SOME HISTORIC REMARKS ON SAMPLING THEOREM

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ABSTRACT

The sampling theorem is a notion linking continuous and discrete signals. Due to that, it is a very important concept in both engineering practice and mathematical theory, closely related with many other basic results. This paper is a brief review of historic development of the sampling theorem for functions on different groups.

1. INTRODUCTION

The sampling theorem is a very fundamental concept in signal processing with many applications and a variety of different extensions and generalizations. We point just few of firstly published related publications, [2], [15], [20], [24], [31], [37], [39], [41], [42], [51], [55], [65], [67], [72], [73], [74].

It is also an important concept in mathematics and, although basically belonging to theory of approximation and theory of interpolation, it can be related with some basic notions in few different areas of mathematics, see for instance [8].

5can be derived from it [8].

Due to that, and as it is often the case with basic concepts, the sampling theorem has been contributed to few authors, and its historical roots have been often discussed, in few historic overviews as, for instance, [14], [27], [32], [43], [60]. Most, if not all, of these overviews are focussed and restricted to discussions of the sampling theorem and generalizations for functions on the real line R. In this paper, we discuss the history of sampling theorem in the framework of abstract harmonic analysis by viewing the real line R as a particular locally compact Abelian group.

2. SAMPLING THEOREM FOR BAND-LIMITED FUNCTIONS

2.1. Work by E.T. Whittaker

The Japanese mathematician K. Ogura formulated in [49] the sampling theorem in the from similar as we know it now, and suggested a way to derive a rigorous proof of it by using the results from calculus of residues presented in the book [40] by Finnish mathematician E. Lindelof, a Professor at the University of Helsinki, from after graduation in 1895 to the retirement in 1938. As it is explained in [14], Ogura contributed the sampling theorem to E.T. Whittaker due to his work in 1915 [68]. In [14], it is remarked that Ogura was the first who stated the sampling theorem and traced the way of a rigorous proof for it, and that erroneously referenced to Whittaker as the initiator of the theorem, since Whittaker did not consider the related applications. It is interesting to note, that the authors of [14] used the term Whittaker-Shannon theorem when discussing mathematical applications of it.

In [68], E.T. Whittaker proposed the function

$$C(x) = \sum_{-\infty}^{\infty} f(a+kw) \frac{\sin \frac{\pi}{w}(x-a-kw)}{\frac{\pi}{w}(x-a-kw)}, \qquad (1)$$

as a solution of the problem of determining a function passing through the points $(a+kw, f(a+kw))$, where $k \in Z$, and w is a complex number. Whittaker searched for the smoothest possible interpolation without singularities and rapid oscillations for given tabular values of $f(x)$, and determined the above interpolation function from the requirement that its Fourier transform does not contain any terms with periods less than $2w$. Since this interpolation function is a unique function with this property, it was called by Whittaker the cardinal function and the corresponding series the cardinal series. In his terminology, the term tabular interval corresponds to the sampling interval, and cotabular set is the set of all possible functions with the same tabular or sampling values.

It is shown, for instance in [14], see also [28], that the Whittaker interpolation formula can be derived as the limiting case of the Lagrange interpolation formula when the number of nodes, i.e., tabular values, tends to infinity. In this setting, notice that the Lagrange interpolation theorem can be viewed as a sampling theorem for band-limited periodic functions [43], since determines a linear combination of sine functions such that it coincides with the approximated function at n equidistant points.

Further important contributions to the theory of cardinal series are due to J.M. Whittaker [69], [70], and in this latter paper there is a result that is characterized in [14] as a weak version of the sampling theorem. This result has not been included in the book [71], which give rise for a conjecture in [14] that J.M. Whittaker himself probably was not thinking about the sampling theorem.

2.2. Kotelnikov Formulation of Sampling Theorem

It is commonly accepted that first precise formulation of the sampling theorem for applications in communication engineering is due to Vladimir Aleksandrovich Kotelnikov

[36]. In his notation and formulation, the sampling theorem is stated as follows.

Theorem 1 [36] *Any function $F(t)$ which consists of frequencies from 0 to f_1 periods per second may be represented by the following series*

$$F(t) = \sum_{-\infty}^{\infty} D_k \frac{\sin w_1 \left(t - \frac{k}{2f_1}\right)}{t - \frac{k}{2f_1}}, \tag{2}$$

where k-integer, $w_1 = 2\pi f_1$, D_k - constant which depends on $F(t)$. Conversely, any function $F(t)$ which is represented by the series in (2) consists of frequencies from 0 to f_1 periods per second.

The paper by Kotelnikov contains seven theorems discussing both lowpass and bandpass signals, and the second theorem is stated as follows.

Theorem 2 [36] *Any function $F(t)$, which consists of frequencies form 0 to f_1, can be transmitted continuously with an arbitrary accuracy, by means of numbers sent at intervals of $\frac{1}{2f_1}$ seconds. Indeed, by measuring of the value $F(t)$ at $t = \frac{n}{2f_1}$ (n integer), we get*

$$F\left(\frac{n}{2f_1}\right) = D_n w_1.$$

Since all terms of the series (2) for this value of t tend to zero, except the term for $k = n$, that, as it can be easily established by calculation of the indefinite point, equals $D_n w_1$. In this way, after each $\frac{1}{2f_1}$ we can determine the next D_k. When these D_k transmitted in a row at each $\frac{1}{2f_1}$ sec., we can from (2) reconstruct $F(t)$ termwise to any degree of accuracy.

2.3. Shannon Formulation of Sampling Theorem

Claude Elwood Shannon published his version of the sampling theorem in [53] in 1948, but it is recorded that he had it written already in 1940, and the long publication time was a consequence of the situation after the World War. The proof of the theorem is given in [54]. In the formulation and notation by Shannon, the sampling theorem presented as the Theorem 13 in [53], is the following.

Theorem 3 [53] *Let $f(t)$ contain no frequencies over W. Then*

$$f(t) = \sum_{-\infty}^{\infty} X_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}, \tag{3}$$

where $X_n = f\left(\frac{n}{2W}\right)$.

It is clear that after some linear transformation and proper settings of parameters, the same as in the case of formulation by Kotelnikov, $f(t) = C(t)$ in the notation by E.T. Whittaker. Shannon was aware of this mathematical work and wrote in [54] the following

Theorem 3 has been given previously in other forms by mathematicians (and put the reference to E.T. Whittaker [68]) but in spite of its evident importance seems not to have appeared in the literature of communication theory.

Notice that the sampling theorem is somewhere called also as the Niqyist-Shannon theorem by referring to the work by Henry Nyquist [47], [48], although the problem of distortionless transmission of telegraphic, i.e., digital, signals and the error-free interpolation of sampling pulses of an analog signals cannot be viewed as the same problem [43]. It is remarked in [43] that *for this reasons these works cannot be regarded as sources for the sampling theorem, especially during the 1920s and 1930s*. The simplest way to clarify the relationships to the work by Nyquist is to recall what Shannon wrote about this in [54]

Niquist [47], [3], however, and more recently Gabor [23], have pointed out that approximately $2TW$ numbers are sufficient, basing their arguments on a Fourier series expansion of the function over thre time interval T. This given TW and $(TW+1)$ cosine terms up to frequency W. The slight discrepancy is due to the fact that the functions obtained in this way will not be strictly limited to the band W but, because of the sudden starting and stopping of the sine and cosine components, contain some frequency content outside of the band. Nyquist pointed out the fundamental importance of the time interval $1/2W$ seconds in connection with telegraphy, and we will call this the Nyquist interval corresponding to the band W.

2.4. Further Rediscovering of the Sampling Theorem

Notice that the sampling theorem has been rediscovered by H. Raabe in his PhD Thesis including discussions of bandpass signals, and considering also practical applications of this theorem. The results were published in [52] in 1939. This result was referenced by W.R. Bennett in [3], and as we see above, this reference is commented by Shannon who wrote in [54]

A result similar to Theorem 3 is established (in [3]), but on a steady-state basis.

The years after, the Japanese scientists Isao Someya discussed the sampling theorem in his book [56] published in 1949. Therefore, in the literature, the term Someya theorem can be found, as well as Someya-Shannon theorem [62], Whittaker-Shannon theorem [5], Whittaker-Kotelnikov-Shannon theorem [13], Whittaker-Koteljnikov-Raabe-Shannon Someya theorem [7], etc.

3. SAMPLING THEOREM FOR TIME-LIMITED FUNCTIONS

Another problem in sampling theory is related to relaxation of the requirement for band-limitedness, which is also related to the reconstruction of duration-limited signals, also called time-limited signals, i.e., signals defined on the real line R but vanishing outside and interval $[a, b]$. Such signals cannot be band-limited, and their study belongs to the theory of reconstruction of non-necessarily band-limited signals in terms of equally spaced samples originated by Charles-Jean Baron de la Vallée Poussin in 1908. In [18], he has shown the interpolation formula for such functions as

$$F_m(x) = \sum_{\alpha_k \in [a,b)} f(\alpha_k) \frac{\sin m(x - \alpha_k)}{m(x - \alpha_k)},$$

where $\alpha_k = \frac{k\pi}{m}$, $k \in Z = \{0, \pm 1, \pm 2, \ldots\}$, $m = n$ or $m = n + 1/2$ with $n \in N = \{1, 2, 3, \ldots\}$. The function F_m interpolate f at the nodes α_n, thus, $F_m(\alpha_n) = f(\alpha_n)$, $n \in Z$. De la Vallée Poussin viewed his considerations as a generalization of the Lagrange interpolation formula for infinite number of nodes and, as pointed in [14] considered the counterpart of Riemann localization principle for Fourier integrals in the case of F_m. In [14], it has been shown, however, that under the additional condition $f(b) = 0$ besides $f(x) = 0$, for $x \neq [a, b]$, the interpolation function F_m can be viewed as a discrete version of the Dirichlet convolution integral, a particular form of the Fourier inversion integral, and the behaviour of F_m for $m \to \infty$ is similar to that of the Fourier inversion integral for f.

A continuation of the work by de la Vallée Poussin can be found in [63], where the author Maria Theis has shown that for the convergence of $F_m(x)$ to $f(x)$ besides continuity of f on $[a, b]$, the condition that $f(x)$ is a function of the bounded variation has to be provides. These results are derived as a modification of more general results in [25] about the convergence of interpolation processes, where the reference to the work by de la Vallée Poussin has been given. For the convergence of $F_m(x)$ for any continuous function $f(x)$, Theis used the kernel function $\phi(x) = (\sin \pi x/\pi x)^2$ which can be viewed as the particular Fejér kernel.

By exploiting the condition of bounded variation for f, J.M. Whittaker studied the case when f does non necessarily vanish outside a finite interval $[a, b]$, and has shown the following.

Theorem 4 [69] *Consider a function f that is Riemann integrable over any finite interval of R and $f(x)/x$ is of bounded variation in (N, ∞) and $(-\infty, N)$ for some $N > 0$. If f is continuous at $x_0 \in R$ and of bounded variation in a neighborhood of x_0, then*

$$f(x_0) = \lim_{m \to \infty} \sum_{k=-\infty}^{\infty} f(\alpha_k) \frac{\sin m(x_0 - \alpha_k)}{m(x_0 - \alpha_k)}.$$

Further contributions to the sampling theory for non-band-limited functions have been provided by P. Weiss [66] and J.L. Brown [4] in 1963 and 1967, respectively. This research has been continued and importantly contributed by P.L. Butzer and his associates at Aachen University of Technology, Aachen, Germany, see for instance [9], [12], [57], [61], [6], and [11] and references therein.

4. SAMPLING THEOREM ON LOCALLY COMPACT ABELIAN GROUPS

The sampling theorem has been defined for function on the real line R, and latter generalized to various classes of functions of real variables functions. Very good reviews of these results can be found in [27], [32], [60].

The real line can be viewed as a particular locally compact Abelian group. A generalization with respect to the domain groups has been provided by Kluvánek [35], who replaced the real line R by an arbitrary locally compact Abelian group and integer multiples in D_k or X_n in Kotelnikov and Shannon notation, respectively, are denoted by Kluvánek as h, by a discrete subgroup.

We denote by G and arbitrary additive locally compact Abelian group, and by Γ its dual group. Consider a discrete subgroup H of G with the discrete annihilator $\Lambda = \{w|\chi(y, w) = 1, \forall y \in H\}$, and the Baier measurable subset Ω of Γ which contains a single element from each coset of Λ, i.e., $\Omega \cap (w + \Lambda)$ contains a single point for each $w \in \Gamma$. With this notation, the sampling theorem is formulated by Kluvánek as follows [35].

Theorem 5 *If $f \in L^2(G)$ and its Fourier transform $S_f(w) = 0$ for almost all $w \in \Omega$, then f is almost everywhere equal to a continuous function, and if f is a continuous function,*

$$f(x) = \sum_{y \in H} f(y)\phi(x - y), \tag{4}$$

where this series converge both uniformly on G as well as in the norm in $L^2(G)$. Further,

$$\|f\|^2 = \sum_{y \in H} |f(y)|^2.$$

The function χ in (4) is defined as

$$\phi(x) = \int_{\Omega} \chi(x, w) dm_\Gamma(w), \tag{5}$$

where m_Γ is the Haar measure on Γ and $\chi(x, w)$ are group characters of G.

As pointed out by Kluvánek [35],

If $G = \Gamma = (-\infty, \infty)$, $\Omega = (-\alpha, \alpha)$ and, consequently, $H = \{\ldots, -2h, -h, 0, h, 2h, \ldots\}$ with $h\alpha = \pi$ for the function from (5) we get $\phi(x) = (\sin \alpha x)/(\alpha x)$. Hence, if $f \in L^2(-\infty, \infty)$ and $\hat{f} = 0$ for $|\gamma| > \alpha$, we obtain (3).

Kluvánek concluded his paper by the statement

If we choose for G the multiplicative group of complex numbers z with $|z| = 1$ for H the group of all roots of the equation $z^n - 1 = 0$, we get a formula due to Cauchy obtained in [16] by the means of the Lagrange interpolation formula.

5. SAMPLING THEOREM ON DYADIC GROUP

Sequency theory [26], that is based on Walsh functions, which are compatible with operation of digital computers, has found some interesting applications in signal processing [1], [30]. Therefore, it was interesting to extend the sampling theorem to Walsh-Fourier analysis [9], [10]. This theorem is often called the dyadic sampling theorem [19]. It sates that any sequency-limited function (defined below) can be reconstructed from its values at equidistant sampling pints. Extensions to random signals with limited sequency and sequency-band-limited nonstationary random processes are given by Maqusi in [44], [45], see also [46], and further elaborated in [19], and several other publications by the research group of Prof. Butzer in Aachen, Germany.

For the illustration, we will show that the sampling theorem in Walsh-Fourier analysis [9] can be derived as a particular case of the Kluvánek sampling theorem. Notice that the sampling theorem in Walsh-Fourier analysis has been for the first time formulated by Pichler in a completely different way [50].

To state this theorem, recall the following basic notions and definitions from Walsh-Fourier analysis.

Any $x \in R_+ = [0, \infty)$ has the dyadic expansion

$$x = \sum_{i=-N(x)}^{\infty} x_i 2^{-i},$$

with $x_i \in \{0, 1\}$, and $N(x) \in Z = \{0, \pm 1, \pm 2, \ldots\}$ is the largest integer i such that $x_{-i} \neq 0$. This representation is unique if x is not dyadically rational, i.e., $x \notin D_+$, where $D_+ = \{x \in R_+ | x = p/2^p, p \in P = \{0, 1, 2, \ldots\}, q \in Z\}$. If x is a dyadic rational, in which case there can be an infinite and a finite expansion for x, the finite expansion is chosen.

Walsh-Fourier analysis for functions on R_+ is defined in terms of the generalized Walsh functions $\psi(y, x)$, $x, y \in R_+$ defined by Fine [21]. The Walsh-Dirichlet kernel, also called the Fine integral, is defined as [21]

$$J(x, r) = \int_0^r \psi_x(s) ds.$$

In particular, when $r = 2^n$, the value of the Walsh-Diriclet kernel is

$$J(x, 2^n) = \begin{cases} 2^n, & 0 \leq x < 2^{-n}, \\ 0, & \text{otherwise}, \end{cases}$$

form where

$$J(1, 2^n x \oplus s) = \begin{cases} 1, & x \in [2^{-n}s, 2^{-n}(s+1), \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 6 [9] *If a given function f and its Walsh-Fourier spectrum $\hat{f}$ belong to $L^1(g)$, and f is continuous on $R_+ \setminus D_+$, and continuous from the right on D_+, and such that $\hat{f}(y) = 0$ for $y \geq 2^n$, $n \in Z$, Z-the set of integers, then*

$$f(x) = \sum_{s=0}^{\infty} f(\frac{s}{2^n}) J(1, 2^n x \oplus s), \quad x \in R_+. \tag{6}$$

Notice that the parameter y in generalized Walsh functions is often called *sequency* by the analogy to the notion of *frequency* in the classical Fourier analysis. In this setting, functions satisfying the conditions of Theorem 6 are called *sequency limited*.

Proof. In the case of Walsh-Fourier analysis, the domain group is the additive group G' of the *dyadic field*, whose dual group is the set of generalized Walsh functions [21]. A discrete subgroup H can be defined as $H = s/2^n$, $s = 0, 1, \ldots$, where n is a fixed integer, and for $s/2^n$ the finite dyadic expansion is taken.

The annihilator Λ for H is isomorphic to the sequences of the form $\Lambda = \ldots, \lambda_{-k}, 0, \ldots, 0, \ldots$, where $\lambda_i \in \{0, 1\}$. Therefore, it follows that for every $x \in H$, it is $\sum \lambda_{1-n} x_n = 0$ and $\Psi(\lambda, x) = 1$, where $\psi(\lambda, x)$ is the generalized Walsh function of the index λ.

We select the set $\Omega = [0, 2^n)$, and define the reconstruction function or the sampling function as

$$\phi(x) = 2^{-n} \int_0^{2^n} \psi(y, x) dy = \begin{cases} 1, & \text{if } x \in (0, 2^{-n}), \\ 0, & \text{otherwise}. \end{cases}$$

Since the conditions for the application of the theorem are satisfied, from the Kluvánek theorem,

$$f(x) = \int_0^{2^n} \hat{f}(y) \phi(y, x) dy.$$

Further, for every $x \in [2^{-n}k, 2^{-n}(k+1))$,

$$f(x) = 2^n \int_{2^{-n}k}^{2^{-n}(k+1)} f(u) du, \quad k \in P,$$

where P is the set of positive integers. Thus, f is a constant on all the intervals $[2^{-n}k, 2^{-n}(k+1))$, $k \in P$, and has the value equal to f in the left limit point. Therefore,

$$f(x) = \sum_{k=0} f(\frac{k}{2^n} \rho[2^{-n}k, 2^{-n}(k+1))(x), \quad x \in R_+,$$

where $\rho(a, b)(x)$ is the *characteristic function* of the interval $[a, b)$ defined as

$$\rho[a, b)(x) = \begin{cases} 1, & \text{if } x \in [a, b), \\ 0, & \text{if } x \notin [a, b). \end{cases}$$

Since for $k \in P$,

$$\begin{aligned} \rho[2^{-n}k, 2^{-n}(k+1))(x) &= \begin{cases} 1, & x \in I_{n,k}, \\ 0, & \text{otherwise}, \end{cases} \\ &= J(1, 2^n x \oplus k), \end{aligned}$$

where $I_{n,k} = [2^{-n}k, 2^{-n}(k+1))$, the proof is complete.

From (6), if the function f is approximated by first N terms, the corresponding truncation error is

$$\begin{aligned} e_N(x) &= f(x) - \sum_{s=0}^{N-1} f(\frac{s}{2^m}) J(1, 2^n x \oplus s) \\ &= \sum_{s=N}^{\infty} f(\frac{s}{2^m}) J(1, 2^n x \oplus s). \end{aligned}$$

From there the bound of the truncation error is

$$|e_N(x)| \leq \sum_{s=N}^{\infty} |f(\frac{s}{2^n})| |J(1, 2^n x \oplus s)| \leq \sum_{s=N} |f(\frac{s}{2^n})|,$$

since $|J(1, 2^n x \oplus s)| \leq 1$ for every s.

The bound thus determined provides impact into behaviour of the error depending on N or x. Two cases should be distinguished, as it has been done in [44] and [46], where the following conclusions have been derived.

If the sampling moment is $x = x_s = \frac{s}{2^n}$, then after some calculations,

$$e_N(x) = \begin{cases} f(\frac{s}{2^n}), & \text{if } s \leq N, \\ 0, & \text{if } s < N, \end{cases}$$

which is a result comparable to the corresponding result for the cardinal series for the sampling theorem in classical Fourier analysis.

If $x \neq x_s = \frac{s}{2^n}$m then [46]

$$|e_N(x)| \leq 2^{n/2} E^{1/2},$$

where E is the *energy* of the signal that is in the case of sequnecy limited signals determined as

$$E = \int_0^\infty f^2(x)dx = \int_0^{2^n} (\hat{f}(y))^2 dy = \frac{1}{2^n} \sum_{s=0}^{\infty} f^2(\frac{s}{2^n}).$$

It can be shown by using properties of the Fine integral that for $x < N/2^n$, it follows $e_N(x) = 0$, which shows advantages of the application of the Walsh-Fourier sampling theorem for sequency limited signals compared to the classical representation by cardinal series. The sampling theorem in Walsh-Fourier analysis has been discussed also in [19], [33], [34], and elsewhere else.

Notice that sequency limitedness of signals is a rather strongly restrictive condition and form the point of view of some authors [9], this theorem is of a limited practical applicability. For that reason the sampling theorem in Walsh-Fourier analysis has been defined also for limited duration signals. Recall that unlike classical Fourier analysis, the finite duration signals may be at the same time sequency limited. An example of such signals is the signal described by the characteristic function $\rho[0, 1)(x)$ for which $\hat{\rho}(y) = \rho[0, 1)(y)$, $y \in R_+$. Another example of such signals can be found in [33].

In Walsh-Fourier analysis,the sampling theorem for finite duration signals has been defined as follows [9].

Theorem 7 *If f is continuous on $R_+ \setminus D_+$ and right continuous on D_+, and such that $f, \hat{f} \in L^1(R_+)$, and $f(x) = 0$ for $x \notin [0, T_e)$, where $T_e \in R_+$, then,*

$$\lim_{n\to\infty} \sum_{s=0}^{K} f(\frac{s}{2^n}) J(1, 2^n x \oplus s) = f(x), \quad x \in R_+,$$

where $K = K(n, T_e) \in P$ is the smallest integer such that $2^{-n}(K + 1) \geq T_e$.

The proof of this theorem can be derived in the same way as for the corresponding theorem in classical Fourier analysis, and can be found in [9].

6. SAMPLING THEOREM ON FINITE ABELIAN GROUPS

In a simplified description, the sampling theorem states that if a function f satisfies certain conditions in the spectral domain, then it is completely specified by its values at a particular suitably determined area of the domain of definition of f. From the application point of view, that reduction of the domain of definition brings many advantages. Therefore, it is interesting to defined the sampling theorem for discrete functions, i.e., functions defined in countably many points or even a finite number of points.

The sampling theorem in finite Walsh-Fourier analysis, i.e., for functions defined on the set B_n of non-negative integers smaller than some $N = 2^n$, n is a natural number, has been defined in [38] and has been proven in the same way as the sampling theorem is proven in classical Fourier analysis. The proof given in [58] directly follows from properties of the discrete Walsh functions and basic results from group theory. The same approach was used to prove the sampling theorem on other finite Abelian and non-Abelian groups [59].

Recall that the set B_n with the addition modulo N forms a group isomorphic to the finite dyadic group C_2^n consisting of binary n-tuples with the componentwise addition modulo 2. Therefore, functions defined in $N = 2^n$ points, can be viewed as functions on finite dyadic group, whose group characters are the discrete Walsh functions [1].

Definition 1 [38] *Consider a function $f(x)$ defined on B_n and a number $M = 2^m$, where $m < n$. The function $f(x)$ is called M-sequency band limited (MBL), if the Walsh transform $\hat{f}(w) = 0$, for $w \geq M$.*

Theorem 8 [38] *An MBL function $f(x)$ can be completely reconstructed from its M values as*

$$f(x) = M^{-1} \sum_{p=0}^{M-1} f(p) d_M(x \oplus p),$$

where $\oplus$ is componentwise addition modulo 2, and $d_M(x)$ is the Walsh-Fourier kernel defined as

$$d_M(x) = \sum_{w=0}^{M-1} wal(w, x) = M \sum_{r=0}^{R-1} \delta(x \oplus r), R = NM^{-1},$$

and $\delta(x) = \delta_{x,0}$ is the discrete delta function.

As noticed above, in [38], this theorem is proven in a way completely analogous to than used to prove the sampling theorem in classical Fourier analysis and in [58] as a particular case of sampling theorem on locally compact Abelian groups.

The sampling theorem for functions on finite dyadic groups has been generalized to functions on an arbitrary Abelian group which can be represented as a direct product of m cyclic groups G_i of orders g_i, $i = 0, 1, \dots, m-1$,

$$\begin{aligned} G &= \times_{i=0}^{m-1} G_i, && (7) \\ g &= \prod_{i=0}^{m-1} g_i, \quad g_0 \le g_1 \le \cdots \le g_{m-1}. \end{aligned}$$

If $p = M = \prod_{i=0}^{k} g_{m-i-1}$, $k \in \{0, 1, \dots, m-1\}$, then the Fourier kernel can be defined as

$$D_M(x) = \sum_{w=0}^{M-1} \chi^*(w, x) = M \sum_{k=0}^{R-1} \delta(kM \circ x^{-1}),$$

where $\circ$ is the group operation on G, $R = g/M$, $\chi(w, x)$ are the group characters of G, χ^* is the complex conjugate of χ, and $\delta(x) = \delta_{x,0}$ is the discrete delta function on G.

As in the case of the sampling theorem on dyadic groups, we impose the requirements on f that should be satisfied in the spectral domain.

Definition 2 *A function f on a finite Abelian group G of order g, is called M-band limited (MBL) if its Fourier transform $\hat{f}(w) = 0$ for $w \ge M$.*

Theorem 9 [59] *An MBL function $f(x)$ on a finite Abelian group G can be reconstructed from its M values as*

$$\begin{aligned} f(x) &= M^{-1} \sum_{p=0}^{M-1} f(p) D_M(p \circ x^{-1}) \\ &= m^{-1} \sum_{p=0}^{M-1} f(p) \left(M \sum_{k=0}^{R-1} \delta(kM \circ (p \circ x^{-1})^{-1}) \right), \end{aligned}$$

where $R = g/M$.

Extensions of the sampling theorem to bandpass functions on finite dyadic groups and finite Abelian groups are also given in [38] and [59], respectively.

Since every finite Abelian group is a locally compact Abeliean group, these theorems can be related to the Kluvánek sampling theorem in abstract harmonic analysis. Formulation of the sampling theorems on finite Abelian groups is much simpler, and therefore, their practical applicability. For instance, these theorems can be used to detect particular properties of multiple-valued functions $f : L^n \to L$, $L = 0, 1, \dots, p-1$, when these functions are viewed as a subset of complex-valued functions on finite Abelian groups. It is assumed that logic values $0, 1, \dots p-1$ are interpreted as the corresponding integers.

The sampling theorem and the requirement to have a band-limited spectrum can be used to formulate the following statement.

Statement 1 *Consider the subset of spectral coefficients $A = \{\hat{f}(w) | w_i = 0\}$ of a multiple-valued logic function $f(x_1, \dots, x_{i-1}, x_i, x_{+1}, \dots x_n)$. Then f is independent on the variable x_i iff all the non-zero Fourier coefficients are in A.*

Therefore, the detection of essential variables in multiple-valued functions can be expressed as the requirement that the functions belong to MPL or MBL classes for some M. The same statement has be derived in a different way in [64], and represents a generalization of the corresponding statement for Boolean functions [29].

7. SAMPLING THEOREM ON FINITE NON-ABELIAN GROUPS

The results in the previous section have been generalized to function on finite non-Abelian groups in [59].

We assume that in the group G representable in the form (7) some of subgroups G_i can be non-Abelian, thus, G is also a non-Abelian group.

The group representations R_w of G can be derived as the Kronecker product of the group representations R_{w_i} of G_i, $i = 0, \dots, m-1$. Therefore, the set of group representations, written in a matrix form, has a block structure analogous to the structure of matrices of group characters for decomposable Abelian groups. In other word, if M_{m-1} is the number of unitary irreducible representations of the subgroup G_{m-1} of order g_{m-1}, then the first M_{m-1} representations of the group G are periodic with the period g_{m-1}, provided that subgroups G_i are ordered such that $g_0 \le g_1 \le \cdots \le g_{m-1}$.

If M_i is the number of irreducible unitary representations of G_i, for a fixed $k \in \{0, 1, \dots, m-1\}$, we define a number $M = \prod_{i=0}^{k} M_{m-i-1}, \quad M < K$, were K is the number of unitary irreducible representations of G.

As in the case of sampling theorems on other groups, some requirements are imposed on f in the transform domain in order to be able to apply the sampling theorem.

Definition 3 *A function $f(x)$ on a finite non-Abelian group representable as in (7), is an M-band limited (MBL) function if its Fourier transform on G, $S_f(w) = 0$ for $w \ge M$.*

Theorem 10 [59] *An MBL function on a finite non-Abelian group G representable as in (7) can be reconstructed from its Q values as*

$$f(x) = \frac{1}{Q} \sum_{u=0}^{Q-1} f(u) D_M(x \circ u^{-1}),$$

where $Q = \prod_{i==}^{k} g_{m-i-1}$, where k is a fixed value determined in definition of M, and

$$d_M(x) = \sum_{w=0}^{M-1} r_w(TrR_w(x)),$$

where TrR_w is the trace of R_w.

The sampling theorem on finite non-Abelian groups can be applied in study of properties of functions on these groups in the same way as in the case of Abelian groups.

8. CLOSING REMARKS

Sampling theorem has an immense in signal processing and is in very foundations of digital signal processing and its applications in practice. In this settings, several results can be consider as predecessors of the sampling theorem from the practical engineering point of view. A condensed overview of such results is given in [43].

As the same time, the sampling theorem, is considered as a very fundamental theoretical result having an unique role in various branches of mathematics, providing a background for relating other fundamental results and to make generalizations [8].

In [8], it is shown that the sampling theorem is essentially equivalent to the Poisson summation formula of Fourier analysis, a particular form of Cauchy integral formula in complex function theory and the Euler-Maclaurin summation formula of numerical analysis.

The sampling theorem can be a starting point to derive these formulas and vice versa, as shown in [8]. It can be also used to generalize the Vandermonde convolution formula for binomial coefficients to fractional and infinite series form, which put links to the Gauss summation theorem in the theory of hypergeometric series. The sampling theorem can be also used to introduce the Stirling functions, as an extension of the Stirling numbers, which links to the Riemann-Zeta function ξ. More about these generalizations can be found in [8].

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