a}(u)$ or $\Box\neg\Box B \in \mathrm{s}(u)$ holds. In the first (second, respectively) case, $\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$ (by $uR_{\mathbf{K}}v$ and Proposition 1.6 (1), respectively). In the last case, $\Box\neg\Box B \in \mathrm{s}(v)$ by $uR'_{\mathbf{S4}}v$. This ends the derivation of $vR_{\mathbf{K5}}w$.

When $L = \mathbf{K5D}$, the fact that $R_{\mathbf{K5}}$ is serial follows from Proposition 5.5 (2). $\Box$

Corollary 5.7. *Suppose $L \in \{\mathbf{K5}, \mathbf{K5D}\}$. The sequent calculus GL and so the logic L are complete with respect to the Kripke frame semantics, and have the finite model property. Though the calculus GL is not cut-free and lacks the subformula property in the original sense, it has an extended subformula property.*

Acknowledgements

The author would like to thank the anonymous referees for their helpful comments on the earlier version of this article.

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