SOME NEW FIXED POINT RESULTS FOR CONVEX CONTRACTIONS IN B-METRIC SPACES

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ABSTRACT

The purpose of this paper is to consider various results for convex contraction mappings in the context of B-metric spaces. We, among other things, generalize, extend, correct and enrich the recent published results from the context of convex contractions defined on an ordinary metric spaces to the ones on the so–called B-metric spaces. One example shows that this generalization is genuine. Let us note that this paper represents only the beginning of our investigation of the properties of convex contractions observed in any general metric space.
In the papers that are to be published, our considerations are applied to cone metric spaces, partial metric spaces, G-metrics, G_B-metrics, extended B-metric spaces and many others.

Keywords: Convex contraction, Fixed point, B-metric space, B-Cauchy sequence

INTRODUCTION AND PRELIMINARIES

Since S. Banach proved his theorem that every contraction on a complete metric space has a unique fixed point, in his doctoral dissertation in 1922, many mathematicians have tried to generalize the famous result. Hundreds of scientific papers have considered this issue.

The previously mentioned statement, leads in two key directions: either the axioms of metric spaces are affected ($d_Y(u, v) = 0$ if and only if $u = v$; $d_Y(u, v) = d_Y(v, u)$; $d_Y(u, w) \leq B(d_Y(u, v) + d_Y(v, w)))$, where d_Y is a function defined on the Cartesian product $Y \times Y$, where Y is a non-empty set with values in $[0, +\infty)$, or it affected the Banach contraction condition ($d_Y(fu, fv) = kd_Y(u, v)$, $\forall u, v \in Y$, where $k \in [0, 1)$). More details about Banach generalization could be seen in (Aleksić et. al., 2019a), (Aleksić et. al., 2019b; Aleksić et. al., 2019c; Alghamdi et. al., 2011; Ampadu, 2017a; Ampadu, 2017b; Andras, 2003; Baakhtin, 1989; Collaco & E Silva, 1997; Istratescu, 1981; Istratescu, 1982; Istratescu, 1983; Jeong & Rhoades, 2005; Kirk et. al., 2003; Kirk & Shahzad, 2014; Rhoades, 1977).

Recently, fixed point theory has been one of the most important research fields in nonlinear and functional analysis. It has wide applications in many disciplines like studying the existence of solutions for nonlinear (algebraic, differential and integral) equations, a system of linear (nonlinear) equations and convergence of many computational methods, economics, sports, medical sciences, etc.

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. We repeat some definitions and results, which will be needed in the sequel.

Definition 1. Let Y be a (non-empty) set and $B \geq 1$ a given real number. A function $d_Y : Y \times Y \to [0, +\infty)$ is said to be a B-metric on Y if the following conditions are satisfied:

(b_1) $d_Y(u, v) = 0$ if and only if $u = v$;
(b_2) $d_Y(u, v) = d_Y(v, u)$ for all $u, v \in Y$;
(b_3) $d_Y(u, w) \leq B(d_Y(u, v) + d_Y(v, w))$ for all $u, v, w \in Y$.

The triplet (Y, d_Y, B) is called a B-metric space with coefficient B.

For more notions such as B-continuous, B-convergence, B-completeness, B-Cauchy sequence, all in the framework of B-metric spaces, the reader is referred to (Aleksić et. al., 2019a; Aleksić et. al., 2019b; Aleksić et. al., 2019c; Ampadu, 2017a; Baakhtin, 1989; Kirk & Shahzad, 2014).

For example, the sequence $\{u_n\}$ in B-metric space (Y, d_Y) converges to the point $u \in Y$ if for each positive number η there is a natural number k_0 such that when $n \geq k_0$ we have $d_Y(u_n, u) < \eta$. This can be also written as $u_n \to u$ when $n \to \infty$ or even as $\lim_{n\to\infty} d_Y(u_n, u) = 0$.

The sequence in B-metric space (Y, d_Y) is B-Cauchy if the following is satisfied: For each positive number η there is a natural number k_0 such that when $m, n \geq k_0$ we have $d_Y(u_n, u_m) < \eta$. We can also write the last expression in the form $\lim_{m,n\to\infty} d_Y(u_n, u_m) = 0$.

We say that the B-metric space is B-complete if every B-Cauchy sequence in it converges.

Let us note that every B-convergent sequence in the B-metric space (Y, d_Y) is also B-Cauchy. Indeed, this fact follows simply from the next inequality:

$$\frac{1}{B} d_Y(u_n, u_m) \leq d_Y(u_n, u) + d_Y(u, u_m).$$

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MATHEMATICS, COMPUTER SCIENCE AND MECHANICS

It is well-known that in every ordinary metric space (M, d^M) the metric d^M is a continuous function with two variables. This means that $d^M(a_n, b_n) \to d^M(a, b)$ when $n \to \infty$ whenever $a_n \to a$ and $b_n \to b$ in the case when $n \to \infty$.

However, it is not the case in B-metric spaces, and this is one of the most important properties that distinguishes B-metric spaces (if $B > 1$) from metric spaces ($B = 1$).

The following, well-known, example in the literature shows this.

Example 2. Let $Y = \mathbb{N} \cup \{\infty\}$ and $d_Y : Y \times Y \to [0, +\infty)$ be defined in the following way

$$
d_Y(p, q) = \begin{cases} 0, & \text{if } p = q, \\ \left| \dfrac{1}{p} - \dfrac{1}{q} \right|, & \text{if } p, q \text{ are even, or } pq = \infty, \\ 5, & \text{if } p, q \text{ are odd, and } p \neq q, \\ 2, & \text{in other cases.} \end{cases}
$$

It is easy to verify that the triangle relation is fulfilled in the context of the B-metric space for $B = 3$, i.e.,

$$
d_Y(p, q) \leq 3 \left(d_Y(p, r) + d_Y(r, q) \right),
$$

for all $p, q, r \in Y$. The other two axioms are obviously satisfied.

We check that the B-metric d_Y is not B-continuous ($B = 3$).

Indeed, if $a_n = 2n$, for each $n \in \mathbb{N}$, then $d_Y(a_n, \infty) = d_Y(2n, \infty) \to 0$, when $n \to \infty$, i.e., $a_n \to \infty$, but it is obvious that $d_Y(a_n, 1) = 2$ is not $d_Y(\infty, 1)$, when $n \to \infty$.

Note that the previous example shows that there exists a B-metric d_Y which is not a metric.

It is important to know how B-metrics can be obtained from the ordinary metric. In that case, the obtained B-metric is a continuous function with two variables. So we have the following:

Let (M, d^M) be a given metric space and let $s > 1$ be a given real number. Then, with $d_M(u, v) = \left(d^M(u, v) \right)^s$ one B-metric is defined on a non-empty set M, where $B = 2^{s-1}$. Obviously, d_M is a continuous B-metric on the set M.

In order to prove a more general result than the previous one, a well-known property of the real function $x \mapsto x^b$, $x > 0$, $b > 1$, is used and it is given in the form

$$
\left(\frac{u + v}{2} \right)^b \leq \frac{u^b + v^b}{2},
$$

whenever u and v are positive numbers and $b > 1$.

Now, let's state this well-known result:

Let (M, d^M) be a given metric space and let $b > 1$, $\lambda \geq 0$, $\alpha > 0$, $u, v \in M$. We define the mapping

$$
d_M(u, v) = \lambda d^M(u, v) + \alpha \left(d^M(u, v) \right)^b.
$$

We have that the function d_M is not the ordinary metric, but (M, d^M) is B- metric space, where coefficient $B = 2^{b-1}$. Indeed, since it is sufficient to check only the triangle relation (the first

two axioms are proved trivial) by applying the previous property of the convex function $x \mapsto x^b$, $x > 0$, $b > 1$, we have

$$
d_M(u, v) = \lambda d^M(u, v) + \alpha \left(d^M(u, v) \right)^b \leq \lambda \left(d^M(u, w) + d^M(w, v) \right) +
$$
$$
+ \alpha \left(\left(d^M(u, w) \right)^b + \left(d^M(w, v) \right)^b \right) \leq \lambda \left(d^M(u, w) + d^M(w, v) \right) +
$$
$$
+ 2^{b-1} \alpha \left(\left(d^M(u, w) \right)^b + \left(d^M(w, v) \right)^b \right) \leq 2^{b-1} \left(d^M(u, w) + d^M(w, v) \right).
$$

Example 3. Let $t \in (0, 1)$ and let

$$
Y = l_t(\mathbb{R}) := \left\{ u = \{u_n\} \subset \mathbb{R} : \sum_{n=1}^{+\infty} |u_n|^t < \infty \right\}.
$$

For $u = \{u_n\}, v = \{v_n\} \in Y$, we define

$$
d_M(u, v) = \left(\sum_{n=1}^{+\infty} |u_n - v_n|^t \right)^{\frac{1}{t}}.
$$

Using the previous property it is easy to show that (Y, d_M) is a B–metric space with coefficient $B = 2^{\frac{1}{t}}$.

At the end of this introductory section, we note the following result that has recently been one of the most popular in the works from the context of the B-metric space:

• If for some sequence $\{u_n\}$ in the B-metric space (Y, d_Y) is satisfied the inequality:

$$
d_M(u_{n+1}, u_n) \leq \mu d_M(u_n, u_{n-1}),
$$

for each $n \in \mathbb{N}$, where $\mu \in [0, 1)$, then the sequence $\{u_n\}$ is B-Cauchy. Otherwise, in the general case, the following auxiliary result is most often used:

• Let (Y, d_Y) be a given B-metric space with a coefficient $B \geq 1$. Suppose that $\{u_n\}$ and $\{v_n\}$ are given sequences that respectively converge to a points u and v. Then the following applies

$$
\frac{1}{B^2} d_Y(u, v) \leq \liminf_{n \to \infty} d_Y(u_n, v_n) \leq \limsup_{n \to \infty} d_Y(u_n, v_n) \leq B^2 d_Y(u, v).
$$

If, in the previous inequality we put that $u = v$, we obtained $\lim_{n \to \infty} d_Y(u_n, v_n) = 0$. Moreover, in that case, for every w we have

$$
\frac{1}{B} d_Y(u, w) \leq \liminf_{n \to \infty} d_Y(u_n, w) \leq \limsup_{n \to \infty} d_Y(u_n, w) \leq B d_Y(u, w).
$$

MAIN RESULTS

In this note we will consider the next contractive condition defined in the context of so-called B-metric spaces. For B-metric spaces see (Aleksić et. al., 2019a; Ampadu, 2017a; Kirk & Shahzad, 2014). Let Y be a non-empty set and d_Y be a metric. Recall from Istratescu (Istratescu, 1981) that a map $I : Y \to Y$ is called a convex contraction mapping of order 2, if for all $u, v \in Y$, $p, q \geq 0$, $p + q < 1$, it holds that

$$
d_Y \left(I^2 u, I^2 v \right) \leq p \, d_Y \left(Iu, Iv \right) + q \, d_Y \left(u, v \right). \tag{1}
$$

Alternatively, one could say $I : Y \to Y$ is a convex type contraction mapping of order 2, if for all $u, v \in Y$ and $0 \leq k \leq \frac{1}{4}$ it holds that

$$
d_Y \left(I^2 u, I^2 v \right) \leq k \left[d_Y \left(Iu, Iv \right) + d_Y \left(u, v \right) \right]. \tag{2}
$$

In the case that (Y, d_Y) is a B-metric space with the coefficient $B \geq 1$ then $p, q \geq 0$, $p + q < \frac{1}{B^2}$, that is, $p + q < \frac{1}{4B^2}$ if eq. (2) holds.

Our following results improve ones from from (Ampadu, 2017a) and (Ampadu, 2017b).

Theorem 4. Let (Y, d_Y) be a B-complete B-metric space with the coefficient $B \geq 1$ and let $I : Y \to Y$ be a B-continuous mapping such that for all $u, v \in Y$ eq. (1) holds, where $p, q \geq 0$, $p + q < \frac{1}{B^2}$. Then I has a unique fixed point (say $z \in Y$) and for all $u \in Y$ the sequence $\{I^n u\}_{n=1}^{+\infty}$ converges to the fixed point z.

Proof. The uniqueness is clear simple and therefore we prove only the existence.

Let z_0 be an arbitrary but fixed point in Y and consider the orbit of z_0 under I, that is, the set $(I^n (z_0))_0^{+\infty}$, $I^0 (z_0) = z_0$. Set $R = \max \left\{ d_Y (z_0, I (z_0)), d_Y \left(I (z_0), I^2 (z_0) \right) \right\}$. Thus for any $m \geq 2$

$$
\begin{aligned}
& d_Y \left(I^{2m+1} (z_0), I^{2m} (z_0) \right) \\
& \leq p\, d_Y \left(I^{2m} (z_0), I^{2m-1} (z_0) \right) \\
& + q\, d_Y \left(I^{2m-1} (z_0), I^{2m-2} (z_0) \right)
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
& d_Y \left(I^{2m-1} (z_0), I^{2m} (z_0) \right) \\
& \leq p\, d_Y \left(I^{2m-2} (z_0), I^{2m-1} (z_0) \right) \\
& + q\, d_Y \left(I^{2m-3} (z_0), I^{2m-2} (z_0) \right).
\end{aligned} \tag{4}
$$

The condition (3) implies the next relations:

$$
\begin{aligned}
d_Y \left(I^3 (z_0), I^2 (z_0) \right) &\leq p\, d_Y \left(I^2 (z_0), I^1 (z_0) \right) \\
&+ q\, d_Y \left(I^1 (z_0), z_0 \right) \leq R (p + q),
\end{aligned}
$$

$$
\begin{aligned}
d_Y \left(I^4 (z_0), I^3 (z_0) \right) &\leq p d_Y \left(I^3 (z_0), I^2 (z_0) \right) \\
+ q\, d_Y \left(I^2 (z_0), I (z_0) \right) &\leq p R (p + q) + q R \leq R (p + q),
\end{aligned}
$$

$$
\begin{aligned}
d_Y \left(I^5 (z_0), I^4 (z_0) \right) &\leq p\, d_Y \left(I^4 (z_0), I^3 (z_0) \right) \\
&+ q\, d_Y \left(I^3 (z_0), I^2 (z_0) \right) \leq p R (p + q) \\
&+ q R (p + q) = R (p + q)^2,
\end{aligned}
$$

because $p + q < \frac{1}{B^2} \leq 1$.

An induction argument shows that

$$
d_Y \left(I^{2m+1} (z_0), I^{2m} (z_0) \right) \leq R (p + q)^m < \frac{R}{B^{2m}} \tag{5}
$$

and the similar by eq. (4) that

$$
d_Y \left(I^{2m-1} (z_0), I^{2m} (z_0) \right) \leq R (p + q)^m < \frac{R}{B^{2m}}, \tag{6}
$$

in the case that $B > 1$. If $B = 1$ the proof follows by (Istratescu, 1981).

Now according (Aleksić et al., 2019a, Remark 2.1.) for $m < n$ follows

$$
\begin{aligned}
d_Y \left(I^m (z_0), I^n (z_0) \right) &\leq B\, d_Y \left(I^m (z_0), I^{m+1} (z_0) \right) \\
&+ B^2\, d_Y \left(I^{m+1} (z_0), I^{m+2} (z_0) \right) + \cdots +
\end{aligned}
$$

$$
\begin{aligned}
&+ B^{n-m-1}\, d_Y \left(I^{n-2} (z_0), I^{n-1} (z_0) \right) \\
&+ B^{n-m-1}\, d_Y \left(I^{n-1} (z_0), I^n (z_0) \right).
\end{aligned} \tag{7}
$$

Further from (7) we obtain that $I^n (z_0)$ is a B-Cauchy sequence in B-metric space (Y, d_Y) with the coefficient $B > 1$.

Indeed, if $m = 2k$, $n = 2l$ ($m = 2k$, $n = 2l - 1$ or $m = 2k - 1$, $n = 2l$), then by using (5), (6) and (7) the result follows. Since (Y, d_Y) is a B-complete B-metric space, then there exists $z \in Y$ such that $u_n \to z$ as $n \to \infty$, i.e., $Iz = z$ because I is B-continuous mapping. The proof of Theorem 4 is complete.

Let for example $m = 4$, $n = 10$, i.e., $k = 2$, $l = 5$. Then

$$
\begin{aligned}
d_Y \left(I^4(z_0), I^{10}(z_0) \right) &\leq B\, d_Y \left(I^4(z_0), I^5(z_0) \right) \\
+ B^2\, d_Y \left(I^5(z_0), I^6(z_0) \right) &+ B^3\, d_Y \left(I^6(z_0), I^7(z_0) \right) \\
+ B^4\, d_Y \left(I^7(z_0), I^8(z_0) \right) &+ B^5\, d_Y \left(I^8(z_0), I^9(z_0) \right) \\
+ B^5\, d_Y \left(I^9(z_0), I^{10}(z_0) \right) &\leq 2B (p + q)^2 \\
+ 2B^3 (p + q)^3 + 2B^5 (p + q)^4 &+ 2B^7 (p + q)^5 = \\
2B (p + q)^2\, \frac{1 - \left[B^2 (p + q) \right]^4}{1 - B^2 (p + q)} &\leq 2B (p + q)^2 (1 + \cdots).
\end{aligned}
$$

Hence, follows the method for general case, for example $m = 2k$, $n = 2l$. ∎

Corollary 5. (Istratescu, 1981, Theorem 1.2.) Let (Y, d_Y) be a complete metric space and let $I : Y \to Y$ be a continuous mapping such that for all $u, v \in Y$ eq. (1) holds, where $p, q \geq 0$, $p + q < 1$. Then I has a unique fixed point (say $z \in Y$) and for all $u \in Y$ the sequence $\{I^n u\}_{n=1}^{+\infty}$ converges to the fixed point z.

Remark 6. It is worth to notice that our proof given in Theorem 4 improves the corresponding from (Ampadu, 2017, Theorem 8). This means that our Theorem 4 is a new result for convex contractions in the context of B-metric spaces given first time in existing literature.

The next is a modification of Example 9 from (Ampadu, 2017).

Example 7. Let $Y = [1, 2]$ and $d_Y (u, v) = |u - v|^2$ and define $I : Y \to Y$ by $I (u) = \frac{u+3}{4}$ for all $u \in Y$. Now observe that with $p = q = \frac{1}{272}$ we have that (1) holds, i.e.,

$$
d_Y \left(I^2 (u), I^2 (v) \right) \leq \frac{1}{272}\, d_Y (Iu, Iv) + \frac{1}{272}\, d_Y (u, v),
$$

for all $u, v \in Y = [1, 2]$.

Remark 8. In the past year, the new class of the B-metric space called Extended B-metric spaces is becoming actual for research. It is defined by applying

$$
(b_3')\quad d_Y(u, w) \leq \Lambda(u, w) \left(d_Y(u, v) + d_Y(v, w) \right)
$$

instead of axiom (b_3) for every $u, v, w \in Y$, where Λ is a function that maps $Y \times Y \to [0, +\infty)$. Obviously, every B-metric space is Extended B-metric.

Let us note that the previous considerations of B-convex contractions can be also treated in other classes of general metric spaces (partial metric spaces, G metric spaces, G_B-metric spaces, B-metric spaces, extended B-metric spaces and many others...).

In fact, even in the context of ordinary metric space, the following important questions have not been considered:

Question 1: What about the existence of the fixed point mapping in the case of complete metric spaces when the right hand side of the expression (1) is:

a) $k \max \left\{ d_Y(u,v), \frac{1}{2}\left[d_Y(u,Iu) + d_Y(v,Iv)\right], \frac{1}{2}\left[d_Y(u,Iv)\right. \right.$ $+ \mathrm{d}_Y(u,v)$ Ćirić general contraction of I order;

b) $k \max \left\{ d_Y(u,v), d_Y(u,Iu), d_Y(v,Iv), \frac{1}{2}\left[d_Y(u,Iv)\right. \right.$ $+ \mathrm{d}_Y(Iu,v)$ Ćirić general contraction of II order;

c) $k \max \left\{ d_Y(u,v), d_Y(u,Iu), d_Y(v,Iv), d_Y(u,Iv), d_Y(Iu,v)\right\}$ Ćirić quasi-contraction.

Question 2: What about the existence of the fixed point mapping in the case of B-metric spaces when the right hand side of the expression (1) is:

a) $k \max \left\{ d_Y(u,v), \frac{1}{2B}\left[d_Y(u,Iu) + d_Y(v,Iv)\right], \frac{1}{2B}\left[d_Y(u,Iv)\right. \right.$ $+ \mathrm{d}_Y(u,v)$;

b) $k \max \left\{ d_Y(u,v), d_Y(u,Iu), d_Y(v,Iv), \frac{1}{2B}\left[d_Y(u,Iv)\right. \right.$ $+ \mathrm{d}_Y(Iu,v)$;

c) $k \max \left\{ d_Y(u,v), d_Y(u,Iu), d_Y(v,Iv), d_Y(u,Iv), d_Y(Iu,v)\right\}$.

Question 3: It is interesting to consider and the case if convex contraction I is not continuous and if it is defined on a metric, that is, on a B-metric space.

Question 4: To consider the existence and uniqueness of the fixed point of continuous convex contraction of the third order, defined both on the ordinary metric and on the B-metric space. In this case the contraindicated condition has a form

$$d_Y(I^3u, I^3v) \le p\, d_Y(I^2u, I^2v) + q\, d_Y(Iu, Iv) + r\, d_Y(u,v),$$

for all $u, v \in Y$, where $p, q, r \in [0, +\infty)$ and $p + q + r < 1$ in the case of the metric space. Find a condition if (Y, d_Y) is a B-metric space with a coefficient $B > 1$.

We conclude our work with another important question for this field.

Question 5: Let M and N be a closed subsets in the complete metric space (Y, d_Y) such that IM is a subset of N and IN is a subset of M. If for all $u \in M$ and for all $v \in N$ it follows that

$$d_Y(I^2u, I^2v) \le p\, d_Y(Iu, Iv) + q\, d_Y(u,v),$$

then the continuous mapping I has a unique fixed point in cross-section of sets M and N. Prove or disprove this claim by example.

Remark 9. Since the termcyclic type mapping was introduced in (Kirk et al., 2003) and thus generalized the famous Banach theorem, many researchers considered the newly introduced phenomenon and obtained corresponding results in various classes of general metric spaces. As far as we know from literature, this type of mapping has not been considered in the case of convex contractions of Istratescu. Therefore, the previous question is interesting for researchers working in this part of the non-linear analysis. Assuming that $M = N = Y$, a positive answer to the last question gives us as a consequence the main result of Istratescu. In other words, we would get a true generalization of Istratescu's result.

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