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A NEW ω-STABLE PLANE

A b s t r a c t. We use a variation on Mason's α-function as a pre-dimension function to construct a not one-based ω-stable plane P (i.e. a simple rank 3 matroid) which does not admit an algebraic representation (in the sense of matroid theory) over any field. Furthermore, we characterize forking in $Th(P)$, we prove that algebraic closure and intrinsic closure coincide in $Th(P)$, and we show that $Th(P)$ fails weak elimination of imaginaries, and has Morley rank ω.

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1. Introduction

In this study we use methods from combinatorial theory and model theory to construct a simple rank 3 matroid that is new from both perspectives. As well-known to experts, the class of simple rank 3 matroids corresponds canonically to the class of linear spaces, or, equivalently, to the class of geometric lattices of rank 3. Matroid theorists refer to simple rank 3 matroid also as *planes*, and so we will adopt this terminology in this study. In [11] we used Crapo's theory of one point-extensions of matroids [6] to construct examples of ω-stable (one of the most important dividing lines in model theory) planes in the context of abstract elementary classes. In the present study we use Mason's α-function of matroid theory and the amalgamation construction known as Hrushovski's construction to build an ω-stable plane in the context of classical first-order logic with an interesting combination of combinatorial and model theoretic properties.

Mason's α-function is a naturally arising notion of complexity for matroids introduced by Mason [16] in his study of so-called gammoids, a now well-known class of matroids arising from paths in graphs. Interestingly, Evans recently showed [7] that the class of strict gammoids corresponds exactly to the class of finite geometries considered by Hrushovski in his celebrated refutation of Zilber's conjecture [10]. A model theoretic analysis of Mason's α-function similar to our approach but quite different in motivation has also appeared in [9, 17][1].

What is referred to as Hrushovski's constructions is a method of constructing model theoretically well-behaved structures via an amalgamation procedure which makes essential use of a certian predimension function. This amalgamation construction results in a countable structure (the so-called "Hrushovski's generic") carrying the additional structure of an infinite dimensional matroid, which controls important model theoretic properties of the structure constructed (which in our case is a simple rank 3 matroid). In this type of constructions, the specifics of the predimension function depend on the case at hand, and in our case the predimension function is a mild but crucial variation of Mason's α function (cf. Definition 4.3).

We believe that our variation on Mason's α-function as a predimension function is of independent interest from various points of view. On the

[1] Our study and [9, 17] were concurrent and both inspired by Evans's work [7].

combinatorial side, we think that the combinatorial consequences of the realization that Mason's α-function is essentially a predimension should be properly explored. On the model theoretic side, we believe that the "collapsed[2] version" of our construction leads to interesting connections with *design theory* (currently explored in works in preparation joint with John Baldwin [2, 3]), and in particular with Steiner k-systems.

Before stating our main theorem we spend few motivating words introducing the model theoretic properties appearing in it. Model theory is the study of classes of structures using logical properties, which are also referred to as dividing lines (since they are often accompanied by a dichotomous behavior). Among the various properties considered by logicians there are certain properties which are more "geometric" in nature. These properties are called "geometric" since they are given by imposing conditions on certain infinite dimensional matroids associated with the structures. The canonical example of this kind of structure is the class of *strongly minimal structures*, where the model theoretic operator of algebraic closure determines an infinite dimensional matroid. This context has later been extended to uncountably categorical structures (one model up to isomorphism in every uncountable cardinality), and even more generally to ω-stable structures (cf. [15, Chapter 6]). In this spirit, one of the geometric properties of the kind mentioned above is the notion of being *one-based*, which on strongly minimal structures corresponds to the natural notion of local modularity of the lattice of closed sets of the associated matroid.

We prove the following theorem:

Theorem 1.1. *There exists a pre-dimension function δ on the class of finite planes (finite simple rank* 3 *matroids) such that the corresponding "Hrushovski's generic" (cf. Definition 4.16) exists, and so it is a plane P (i.e. a simple rank* 3 *matroid, cf. Definition 3.1), and it satisfies the following conditions:*

(1) P contains the "non-Desarguesian" matroid (cf. Figure 1, or [20, pg. 139]), and so it is not algebraic (in the sense of matroid theory);

[2] The so-called "collapse" is a technical variation of the Hrushovki's construction which ensures the satisfaction of further important model-theoretic properties, as e.g. uncountable categoricity: only one model up to isomorphism in every uncountable cardinalily.

(2) *in $Th(P)$ intrinsic closure and algebraic closure coincide (cf. Definition 4.22);*

(3) *$Th(P)$ does* not *have weak elimination of imaginaries (cf. Definition 4.30);*

(4) *$Th(P)$ is not one-based (cf. Definition 4.32);*

(5) *$Th(P)$ is ω-stable and has Morley rank ω (cf. [15, Chapter 6]);*

(6) *over algebraically closed sets forking in $Th(P)$ corresponds to the canonical amalgamation introduced in [11, Theorem 4.2] (cf. Remark 4.11).*

As mentioned above, properties (2)-(6) of Theorem 1.1 are important dividing lines in model theory, and their satisfaction shows that our object is particularly well-behaved from this perspective; for an introduction to these notions see e.g. [15, Chapter 6]. In combination with these properties, the fact that our plane P is not algebraic (a matroid theoretic notion related to fields, cf. Definition 1.1) makes our plane particularly exotic. Non-algebraic planes are somewhat rare in nature, and in fact the existence of non-algebraic planes is a non-trivial fact due to Lindström [13, 14], who constructed in [13] an infinite familiy of non-algebraic finite planes. Furthermore, this shows that our variation on Mason's α-function is crucial, since the class of finite simple matroids M with $\alpha(M) \geqslant 0$ is the already mentioned class of strict gammoids (see [1, Chapter 7, Section 4] or [7]), and these structures are known to be linear (a matroid theoretic notion related to vector spaces, cf. Definition 1.1 and [1, Corollary 7.75]), and thus in particular algebraic. On the other hand, in [18] we constructed a simple rank 3 matroid with strong homogeneity properties with $\wedge$-embeds all the finite simple rank 3 matroids, and so in particular it is *not* algebraic, but that structure has the so-called *independence property*, and so it is in a completely different region of the model theoretic universe. In fact, we stress once again that what is interesting about the structure constructed in this paper is the combination of the failure of algebraicity together with the satisfaction of ω-stability, and of the other model theoretic properties of Theorem 1.1.

Concerning the structure of the paper: in Section 2 we give a quick introduction to matroid theory and recall the definition of Mason's α-function;

in Section 4 we introduce the construction at the core of this paper and prove Theorem 1.1.

We stress that in our construction the fact that we use a modification of the α-function (and not the α-function per se) is crucial, since the δ which we define in Section 4 is submodular, while the α-function is not submodular, cf. Remark 4.5.

We refer the reader to [2] for further discussion concerning how our predimension function δ differs from the previously considered predimensions in the literature, and for further parallelisms with the current literature on Hrushovski's constructions.

2. Matroid Theory Background

In this section we give a quick background on notions from matroid theory which are relevant for the present paper. For an introduction to matroid theory directed to model theorists see also e.g. [11, Section 2]. We will first give the definition of a simple matroid (a.k.a. combinatorial geometry) as a set with a closure operator (cf. Definition 2.1), then give the definition which takes as primary the collection of dependent sets (cf. Definition 2.3), and then observe that the two definitions are equivalent (cf. Fact 2.4). We will then lay the correspondence between geometric lattices and simple matroids, and finally define Mason's α-function.

Definition 2.1. We say that (M, cl_M) is a combinatorial geometry (or a simple matroid) of finite rank if the following conditions are met:

(1) if $A \subseteq B$, then $cl_M(A) \subseteq cl_M(B) = cl_M(cl_M(B))$;

(2) $cl_M(\emptyset) = \emptyset$, and $cl_M(\{a\}) = \{a\}$, for every $a \in M$;

(3) if $a \in cl_M(A \cup \{b\}) - cl_M(A)$, then $b \in cl_M(A \cup \{a\})$;

(4) if $a \in \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$;

(5) there exists finite $A \subseteq M$ such that $cl_M(A) = M$.

Remark 2.2. We make a comment on terminology concerning Definition 2.1. In Definition 2.1 we defined what is a *simple matroid*, relaxing condition (2) we obtain the more general notion of *matroid*. Model theorists often refer to objects as in Definition 2.1 as *finite dimensional (combinatorial) geometries*. We have the following correspondence between matroid theoretic and model theoretic terminology:

$$\text{matroid : simple matroids = pregeometries : geometries.}$$

In this paper we follow the matroid theoretic termimonology.

We write $A \subseteq_\omega B$ for $A \subseteq B$ and $|A| < \omega$.

Definition 2.3. Let M be a set and $\mathcal{D}$ a collection of non-empty finite subsets of M . We say that $(M, \mathcal{D})$ is a simple matroid, and refer to sets in $\mathcal{D}$ as dependent sets, when we have:

(1) if D is dependent, then $|D| > 2$;

(2) if D is dependent and $D \subseteq D' \subseteq_\omega M$, then D' is dependent;

(3) if $D_1, D_2 \subseteq M$ are dependent and $D_1 \cap D_2$ is not dependent, then for every $a \in M$ we have that $D_1 \cup D_2 - \{a\}$ is dependent;

(4) there is $n < \omega$ such that if $D \subseteq_\omega M$ and $|D| \geqslant n$, then D is dependent.

Fact 2.4 (see e.g. [5])**.** *(1) Let (M, cl_M) be a combinatorial geometry of finite rank (cf. Definition 2.1), and call a finite subset $D \subseteq M$ dependent if there is $a \in D$ such that $a \in cl_M(D - \{a\})$. Then, denoting by $\mathcal{D}$ the set of dependent sets of (M, cl_M), we have that $(M, \mathcal{D})$ is a simple matroid (cf. Definition 2.3).*

(2) Let $(M, \mathcal{D})$ be a simple matroid, and define the following operator cl_M on M:

(a) if $A \subseteq M$ is finite, then $a \in cl_M(A)$ if $A \cup \{a\}$ is dependent;

(b) if $A \subseteq M$ is infinite, then $cl_M(A) = \bigcup_{B \subseteq_\omega M} cl_M(B)$.

Then (M, cl_M) is a combinatorial geometry of finite rank.

The following convention is adopted in this paper, where by definitions of the notions occurring in it we consider the definitions given in this paper.

Convention 2.5. *In virtue of Fact 2.4, when talking about simple matroids we will not distinguish between the formalisms of Definitions 2.1 and 2.3.*

We now define the notion of *rank* of simple matroid.

Definition 2.6. Let (M, cl_M) be a simple matroid. We let the *rank* of M, denoted as $rk(M)$, to be the least n such that every set of size $n+1$ is dependent (cf. Definition 2.3). By convention if $|M| \in \{0, 1, 2\}$, then $rk(M) = |M|$.

We now come to geometric lattices and their correspondence with simple matroids. Recall that a lattice is a partial order $(L, \leqslant)$ such that any two elements a, b have a least upper bound and a greatest lower bound, denoted by $a \wedge b$ and $a \vee b$. A chain in a lattice $(L, \leqslant)$ is a subset $X \subseteq L$ such that $(X, \leqslant)$ is a linear order.

Assumption 2.7. *In this paper all lattices have a maximum element* 1 *and a minimum element* 0*. Furthermore, any chain between any two elements is finite.*

Given a lattice $(L, \leqslant)$ and $x \in L$, we let $h(x)$, the height of x, to be the length of the longest maximal chain between 0 and x. Furthermore, given $a, b \in L$, we say that a is *covered* by b, for short $a \lessdot b$, if $a < b$ and for every $a \leqslant c \leqslant b$ we have that either $a = c$ or $c = b$. Finally, we say that a is an *atom* if it covers 0.

Definition 2.8. Let $(L, \leqslant)$ be a lattice.

i) We say that $(L, \leqslant)$ is *semimodular* if for every $a, b \in L$ we have that
$$a \wedge b \lessdot a \;\Rightarrow\; b \lessdot a \vee b.$$

ii) We say that $(L, \leqslant)$ is a *point lattice* if every $a \in L$ is a supremum of atoms.

iii) We say that $(L, \leqslant)$ is *geometric* if $(L, \leqslant)$ is a semimodular point lattice such that its greatest element 1 exists and it is equal to a finite set of atoms.

Remark 2.9. Notice that Definition 2.8(3) implies in particular that geometric lattices are complete. In the context of geometric lattices, atoms are often referred to as points, we will follow this convention throughout the entire paper.

Remark 2.10. Concerning Theorem 2.11, recall that in our setting Definition 2.1(5) holds, or, following the model theoretic terminology mentioned in Remark 2.2, we only consider finite dimensional geometries.

Theorem 2.11 (Birkhoff-Whitney)**.** *i) Let* (M, cl) *be a (simple) matroid and let* $G(M)$ *the set of closed subsets of* M *(later referred to as flats), i.e. the* $X \subseteq M$ *such that* $\mathrm{cl}(X) = X$*. Then* $(G(M), \subseteq)$ *is a geometric lattice.*

ii) Let $(G, \leqslant) = (G, 0, 1, \vee, \wedge)$ *be a geometric lattice with point set* M *and for* $A \subseteq M$ *let:*

$$\mathrm{cl}(A) = \left\{ p \in M \,|\, p \leqslant \bigvee A \right\}.$$

Then (G, cl) *is a simple matroid. Furthermore, the function* $\phi : G \to G(M)$ *such that* $\phi(x) = \{p \in M \,|\, p \leqslant x\}$ *is a lattice isomorphism.*

We will also need the following definition (which is used in Fact 4.10).

Definition 2.12. Let $M = (M, cl)$ and $N = (N, cl)$ be simple matroids. We say that M is a $\wedge$-subgeometry of N if M is a subgeometry of N (i.e. $M \subseteq N$ and $cl_M(X) = cl_N(X) \cap M$) and the inclusion map $i_M : M \to N$ induces an embedding (with respect to both $\vee$ and $\wedge$) of $G(M)$ into $G(N)$ (cf. [11, Section 2]).

Concerning the notion of algebraic matroid occurring in item (1) of Theorem 1.1:

Definition 2.13. Let M be a matroid.

(1) We say that M is linear if there is a field K, a K-vector space V, and an injective map $f : M \to V$ such that $X \subseteq M$ is independent in M if and only if $f(X)$ is linearly independent in V.

(2) We say that M is algebraic if there exists an algebraically closed field K and an injective map $f : M \to K$ such that $X \subseteq M$ is independent in M if and only if $f(X)$ is algebraically independent in K.

We introduce some useful notations and terminology.

Notation 2.14. *Let* $M = (M, cl)$ *be a simple matroid.*

(1) We refer to closed subsets of M *(i.e. subsets* $F \subseteq M$ *of the form* $cl_M(F) = F$*) as* flats *of* M*, or* M*-flats.*

(2) *Given two subsets F and X of M we use the notation*[3] $F \preccurlyeq X$ *(resp. $F \prec X$) to mean that F is a subset of X (resp. a proper subset) and F is a flat of M.*

We finally define the α-function.

Definition 2.15 (Mason's α-function [16])**.** Let M be a finite simple matroid. For each subset X of M we define recursively:

$$\alpha(X) = |X| - rk(X) - \sum_{F \prec X} \alpha(F).$$

Definition 2.16. Let M be a finite simple matroid and F an M-flat. We define the nullity of F as follows:

$$\mathbf{n}(F) = |F| - rk(F).$$

The following conventions will simplify a great deal the computations of Section 4. Its use will be limited to Proposition 4.7 and Lemma 4.8.

Convention 2.17. *Let $M = (M, cl)$ and $N = (N, cl)$ be finite simple matroids and suppose that M is a subgeometry of N. If F is an N-flat, then:*

(1) *we denote by $|F|_M$ the number $|F \cap M|$;*

(2) *we denote by $\mathbf{n}_M(F)$ the number $\mathbf{n}(F \cap M)$, considering $F \cap M$ computed in M as an M-flat, and by $\mathbf{n}_N(F)$ the number $\mathbf{n}(F)$ computed in N as an N-flat.*

Convention 2.18. *Let $M = (M, cl)$ be a simple matroid. Then:*

(1) *M-flats of rank 2 are referred to as* lines*;*

(2) *we denote by $L(M)$ the set of lines of M.*

(3) *For $N \subseteq M$ and $\ell \in L(M)$, we say that ℓ is based in N if $|\ell \cap N| \geqslant 2$.*

(4) *For $N \subseteq M$, we let $L_M(N)$ to be the set of $\ell \in L(M)$ which are based in N. Since $L(N)$ and $L_M(N)$ are in canonical bijection we will be sloppy in distinguishing between them, and often write $L(N)$ instead of $L_M(N)$.*

[3] This notation is taken from [16] where the notion of α-function was introduced.

3. Our Context

In the present paper we will actually only be interested in matroids of rank $\leqslant 3$. In this case the definitions from the previous section can be greatly simplified. We will now give a direct definition of a simple rank 3 matroid as a simple combinatorial structure, and then argue why this is coherent with the notions from Section 2. For a more extensive discussion of this definition see also [2, Section 2].

Definition 3.1. A *simple matroid of rank* $\leqslant 3$ is a 3-hypergraph $M = (V, R)$ whose adjacency relation is irreflexive, symmetric[4] and satisfies the following axiom:

(Ax) if $R(a, b, c)$ and $R(a, b, d)$, then $\{a, b, c, d\}$ is an R-clique.

This definition of matroids is formally not a complete definition, since it does not specify the dependent sets of cardinality different than 3 (so with respect to Definition 2.3). The point is that if M is a simple matroid of rank $\leqslant 3$, then:

(1) every set of size < 3 is not dependent;

(2) every set of size > 3 is dependent.

Hence, every structure $M = (V, R)$ as in the current definition admits canonically the structure of a matroid of rank $\leqslant 3$ simply by letting:

(a) every set of size < 3 as independent;

(b) every set of size > 3 as dependent;

(c) if $X = \{a, b, c\} \subseteq M$ has size 3, then X is dependent iff $M \models R(a, b, c)$.

The following remark gives an explicit characterization of $\alpha(M)$ in the case M is of rank 3. For the purposes of the present paper this characterization suffices, and thus we could have avoided the general definition of the α-function; we chose not to do so because we wanted to motivate the naturality of the predimension function of Definition 4.3 (from Section 4) and make explicit its relation to the α-function.

[4] Explictly, we have: (i) $M \models R(a, b, c)$ implies $|\{a, b, c\} = 3|$, and (ii) if $M \models R(a_1, a_2, a_3)$ and σ is a bijection of $\{1, 2, 3\}$ then $M \models R(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$.

Remark 3.2. Let M be a finite simple matroid of rank 3, then:

$$\alpha(M) = |M| - 3 - \sum_{\ell \in L(M)} \mathbf{n}(\ell).$$

Convention 3.3. *Given $M = (V, R)$ as in Definition 3.1 and $\{a, b, c\} \subseteq M$ a subset of size 3 we say that a is under the line $b \vee c$ to mean that $M \models R(a, b, c)$.*

4. The Construction

We follow the general framework of [4], and refer to proofs from there when minor changes to the arguments are needed in order to establish our claims. It is strongly advised to have a copy of [4] and [22] while reading this section.

Notation 4.1. *Let $\mathbf{K}_0^*$ be the class of finite simple matroids of rank $\leqslant 3$ seen as structures in a language with a ternary predicate R for dependent sets of size 3 (cf. Definition 3.1). Recall that we refer to elements $A \in \mathbf{K}_0^*$ as* planes *(if $rk(A) < 3$ we say that A is degenerate). We say that $A \in \mathbf{K}_0^*$ has positive α if $\alpha(A) \geqslant 0$.*

Convention 4.2. *Throughout the rest of the paper model theoretically we will consider our planes only in the language of Notation 4.1. In particular, if P is a plane seen as an L-structure, then the lines of P (in the sense of the associated geometric lattice $G(P)$) are not elements of P, but only definable subsets of P.*

Definition 4.3. For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}(\ell).$$

Definition 4.4. Let:

$$\mathbf{K}_0 = \{A \in \mathbf{K}_0^* \text{ such that for any } A' \subseteq A, \delta(A') \geqslant 0\},$$

and $(\mathbf{K}_0, \leqslant)$ be as in [4, Definition 3.11], i.e. we let $A \leqslant B$ if and only if:

$$A \subseteq B \wedge \forall X (A \subseteq X \subseteq B \Rightarrow \delta(X) \geqslant \delta(A)).$$

Finally, we write $A < B$ to mean that $A \leqslant B$ and A is a proper subset of B.

Remark 4.5 and Example 4.6 are connected to the discussion in the introduction after the statement of Theorem 1.1.

Remark 4.5. Notice that, by Remark 3.2, if $A \in \mathbf{K}_0^*$ has rank 3, then:

$$\delta(A) = \alpha(A) + 3.$$

And so our δ is just a natural variation of Mason's α-function. Despite this, our variation is crucial, since, as we observed in the introduction, the class of finite simple matroids of positive α is the already mentioned class of strict gammoids, and these structures are known to be linear (see e.g. [1, Corollary 7.75]), while, as shown in Example 4.6, there exists a non-algebraic $A \in \mathbf{K}_0^*$ such that $\delta(A) \geqslant 0$.
Even more interestingly, although as a consequence of Lemma 4.8, δ is submodular, the α-function is *not* submodular, i.e. there exists $A, B \in \mathbf{K}_0^*$ such that:

$$\alpha(A \cup B) > \alpha(A) + \alpha(B) - \alpha(A \cap B).$$

In fact letting $A = \{a, b, c\}$ and $B = \{a, b, d\}$ be two copies of the three element simple matroid of rank 3 we have that:

$$\alpha(A \cup B) = 1 > \alpha(A) + \alpha(B) - \alpha(A \cap B) = 0 + 0 + 0.$$

Example 4.6. *Let A be the "non-Desarguesian" matroid (cf. Figure 1, for another representation of this matroid see [20, pg. 139]). Then, $\delta(A) = 1$, since A has 10 points and exactly 9 non-trivial lines, each of nullity 1 (i.e. each has size 3). Furthermore, inspection of Figure 1 shows that for every $B \subseteq A$, we have that $\delta(B) \geqslant 0$. The "non-Desarguesian" matroid was shown not to be algebraic in [14, Corollary, pg. 238]. This will be relevant for the proof of Theorem 1.1(1). Finally, notice on the other hand that $\alpha(A) < 0$ (where α is Mason's α-function from Def. 2.15), and so the class of planes with positive δ but negative α is non-trivial, as in fact all the matroids with non-negative α are linear (as they are gammoids).*

The following two claims constitute the computational core of the paper, and aim at proving that our function δ is lower semimodular. Proposition 4.7 is used to prove Lemma 4.8, which in turn is used to draw Conclusion 4.9. In Proposition 4.7 and Lemma 4.8 we will make a crucial use of Conventions 2.17-2.18.

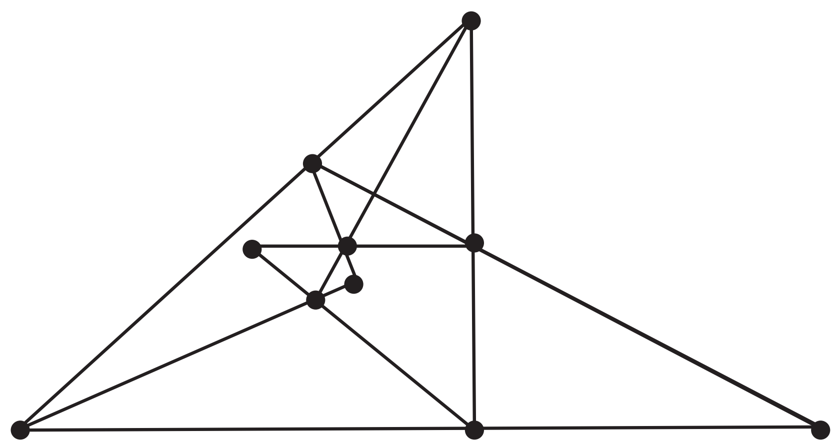

Figure 1: The "non-Desarguesian" matroid.

Proposition 4.7. *Let A and B be disjoint subsets of a matroid $C \in \mathbf{K}_0^*$ (so, in particular A and B are submatroids of the matroid C, or, equivalently, substructures in the sense of Notation 4.1). Then:*

(1) if $\ell \in L(B)$, then $\mathbf{n}_{AB}(\ell) - \mathbf{n}_B(\ell) = |\ell|_A$ (clearly, if $\ell \in L(B)$, then $\ell \in L(AB)$);

(2) $\delta(A/B) := \delta(AB) - \delta(B)$ is equal to:

$$|A| - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \notin L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \in L(B)}} |\ell|_A - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(B) \\ \ell \notin L(A)}} |\ell|_A.$$

Proof. Concerning item (1), for $\ell \in L(AB)$ and $\ell \in L(B)$ we have:

$$\begin{aligned} \mathbf{n}_{AB}(\ell) - \mathbf{n}_B(\ell) &= |\ell|_{AB} - rk(\ell) - |\ell|_B + rk(\ell) \\ &= |\ell|_A + |\ell|_B - |\ell|_B \\ &= |\ell|_A. \end{aligned}$$

Concerning item (2), we have that $\delta(A/B)$ is:

$$\begin{aligned}
&= |AB| - \sum_{\ell \in L(AB)} \mathbf{n}_{AB}(\ell) - |B| + \sum_{\ell \in L(B)} \mathbf{n}_B(\ell) \\
&= |A| + |B| - \sum_{\ell \in L(AB)} \mathbf{n}_{AB}(\ell) - |B| + \sum_{\ell \in L(B)} \mathbf{n}_B(\ell) \\
&= |A| - \sum_{\ell \in L(AB)} \mathbf{n}_{AB}(\ell) + \sum_{\ell \in L(B)} \mathbf{n}_B(\ell) \\
&= |A| - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \notin L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \in L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(B) \\ \ell \notin L(A)}} \mathbf{n}_{AB}(\ell) \\
&\quad + \sum_{\ell \in L(B)} \mathbf{n}_B(\ell) \\
&= |A| - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \notin L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \in L(B)}} (\mathbf{n}_{AB}(\ell) - \mathbf{n}_B(\ell)) \\
&\quad - \textstyle\sum_{\substack{\ell \in L(AB) \\ \ell \in L(B) \\ \ell \notin L(A)}} (\mathbf{n}_{AB}(\ell) - \mathbf{n}_B(\ell)) \\
&= |A| - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \notin L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \in L(B)}} |\ell|_A - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(B) \\ \ell \notin L(A)}} |\ell|_A.
\end{aligned}$$

Concerning the passage from the third equation to the fourth equation notice that if $\ell \in L(AB) - (L(A) \cup L(B))$, then $\mathbf{n}_{AB}(\ell) = 0$. $\square$

Lemma 4.8. *Let $A, B, C \subseteq D \in \mathbf{K}_0^*$, with $A \cap C = \emptyset$ and $B \subseteq C$. Then:*

$$\delta(A/B) \geqslant \delta(A/C).$$

Proof. Let A, B, C be subsets of a matroid D and suppose that $B \subseteq C$ and $A \cap C = \emptyset$. Notice that by Proposition 4.7 we have:

$$-\delta(A/C) = -|A| + \sum_{\substack{\ell \in L(AC) \\ \ell \in L(A) \\ \ell \notin L(C)}} \mathbf{n}_{AC}(\ell) + \sum_{\substack{\ell \in L(AC) \\ \ell \in L(A) \\ \ell \in L(C)}} |\ell|_A + \sum_{\substack{\ell \in L(AC) \\ \ell \in L(C) \\ \ell \notin L(A)}} |\ell|_A, \qquad (\star_1)$$

$$\delta(A/B) = |A| - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \notin L(B)}} \mathbf{n}_{AB}(\ell) - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(A) \\ \ell \in L(B)}} |\ell|_A - \sum_{\substack{\ell \in L(AB) \\ \ell \in L(B) \\ \ell \notin L(A)}} |\ell|_A. \qquad (\star_2)$$

Notice now that for $\ell \in L(AC)$ we have:

(a) if $\ell \in L(A)$, $\ell \notin L(B)$ and $\ell \notin L(C)$, then ℓ occurs in the first sum of $(\star_1)$ and in the first sum of $(\star_2)$, and clearly $\mathbf{n}_{AC}(\ell) \geqslant \mathbf{n}_{AB}(\ell)$;

(b) if $\ell \in L(A)$ and $\ell \in L(B)$, then $\ell \in L(A)$ and $\ell \in L(C)$, and so ℓ occurs in the second sum of $(\star_1)$ and in the second sum of $(\star_2)$;

(c) if $\ell \in L(B)$ and $\ell \notin L(A)$, then $\ell \in L(C)$ and $\ell \notin L(A)$, and so ℓ occurs in the third sum of $(\star_1)$ and in the third sum of $(\star_2)$;

(d) if $\ell \in L(A)$, $\ell \notin L(B)$ and $\ell \in L(C)$, then ℓ occurs in the second sum of $(\star_1)$ and in the first sum of $(\star_2)$, and furthermore we have:

$$\mathbf{n}_{AB}(\ell) \leqslant \mathbf{n}_A(\ell) + 1 < \mathbf{n}_A(\ell) + 2 = |\ell|_A.$$

Since, clauses (a)-(d) above cover all the terms occurring in $(\star_2)$, we conclude that $\delta(A/B) \geqslant \delta(A/C)$, as wanted. □

Conclusion 4.9. $(\mathbf{K}_0, \leqslant)$ *satisfies Axiom A1-A6 from [4, Axioms Group A], i.e.:*

(1) if $A \in \mathbf{K}_0$, then $A \leqslant A$;

(2) if $A \leqslant B$, then $A \subseteq B$;

(3) if $A, B, C \in \mathbf{K}_0$ and $A \leqslant B \leqslant C$, then $A \leqslant C$;

(4) if $A, B, C \in \mathbf{K}_0$, $A \leqslant C$, $B \subseteq C$, and $A \subseteq B$, then $A \leqslant B$;

(5) $\emptyset \in \mathbf{K}_0$ and $\emptyset \leqslant A$, for all $A \in \mathbf{K}_0$;

(6) if $A, B, C \in \mathbf{K}_0$, $A \leqslant B$, and C is a substructure of B, then $A \cap C \leqslant C$.

Proof. As in e.g. [4, Theorem 3.12], this is easy to establish using Lemma 4.8. □

Fact 4.10 ([11, Theorem 4.2])**.** *Let $A, B, C \in \mathbf{K}_0$ with C a $\wedge$-subgeometry (cf. Definition 2.12) of A and B and $A \cap B = C$. Then there exists a canonical amalgam of A and B over C, which we denote as $A \oplus_C B$. In the next remark we give an explicit characterization of $A \oplus_C B$ as an L-structure, i.e. we simply translate the lattice theoretic definition of $A \oplus_C B$ from [11] into the language of L-structures.*

Remark 4.11. The amalgam $D := A \oplus_C B$ of Fact 4.10 can be characterized as the following L-structure:

(1) the domain of D is $A \cup B$;

(2) $R^D = R^A \cup R^B \cup \{\{a, b, c\} : a \vee b \vee c = a' \vee b' \text{ for some } \{a', b'\} \subseteq C\}$.

Where $\vee$ refers to the canonically associated geometric lattice $G(D)$. A more transparent way to define the amalgam $A \oplus_C B$ is by defining the domain of $A \oplus_C B$ to be simply $A \cup B$, and the lines of $A \oplus_C B$ to be the lines coming from A, those coming from B, modulo identifying the lines from C, plus the obvious trivial lines.

Lemma 4.12. *(1) If $A \leqslant B \in \mathbf{K}_0$, then A is a $\wedge$-subgeometry of B.*

(2) $(\mathbf{K}_0, \leqslant)$ has the amalgamation property.

Proof. Concerning (1), suppose that $A, B \in \mathbf{K}_0$, and A is not a $\wedge$-subgeometry of B, then there exists $p \in B - A$ and $\ell_1 \neq \ell_2 \in L(A)$ such that p is incident with both ℓ_1 and ℓ_2. Thus, $\delta(Ap) < \delta(A)$ and so $A \nleqslant B$. Concerning (2), let $A, B, C \in \mathbf{K}_0$ and suppose that $C \leqslant A, B$ with $A \cap B = C$ (without loss of generality). Let $A \oplus_C B := D$ (recall Notation 4.10), which exists by (1). Using e.g. Remark 4.11, it is easy to see that:

$$\delta(D) = \delta(A) + \delta(B) - \delta(C). \tag{$\star_3$}$$

Furthermore, for every $C \subseteq X \subseteq D$ we have that $X = (A \cap X) \oplus_{C \cap X} (B \cap X)$. Thus, it is immediate to infer that $D \in \mathbf{K}_0$ and $B, C \leqslant D$, as wanted. $\square$

We now introduce several technical notions of amalgamation, in particular sharp and uniform amalgamation. We are only interested in sharp amalgamation as a sufficient condition for uniform amalgamation, and we are only interested in the latter as a sufficient condition for ω-stability, see Conclusion 4.28.

Definition 4.13. Let $(\mathbf{L}_0, \leqslant)$ be a class of relational structures of the same vocabulary satisfying the conditions in Conclusion 4.9 and let $A, B, C \in \mathbf{L}_0$.

(1) For $k < \omega$, we say that A is k-strong in B, denoted $A \leqslant^k B$, if for any B' with $A \subseteq B' \subseteq B$ and $|B' - A| \leqslant k$ we have $A \leqslant B'$ (cf. [4, Definition 2.26]).

(2) We say that B is a primitive extension of A if $A \leqslant B$ and there is no $A \subsetneq B_0 \subsetneq B$ such that $A \leqslant B_0 \leqslant B$ (cf. [4, Definition 2.30]).

(3) Given $C \leqslant A, B$ with $A \cap B = C$, we let $A \otimes_C B$ denote the free amalgam of A and B over C, i.e. the structure with domain $A \cup B$ and no additional relations apart from the ones in A and the ones in B.

(4) We say that $(\mathbf{L}_0, \leqslant)$ has the sharp amalgamation property if for every $A, B, C \in \mathbf{L}_0$, if $C \leqslant A$ is primitive and $C \leqslant^{|A|-|C|} B$, then either $A \otimes_C B \in \mathbf{L}_0$ or there is a $\leqslant$-embedding of A into B over C (cf. [4, Definition 2.31]).

(5) We say that $(\mathbf{L}_0, \leqslant)$ has the uniform amalgamation property if the following condition holds: for every $A \leqslant B \in \mathbf{L}_0$, and for every $m < \omega$ there is an $n = f_B(m)$ such that if $A \leqslant^n C$, then there is a D, a strong embedding of C into D and an m-strong embedding of B into D that completes a commutative diagram with the given embeddings of A into B and C.

Proposition 4.14. *Let $A \leqslant B \in \mathbf{K}_0$ be primitive. Then either $|B - A| \leqslant 1$, or for every $p \in B - A$ we have that p is not incident with a line $\ell \in L(A)$. Furthermore, in the first case we have that $\delta(B/A) \leqslant 1$.*

Proof. Suppose that there exists $p \in B - A$ such that p is incident with a line $\ell \in L(A)$ (and thus under no other line $\ell' \in L(A)$, cf. Lemma 4.12 and recall Convention 3.3). Then we have $\delta(A) = \delta(Ap)$, and so if $|B| - |A| > 1$ we have $\delta(A) = \delta(Ap) \leqslant \delta(B)$, and thus $A < Ap < B$, contradicting the assumptions of the proposition. The furthermore part is immediate from the definition of δ. □

Lemma 4.15. *(1) $(\mathbf{K}_0, \leqslant)$ has the sharp amalgamation property.*

(2) In (1) we can replace $|A| - |C|$ with 1, i.e. the conclusion of Definition 4.13(4) is true for the all the extensions of the form $C \leqslant^1 B$, not only for the extensions of the form $C \leqslant^{|A|-|C|} B$, as required by Definition 4.13(4)).

(3) $(\mathbf{K}_0, \leqslant)$ has the uniform amalgamation property (cf. Definition 4.13(5)).

Proof. The general deduction of (3) from (1) is by [4, Lemma 2.32]. We prove (1) and (2). Let $A, B, C \in \mathbf{K}_0$ and suppose that $C < A$ is primitive, $C \leqslant^1 B$ and $A \cap B = C$ (without loss of generality). By Proposition 4.14, either every $p \in A - C$ is not incident with a line $\ell \in L(C)$ or $C - A = \{p\}$ and there exists a line $\ell \in L(C)$ such that p is incident with ℓ. Suppose the first, then by Remark 4.11 the canonical amalgam $A \oplus_C B$ (cf. Notation 4.10) coincides with the free amalgam $A \otimes_C B$ (cf. Definition 4.13(3)), and so we are done. Suppose the second and let p and ℓ witness it. If every $p' \in B - C$ is not incident with the line ℓ, then also in this case $A \oplus_C B = A \otimes_C B$, and so we are done. Finally, if there exists $p' \in B - C$ such that p is incident with ℓ, then clearly $A = Cp$ is such that it $\leqslant$-embeds into B over C, since $\delta(C) = \delta(Cp') = \delta(Cp)$. □

Definition 4.16. Let $(\mathbf{L}_0, \leqslant)$ be a class of relational structures in the language L satisfying the conditions in Conclusion 4.9. A countable L-model M is said to be $(\mathbf{L}_0, \leqslant)$-*generic* when:

(1) if $A \leqslant M, A \leqslant B \in \mathbf{L}_0$, then there exists $B' \leqslant M$ such that $B \cong_A B'$;

(2) M is a union of finite substructures.

Fact 4.17 ([4, Theorem 2.12])**.** *Let* $(\mathbf{L}_0, \leqslant)$ *be a class of relational structures of the same vocabulary satisfying the conditions in Conclusion 4.9, and suppose that* $(\mathbf{L}_0, \leqslant)$ *has the amalgamation property. Then there exists a* $(\mathbf{L}_0, \leqslant)$*-generic model, and this model is unique up to isomorphism.*

Corollary 4.18. *The* $(\mathbf{K}_0, \leqslant)$*-generic model exists.*

Proof. By Fact 4.17 and Lemma 4.12. □

Notation 4.19.

(1) Let P be the generic model for $(\mathbf{K}_0, \leqslant)$ *(cf. Corollary 4.18), and let $\mathfrak{M}$ be the monster model of $Th(P)$.*

(2) Given $A, B, C \subseteq \mathfrak{M}$ we write $A \equiv_C B$ to mean that there is an automorphism of $\mathfrak{M}$ fixing C pointwise and mapping A to B.

We recall that we write $A \subseteq_\omega B$ to mean that $A \subseteq B$ and $|A| < \aleph_0$.

Definition 4.20. Let $M \models Th(P)$.

(1) Given $A \subseteq_\omega M$, we let:

$$d(A) = inf\{\delta(B) : A \subseteq B \subseteq_\omega M\}.$$

(2) Given $A \subseteq_\omega M$, we let $A \leqslant M$ if $d(A) = \delta(A)$.

(3) Given $A, B, C \subseteq_\omega M$ with $C \leqslant A, B \leqslant M$ and $A \cap B = C$, we let $A \mathop{\smile\!\!\!\!\!\!\mid}^d_C B$ if:

$$d(A/C) = d(A/B).$$

Fact 4.21. *Let $M \models Th(P)$ and $A \subseteq_\omega M$. Then there exists a unique finite $B_A \subseteq_\omega M$ such that $A \subseteq B_A \leqslant M$ and B_A is minimal with respect to inclusion. Furthermore, $B_A \subseteq acl_M(A)$ (where $acl_M(A)$ is the algebraic closure of A in M).*

Proof. By [4, Theorem 2.23], since clearly $\mathbf{K}_0$ has finite closure. □

Definition 4.22. Following [4] we denote the set B_A from Fact 4.21 by $icl_M(A)$, and we call it the intrinsic closure of A in M.

Lemma 4.23. *Let $A \subseteq_\omega P$. Then $acl_P(A) \subseteq icl_P(A)$.*

Proof. Let $A \subseteq_\omega P$, $b \in P - icl_P(A)$, $A' = icl_P(A)$ and $B' = icl_P(Ab)$. Now, for every $1 < k < \omega$, we can find $D \leqslant P$ such that:

$$D \cong_{A'} \underbrace{B' \oplus_{A'} B' \oplus_{A'} \cdots \oplus_{A'} B'}_{k\text{-times}} := F,$$

since $A' \leqslant B' \leqslant F \in \mathbf{K}_0$ and P is generic (cf. [4, Definition 2.11]). Thus, by the homogeneity of P, we can find infinitely many elements of P with the same type as b over A'. Hence, $b \notin acl_P(A)$. □

Conclusion 4.24. *Let $A \subseteq_\omega M \models Th(P)$, then $icl_M(A) = acl_M(A)$, i.e. intrinsic closure and algebraic closure coincide in M.*

Proof. The inclusion $icl_M(A) \subseteq acl_M(A)$ is by Fact 4.21. For the other inclusion argue as in [4, Theorem 4.5] using Lemma 4.23. □

Proposition 4.25. *Let $A, B, C \subseteq_\omega \mathfrak{M}$ with $C \leqslant A, B \leqslant \mathfrak{M}$ and $A \cap B = C$. If $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$ (cf. Definition 4.20(3)), then $AB \leqslant \mathfrak{M}$.*

Proof. As in [4, Theorem 3.31]. □

Lemma 4.26. *Let $A, B, C \subseteq_\omega \mathfrak{M}$ with $C \leqslant A, B \leqslant \mathfrak{M}$ and $A \cap B = C$. Then the following are equivalent:*

(1) $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$ (cf. Definition 4.20(3));

(2) $AB = A \oplus_C B$ (cf. Notation 4.10).

Proof. Easy to see using Proposition 4.25 and Remark 4.11. □

Lemma 4.27. *Let $A, B, C \subseteq_\omega \mathfrak{M}$ with $C \leqslant A, B \leqslant \mathfrak{M}$ and $A \cap B = C$. Then:*

(1) (Existence) there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$;

(2) (Stationarity) $A \equiv_C A'$, $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$ and $A' \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$, then $A \equiv_B A'$.

Proof. Immediate from Lemma 4.26 and Remark 4.11. □

Conclusion 4.28. *P is ω-stable.*

Proof. As observed in Fact 4.21, the class $\mathbf{K} = Mod(Th(P))$ has finite closures. Thus, the result follows from Lemma 4.15, [4, Theorem 2.28], [4, Theorem 2.21], [4, remark right after 2.20] and [4, Theorem 3.34], where the argument in [4, Theorem 3.34] goes through by Lemma 4.27. □

Corollary 4.29. *Let $A, B, C \subseteq_\omega \mathfrak{M}$ with $C \leqslant A, B \leqslant \mathfrak{M}$ and $A \cap B = C$. Then the following are equivalent:*

(1) $A \mathop{\smile\!\!\!\!\!\mid}_{C} B$ (in the forking sense, cf. e.g. [15, Chapter 6]);

(2) $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B$ (cf. Definition 4.20(3));

(3) $AB = A \oplus_C B$ (cf. Notation 4.10).

Proof. The equivalence (1) $\Leftrightarrow$ (2) is as in [4, Lemma 3.38] using Lemma 4.27, the equivalence (2) $\Leftrightarrow$ (3) is Lemma 4.26. □

Definition 4.30 ([19, Exercise 8.4.2])**.** Let T be a first-order theory. We say that T has *weak elimination of imaginaries* if for every model $M \models T$ and definable set D over $A \subseteq M$ there is a smallest algebraically closed set over which D is definable.

Corollary 4.31. *$Th(P)$ does* not *have weak elimination of imaginaries (Def. 4.30).*

Proof. Let $\{a, b, a', b'\} \subseteq \mathfrak{M}$ be such $|\{a, b, a', b'\}| = 4$, $\{a, b, a', b'\} \leqslant \mathfrak{M}$ and $\{a, b, a', b'\}$ forms an R-clique (i.e. the points a, b, a', b' are collinear). Consider now the definable set $X = \{a, b\} \cup \{c \in \mathfrak{M} : \mathfrak{M} \models R(a, b, c)\}$ in $\mathfrak{M}$. Then in $\mathfrak{M}$ there is no smallest algebraically closed set over which X is definable, since clearly $X = \{a', b'\} \cup \{c \in \mathfrak{M} : \mathfrak{M} \models R(a, b, c)\}$ and both $\{a, b\}$ and $\{a', b'\}$ are algebraically closed in $\mathfrak{M}$ (recall Conclusion 4.24). $\square$

We now introduce the notion of a theory being one-based, a crucial property in geometric model theory.

Definition 4.32. Let T be an ω-stable first-order theory, and let $\mathfrak{M}$ be its monster model. We say that T is one-based if for every $A, B \subseteq \mathfrak{M}$ such that $A = acl(A)$ and $B = acl(B)$ we have that $A \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} B$.

Proposition 4.33. *$Th(P)$ is* not *one-based.*

Proof. Let $C \leqslant \mathfrak{M}$ be a simple rank 3 matroid with domain $\{p_1, p_2, p_3\}$. Let $B \leqslant \mathfrak{M}$ be an extension of C with a generic point q_1 (i.e. q_1 is not incident with any line from C). Let $D \leqslant \mathfrak{M}$ be an extension of C with a new point q_2 under the line $p_1 \vee q_1$. Notice now that the the submatroid A of D with domain $\{p_1, p_2, p_3, q_2\}$ is such that $A \leqslant D$, since $\delta(A) = \delta(D)$. Thus, $A, B, C \leqslant \mathfrak{M}$, $A \cap B = C$ and $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C B$ (by Corollary 4.29). $\square$

The following four items are an adaptations of items 4.6, 4.8, 4.9, 4.10 of [22]. We will use them to show that $\mathfrak{M}$ has Morley rank ω, using the argument laid out in [22, Proposition 4.10].

Lemma 4.34. *Let $B \leqslant C \in \mathbf{K}_0$ be a primitive extension (cf. Definition 4.13(2)). Then there are two cases:*

(1) $\delta(C/B) = 1$ and $C = B \cup \{c\}$;

(2) $\delta(C/B) = 0$.

Proof. Suppose that $B \leqslant C \in \mathbf{K}_0$, $\delta(C/B) > 0$ and $c_1 \neq c_2 \in C - B$. We make a case distinction:
Case 1. c_1 or c_2 is not incident with any line from B.
Without loss of generality c_1 is not incident with any line from B. Then, $\delta(Bc_1) = \delta(B)+1 \leqslant \delta(C)$, where the second inequality is because $\delta(C/B) > 0$, and so $B < Bc_1 < C$. Hence, in this case we have that $B \leqslant C$ is *not* primitive.
Case 2. c_1 and c_2 are both incident with a line from B.
Then $\delta(B) = \delta(Bc_1) \leqslant \delta(C)$ and so $B < Bc_1 < C$. Hence, also in this case we have that $B \leqslant C$ is *not* primitive.
Thus, from the above argument we see that if $B \leqslant C$ is primitive and $\delta(C/B) > 0$, then $C = B \cup \{c\}$, and so $\delta(C/B) = 1$ (cf. Proposition 4.14). $\square$

Remark 4.35. Notice that it is possible that $B \leqslant C \in \mathbf{K}_0$ is primitive, $\delta(C/B) = 0$ and $|C - B| \geqslant 2$. To see this, consider the plane whose geometric lattice is represented in Figure 2 and let $B = \{a, b, c\}$ and $C = \{a, b, c, d, e, f\}$. To be more clear we explain how to read Figure 2. The element $\emptyset$ on the first line of the diagram is the element 0 of the lattice. The elements a, b, c, d, e, f on the second line of the diagram represent the points of the matroid and thus the elements in the domain of the structure $M = (V, R)$ as in Definition 3.1. The elements on the third line of the diagram represent the lines of the geometric lattice, i.e. the closed subsets of rank 2. Finally the element in the fourth line represent the 1 of the lattice.

$$abcdef$$

$$ab \quad ac \quad bc \quad adf \quad bd \quad cde \quad ae \quad bef \quad cf$$

$$a \quad b \quad c \quad d \quad e \quad f$$

$$\emptyset$$

Figure 2: An example.

Lemma 4.36. *Let $B \leqslant C \in \mathbf{K}_0$ be primitive, $C \leqslant \mathfrak{M}$, and suppose that $\delta(C/B) = 0$. Then $tp(C/B)$ is isolated and strongly minimal.*

Proof. As in the proof of [22, Lemma 4.8] replacing the free amalgam $A \otimes_C B$ with the canonical amalgam $A \oplus_C B$ (cf. Notation 4.10). □

Corollary 4.37. *Let $B \leqslant C \in \mathbf{K}_0$, $C \leqslant \mathfrak{M}$, and suppose that $\delta(C/B) = 0$. Then:*

(1) $tp(C/B)$ has finite Morley rank;

(2) the Morley rank of $tp(C/B)$ is at least the length of a decomposition of C/B into primitive extensions.

Proof. Exactly as in [22, Corollary 4.9]. □

Proposition 4.38. *There exists finite $B \leqslant \mathfrak{M}$ and elements q_k, for $k < \omega$, such that $d(q_k/B) = 0$, and the extension $cl(Bq_k)$ has decomposition length k.*

Proof. Let $B \leqslant \mathfrak{M}$ be a simple rank 3 matroid with domain $\{p_1, p_2, p_3\}$. By induction on $k < \omega$, we define $B \leqslant Q_k \leqslant \mathfrak{M}$ such that $q_k \in Q_k$. For $k = 0$, let $Q_0 \leqslant \mathfrak{M}$ be an extension of B with a new point q_0 under the line $p_1 \vee p_2$. For $k = m + 1$, let $Q_k \leqslant \mathfrak{M}$ be an extension of Q_m with a new point q_k under the line $p_2 \vee q_m$ if m is even, and under the line $p_1 \vee q_m$ if m is odd. Then clearly $d(q_k/B) = 0$, and the extension $cl(Bq_k) = Q_k$ has decomposition length k. □

We now restate our main theorem and point out where we have proved the various items.

Theorem 1.2. *There exists a pre-dimension function δ on the class of finite planes (finite simple rank 3 matroids) such that the corresponding "Hrushovski's generic" (cf. Definition 4.16) exists, and so it is a plane P (i.e. a simple rank 3 matroid, cf. Definition 3.1), and it satisfies the following conditions:*

(1) P contains the "non-Desarguesian" matroid (cf. Figure 1, or [20, pg. 139]), and so it is not algebraic (in the sense of matroid theory);

(2) in $Th(P)$ intrinsic closure and algebraic closure coincide (cf. Definition 4.22);

(3) $Th(P)$ does not *have weak elimination of imaginaries (cf. Definition 4.30);*

(4) $Th(P)$ is not one-based (cf. Definition 4.32);

(5) $Th(P)$ is ω-stable and has Morley rank ω (cf. [15, Chapter 6]);

(6) over algebraically closed sets forking in $Th(P)$ corresponds to the canonical amalgamation introduced in [11, Theorem 4.2] (cf. Remark 4.11).

Proof. Concerning item (1), notice that if a matroid is algebraic, then so is any of its submatroids. Thus, P is not algebraic since it contains the "non-Desarguesian" matroid from Example 4.6, which is explicitly shown not to be algebraic in [14, Corollary, pg. 238]. Item (2) is Conclusion 4.24. Item (3) is Corollary 4.31. Item (4) is by Proposition 4.33 and Conclusion 4.24. Concerning item (5), argue as in [22, Proposition 4.10] using Corollary 4.37 and Proposition 4.38. Item (6) is by Corollary 4.29 and Conclusion 4.24. □

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