Path-Space Differentiable Rendering

Supplemental Document

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1 DIFFERENTIATING INTEGRALS OVER EVOLVING SURFACES

Our derivation of differential path integrals in §5 of the paper relies heavily on mathematical preliminaries described in §3.1 and §3.2. Specifically, in §3.1 of the paper, we present mathematical tools for expressing the evolution of surfaces, local velocities of surface/curve points driven by such evolutions, and scene derivatives of scalar fields defined on evolving surfaces. In §3.2, we further show a transport relation proposed by Cermelli et al. [2005] that calculates derivatives of integrals defined on evolving surfaces.

In what follows, we use a few simple examples to demonstrate how these mathematical concepts and tools work.

1.1 Translating Square

We first consider a unit square that evolves by translating along the direction $(1, 1, 1)$. That is,

$$\mathcal{M}(\pi) = \left\{(x, y, \pi) \in \mathbb{R}^3 \;:\; \pi < x, y < 1 + \pi\right\}, \tag{1}$$

for all $\pi \in \mathbb{R}$. Eq. (1) effectively gives the *trajectory* of the translating square:

$$\begin{aligned}\mathcal{T} &\coloneqq \{(\boldsymbol{x}, \pi) \;:\; \boldsymbol{x} \in \mathcal{M}(\pi)\}\\ &= \left\{(x, y, \pi, \pi) \in \mathbb{R}^4 \;:\; \pi < x, y < 1 + \pi\right\}.\end{aligned} \tag{2}$$

There exist infinitely many combinations of *reference configuration* $\mathcal{B}$ and *motion* X that produce this trajectory. For instance, if we pick

$$\mathcal{B} \coloneqq \left\{(x, y, 0) \in \mathbb{R}^3 \;:\; 0 < x, y < 1\right\}, \tag{3}$$

it is easy to verify that the motion

$$\mathsf{X}(\boldsymbol{p}, \pi) = \boldsymbol{p} + (\pi, \pi, \pi), \tag{4}$$

for all *material points* $\boldsymbol{p} \in \mathcal{B}$, gives the trajectory (2). The corresponding *reference map* that transforms $\mathcal{M}(\pi)$ back to the reference $\mathcal{B}$ is

$$\mathsf{P}(\boldsymbol{x}, \pi) = \boldsymbol{x} - (\pi, \pi, \pi), \tag{5}$$

for all $\boldsymbol{x} \in \mathcal{M}(\pi)$.

Velocities of surface points. The motion in Eq. (4) induces a *global parameterization* $\hat{x}^{\text{global}}(\boldsymbol{\xi}, \pi)$ of the translating square via

$$\hat{\boldsymbol{x}}^{\text{global}}(\boldsymbol{\xi}, \pi) = \mathsf{X}((\xi_1, \xi_2, 0), \pi) = (\xi_1 + \pi,\ \xi_2 + \pi,\ \pi), \tag{6}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in (0, 1)^2$. Under this surface parameterization, a *spatial point* $(x, y, \pi) \in \mathcal{M}(\pi)$ has *local coordinates* $(x - \pi,\ y - \pi)$. Further, it holds that, for any spatial point $\boldsymbol{x} \in \mathcal{M}(\pi)$ with local coordinates $\boldsymbol{\xi}$,

$$\boldsymbol{v}^{\text{global}}(\boldsymbol{x}, \pi) \coloneqq \frac{\partial \hat{\boldsymbol{x}}^{\text{global}}}{\partial \pi}(\boldsymbol{\xi}, \pi) = \frac{\partial}{\partial \pi}(\xi_1 + \pi,\ \xi_2 + \pi,\ \pi) = (1, 1, 1). \tag{7}$$

In other words, the *local velocity* of all (spatial) points on the square equals $(1, 1, 1)$.

Alternatively, we can also parameterize the square *locally* without relying on the motion in Eq. (4). One possibility is to set

$$\hat{\boldsymbol{x}}^{\text{local}}(\boldsymbol{\xi}, \pi') = (\xi_1,\ \xi_2,\ \pi'), \tag{8}$$

with respect to any fixed π and spatial point $(x, y, \pi) \in \mathcal{M}(\pi)$. Under this parameterization, the spatial point has local coordinates (x, y) and local velocity

$$\boldsymbol{v}^{\text{local}}(\boldsymbol{x}, \pi) = \frac{\partial \hat{\boldsymbol{x}}^{\text{local}}}{\partial \pi'}(\boldsymbol{\xi}, \pi') = \frac{\partial}{\partial \pi'}(\xi_1,\ \xi_2,\ \pi') = (0, 0, 1). \tag{9}$$

Since the surface normal $\boldsymbol{n}$ of the evolving square remains constantly $(0, 0, 1)$, the *scalar normal velocity* of all $\boldsymbol{x} \in \mathcal{M}(\pi)$ equals

$$V = \boldsymbol{v} \cdot \boldsymbol{n} = \underbrace{(1, 1, 1)}_{\text{Eq. (7)}} \cdot (0, 0, 1) = \underbrace{(0, 0, 1)}_{\text{Eq. (9)}} \cdot (0, 0, 1) = 1, \tag{10}$$

confirming its parameterization-independence. Additionally, the global parameterization (6) gives the *local tangential velocity*

$$\boldsymbol{v}_{\tan}^{\text{global}} = \boldsymbol{v}^{\text{global}} - V\,\boldsymbol{n} = (1, 1, 1) - 1\,(0, 0, 1) = (1, 1, 0), \tag{11}$$

for all $\boldsymbol{x} \in \mathcal{M}(\pi)$. The local parameterization (8), on the contrary, provides zero local tangential velocity. That is,

$$\boldsymbol{v}_{\tan}^{\text{local}} = (0, 0, 0). \tag{12}$$

Scene derivatives. We consider a scalar field

$$\varphi(\boldsymbol{x}, \pi) = x^2 + y^2 + z^2, \tag{13}$$

where $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$.

Under the global parameterization (6), a spatial point $\boldsymbol{x} = (x, y, \pi) \in \mathcal{M}(\pi)$ has local coordinates $\boldsymbol{\xi} = (x - \pi,\ y - \pi)$. The *scene derivative* given by this parameterization for any fixed π and $\boldsymbol{x}$ is

$$\begin{aligned}\dot{\varphi}^{\text{global}}(\boldsymbol{x},\pi) &= \frac{\partial}{\partial\pi'}\varphi(\hat{\boldsymbol{x}}^{\text{global}}(\xi,\pi'),\pi')\Big|_{\pi'=\pi} \\ &= \frac{\partial}{\partial\pi'}\varphi((x-\pi+\pi',\ y-\pi+\pi',\ \pi'),\ \pi')\Big|_{\pi'=\pi} \\ &= \frac{\partial[(x-\pi+\pi')^2+(y-\pi+\pi')^2+(\pi')^2]}{\partial\pi'}\Big|_{\pi'=\pi} \\ &= 2(x+y+\pi).\end{aligned} \tag{14}$$

As the surface gradient of φ at $\boldsymbol{x}$ equals $(\partial\varphi/\partial x,\ \partial\varphi/\partial y,\ 0) = (2x, 2y, 0)$, the corresponding *normal scene derivative* is

$$\begin{aligned}\overset{\square}{\varphi}(\boldsymbol{x},\pi) &= \dot{\varphi}^{\text{global}}(\boldsymbol{x},\pi) - \underbrace{\boldsymbol{v}_{\tan}^{\text{global}}(\boldsymbol{x},\pi)}_{\text{Eq. (11)}}\cdot(\text{grad}_{\mathcal{M}}\varphi)(\boldsymbol{x},\pi) \\ &= 2(x+y+\pi) - (1,1,0)\cdot(2x,2y,0) \\ &= 2\pi.\end{aligned} \tag{15}$$

Under the local parameterization (8), a spatial point $\boldsymbol{x} = (x, y, \pi) \in \mathcal{M}(\pi)$ has local coordinates $\boldsymbol{\xi} = (x, y)$. The scene derivative for any fixed π and $\boldsymbol{x}$, in this case, equals

$$\begin{aligned}\dot{\varphi}^{\text{local}}(\boldsymbol{x},\pi) &= \frac{\partial}{\partial\pi'}\varphi(\hat{\boldsymbol{x}}^{\text{local}}(\xi,\pi'),\pi')\Big|_{\pi'=\pi} \\ &= \frac{\partial}{\partial\pi'}\varphi((x,\ y,\ \pi'),\ \pi')\Big|_{\pi'=\pi} \\ &= \frac{\partial[x^2+y^2+(\pi')^2]}{\partial\pi'}\Big|_{\pi'=\pi} \\ &= 2\pi.\end{aligned} \tag{16}$$

Computing the corresponding normal scene derivative yields

$$\overset{\square}{\varphi}(\boldsymbol{x},\pi) = \dot{\varphi}^{\text{local}}(\boldsymbol{x},\pi) - \underbrace{\boldsymbol{v}_{\tan}^{\text{local}}(\boldsymbol{x},\pi)}_{\text{Eq. (12)}}\cdot(\text{grad}_{\mathcal{M}}\varphi)(\boldsymbol{x},\pi) = 2\pi, \tag{17}$$

which agrees with Eq. (15). This is expected since normal scene derivatives are parameterization-independent.

Differentiating surface integrals. We now consider the problem of differentiating the surface integral of $\varphi(\boldsymbol{x},\pi)$ in Eq. (13) over the translating square $\mathcal{M}(\pi)$.

Given the Eqs. (1) and (13), it holds that, for fixed $\pi \in \mathbb{R}$,

$$\begin{aligned}&\int_{\mathcal{M}}\varphi(\boldsymbol{x},\pi)\,\mathrm{d}A(\boldsymbol{x}) \\ &= \int_{\pi}^{\pi+1}\int_{\pi}^{\pi+1}\left(x^2+y^2+\pi^2\right)\mathrm{d}x\,\mathrm{d}y \\ &= 3\pi^2+2\pi+\frac{2}{3}.\end{aligned} \tag{18}$$

Thus, we know that

$$\frac{\mathrm{d}}{\mathrm{d}\pi}\int_{\mathcal{M}}\varphi(\boldsymbol{x},\pi)\,\mathrm{d}A(\boldsymbol{x}) = \frac{\mathrm{d}}{\mathrm{d}\pi}\left(3\pi^2+2\pi+\frac{2}{3}\right) = 6\pi+2. \tag{19}$$

In what follows, we use the transport relation expressed in Eq. (11) from §3.2 of the paper to calculate the derivative of Eq. (19). Since φ is continuous and the plane has zero curvature, we have

$$\frac{\mathrm{d}}{\mathrm{d}\pi}\int_{\mathcal{M}}\varphi\,\mathrm{d}A = \overset{\text{interior}}{\boxed{\int_{\mathcal{M}}\overset{\square}{\varphi}\,\mathrm{d}A}} + \overset{\text{boundary}}{\boxed{\int_{\partial\mathcal{M}}\varphi\,V_{\partial\mathcal{M}}\,\mathrm{d}\ell}}, \tag{20}$$

where $V_{\partial\mathcal{M}}$ indicates the normal velocity of a point on the square's boundary.

Given Eq. (15), it holds that the *interior* term equals

$$\int_{\mathcal{M}}\overset{\square}{\varphi}\,\mathrm{d}A = \int_{\mathcal{M}}2\pi\,\mathrm{d}A = 2\pi. \tag{21}$$

To calculate the remaining *boundary* term, we need to have available the normal velocity $V_{\partial\mathcal{M}}$. Since this quantity is parameterization-independent, we parameterize the boundary curve $\partial\mathcal{M}$ globally using Eq. (6) so that all boundary points have local velocity $(1, 1, 1)$. Then, $V_{\partial\mathcal{M}}$ remains constant on each edge of the square:

- All points on the edges with unit normal $(0, -1, 0)$ and $(-1, 0, 0)$ have $V_{\partial\mathcal{M}} = -1$;
- Those on the edges with unit normal $(0, 1, 0)$ and $(1, 0, 0)$ have $V_{\partial\mathcal{M}} = 1$.

It follows that

$$\begin{aligned}&\int_{\partial\mathcal{M}}\varphi\,V_{\partial\mathcal{M}}\,\mathrm{d}\ell \\ &= \int_{\pi}^{\pi+1} -\left[x^2+\pi^2+\pi^2\right]\mathrm{d}x + \int_{\pi}^{\pi+1}\left[x^2+(\pi+1)^2+\pi^2\right]\mathrm{d}x + \\ &\quad \int_{\pi}^{\pi+1} -\left[\pi^2+y^2+\pi^2\right]\mathrm{d}y + \int_{\pi}^{\pi+1}\left[(\pi+1)^2+y^2+\pi^2\right]\mathrm{d}y \\ &= \int_{\pi}^{\pi+1}\left[(\pi+1)^2-\pi^2\right]\mathrm{d}x + \int_{\pi}^{\pi+1}\left[(\pi+1)^2-\pi^2\right]\mathrm{d}y \\ &= (2\pi+1)+(2\pi+1) = 4\pi+2.\end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\pi}\int_{\mathcal{M}}\varphi\,\mathrm{d}A &= \underbrace{\int_{\mathcal{M}}\overset{\square}{\varphi}\,\mathrm{d}A}_{\text{Eq. (21)}} + \underbrace{\int_{\partial\mathcal{M}}\varphi\,V_{\partial\mathcal{M}}\,\mathrm{d}\ell}_{\text{Eq. (22)}} \\ &= 2\pi+(4\pi+2) = 6\pi+2,\end{aligned} \tag{23}$$

which agrees with Eq. (19).

1.2 Scaling Sphere

Our second example involves a scaling sphere centered at the origin. Specifically,[1]

$$\mathcal{M}(r) = \left\{(x,y,z)\in\mathbb{R}^3 \ :\ x^2+y^2+z^2=r^2\right\}, \tag{24}$$

for all $r > 0$.

Let the reference configuration be the unit sphere:

$$\mathcal{B} = \mathbb{S}^2 := \left\{(x,y,z)\in\mathbb{R}^3 \ :\ x^2+y^2+z^2=1\right\}. \tag{25}$$

Then, it is easy to verify that the motion

$$\mathsf{X}(\boldsymbol{p},r) = r\,\boldsymbol{p}, \tag{26}$$

[1] We use "r" instead of "π" as the parameter in this example since we will be using the latter to indicate Archimedes' constant (i.e., the ratio of a circle's circumference to its diameter).

for any fixed r, gives a smooth one-to-one mapping from $\mathcal{B}$ to the scaling sphere $\mathcal{M}(r)$. This motion further induces a global parameterization of the sphere such that, a spatial point $\boldsymbol{x} \in \mathcal{M}(r)$ has local velocity

$$\boldsymbol{v}^{\text{global}}(\boldsymbol{x}, r) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{x}}{r}, \tag{27}$$

which has zero tangential component (i.e., $\boldsymbol{v}^{\text{global}}_{\text{tan}} = \boldsymbol{0}$). Since the surface normal at $\boldsymbol{x} \in \mathcal{M}(r)$ on the sphere is $\boldsymbol{x}/r$, the scalar normal velocity at $\boldsymbol{x}$ is

$$V = \boldsymbol{v}^{\text{global}} \cdot \boldsymbol{n} = \frac{\boldsymbol{x}}{r} \cdot \frac{\boldsymbol{x}}{r} = \frac{\boldsymbol{x} \cdot \boldsymbol{x}}{r^2} = 1. \tag{28}$$

Additionally, for $\varphi(\boldsymbol{x}, r)$ defined in Eq. (13),

$$\overset{\square}{\varphi}(\boldsymbol{x}, \pi) = \dot{\varphi}^{\text{local}}(\boldsymbol{x}, r) = \frac{\partial}{\partial r'} \underbrace{\varphi(\mathsf{X}(\boldsymbol{p}, r'), r')}_{= (r')^2} \Big|_{r'=r} = 2r. \tag{29}$$

Differentiating surface integrals. We now consider the problem of differentiating the surface integral of $\varphi(\boldsymbol{x}, r)$ over the scaling square. Since $\varphi(\boldsymbol{x}, r) = r^2$ for all $\boldsymbol{x} \in \mathcal{M}(r)$, it holds that

$$\int_{\mathcal{M}} \varphi \, \mathrm{d}A = 4\pi r^4, \tag{30}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r} \int_{\mathcal{M}} \varphi \, \mathrm{d}A = 16\pi r^3. \tag{31}$$

We note that the scaling square has no boundary and that the total curvature κ equals $-2/r$ for all $\boldsymbol{x} \in \mathcal{M}(r)$. Hence, when differentiating the surface integral using the transport relation from Eq. (11) of the paper, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\pi} \int_{\mathcal{M}} \varphi \, \mathrm{d}A &= \int_{\mathcal{M}} \left(\overset{\square}{\varphi} - \varphi \, \kappa \, V \right) \mathrm{d}A \\ &= \int_{\mathcal{M}} \left(2r - r^2 \frac{-2}{r} \right) \mathrm{d}A \\ &= \int_{\mathcal{M}} 4r \, \mathrm{d}A \\ &= (4\pi r^2)(4r) = 16\pi r^3, \end{aligned} \tag{32}$$

which agrees with Eq. (31).

2 CHANGE-OF-VARIABLE JACOBIAN

When introducing our *material-form* reparameterization of the direct-illumination integral in §4.1 of the paper, a key ingredient is the Jacobian determinant capturing the change of variable from *spatial points* $\boldsymbol{x}$ to *material points* $\boldsymbol{p}$.

Precisely, at $\boldsymbol{x} = \mathsf{X}(\boldsymbol{p}, \pi)$, this term captures the ratio of the infinitesimal area $\Delta A_{\boldsymbol{x}}$ spanned by $\boldsymbol{d}_\alpha = \mathsf{X}(\boldsymbol{p} + \alpha \boldsymbol{d}, \pi) - \boldsymbol{x}$ and $\boldsymbol{e}_\alpha = \mathsf{X}(\boldsymbol{p} + \alpha \boldsymbol{e}, \pi) - \boldsymbol{x}$ to the area $\Delta A_{\boldsymbol{p}}$ spanned by $\alpha \boldsymbol{d}$ and $\alpha \boldsymbol{e}$:

$$J(\boldsymbol{p}) = \left| \frac{\mathrm{d}A(\boldsymbol{x})}{\mathrm{d}A(\boldsymbol{p})} \right| = \lim_{\alpha \to 0} \frac{\Delta A_{\boldsymbol{x}}}{\Delta A_{\boldsymbol{p}}} = \lim_{\alpha \to 0} \frac{|\boldsymbol{d}_\alpha \times \boldsymbol{e}_\alpha|}{|\alpha \boldsymbol{d} \times \alpha \boldsymbol{e}|}, \tag{33}$$

where "$\times$" denotes vector cross product (see Figure 1).

We consider a special case where the reference configuration $\mathcal{B}$ is a triangle with vertices $\boldsymbol{p}_i \in \mathbb{R}^3$ for $i = 1, 2, 3$. For any $\boldsymbol{p} \in \mathcal{B}$ with barycentric coordinates (s, t), that is,

$$\boldsymbol{p} = (1 - s - t) \, \boldsymbol{p}_1 + s \, \boldsymbol{p}_2 + t \, \boldsymbol{p}_3. \tag{34}$$

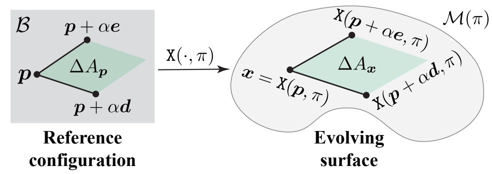

Fig. 1. Given a deformation $\mathsf{X}(\cdot, \pi)$ with fixed $\pi \in \mathbb{R}$, the **Jacobian term** of Eq. (33) equals the ratio of the infinitesimal areas $\Delta A_{\boldsymbol{x}}$ and $\Delta A_{\boldsymbol{p}}$.

Then, we can set $\boldsymbol{d} = \boldsymbol{p}_2 - \boldsymbol{p}_1$ and $\boldsymbol{e} = \boldsymbol{p}_3 - \boldsymbol{p}_1$ (for all $\boldsymbol{p}$). Further,

$$\boldsymbol{p} + \alpha \boldsymbol{d} = (1 - s - t - \alpha) \, \boldsymbol{p}_1 + (s + \alpha) \, \boldsymbol{p}_2 + t \, \boldsymbol{p}_3, \tag{35}$$

$$\boldsymbol{p} + \alpha \boldsymbol{e} = (1 - s - t - \alpha) \, \boldsymbol{p}_1 + s \, \boldsymbol{p}_2 + (t + \alpha) \, \boldsymbol{p}_3. \tag{36}$$

Given $\dot{\boldsymbol{p}}_i \in \mathbb{R}^3$ for $i = 1, 2, 3$, let

$$\begin{aligned} \mathsf{X}(\boldsymbol{p}, \pi) &= (1 - s - t) \, (\boldsymbol{p}_1 + \pi \, \dot{\boldsymbol{p}}_1) + s \, (\boldsymbol{p}_2 + \pi \, \dot{\boldsymbol{p}}_2) + t \, (\boldsymbol{p}_3 + \pi \, \dot{\boldsymbol{p}}_3) \\ &= \boldsymbol{p} + \pi [(1 - s - t) \, \dot{\boldsymbol{p}}_1 + s \, \dot{\boldsymbol{p}}_2 + t \, \dot{\boldsymbol{p}}_3], \end{aligned} \tag{37}$$

for all $\boldsymbol{p}$ with barycentric coordinates (s, t). It follows that

$$\boldsymbol{d}_\alpha = \mathsf{X}(\boldsymbol{p} + \alpha \boldsymbol{d}, \pi) - \mathsf{X}(\boldsymbol{p}, \pi) = \alpha(\boldsymbol{d} + \pi \, \dot{\boldsymbol{d}}), \tag{38}$$

$$\boldsymbol{e}_\alpha = \mathsf{X}(\boldsymbol{p} + \alpha \boldsymbol{e}, \pi) - \mathsf{X}(\boldsymbol{p}, \pi) = \alpha(\boldsymbol{e} + \pi \, \dot{\boldsymbol{e}}), \tag{39}$$

where $\dot{\boldsymbol{d}} \coloneqq \dot{\boldsymbol{p}}_2 - \dot{\boldsymbol{p}}_1$ and $\dot{\boldsymbol{e}} \coloneqq \dot{\boldsymbol{p}}_3 - \dot{\boldsymbol{p}}_1$. Thus, the change-of-variable Jacobian determinant of Eq. (33) becomes

$$J(\boldsymbol{p}) = \frac{\left| \boldsymbol{d} \times \boldsymbol{e} + \pi(\boldsymbol{d} \times \dot{\boldsymbol{e}} + \dot{\boldsymbol{d}} \times \boldsymbol{e}) + \pi^2 (\dot{\boldsymbol{d}} \times \dot{\boldsymbol{e}}) \right|}{|\boldsymbol{d} \times \boldsymbol{e}|}, \tag{40}$$

which is independent of the barycentric coordinates (s, t) of $\boldsymbol{p}$. In other words, under the motion of Eq. (37), J remains constant for all material points $\boldsymbol{p} \in \mathcal{B}$.

When $\pi = 0$, it is easy to verify that $J(\boldsymbol{p})$ given by Eq. (40) equals one. On the contrary, the derivative $\partial J / \partial \pi$, which can be calculated using automatic differentiation, is generally non-zero.

REFERENCES

Paolo Cermelli, Eliot Fried, and Morton E Gurtin. 2005. Transport relations for surface integrals arising in the formulation of balance laws for evolving fluid interfaces. *Journal of Fluid Mechanics* 544 (2005), 339–351.