The Relationship Between Flux Density and Brightness Temperature

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June 3, 2015

Contents

1 The Answer

In this document I derive the general relationship between the flux density of a source and its image brightness temperature for an antenna beam measuring two different source geometries, uniform disk and gaussian. For the impatient I give the answer here. For those interested in the details, subsequent sections provide those details. For a point source, measured with a gaussian antenna beam from an antenna with main reflector diameter D, the relationship between the source's flux density ($S_\nu^{point}(Jy)$) and its Rayleigh-Jeans equivalent brightness temperature (T_A) is given by:

$$\boxed{\begin{aligned} S_\nu^{point}(Jy) &= \frac{\pi k}{2\left(\ln(2) D\right)^2}\frac{T_A}{\eta_{mb}} \\ &\simeq 4513.90\frac{T_A}{D^2(m)\eta_{mb}}\ (K) \end{aligned}} \tag{1}$$

2 Introduction

In the following I derive the relationship between the flux density of a source and its brightness temperature for an antenna beam measuring two different source geometries, uniform disk and gaussian. The general relation between the flux density of a source and its brightness temperature is

$$S_\nu = \frac{2k}{\lambda^2} \int T_B(\Omega) d\Omega \tag{2}$$

Note that Equation 2 assumes that the Rayleigh-Jeans approximation ($h\nu \ll kT$) applies. In a way, then, this is a fictitious temperature which is useful for describing the output power from a radio source using a relation that is linearly proportional to temperature. For a source with brightness temperature distribution $\psi(\xi, \eta)$, we can define the source solid angle as follows

$$\Omega_s = \int\int \psi(\xi, \eta) d\xi d\eta. \tag{3}$$

The solid angle of the source normalized by the primary beam pattern of the antenna ($f(\xi, \eta)$), sometimes called the "effective source solid angle", is given by

$$\Omega_s' = \int\int f(\xi, \eta)\psi(\xi, \eta) d\xi d\eta, \tag{4}$$

while the main beam solid angle is given by

$$\Omega_m = \int\int_{mb} f(\xi, \eta) d\xi d\eta, \tag{5}$$

and the full beam solid angle is given by

$$\Omega_A = \int\int_b f(\xi, \eta) d\xi d\eta. \tag{6}$$

As the main beam full-width at half-maximum (FWHM) beam width θ_B is defined as follows (see Baars 2007, "The Paraboloidal Reflector Antenna in Radio Astronomy and Communication", Chapter 4):

$$\theta_B = \frac{b\lambda}{D}, \tag{7}$$

it is convenient in the following to parameterize the main beam solid angle in terms of the illumination taper employed by the measurements:

$$\begin{aligned} \Omega_m &= \frac{\pi}{4}\left(\frac{b\lambda}{D}\right)^2 \\ &= \frac{\pi b^2 \theta_{Ba}\theta_{Bb}}{4}, \end{aligned} \tag{8}$$

where b is the illumination taper factor. For a Gaussian beam $b = 1/\sqrt{\ln(2)} \simeq 1.2$.

With measurements of the solid angle of the source normalized by the primary beam pattern of the antenna (Ω_s'), the primary (main) beam (Ω_m), and the main beam efficiency (η_{mb}), we can write the relation between the measured antenna temperature (T_A) and the source brightness temperature (T_B) as follows

$$T_A = \frac{\Omega_s'}{\Omega_m} T_B \eta_{mb} \tag{9}$$

Substituting for the quantity $\int T_B d\Omega_s = T_B \Omega_s$ in Equation 2 results in the following general relation between flux density and measured antenna temperature

$$\begin{aligned} S_\nu &= \frac{2k}{\lambda^2} T_A \Omega_A \frac{\Omega_s}{\Omega'_s} \\ &= \frac{2k}{\lambda^2} \frac{T_A}{\eta_{mb}} \Omega_m \frac{\Omega_s}{\Omega'_s} \end{aligned} \tag{10}$$

where we have defined the main beam efficiency η_{mb} as follows:

$$\eta_{mb} \equiv \frac{\Omega_m}{\Omega_A} \tag{11}$$

Note that the term $\frac{\Omega_s}{\Omega'_s}$ defines the coupling of the measured source to the beam of the antenna.

In the following I calculate $\frac{\Omega_s}{\Omega'_s}$ for two standard source distributions. For these calculations I use the simplification that the antenna beam is circular ($\theta_{Ba} = \theta_{Bb} = \theta_B$). This is a common situation in practice, and avoids the need to include a position angle term in our beam and source major (θ_{Ba} and θ_{Sa}) and minor (θ_{Bb} and θ_{Sb}) axis terms[1].

3 Elliptical Gaussian Source

For an elliptical Gaussian source with dimensions θ_{Sa} θ_{Sb} and an elliptical beam with FWHM θ_B,

$$\begin{aligned} \Omega_S &= \int\int_{-\infty}^{\infty} \exp\left[-4\ln(2)\left(\frac{\xi^2}{\theta_{Sa}^2} + \frac{\eta^2}{\theta_{Sb}^2}\right)\right] d\xi d\eta \\ &= \frac{\pi\theta_{Sa}\theta_{Sb}}{4\ln(2)} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \Omega_m &= \int\int_{-\infty}^{\infty} \exp\left[-\frac{4}{b^2}\left(\frac{\xi^2}{\theta_{Ba}^2} + \frac{\eta^2}{\theta_{Bb}^2}\right)\right] \\ &= \frac{\pi b^2 \theta_B^2}{4}. \end{aligned} \tag{13}$$

With Equations 12 and 13 the solid angle of the source multiplied by the primary beam pattern of the antenna becomes

$$\begin{aligned} \Omega'_S &= \int\int_{-\infty}^{\infty} \exp\left[-4\ln(2)\left(\frac{\xi^2}{\theta_{Sa}^2} + \frac{\eta^2}{\theta_{Sb}^2}\right)\right] \exp\left[-\frac{4}{b^2}\left(\frac{\xi^2+\eta^2}{\theta_B^2}\right)\right] d\xi d\eta \\ &= \frac{\pi b \theta_B^2}{4\sqrt{\ln(2)}} \left(\frac{\theta_{Sa}^2}{\theta_{Sa}^2+\theta_B^2}\right)^{1/2} \left(\frac{\theta_{Sb}^2}{\theta_{Sb}^2+\theta_B^2}\right)^{1/2}, \end{aligned} \tag{14}$$

[1]This entails including the scaling factor $\cos PA$ in all equations that include major and minor axis variables.

where I have used the fact that

$$\int_0^\infty \exp(-r^2x^2)dx = \frac{\sqrt{\pi}}{2r}. \tag{15}$$

We can now write the source correction factor defining the coupling between a Gaussian source and a Gaussian beam as follows

$$\begin{aligned}
\frac{\Omega_s}{\Omega'_s} &= \frac{\frac{\pi}{4\ln(2)}\left(\theta_{Sa}\theta_{Sb}\right)}{\frac{\pi b\theta_B^2}{4\sqrt{\ln(2)}}\left(\frac{\theta_{Sa}^2}{\theta_{Sa}^2+\theta_B^2}\right)^{1/2}\left(\frac{\theta_{Sb}^2}{\theta_{Sb}^2+\theta_B^2}\right)^{1/2}} \\
&= \frac{1}{b\sqrt{\ln(2)}}\left(\frac{\theta_{Sa}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\left(\frac{\theta_{Sb}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}.
\end{aligned} \tag{16}$$

Inserting this value for $\frac{\Omega_s}{\Omega'_s}$ and using $\Omega_m = \frac{\pi b^2\theta_B^2}{4}$ in Equation 10 yields

$$\begin{aligned}
S_\nu(Jy) &= \frac{2k\nu^2}{c^2}\frac{\pi b^2\theta_B^2}{4}\left(\frac{\theta_{Sa}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\left(\frac{\theta_{Sb}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\frac{T_A}{\eta_{mb}} \\
&= \frac{2k}{c^2}10^{23}\left[\frac{(2\pi)(10^9)}{(360)(3600)}\right]^2\frac{\pi b^2\nu^2(GHz)\theta_B^2(arcsec)}{4} \\
&\quad \left(\frac{\theta_{Sa}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\left(\frac{\theta_{Sb}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\frac{T_A}{\eta_{mb}}\ (K) \\
&= 5.669\times 10^{-7}b^2\nu^2(GHz)\theta_B^2(arcsec) \\
&\quad \left(\frac{\theta_{Sa}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\left(\frac{\theta_{Sb}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\frac{T_A}{\eta_{mb}}\ (K) \\
S_\nu^{gauss}(Jy) &\simeq 0.750\left(\frac{b\nu(GHz)}{115}\right)^2\frac{\theta_B^2(arcsec)}{100} \\
&\quad \left(\frac{\theta_{Sa}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\left(\frac{\theta_{Sb}^2+\theta_B^2}{\theta_B^2}\right)^{1/2}\frac{T_A}{\eta_{mb}}\ (K).
\end{aligned} \tag{17}$$

Note that for a point source $\theta_{Sa}, \theta_{Sb} \ll \theta_B$, and Equation 17 becomes:

$$\begin{aligned}
S_\nu(Jy) &= \frac{2k\nu^2}{c^2}\frac{\pi b^2\theta_B^2}{4}\frac{T_A}{\eta_{mb}} \\
&= \frac{2k}{c^2}10^{23}\left[\frac{(2\pi)(10^9)}{(360)(3600)}\right]^2\frac{\pi b^2\nu^2(GHz)\theta_B^2(arcsec)}{4}\frac{T_A}{\eta_{mb}}\ (K) \\
&= 5.669\times 10^{-7}b^2\nu^2(GHz)\theta_B^2(arcsec)\frac{T_A}{\eta_{mb}}\ (K) \\
S_\nu^{point}(Jy) &\simeq 0.750\left(\frac{b\nu(GHz)}{115}\right)^2\frac{\theta_B^2(arcsec)}{100}\frac{T_A}{\eta_{mb}}\ (K).
\end{aligned} \tag{18}$$

Still assuming a point source, If you prefer to use the reflector diameter (D) rather than the beam size (θ_B), Equation 18 can be written as follows

$$\begin{aligned} S_\nu^{point}(Jy) &= \frac{2k\nu^2}{c^2}\frac{\pi b^2}{4}\left(\frac{cb}{\nu D}\right)^2\frac{T_A}{\eta_{mb}} \\ &= \frac{\pi k b^4}{2D^2}\frac{T_A}{\eta_{mb}} \\ &= \frac{\pi k b^4}{2D^2(m)}\frac{10^{23}}{(100)^2}\frac{T_A}{\eta_{mb}}\ (K) \\ &\simeq 2168.718\frac{b^4 T_A}{D^2(m)\eta_{mb}}\ (K). \end{aligned} \tag{19}$$

4 Uniform Disk Source

For a uniform disk source, such as a planet or asteroid, with equatorial and poloidal angular sizes θ_{eq} and θ_{pol}, and whose area, for a source radius R_s, is given by $\int_0^{R_s} 2\pi r dr$:

$$\begin{aligned} \Omega_S' &= \int_0^{R_s} 2\pi r\left[\exp\left(-\frac{4r^2}{b^2\theta_B^2}\right)\right]dr \\ &= \frac{\pi b^2\theta_B^2}{4}\left[1-\exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right], \end{aligned} \tag{20}$$

where I have used the fact that

$$\frac{d}{dr}\left[\exp\left(-\frac{4r^2}{b^2\theta_B^2}\right)\right] = -\frac{8r}{b^2\theta_B^2}\left[\exp\left(-\frac{4r^2}{b^2\theta_B^2}\right)\right]. \tag{21}$$

Since the integral over the source is given by $\Omega_s = \int_0^{R_s} 2\pi r = \pi R_s^2$, we can write the source correction factor defining the coupling between a disk source and a Gaussian beam as follows

$$\begin{aligned} \frac{\Omega_s}{\Omega_s'} &= \frac{\pi R_s^2}{\frac{\pi b^2\theta_B^2}{4}}\left[1-\exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right]^{-1} \\ &= \frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\left[1-\exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right]^{-1}. \end{aligned} \tag{22}$$

Inserting this value for $\frac{\Omega_s}{\Omega'_s}$ and using $\Omega_m = \frac{\pi b^2 \theta_B^2}{4}$ in Equation 10 yields

$$\begin{aligned}
S_\nu(Jy) &= \frac{2k\nu^2}{c^2}\frac{\pi b^2\theta_B^2}{4}\frac{1}{b^2}\frac{\theta_{eq}\theta_{pol}}{\theta_B^2} \\
&\quad \left[1 - \exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right]^{-1}\frac{T_A}{\eta_{mb}} \\
&= \frac{2k}{c^2}10^{23}\left[\frac{(2\pi)(10^9)}{(360)(3600)}\right]^2\frac{\pi\nu^2(GHz)\theta_{eq}(arcsec)\theta_{pol}(arcsec)}{4} \\
&\quad \left[1 - \exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right]^{-1}\frac{T_A}{\eta_{mb}}\ (K) \\
&= 5.669\times 10^{-7}\nu^2(GHz)\theta_{eq}(arcsec)\theta_{pol}(arcsec) \\
&\quad \left[1 - \exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right]^{-1}\frac{T_A}{\eta_{mb}}\ (K) \\
S_\nu^{disk}(Jy) &\simeq 0.749\left(\frac{\nu(GHz)}{115}\right)^2\frac{\theta_{eq}(arcsec)\theta_{pol}(arcsec)}{100} \\
&\quad \left[1 - \exp\left(-\frac{\theta_{eq}\theta_{pol}}{b^2\theta_B^2}\right)\right]^{-1}\frac{T_A}{\eta_{mb}}\ (K). \qquad (23)
\end{aligned}$$

Note that for a point source $\theta_{eq}, \theta_{pol} \ll \theta_B$, and Equation 23 becomes:

$$\begin{aligned}
S_\nu(Jy) &= \frac{2k\nu^2}{c^2}\frac{\pi b^2\theta_B^2}{4}\frac{T_A}{\eta_{mb}} \\
&= \frac{2k}{c^2}10^{23}\left[\frac{(2\pi)(10^9)}{(360)(3600)}\right]^2\frac{\pi b^2\nu^2(GHz)\theta_B^2(arcsec)}{4}\frac{T_A}{\eta_{mb}}\ (K) \\
&= 5.669\times 10^{-7}b^2\nu^2(GHz)\theta_B^2(arcsec)\frac{T_A}{\eta_{mb}}\ (K) \\
S_\nu^{point}(Jy) &\simeq 0.750\left(\frac{b\nu(GHz)}{115}\right)^2\frac{\theta_B^2(arcsec)}{100}\frac{T_A}{\eta_{mb}}\ (K), \qquad (24)
\end{aligned}$$

which is exactly the same relation derived from the assumption of an elliptical gaussian source and beam in Equation 18 (as it should be).