PLANES AND AXIOMS

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Abstract. Planes are familiar mathematical objects which lie at the subtle boundary between continuous geometry and discrete combinatorics. A plane is geometrical, certainly, but the ways that two planes can interact break cleanly into discrete sets: the planes can intersect or not. Here we review how oriented matroids can be used to try to capture the combinatorial aspect, giving a way to encode with finite sets all the ways that n planes can interact. We mention how the one-to-one correspondence breaks down in 2 dimensions for 9 lines, and in 3D for 8 planes. We include illustrations of all the types of plane arrangements using $n = 4$ and 5.

1. Introduction: Linear algebra

The number of solutions to a system of linear equations can only be 0, 1 or ∞. The geometric explanation of this fact is that each linear equation in n variables determines a hyperplane in $\mathbb{R}^n$: a point in $\mathbb{R}^1$, a line in $\mathbb{R}^2$, a plane in $\mathbb{R}^3$, and so on. Several distinct hyperplanes can: (1) avoid mutual intersection, (2) mutually intersect in one point (we'll need at least n for that), or (3) have an $(n-2)$-dimensional intersection. We say *essential* arrangements are those which have one common point of intersection (at the origin.) We call an arrangement *central* when it has a nonempty intersection of all the hyperplanes. The pictures of two lines which can be parallel or intersecting—and three planes, which have 5 different possibilities, shown in Figure 2—are familiar from the first pages of books on linear algebra.

For several years it has been a mystery to this author why the pictures stop there: why are there no collections of pictures showing all the ways that four or five or more planes can interact? We fix that here: see Figure 1 and then Figures 6- 11. Related questions motivated this paper. We begin with how oriented matroids can be used to encode the combinatorial classes of plane arrangements using finite sets. We present a gentle introduction of the axioms, and review the historical discoveries (from the ancient days of the late 1970's and early 80's). The oriented matroid correspondence breaks down in 2 dimensions for 9 lines, and in 3D for 8 planes, leaving us the consolation prize of upper bounds. Something about the geometry gets in the way of a clean representation via strings of symbols—at least as far as our current understanding can tell. We hope to motivate some improvements! The only new material here is that we (using numbers calculated by Lukas Finschi) sum up the possible abstract ways for $n = 4$ and 5 planes to be arranged in three Euclidean dimensions, and then illustrate them in order to explicitly show that the abstract possibilities are all realized by actual planes; this was predicted for $n \leq 7$

2000 *Mathematics Subject Classification.* 05B35, 52B40, 52C40.
Key words and phrases. hyperplanes, oriented matroids.

by Goodman and Pollack. Some interesting corollaries are seen, like the fact that choosing one of 10 polytopal complexes with 2 or 3 chambers determines an arrangement of 5 hyperplanes in $\mathbb{R}^3$; there are 64 other arrangements determined by other facts. We include some exercises and open problems.

By five ways of intersecting 3 planes in $\mathbb{R}^3$ we mean five combinatorially inequivalent arrangements of hyperplanes. Two arrangements of hyperplanes in n-dimensional Euclidean space are combinatorially isomorphic (or combinatorially equivalent) if they have *isomorphic posets of faces*. Intuitively, this means that a collection of planes in space chop up the space into regions and also intersect each other in various ways. If we label all those regions and intersections (we call them labeled *faces*) then we can list those faces in a hierarchy of boundaries. The highest dimensional faces are regions of the ambient space; these are called *chambers* and have boundaries made of pieces of the hyperplanes. Those pieces are in turn bounded by lower dimensional intersections, and so on. For example we label some faces in Figure 3: the regions X, Z, Q, and Y, line segments S and P, and points R and W. This makes the set of faces partially ordered by inclusion: a boundary face A is less than any face B it is included in (is a boundary of), but not all pairs of faces are thus related. For faces in Figure 3, $R < P < X$. If two such posets are in bijection, and that bijection (and its inverse) respects the ordering, then they are isomorphic.

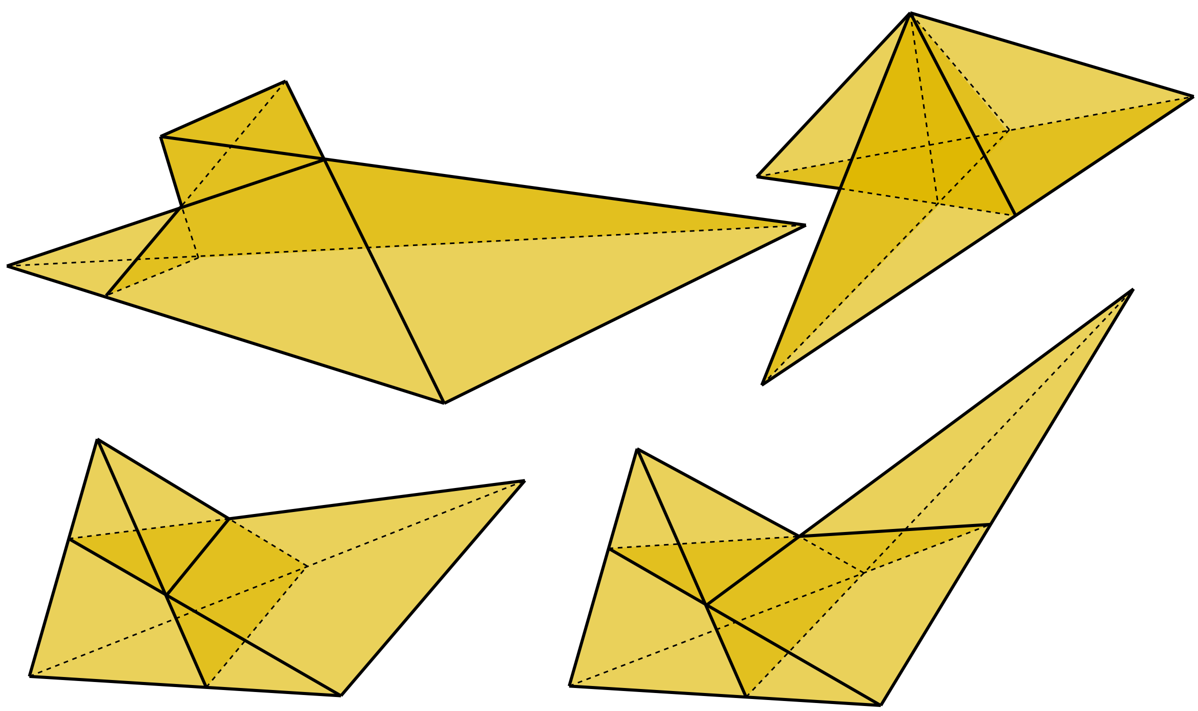

FIGURE 1. Three of the 74 arrangements of five planes have a trio of closed chambers, and the last has four closed chambers. That last is the unique arrangement in general position: all ten subsets of three planes each determine a 0-dimensional intersection.

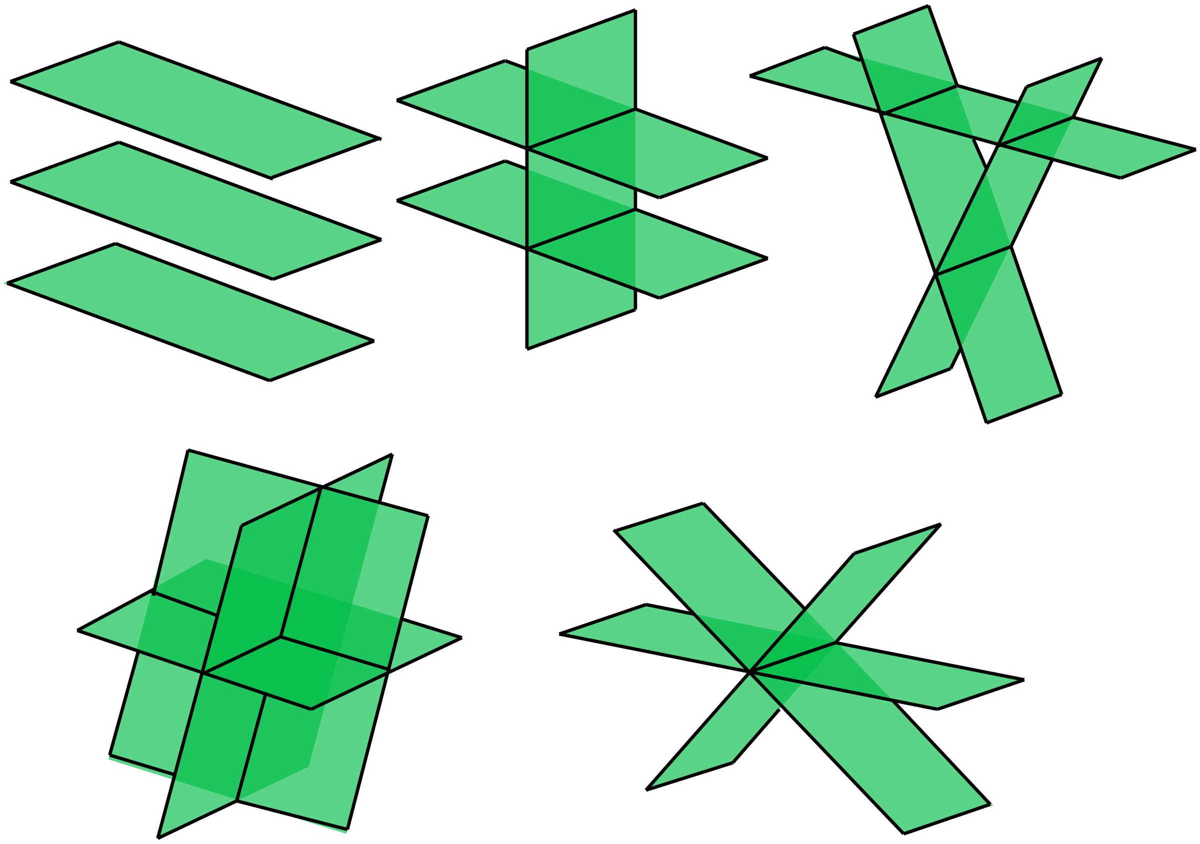

FIGURE 2. Here are the five ways that three planes can intersect. The top row shows the cases in which there is no simultaneous solution to the three (affine) linear equations.

Isomorphism is an equivalence relation. Counting the number of equivalence classes of arrangements, given n distinct hyperplanes in (affine) space, is definitely a hard problem. Note that the number of polytopes with n facets is a subproblem. Some partial answers to this question are out there. For instance, sequence A241600 in [19] counts the number of arrangements of lines in the affine plane, up to $n = 7$, agreeing with the counts in [8, 9]. Peter Shor points out that sequence A241600 is defined differently than via combinatorial equivalence, rather it uses parameterized equivalence, where the homotopy between equivalent arrangements must preserve straight lines and their intersections [18].

2. Oriented matroids from hyperplane arrangements

To get a grasp of how many ways a bunch of hyperplanes might interact, mathematicians have looked hard at the finite poset of faces to see if it has some features which really correspond nicely to geometry. First there is a product structure called composition, where any two faces X and Y produce a third face $Z = X \circ Y$. The way this composition arises via geometry is that you start by choosing any point in the interior of X and moving a tiny positive distance $d < \epsilon$

towards any point interior to Y. For ϵ small enough, this movement lands in the same face every time, which is defined to be $Z = X \circ Y$. Either we see $Z = X$, or the composition is a higher dimensional face Z with $X < Z$. This is not commutative; $W = Y \circ X$ is a different face in most cases. It is not a group; there is usually no inverse face. However, if we restrict our attention to the essential arrangements it is a right-unital monoid, using the origin as the right identity: $X \circ \mathbf{0} = X$.

Another structure on the faces of an arrangement is similar to the composition, but is only defined for certain ordered pairs of faces. If X and Y are completely separated by at least one hyperplane, but Y is a part of any hyperplanes that X is part of, we can define the restriction X_Y of X towards Y as the subset of boundary faces of X on the "side towards" Y. That is, X_Y is the set of boundary faces of X whose points lie on a line connecting some point in the interior of X with some point in the interior of Y. Both structures are illustrated in Example 1, referencing Figure 3.

Noticing the structure (composition, restriction) on the faces of a hyperplane arrangement allows us to generalize by defining a certain kind of collection of vectors on a set, called an *oriented matroid*. We start with a finite *ground set*, usually $[n] = \{1, \ldots n\}$ which corresponds to the planes, numbered 1 to n. Then we have a set of n-tuples, called *sign vectors* (which correspond to the faces), which are ordered lists of length n made of the symbols $+, -, 0$. It is easy to see how to create sign vectors from a hyperplane arrangement. Each hyperplane is given an orientation arbitrarily, and each face X can be described by a sign vector also called X: the value of the component X_i is determined by whether it sits on plane i (in which case $X_i = 0$),

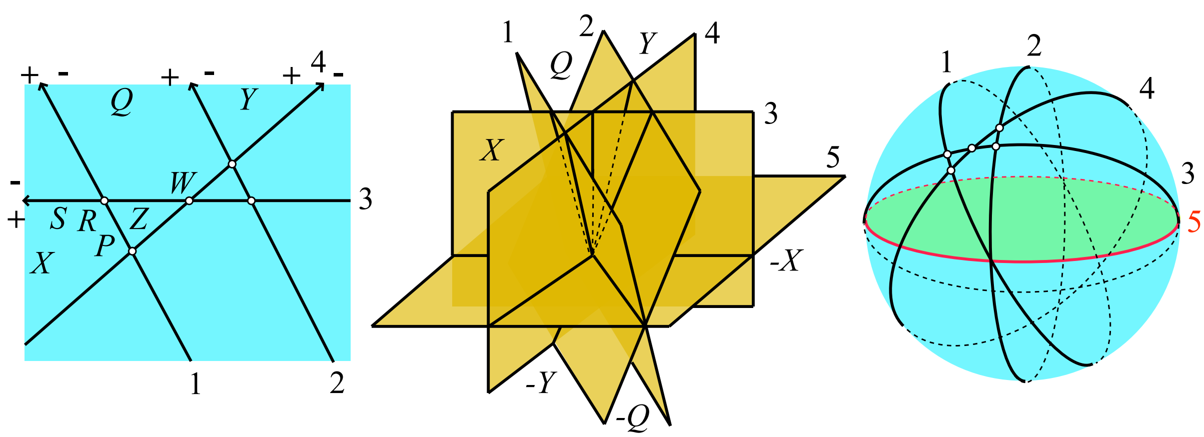

Figure 3. The arrangement of 4 affine lines in the plane shown at left corresponds to the signed suspension, the arangement of 5 planes in the center. The great circles of the sphere on the right are another way to show the same arrangement, and often the equator is seen as a projective "line at infinity." Some opposite faces are shown, and compositions and restrictions of the labeled faces are seen in Example 1.

or on the plus side, or on the minus side. Not all vector sets are called oriented matroids: there are required structures and properties. Sign vectors are also partially ordered. We say $X \leq Y$ when $X_i \neq 0 \Rightarrow X_i = Y_i$. Therefore strict inequality, $X < Y$, means that X is made by turning some of the non-zero components of Y into 0's. We define the opposite, or negative $-X$, of a vector x in the obvious way: switch all the plus signs to minus, and vice versa, leaving 0's as is. That is, $(-X)_i = -(X_i)$ for $i = 1, \ldots, n$. The composition and restriction of any two sign vectors are defined as follows:

$$(X \circ Y)_i = \begin{cases} X_i \ , & X_i \neq 0 \\ Y_i \ , & X_i = 0. \end{cases}$$

$$X_Y = \{Z < X \mid X_i \neq -Y_i \Rightarrow Z_i = X_i\}$$

Example 1. *By looking at some examples in Figure 3, we can see that these operations are designed to mirror the geometric definitions of composition and restriction. For $n = 5$ let $X = (+ + + + +)$, $Y = (- - - + +)$, and $W = (- + 0\ 0\ +)$ so that all three faces can be seen on the left of Figure 3, as an above view of the 3D picture. Then we have:*

$$\begin{aligned}
&W \circ Y = (- + - + +) = Q \\
&Y \circ W = (- - - + +) = Y \\
&W \circ X = (- + + + +) = Z \\
&X_W = \{(0\ + + + +)\} = \{P\} \\
&X_Y = \{(0\ + + + +), (+ +\ 0\ + +), (0\ +\ 0\ + +), \\
&\qquad (+\ 0\ + ++), (0\ 0\ + ++), (+\ 0\ 0\ + +), (0\ 0\ 0\ + +)\} = \{P, S, R, \ldots\}
\end{aligned}$$

Notice that three of the sign vectors in X_Y are not seen as faces in the arrangement! The three that are seen, P, S, and R, fit our geometric description. Now we can list the axioms:

Definition 2. *An oriented matroid on $E = [n]$ is a set V of length n sign vectors: $V \subseteq \{+, -, 0\}^E$, obeying for all $i \in E$ and $X, Y \in V$:*

(SV0) $\mathbf{0} = (0, \ldots, 0) \in V$.

(SV1) $-X \in V$.

(SV2) $X \circ Y \in V$.

(SV3) $((X_i = 0 \Rightarrow Y_i = 0)$ *and* $(\exists j \in E$ *s.t.* $X_j = -Y_j \neq 0)) \Rightarrow X_Y \cap V \neq \emptyset$.

For some easy examples, note that $V = \{\mathbf{0}\}$ and $V = \{+, -, 0\}^E$ are oriented matroids for all n. Thus the size of V is between 1 and 3^n. The first two axioms are for convenience, requiring the matroid to contain $\mathbf{0}$ and all opposite vectors. Axioms $SV2$ and $SV3$ say that compositions and restrictions must exist in V (always for composition, but restriction only for certain ordered pairs.) Axiom $SV3$ we include here is equivalent to the one used by Edmonds and Mandel [15], and labeled $V3''$ in [7]. Several other slightly different versions of that last axiom are used in other publications, such as [17] in [12].

Oriented matroids turn out to describe lots of other mathematical situations, like directed graphs and zonotopes (certain polytopes with parallel facets.) However the important thing about them here is that they are precisely represented by arrangements of planes—with an important caveat. Every combinatorially unique plane arrangement (after labelling the planes $1, \ldots, n$) gives a unique oriented matroid. We have already described how to attach a sign vector to each face. However, there are extra oriented matroids, which you can still draw an arrangement for, but for which some of the planes must be replaced by curved surfaces! We show the famous examples in the next section. It is an open question whether there is some extra requirement of the vectors that will eliminate these extra oriented matroids, leaving only the ones that can be represented by perfectly flat planes. That problem is tempting but probably very hard: the first guess would be to look for some finite list of forbidden sub-arrangements whose presence would obstruct any possible representation via flat hyperplanes. However, this finite list has been shown not to exist; see [3] for more details.

Since we don't have that answer yet, one way to use the oriented matroids for counting plane arrangements is to produce an upper bound. First we exhaustively find all the oriented matroids for a given n and dimension d, and then we try to hit that upper bound by producing each plane arrangement explicitly.

3. Counting oriented matroids and counting pseudoplane arrangements: Stretchability and representability

A *loop* in an oriented matroid is an element $k \in E$ of the ground set for which all the sign vectors have a 0 component. The famous Topological Representability theorem of Folkman and Lawrence, [10], says that equivalence classes of loop-free oriented matroids are in bijection with equivalence classes of arrangements of *pseudohyperplanes*. The latter, including pseudolines and pseudoplanes, are deformations of straight lines and planes, but are required to obey the usual laws of intersection: for instance two pseudolines can intersect at most once.

This leads us to a pair of counting problems. Counting the number of rank d oriented matroids on $[n]$, so the number of arrangements of n pseudohyperplanes, is hard in itself. An open problem is to find a good formula. However it can be done in finite time on a computer, and many of the smaller values were calculated in 2001, in the thesis of Lukas Finschi [8]. We list some of these in Table 1, with values taken directly from that source, but listed by number of hyperplanes and with a new row of totals. The second problem is to count the number of actual hyperplane arrangements, but this is much harder since we don't have a perfect set-theoretical model for them. We do know when the enumerations separate into two distinct problems, at least for dimensions 2 and 3.

The Pappus arrangement of 9 lines helps us find the first counterexample showing that not all oriented matroids are represented as hyperplane arrangements. This is the lefthand arrangement

$d =$	$n =$	1	2	3	4	5	6	7	8	9	10
1		1	1	1	1	1	1	1	1	1	1
2			1	3	8	46	790	37829	4134939	?	?
3				1	5	27	1063	1434219	?	?	?
$d \leq 3$		1	2	5	14	74	1854	1472049	?	?	?

TABLE 1. Numbers of equivalence classes of loop-free oriented matroids of rank $d \leq 3$, also called abstract dissection types in [8]. The values in the last row are also the numbers of plane arrangements, for $n \leq 7$.

in Figure 4, and the key feature is that points of intersection on the top and bottom horizontal lines force the three points in the middle to be colinear. That suggests the second picture: it is an arrangement of pseudolines which thus cannot be straightened while preserving their face structure—we call this situation *nonstretchable*. It was found by Levi, and Goodman and Pollack proved Grunbaum's conjecture that no arrangement of 8 or fewer pseudolines is nonstretchable [13]. Thus the total number of arrangements of 8 lines is 41349340. For 9 lines, the number is unknown: but Richter-Gebert showed that the example in Figure 4 is the unique nonstretchable case [16].

Figure 5 is an example of a non-stretchable oriented matroid where $n = 8$ in dimension 3, found by Goodman and Pollack in [14]. The 8 planes in the first picture are the four planes in the tetrahedron $OABC$, the three planes made by the triangles inside it: $\Delta A'B'C$, $\Delta AB'C'$, and $\Delta A'BC'$, and the plane which contains O, P, Q, and R. This eighth plane is determined: P is the intersection of lines $\overline{BC} \cap \overline{B'C'}$, $Q = \overline{AC} \cap \overline{A'C'}$, and $R = \overline{AB} \cap \overline{A'B'}$. Goodman and Pollack show that the version where we bend the eighth plane enough to miss the point P is a non-stretchable pseudoplane arrangement. They also show that eight pseudoplanes are required to make such an example, so that all arrangements of 7 or fewer planes are stretchable. Thus we can conclude by finding all the arrangements of 4 or 5 planes in $\mathbb{R}^3$, since the total numbers coincide with the total numbers of oriented matroids.

4. Illustrating plane arrangements: n=4 and n=5 planes

The center picture of Figure 3 is an example of a hyperplane arrangement of 5 planes in $\mathbb{R}^3$. It is a *signed suspension* of the affine line arrangement to its left. The signed suspension (or coning, or just suspension) is found by placing the lower dimensional (here, 2D) arrangement in the Euclidean hyperplane at constant height on the n^{th} axis (here $z = 1$), and then taking as the new hyperplanes the spans of the original lines, plus the plane at $z = 0$. The suspensions of two different line arrangements can be equivalent. This highlights the difficulty of counting the actual number of ways to arrange n planes in $\mathbb{R}^3$. The arrangements are of several types:

1) The trivial arrangement of n parallel planes

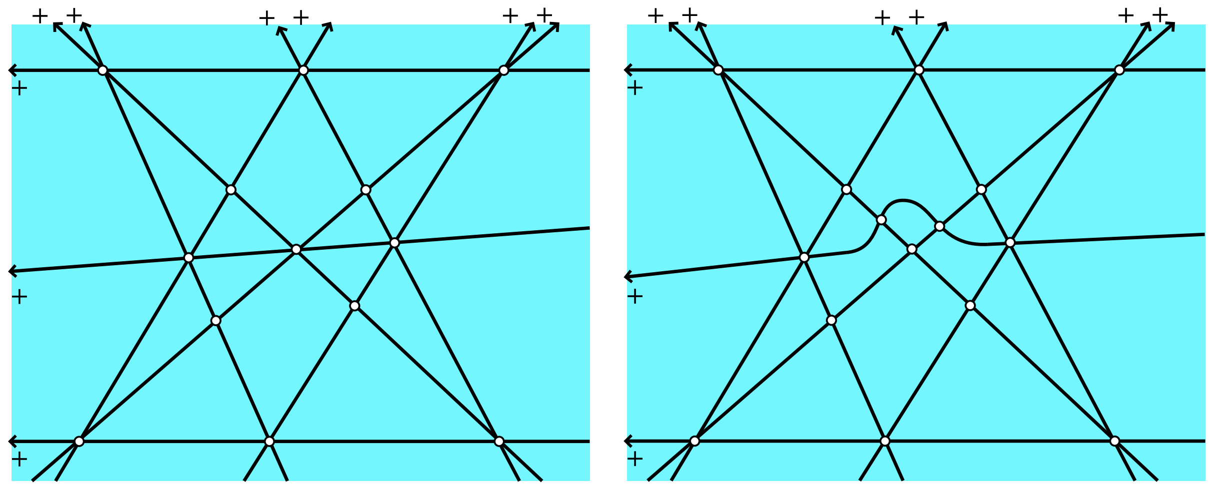

FIGURE 4. The Pappus arrangement on the left, and a nonstretchable pseudoline arrangement on the right.

2) The cross product of any nontrivial n-line arrangement in the xy-plane with the z-axis. If the line arrangement is essential, then the cross-product is central. There are 8 of these using 4 planes, as seen in Figure 6. There are 46 of these with 5 planes, as seen in Table 1 (where the arrangements of 5 lines are listed). For pictures of the 46 line arrangements see [8].

3) The signed suspensions of any line arrangements of $n-1$ lines. Distinct affine line arrangements can produce equivalent signed suspensions. All signed suspensions of affine arrangements will be essential. These are counted directly, checking for duplicates. Two of these are seen using 4 planes in Figure 7. We find 3 of these using five planes, as seen in Figure 8. We leave it as an exercise to the reader to see which of the three the other line arrangements suspend to become! That there are only those three is also seen indirectly once we list the rest of the 5-plane arrangements to fill out the 27 predicted by Table 1.

Exercise 3. *For each of the 8 nontrivial line arrangements of 4 lines, find the signed suspensions (each will have 5 planes). One is shown in Figure 3, but it is a duplicate of one of the three shown in Figure 8. Classify all 8 into the three combinatorial classes.*

Problem 4. *Find invariants for hyperplane arrangements that can predict when their signed suspensions will be combinatorially inequivalent. Test these on the 46 line arrangements of five lines shown in [8].*

4) The cross-product of an $(n-1)$-line arrangement with the z-axis, together with the xy-plane itself (or any plane perpendicular to the z-axis.) Two of these are seen using 4 planes in Figure 7. There are 8 of these using 5 planes, since using the trivial arrangement of 4 lines yields a duplicate of a cross product from 2. Figure 9 shows an example.

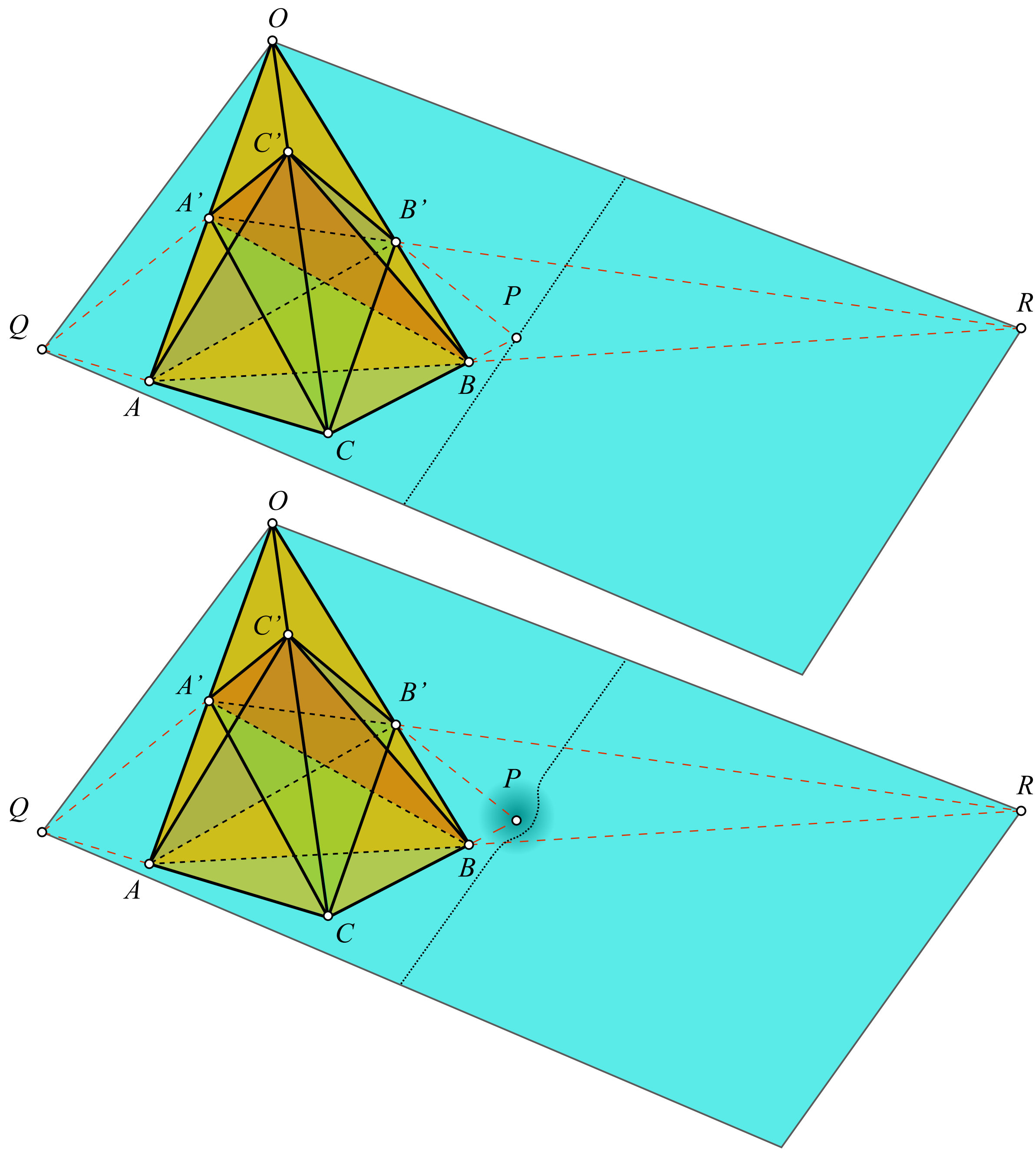

FIGURE 5. Top: an arrangement of 8 planes described by Goodman and Pollack in [14]. Bottom: the associated nonstretchable arrangement of 8 pseudoplanes, where point P is not on the pseudoplane containing ΔOQR.

5) "New" affine arrangements (not using cross-products or suspensions). These can be counted abstractly by suspending them into $\mathbb{R}^4$, to produce essential arrangements of 4D hyperplanes. In practice, we find most of them by focusing on closed polytopal complexes, arrangements with various inclusions of faces that are closed 3D chambers. One of these is seen using 4 planes in

Figure 7. Using 5 planes there is only one of these that has no closed chambers, in Figure 9. That leaves 15 to finish the count of 74 using 5 planes, all are shown in Figures 10, 11, and 1.

For more elementary reading we recommend first the excellent introductions to arrangements and oriented matroids in [12]. Perhaps the most exciting new developments are the specializations of oriented matroids, by adding extra requirements like *purity* and *positivity*, for the added value in studying networks. The connections between pure oriented matroids and positroids are discussed by Galashin and Postnikov [11]. The connections between realizability, positively oriented matroids, and positroids are described by Ardila, Rincón and Williams in [3, 4]. For more algebraic structures on the faces of hyperplanes, including some category theory and Hopf algebras, we recommend the new books by Aguiar and Mahajan: [1] and [2]. For really ambitious readers who appreciate the positroids and related positive geometry, there is the new Amplituhedron approach to scattering matrices of Arkani-Hamed and collaborators, in [6] and [5], among many other papers.

Knowing that there are precisely 74 arrangements of 5 hyperplanes in $\mathbb{R}^3$ gives us lots of power. We have drawn all 27 (excluding the 47 found by crossing a line arrangement with the z-axis) and now since we can show that our pictures are distinct combinatorially, then there are no further combinatorial possibilities. That allows other questions to be answered. For instance, we see that the arrangement of 2 or more closed chambers completely determines the 5 hyperplane arrangement in $\mathbb{R}^3$. We close with some more exercises and open problems, neither guaranteed to be easy!

Exercise 5. *Choosing any of the illustrated hyperplane arrangements in this paper, analyze the associated matrices. For instance, an arrangement from Figure 6 or 7 comes from a matrix equation $A\mathbf{x} = \mathbf{b}$ where A is 4×3, $\mathbf{x}$ is a proposed solution in $\mathbb{R}^3$, and $\mathbf{b} \in \mathbb{R}^4$. What are the rank and nullity of A? Does a solution $\mathbf{x}$ exist? Can $\mathbf{b}$ be the 0-vector? Numbering the rows of A for the 4 planes, which sets of rows are linearly independent?*

Exercise 6. *Lots of other pictures of 5-plane arrangements can be constructed: the challenge is to always find their combinatorial equivalent in the listing. For instance, Figure 9 shows adding two parallel planes to one of the five arrangements in Figure 2. Do the same for the other four, but where are they in the list already?*

Exercise 7. *The number of arrangements of n planes that have a single closed tetrahedral chamber (and no other closed chamber) is $0, 1, 2, \ldots,$ for $n = 3, 4, 5, \ldots$. What is the number for $n = 6$? Hint: notice in Figure 10 the location of the fifth plane, the one not forming a side of a tetrahedron.*

Problem 8. *The number of arrangements of n planes that have a single closed tetrahedral chamber (and no other closed chamber) is $0, 1, 2, \ldots,$ for $n = 3, 4, 5, \ldots$. What is the general formula for this sequence?*

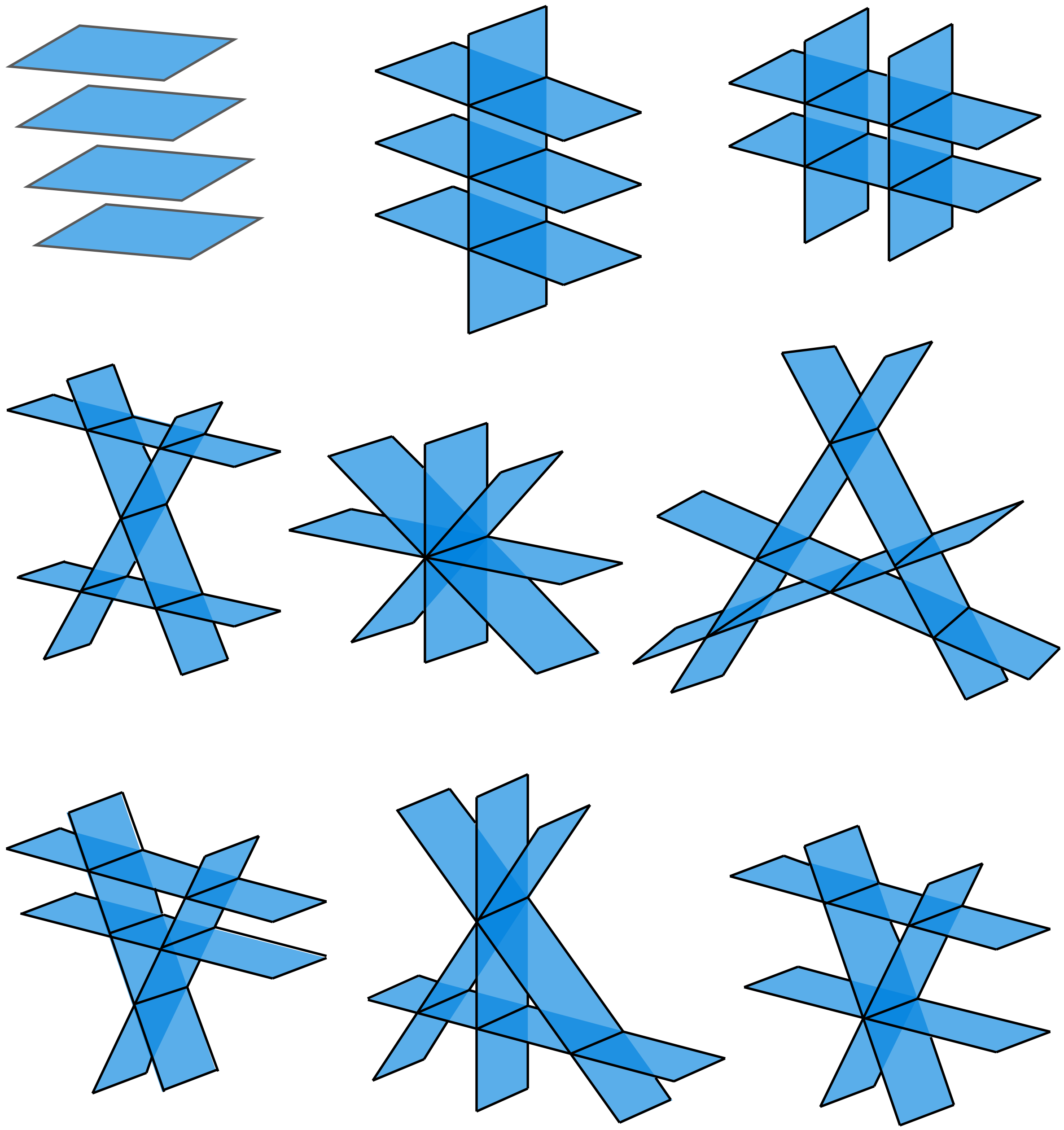

FIGURE 6. Nine of the 14 plane arrangements of 4 planes in $\mathbb{R}^3$. These are the nine found as an arrangement of lines in the plane, crossed with the z-axis.

Problem 9. *The number of arrangements of n planes that have only a single closed chamber (of any shape) is $0, 1, 4, \ldots,$ for $n = 3, 4, 5, \ldots$. What is the general formula for this sequence?*

Problem 10. *The number of arrangements of n planes that have only exactly two closed chambers (of any shape) is $0, 0, 7, \ldots,$ for $n = 3, 4, 5, \ldots$. What is the general formula for this sequence?*

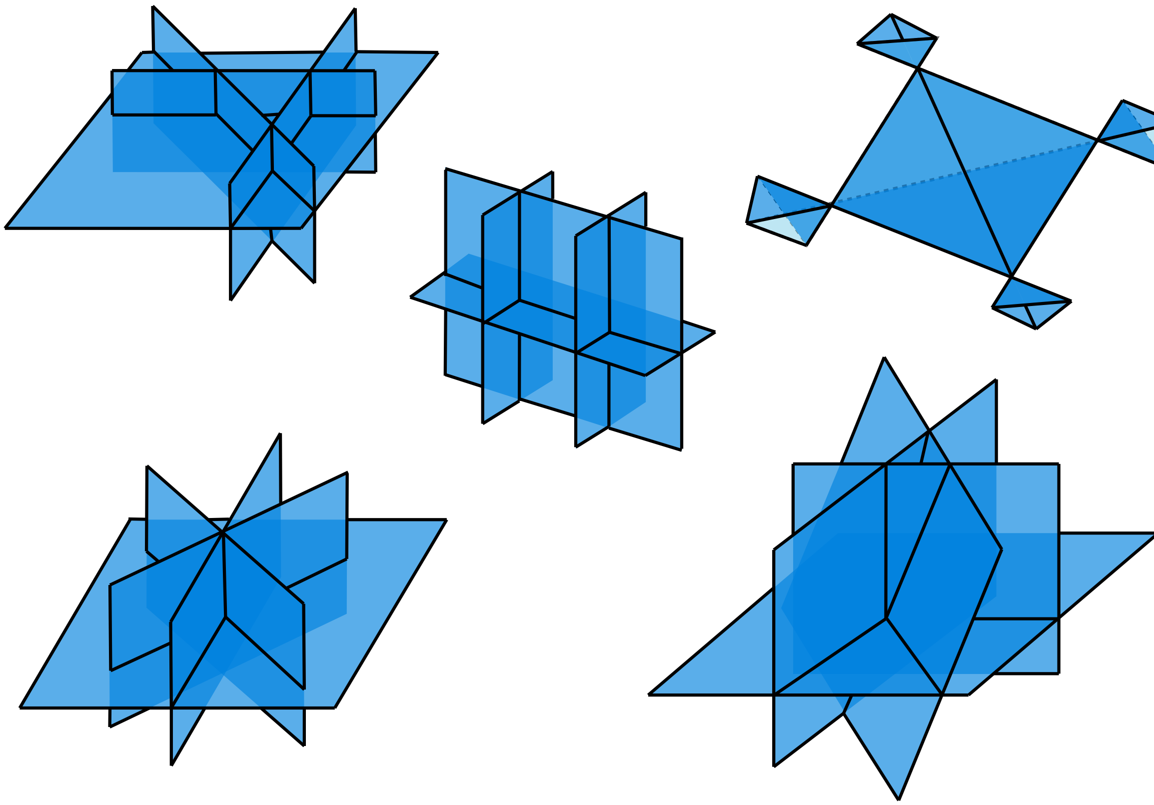

FIGURE 7. Five of the 14 plane arrangements of 4 planes in $\mathbb{R}^3$. The first two on the top row are found by adding a perpendicular plane to an arrangement of 3 planes. The third on the top row is a tetrahedral chamber (the four planes are extended visually a bit at each vertex). The bottom row are the two essential arrangements, found as signed suspensions of arrangements of 3 lines.

Problem 11. *The number of arrangements of n planes that have any number of closed chambers (of any shape) is $0, 1, 14, \dots,$ for $n = 3, 4, 5, \dots$. What is the general formula for this sequence?*

We know that the maximum number of 0-dimensional faces (points) of an arrangement of n hyperplanes in $\mathbb{R}^k$ is $\binom{n}{k}$. From Zaslavsky [20] we have that the maximum number of chambers, both closed and open, is $\sum_{i=0}^{k} \binom{n}{i}$ (Sequence A008949 in [19].)

Problem 12. *The maximum number of closed chambers in an arrangement of n planes is $0, 1, 4, \dots,$ for $n = 3, 4, 5, \dots$. What is the general formula for this sequence?*

Problem 13. *The number of arrangements of n planes that have exactly n 0-dimensional faces is $0, 1, 4, \dots,$ for $n = 3, 4, 5, \dots$. What is the general formula for this sequence?*

Problem 14. *What is the smallest value n for which parameterized equivalence of line arrangements does not give the same classes as combinatorial equivalence? If $n > 8$ then the sequence A241600 can be extended by the value 4134940 as seen in [8].*

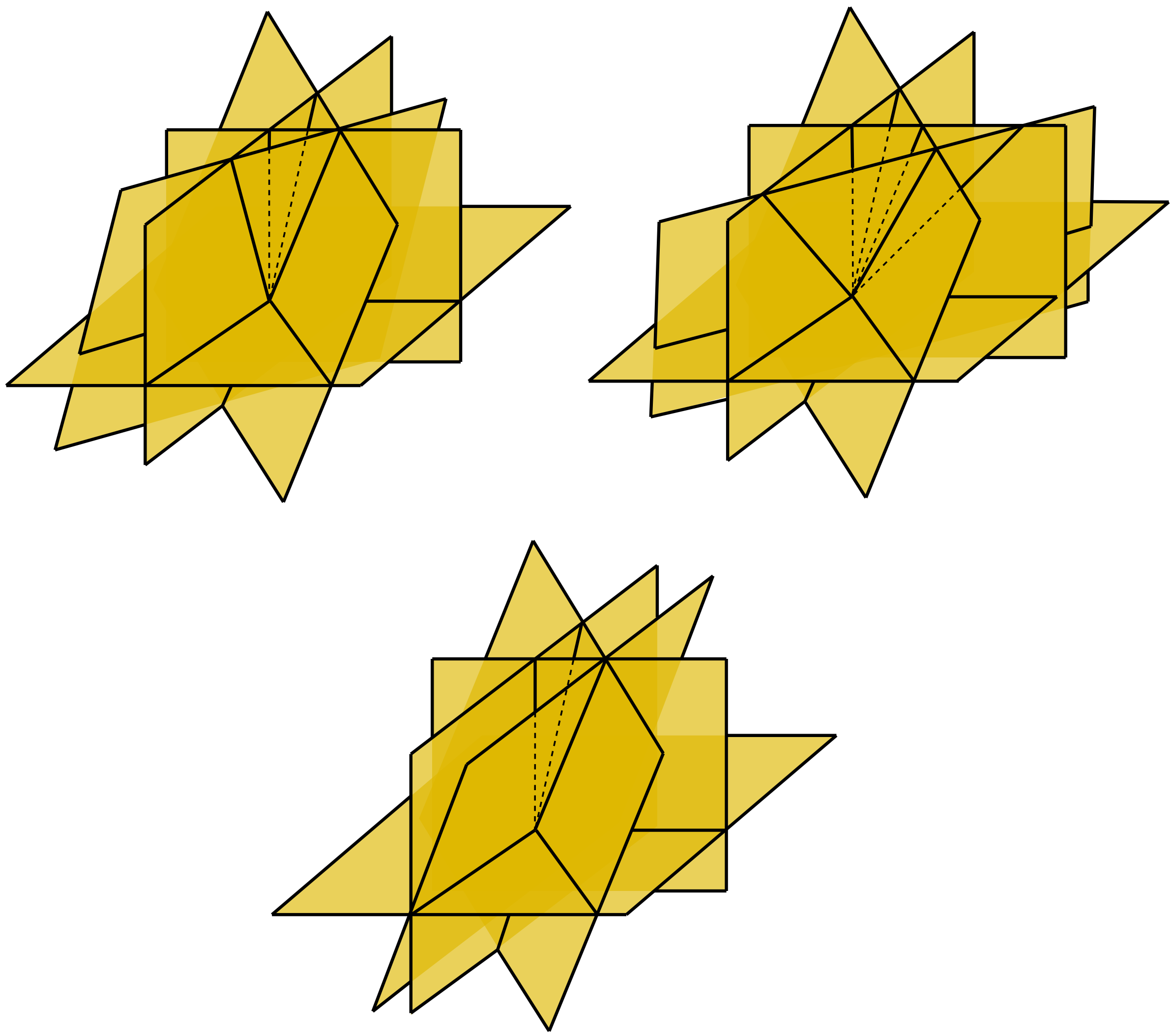

FIGURE 8. Three of the 74 arrangements of five planes are essential, and all are found as signed suspensions of 4 lines. Notice that the example in Figure 3 at first appears to be missing, but it is here!

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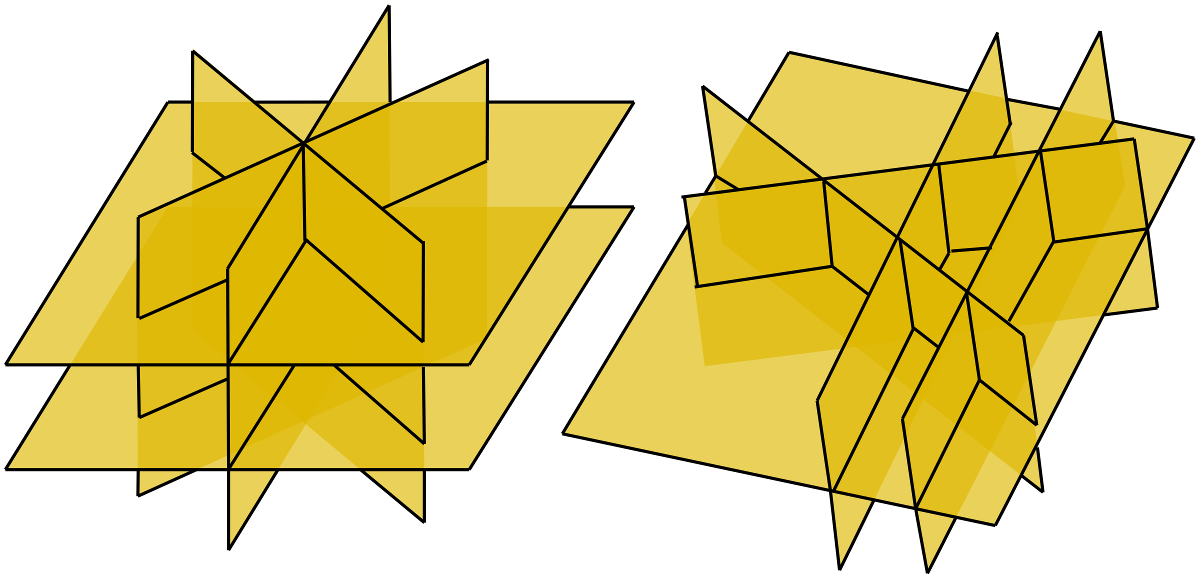

FIGURE 9. 2 more of the 74 arrangements of five planes. On the left is an arrangement found by adding two planes both perpendicular to a central arrangement of three planes. On the right is a representative of the 8 arrangements found by adding one perpendicular plane to one of the 8 nontrivial arrangements from Figure 6.

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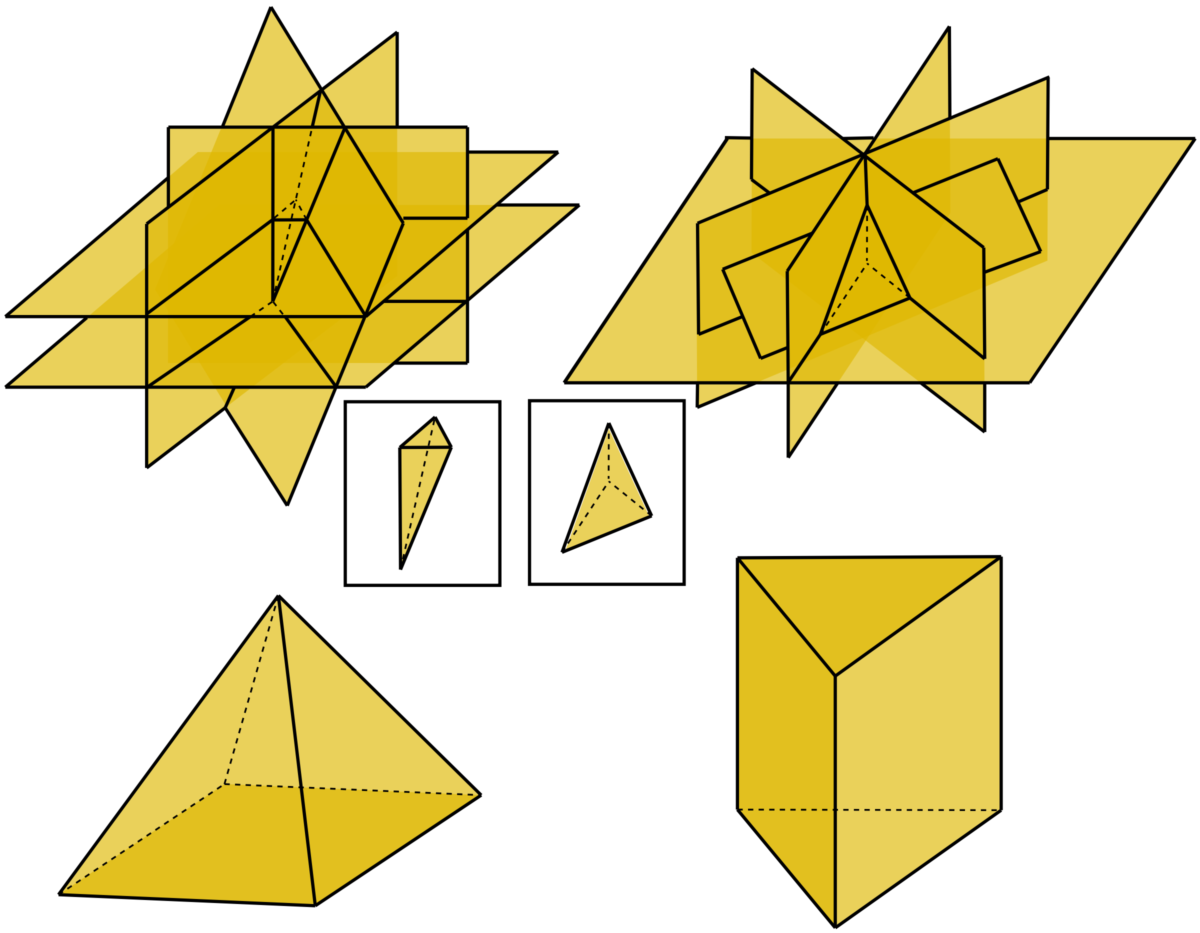

Figure 10. Four of the 74 arrangements of five planes have a single closed chamber: two of them have a tetrahedral chamber, one a square pyramid and one a triangular prism.

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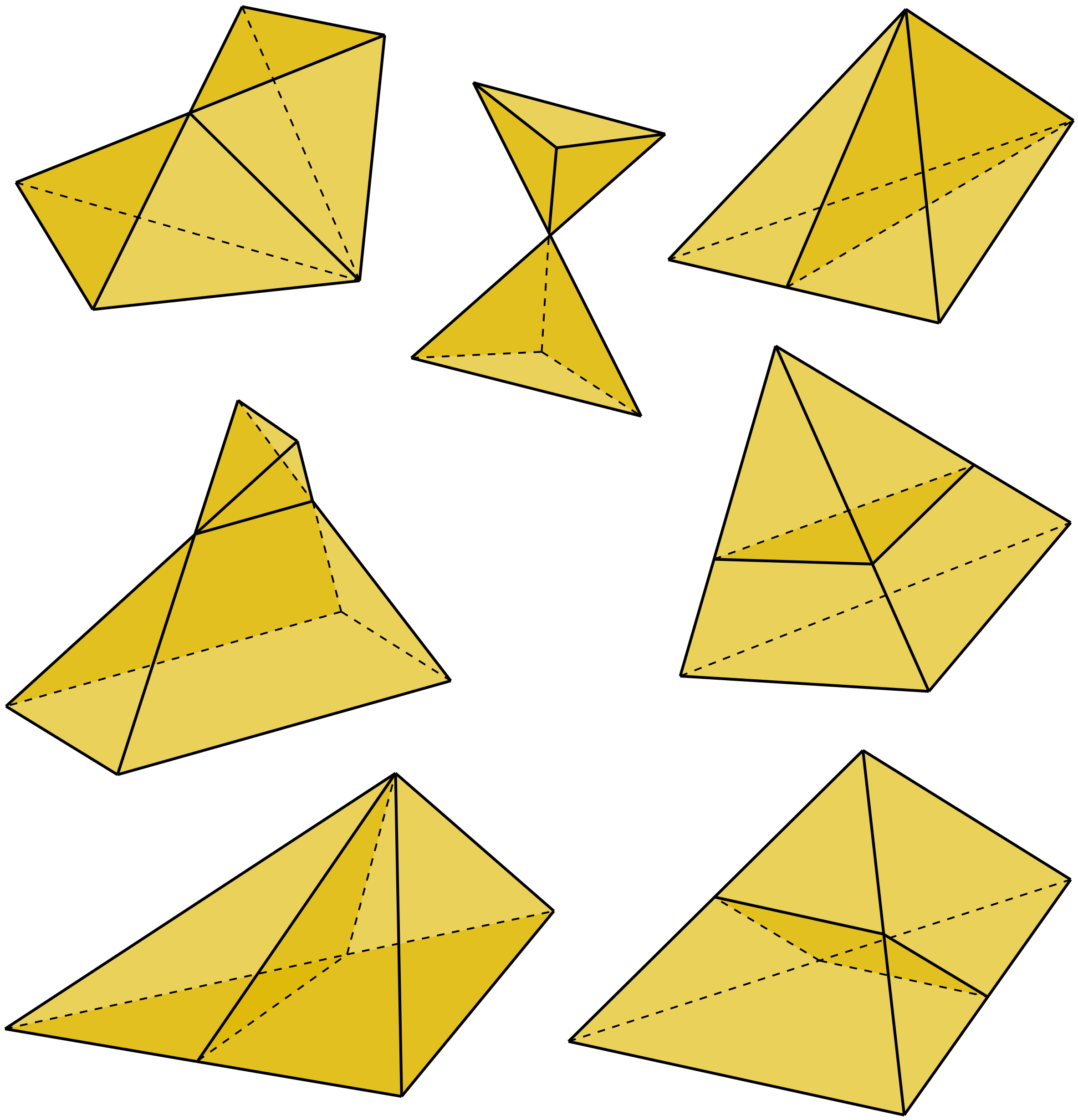

Figure 11. Seven of the 74 arrangements of five planes have a pair of closed chambers.

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