Asymptotics of Closed Fork and Join Queues with Subexponential Service Times[1]

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Abstract

We consider a closed fork and join queueing network where several lines feed a single assembly station. Under the assumption that at least one service time distribution is subexponential, we obtain the tail asymptotics of transient cycle times and waiting times. We also discuss under which conditions these results can be generalized to the tail asymptotics of stationary cycle times and waiting times.

Keywords: cycle time, waiting time, subexponential asymptotics

1 Introduction

Recent research has shown that in many queueing networks service times have subexponential distributions. For example in telecommunications setting, Fowler [15] argue that FTP (File Transfer Protocol) transfers have session sizes and session durations with subexponential distributions. Similar observations are made for the TELNET sessions in Paxson and Floyd [20] even though TELNET is an application qualitatively quite different from FTP. Feldmann, Gilbert, Willinger and Kurtz [14] argue that these observations remain valid for today's World Wide Web (WWW) applications. Similarly, Arlitt and Williamson [1], Crovella and Bestavros [11] and Crovella and Lipsky [12] have shown evidence that the file sizes in Web have subexponential distributions.

In this paper, we focus on a closed fork and join queueing network with subexponential service time distributions. Fork and join queues arise in many telecommunication and manufacturing applications (see Ko and Serfozo [18] for an excellent review of the literature on these networks). We consider a cyclic fork and join type queueing network with L lines feeding a single assembly station (see Figure 1). Line l ($l = 1, \ldots, L$) has K_l stations and the $(K_l + 1)^{\text{th}}$ station in each line

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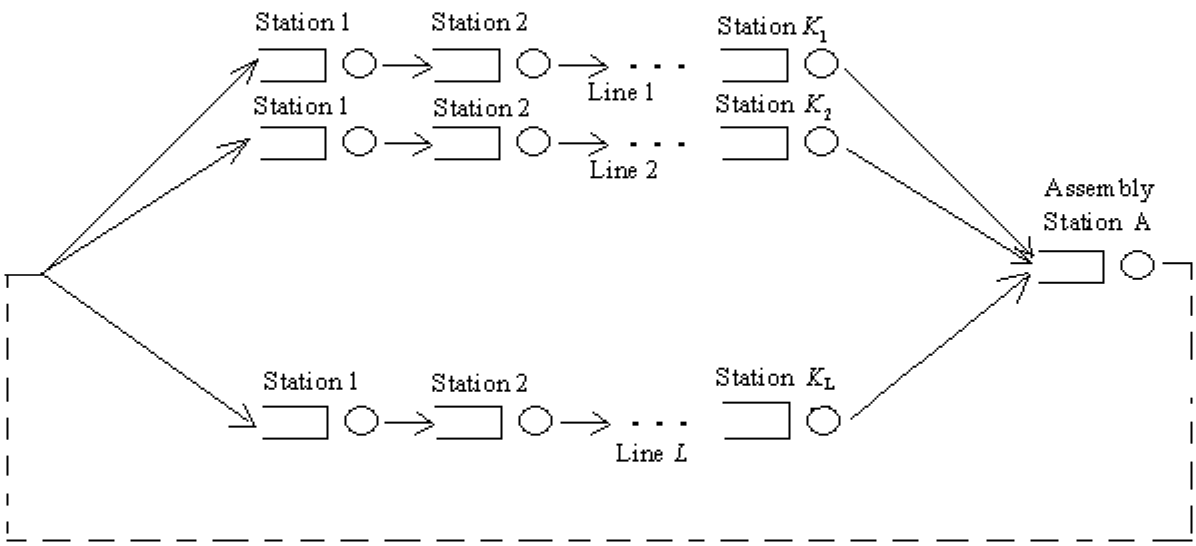

Figure 1: Closed fork and join queueing network.

is the assembly station which will be denoted by A. There is a single server at each station and the service discipline at all stations is First Come First Served. The capacity of the buffer between any two consecutive stations is infinite. There are $N \geq 1$ customers in each line $l \in \{1, \ldots, L\}$, who sequentially visit station 1 to station $K_l + 1$ in line l. Note that for the assembly operation to take place, we need at least one customer from each line to be present in front of the assembly station. We assume that at time zero there are $N_{k,l}$ customers in front of station $k \in \{1, \ldots, K_l\}$ in line l and there are $N_{A,l}$ customers waiting in front of the assembly station who come from line $l \in \{1, \ldots, L\}$. Hence, $\sum_{k=1}^{K_l} N_{k,l} + N_{A,l} = N$ for all $l \in \{1, \ldots, L\}$. Service times at station $k \in \{1, \ldots, K_l\}$ in line l are independent and identically distributed random variables $\{B_n^{k,l}\}$ with distribution function $B_{k,l}(\cdot)$ and the service times at the assembly station are independent identically distributed random variables $\{B_n^A\}$ with distribution function $B_A(\cdot)$. The sequence of service times at each station is independent of the service times at the other stations. Furthermore, we assume that there exists a subexponential distribution $F(\cdot)$ ($F \in \mathcal{S}$) and there exist constants $c_{k,l}, c_A \in [0, \infty)$ with $c_A + \sum_{l=1}^{L} \sum_{k=1}^{K_l} c_{k,l} > 0$ such that for all $k \in \{1, \ldots, K_l\}$ and $l \in \{1, \ldots, L\}$

$$\lim_{x \to \infty} \frac{\overline{B}_{k,l}(x)}{\overline{F}(x)} = c_{k,l},$$

and

$$\lim_{x \to \infty} \frac{\overline{B}_A(x)}{\overline{F}(x)} = c_A$$

where $\overline{F}(x) = 1 - F(x)$. In the remainder of the paper, if $\lim_{x\to\infty} f(x)/g(x) = 1$ for two functions $f(\cdot)$ and $g(\cdot)$, we will denote this by $f(x) \sim g(x)$ as $x \to \infty$.

For the network described above, we are interested in the tail behavior of transient and stationary cycle times (time between successive departures of the same customer from a given station) and waiting times at station $k \in \{1, \ldots, K_l\}$ in line $l \in \{1, \ldots, L\}$ and at the assembly station. We start our analysis by obtaining upper and lower bounds on the transient cycle times and waiting times. These bounds are sufficient to compute the tail asymptotics of these two performance measures. Since the fork and join network depicted in Figure 1 is an example of an autonomous (max,+) linear system, using the results on the existence of a stationary regime for autonomous (max,+) linear systems in Chapter 7 of Baccelli, Cohen, Olsder and Quadrat [7], we show that under certain conditions on service times, tail asymptotics for transient characteristics also hold for their stationary counter parts. We would like to point out that general (max,+) linear systems and (max,+) algebra are beyond the scope of this paper. The interested reader is referred to Baccelli, Cohen, Olsder and Quadrat [7] for more details on this formalism.

In the last few decades, there has been a growing interest in queues with subexponential service times. However, majority of the existing research has focused on single stage queues; see for example Asmussen, Kluppelberg and Sigman [2], Asmussen and Moller [3], Asmussen, Schmidli and Schmidt [4], Embrechts and Veraverbeke [13], Jelenkovic and Lazar [17], Pakes [19], Rolski, Schmidli, Schmidt and Teugels [21], Willekens and Teugels [23] and Xia and Liu [24].

There are not many existing results on the asymptotics of queueing networks with subexponential service times. Baccelli, Schlegel and Schmidt [10] consider the tail behavior of stationary response times in open (max,+) linear systems. In a similar paper, Huang and Sigman [16] focus on the asymptotics of sojourn times and queue lengths in open tandem queues and split-match queues. Baccelli and Foss [8] compute upper and lower bounds for the tail asymptotics of the stationary maximal dater in more general monotone-separable stochastic networks. Baccelli, Foss and Lelarge [9] compute the exact tail asymptotics of stationary response times for both irreducible and reducible open stochastic event graphs under the assumptions that the arrival process is a renewal process and the service times have subexponential distributions. To the best of our knowledge, the only paper that studies closed networks with subexponential processing times is the one by Ayhan, Palmowski and Schlegel [5]. In [5], the authors analyze the tail distribution of transient and stationary cycle times and waiting times in closed tandem queues with subexponential service times. Our objective is to generalize these results to the more complicated closed fork and join network described above. Clearly, when $L = 1$, our results reduce to the ones given in [5].

The paper is organized as follows. In Section 2, we introduce the notation used in our analysis. In Section 3, we derive upper and lower bounds on transient cycle times and waiting times. Using these bounds, we obtain the tail asymptotics of transient cycle times and waiting times in Section 4. In Section 5, we first argue that under certain conditions on service times a stationary regime exists and the results of Section 4 can be generalized to stationary cycle times and waiting times. For the sake of completeness, in the Appendix we provide some properties of subexponential distributions that are used in our analysis.

2 Notation

In this section, we introduce the notation used throughout our developments. Let $X_n^{k,l}$ be the departure time of the n^{th} customer from station $k \in \{1, \dots, K\}$ in line $l \in \{1, \dots, L\}$. Similarly, X_n^A denotes the departure time of the n^{th} customer from the assembly station. Moreover, we set

$$X_n^{[j],l} = \begin{cases} X_n^A & \text{if } j \bmod K_l + 1 = 0 \\ X_n^{j \bmod K_l+1,l} & \text{if } j \bmod K_l + 1 \neq 0 \end{cases},$$

$$B_n^{[j],l} = \begin{cases} B_n^A & \text{if } j \bmod K_l + 1 = 0 \\ B_n^{j \bmod K_l+1,l} & \text{if } j \bmod K_l + 1 \neq 0 \end{cases},$$

and

$$N_{[j],l} = \begin{cases} N_{A,l} & \text{if } j \bmod K_l + 1 = 0 \\ N_{j \bmod K_l+1,l} & \text{if } j \bmod K_l + 1 \neq 0 \end{cases}.$$

Recall that K_l is the number of stations in line l. One can easily see that for the type of closed fork and join network that we consider for all $n \geq 1$

$$X_n^{k,l} = \max\{X_{n-1}^{k,l}, X_{n-N_{k,l}}^{[k-1],l}\} + B_n^{k,l} \text{ for all } k \in \{1, \dots, K_l\} \text{ and } l \in \{1, \dots, L\}, \tag{1}$$

$$X_n^A = \max\{X_{n-1}^A, \max_{l=1,\dots,L} X_{n-N_{A,l}}^{K_l,l}\} + B_n^A. \tag{2}$$

Throughout our developments we set $X_n^{k,l} = X_n^A = 0$ and $B_n^{k,l} = B_n^A = 0$ for $n \leq 0$.

Note that the performance measures that we are interested in, namely, the cycle time and the waiting time of the n^{th} customer at each station can be expressed in terms of the departure times. Let $C_n^{k,l}$ denote the n^{th} cycle time at station $k \in \{1, \dots, K_l\}$ in line $l \in \{1, \dots, L\}$ and similarly let C_n^A denote the n^{th} cycle time at the assembly station. Since the cycle time is the time between two successive departures of the same customer from a given station, cycle time expressions are given as

$$C_n^{k,l} = X_{n+N}^{k,l} - X_n^{k,l} \text{ for all } l \in \{1, \dots, L\} \text{ and } k \in \{1, \dots, K_l\}, \tag{3}$$

$$C_n^A = X_{n+N}^A - X_n^A \tag{4}$$

for all $n \geq 1$. Similarly, let $W_n^{k,l}$ denote the waiting time of the n^{th} customer until the start of his service at station $k \in \{1, \dots, K_l\}$ in line $l \in \{1, \dots, L\}$ and let $W_n^{A,l}$ denote the waiting time of the n^{th} arriving customer from line $l \in \{1, \dots, L\}$ at the assembly station. Then for $n \geq 1$

$$W_n^{k,l} = \max\{X_{n-1}^{k,l} - X_{n-N_{k,l}}^{[k-1],l}, 0\} \text{ for all } k \in \{1, \dots, K_l\} \text{ and } l \in \{1, \dots, L\}, \tag{5}$$

$$W_n^{A,l} = \max\{X_{n-1}^A - X_{n-N_{A,l}}^{K_l,l}, \max_{\substack{i=1,\dots,L \\ i \neq l}} X_{n-N_{A,i}}^{K_i,i} - X_{n-N_{A,l}}^{K_l,l}, 0\} \text{ for all } l \in \{1, \dots, L\}. \tag{6}$$

While obtaining upper and lower bounds on transient cycle times and waiting times, we use $p_n^{k,l}$ to denote the n^{th} customer served at station $k \in \{1, \dots, K_l\}$ in line $l \in \{1, \dots, L\}$ and p_n^A to denote the n^{th} customer served at the assembly station. Finally, we have the following expressions for sojourn times

$S_n^{k,l}$ sojourn time of customer $p_n^{k,l}$ at station $k \in \{1, \ldots, K_l\}$ in line $l \in \{1, \ldots, L\}$,

S_n^A sojourn time of customer p_n^A at the assembly station.

3 Bounds on Cycle Times and Waiting Times

In this section, we provide upper and lower bounds on transient cycle times and waiting times. We start with an upper bound on the waiting time of the n^{th} customer at each station.

Lemma 3.1 *For $l \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K_l\}$ and $n \geq 1$*

$$W_n^{k,l} \leq \sum_{r=n-N+1}^{n-1} B_r^{k,l}, \tag{7}$$

with the convention that summation over an empty set is 0 and for $n \geq 1$

$$W_n^{A,l} \leq \sum_{r=n-N+1}^{n-1} B_r^A + \sum_{\substack{i=1 \\ i \neq l}}^{L} \sum_{k=1}^{K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i} \tag{8}$$

with the convention that summation over an empty set is 0.

Proof The bound on $W_n^{k,l}$ is obvious since the n^{th} arriving customer at station k can find at most $N-1$ customers (namely, $p_{n-N+1}^{k,l}, \ldots, p_{n-N}^{k,l}$) in front of him. Hence,

$$W_n^{k,l} \leq \sum_{r=n-N+1}^{n-1} B_r^{k,l}.$$

In order to see the upper bound on $W_n^{A,l}$, note that the n^{th} arriving customer from line l is also the n^{th} customer served at the assembly station (i.e. p_n^A). But $p_n^A = p_{n-\sum_{q=k}^{K_l} N_{[q+1],l}}^{k,l}$ for all $k \in \{1, \ldots, K_l\}$ and $l \in \{1, \ldots, L\}$. Then, since the n^{th} arriving customer from line l can find at most $N-1$ customers (namely, $p_{n-N+1}^A, \ldots, p_{n-N}^A$) ahead of him at the time of his arrival

$$\begin{aligned} W_n^{A,l} &\leq \max_{\substack{i=1,\ldots,L \\ i \neq l}} \sum_{k=1}^{K_i} S_{n-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i} + \sum_{r=n-N+1}^{n-1} B_r^A \\ &\leq \sum_{\substack{i=1 \\ i \neq l}}^{L} \sum_{k=1}^{K_i} S_{n-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i} + \sum_{r=n-N+1}^{n-1} B_r^A. \end{aligned} \tag{9}$$

We know that

$$\begin{aligned} S_{n-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i} &= W_{n-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i} + B_{n-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i} \\ &\leq \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i} \end{aligned} \tag{10}$$

where the inequality follows from (7). Plugging (10) into (9), we obtain (8). $\square$

In the next lemma, we provide an upper bound on $C_n^{k,l}$ for $l \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K_l\}$ and C_n^A for all $n \geq 1$.

Lemma 3.2 *For $l \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K_l\}$ and $n \geq 1$*

$$C_n^{k,l} \leq U(C_n^{k,l}) \tag{11}$$

where

$$U(C_n^{k,l}) = \sum_{j=1}^{k} \sum_{r=n+1-\sum_{q=j+1}^{k} N_{q,l}}^{n+N-\sum_{q=j+1}^{k} N_{q,l}} B_r^{j,l} + \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}}^{n+N-\sum_{q=1}^{k} N_{q,l}} B_r^A + \sum_{j=k+1}^{K_l} \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} B_r^{j,l} + \sum_{\substack{i=1 \\ i\neq l}}^{L} \sum_{j=1}^{K_i} \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} B_r^{j,i}$$

with the convention that summation over an empty set is 0 and for $n \geq 1$

$$C_n^A \leq U(C_n^A) \tag{12}$$

where

$$U(C_n^A) = \sum_{r=n+1}^{n+N} B_r^A + \sum_{l=1}^{L} \sum_{k=1}^{K_l} \sum_{r=n+1-\sum_{q=k}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} B_r^{k,l}$$

with the convention that summation over an empty set is 0.

Proof Note that

$$p_n^{k,l} = p_{n+\sum_{q=k}^{K_l} N_{[q+1],l}+\sum_{q=1}^{j} N_{q,l}}^{j,l} = p_{n+N-\sum_{q=j+1}^{k} N_{q,l}}^{j,l} \text{ for all } j \in \{1, \ldots, k\}$$

and

$$p_n^{k,l} = p_{n+\sum_{q=k}^{j-1} N_{[q+1],l}}^{j,l} = p_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}^{j,l} \text{ for all } j \in \{k+1, \ldots, K_l\}.$$

Similarly, $p_n^{k,l} = p_{n+\sum_{q=k}^{K_l} N_{[q+1],l}}^A = p_{n+N-\sum_{q=1}^{k} N_{q,l}}^A$. Then it follows from (3) that

$$C_n^{k,l} = \sum_{j=k+1}^{K_l} S_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}^{j,l} + S_{n+N-\sum_{q=1}^{k} N_{q,l}}^A + \sum_{j=1}^{k} S_{n+N-\sum_{q=j+1}^{k} N_{q,l}}^{j,l}. \tag{13}$$

From (7), we have

$$\begin{aligned} S_{n+N-\sum_{q=j+1}^{k} N_{q,l}}^{j,l} &= W_{n+N-\sum_{q=j+1}^{k} N_{q,l}}^{j,l} + B_{n+N-\sum_{q=j+1}^{k} N_{q,l}}^{j,l} \\ &\leq \sum_{r=n+1-\sum_{q=j+1}^{k} N_{q,l}}^{n+N-\sum_{q=j+1}^{k} N_{q,l}} B_r^{j,l} \text{ for all } j \in \{1, \ldots, k\} \end{aligned} \tag{14}$$

and

$$S^{j,l}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} = W^{j,l}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} + B^{j,l}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}$$
$$\leq \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} B^{j,l}_r \text{ for all } j \in \{k+1,\ldots,K_l\}. \quad (15)$$

Finally, from (8), we know that

$$S^A_{n+N-\sum_{q=1}^{k} N_{q,l}} = W^A_{n+N-\sum_{q=1}^{k} N_{q,l}} + B^A_{n+N-\sum_{q=1}^{k} N_{q,l}}$$
$$\leq \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}}^{n+N-\sum_{q=1}^{k} N_{q,l}} B^A_r + \sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{j=1}^{K_i} \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} B^{j,i}_r. \quad (16)$$

Plugging (14), (15) and (16) into (13) gives (11).

Next we obtain an upper bound on C^A_n. Note that

$$p^A_n = p^{k,l}_{n+\sum_{q=1}^{k} N_{q,l}} = p^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} \text{ for all } k \in \{1,\ldots,K_l\},\ l \in \{1,\ldots,L\}.$$

Then it follows from (4) that

$$C^A_n = \max_{l=1,\ldots,L}\{\sum_{k=1}^{K_l} S^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} + W^{A,l}_{n+N}\} + B^A_{n+N}.$$

From (7), we have

$$S^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} = W^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} + B^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}}$$
$$\leq \sum_{r=n+1-\sum_{q=k}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} B^{k,l}_r \text{ for all } k \in \{1,\ldots,K_l\},\ l \in \{1,\ldots,L\}.$$

Similarly, we know from (8) that

$$W^{A,l}_{n+N} \leq \sum_{r=n+1}^{n+N-1} B^A_r + \sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{k=1}^{K_i} \sum_{r=n+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=k}^{K_i} N_{[q+1],i}} B^{k,i}_r \text{ for all } l \in \{1,\ldots,L\}.$$

Then

$$C^A_n \leq \max_{l=1,\ldots,L}\{\sum_{k=1}^{K_l} \sum_{r=n+1-\sum_{q=k}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} B^{k,l}_r + \sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{k=1}^{K_i} \sum_{r=n+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=k}^{K_i} N_{[q+1],i}} B^{k,i}_r\} + \sum_{r=n+1}^{n+N} B^A_r$$

$$= \max_{l=1,\dots,L}\{\sum_{i=1}^{L}\sum_{k=1}^{K_i}\sum_{r=n+1-\sum_{q=k}^{K_i}N_{[q+1],i}}^{n+N-\sum_{q=k}^{K_i}N_{[q+1],i}} B_r^{k,i}\} + \sum_{r=n+1}^{n+N} B_r^A$$

$$= \sum_{i=1}^{L}\sum_{k=1}^{K_i}\sum_{r=n+1-\sum_{q=k}^{K_i}N_{[q+1],i}}^{n+N-\sum_{q=k}^{K_i}N_{[q+1],i}} B_r^{k,i} + \sum_{r=n+1}^{n+N} B_r^A.$$

□

In the next lemma, we provide a lower bound for $C_n^{k,l}$ for $l \in \{1,\dots,L\}$, $k \in \{1,\dots,K_l\}$ and C_n^A for all $n \geq 1$.

Lemma 3.3 *For $l \in \{1,\dots,L\}$, $k \in \{1,\dots,K_l\}$ and $n \geq 1$*

$$C_n^{k,l} \geq \max\{\max_{j=1,\dots,k}\sum_{r=n+1-\sum_{q=j+1}^{k}N_{q,l}}^{n+N-\sum_{q=j+1}^{k}N_{q,l}} B_r^{j,l}, \sum_{r=n+1-\sum_{q=1}^{k}N_{q,l}}^{n+N-\sum_{q=1}^{k}N_{q,l}} B_r^A, \max_{j=k+1,\dots,K_l}\sum_{r=n+1-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_l}N_{[q+1],l}}^{n+N-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_l}N_{[q+1],l}} B_r^{j,l},$$
$$\max_{\substack{i=1,\dots,L\\ i\neq l}}\max_{j=1,\dots,K_i}\sum_{r=n+1-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_i}N_{[q+1],i}}^{n+N-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_i}N_{[q+1],i}} B_r^{j,i}\} - U(C_{n-N}^{k,l}) - U(C_{n-2N}^{k,l}) \quad (17)$$

with the convention that the maximum of an empty set is $-\infty$ and the summation over an empty set is 0. For $n \geq 1$

$$C_n^A \geq \max\{\sum_{r=n+1}^{n+N} B_r^A, \max_{l=1,\dots,L}\max_{k=1,\dots,K_l}\sum_{r=n+1-\sum_{q=k}^{K_l}N_{[q+1],l}}^{n+N-\sum_{q=k}^{K_l}N_{[q+1],l}} B_r^{k,l}\} - U(C_{n-N}^A) \quad (18)$$

with the convention that the maximum of an empty set is $-\infty$ and the summation over an empty set is 0.

Proof From (1) and (2), one can deduce that

$$X_{n+N}^{k,l} \geq X_{n+N-\sum_{q=j}^{k-1}N_{[q+1],l}}^{j,l} \text{ for all } j \in \{1,\dots,k\}$$
$$X_{n+N}^{k,l} \geq X_{n+N-\sum_{q=1}^{k}N_{q,l}}^{A}$$
$$X_{n+N}^{k,l} \geq X_{n+N-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_l}N_{[q+1],l}}^{j,l} \text{ for all } j \in \{k+1,\dots,K_l\}$$
$$X_{n+N}^{k,l} \geq X_{n+N-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_i}N_{[q+1],i}}^{j,i} \text{ for all } j \in \{1,\dots,K_i\} \text{ and } i \in \{1,\dots,L\}\setminus\{l\}.$$

Similarly, we can obtain from (1) and (2) that for $k \in \{1,\dots,K_l\}$ and $l \in \{1,\dots,L\}$

$$X_{n-\sum_{q=j}^{k-1}N_{[q+1],l}}^{j,l} \geq X_{n-N}^{k,l} \text{ for all } j \in \{1,\dots,k\}$$
$$X_{n-\sum_{q=1}^{k}N_{q,l}}^{A} \geq X_{n-N}^{k,l}$$
$$X_{n-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_l}N_{[q+1],l}}^{j,l} \geq X_{n-N}^{k,l} \text{ for all } j \in \{k+1,\dots,K_l\}$$
$$X_{n-\sum_{q=1}^{k}N_{q,l}-\sum_{q=j}^{K_i}N_{[q+1],i}}^{j,i} \geq X_{n-2N}^{k,l} \text{ for all } j \in \{1,\dots,K_i\} \text{ and } i \in \{1,\dots,L\}\setminus\{l\}.$$

In the time interval from $X_n^{k,l}$ to $X_{n+N}^{k,l}$, customers $p_{n+1}^{k,l}$, $p_{n+2}^{k,l}, \ldots,$ $p_{n+N}^{k,l}$ are served at station k in line l. Therefore, for all $k \in \{1, \ldots, K_l\}$ and $l \in \{1, \ldots, L\}$,

$$X_{n+N}^{k,l} - X_n^{k,l} \geq \sum_{r=n+1}^{n+N} B_r^{k,l}.$$

Similarly, in the interval from X_n^A to X_{n+N}^A, customers p_{n+1}^A, $p_{n+2}^A, \ldots,$ p_{n+N}^A are served at the assembly station and we have

$$X_{n+N}^A - X_n^A \geq \sum_{r=n+1}^{n+N} B_r^A. \tag{19}$$

Putting all these together and using the cycle time expression in (3), we obtain the following bounds. For all $j \in \{1, \ldots, k\}$

$$\begin{aligned} C_n^{k,l} &\geq X^{j,l}_{n+N-\sum_{q=j}^{k-1} N_{[q+1],l}} - X^{j,l}_{n-\sum_{q=j}^{k-1} N_{[q+1],l}} - (X_n^{k,l} - X^{j,l}_{n-\sum_{q=j}^{k-1} N_{[q+1],l}}) \\ &\geq X^{j,l}_{n+N-\sum_{q=j}^{k-1} N_{[q+1],l}} - X^{j,l}_{n-\sum_{q=j}^{k-1} N_{[q+1],l}} - (X_n^{k,l} - X_{n-N}^{k,l}) \\ &\geq \sum_{r=n+1-\sum_{q=j+1}^{k} N_{q,l}}^{n+N-\sum_{q=j+1}^{k} N_{q,l}} B_r^{j,l} - U(C_{n-N}^{k,l}) \end{aligned} \tag{20}$$

and

$$\begin{aligned} C_n^{k,l} &\geq X^A_{n+N-\sum_{q=1}^{k} N_{q,l}} - X^A_{n-\sum_{q=1}^{k} N_{q,l}} - (X_n^{k,l} - X^A_{n-\sum_{q=1}^{k} N_{q,l}}) \\ &\geq X^A_{n+N-\sum_{q=1}^{k} N_{q,l}} - X^A_{n-\sum_{q=1}^{k} N_{q,l}} - (X_n^{k,l} - X_{n-N}^{k,l}) \\ &\geq \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}}^{n+N-\sum_{q=1}^{k} N_{q,l}} B_r^A - U(C_{n-N}^{k,l}). \end{aligned} \tag{21}$$

For all $j \in \{k+1, \ldots, K_l\}$,

$$\begin{aligned} C_n^{k,l} &\geq X^{j,l}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} - X^{j,l}_{n-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} - (X_n^{k,l} - X^{j,l}_{n-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}) \\ &\geq X^{j,l}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} - X^{j,l}_{n-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} - (X_n^{k,l} - X_{n-N}^{k,l}) \\ &\geq \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} B_r^{j,l} - U(C_{n-N}^{k,l}). \end{aligned} \tag{22}$$

Finally, for $j \in \{1, \ldots, K_i\}$ and $i \in \{1, \ldots, L\} \setminus \{l\}$

$$
\begin{aligned}
C_n^{k,l} &\geq X^{j,i}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} - X^{j,i}_{n-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} - (X_n^{k,l} - X^{j,i}_{n-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}}) \\
&\geq X^{j,i}_{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} - X^{j,i}_{n-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} - (X_n^{k,l} - X_{n-2N}^{k,l}) \\
&\geq \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} B_r^{j,i} - (X_n^{k,l} - X_{n-N}^{k,l} + X_{n-N}^{k,l} - X_{n-2N}^{k,l}) \\
&\geq \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} B_r^{j,i} - U(C_{n-N}^{k,l}) - U(C_{n-2N}^{k,l}).
\end{aligned}
\tag{23}
$$

The lower bound in (17) then follows from (20), (21), (22) and (23). Next we obtain the bound in (18). From (2), we have

$$
X_{n+N}^{A} \geq X^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} \text{ for all } k \in \{1,\ldots,K_l\},\ l \in \{1,\ldots,L\},
$$

and from (1), we have

$$
X^{k,l}_{n-\sum_{q=k}^{K_l} N_{[q+1],l}} \geq X_{n-N}^{A} \text{ for all } k \in \{1,\ldots,K_l\},\ l \in \{1,\ldots,L\}.
$$

Then using the cycle time expression in (4), for all $k \in \{1,\ldots,K_l\}$, $l \in \{1,\ldots,L\}$ we obtain

$$
\begin{aligned}
C_n^{A} &\geq X^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} - X^{k,l}_{n-\sum_{q=k}^{K_l} N_{[q+1],l}} - (X_n^{A} - X^{k,l}_{n-\sum_{q=k}^{K_l} N_{[q+1],l}}) \\
&\geq X^{k,l}_{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} - X^{k,l}_{n-\sum_{q=k}^{K_l} N_{[q+1],l}} - (X_n^{A} - X_{n-N}^{A}) \\
&\geq \sum_{r=n+1-\sum_{q=k}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} B_r^{k,l} - U(C_{n-N}^{A}).
\end{aligned}
\tag{24}
$$

The result then follows from (19) and (24). □

Finally, the next lemma provides a lower bound on $W_n^{k,l}$ for $l \in \{1,\ldots,L\}$, $k \in \{1,\ldots,K_l\}$ and W_n^A for all $n \geq 1$.

Lemma 3.4 *For $l \in \{1,\ldots,L\}$ and $n \geq 1$*

$$
W_n^{1,l} \geq \sum_{r=n-N+1}^{n-1} B_r^{1,l} - U(C^{A}_{n-N-N_{1,l}}) \tag{25}
$$

and for $l \in \{1,\ldots,L\}$, $k \in \{2,\ldots,K_l\}$ and $n \geq 1$

$$
W_n^{k,l} \geq \sum_{r=n-N+1}^{n-1} B_r^{k,l} - U(C^{k-1,l}_{n-N-N_{k,l}}) \tag{26}
$$

with the convention that the summation over an empty set is 0. For $l \in \{1, \ldots, L\}$ and $n \geq 1$

$$W_n^{A,l} \geq \max\{\sum_{r=n-N+1}^{n-1} B_r^A, \max_{\substack{i=1,\ldots,L \\ i\neq l}} \max_{k=1,\ldots,K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i}\}$$
$$-\sum_{i=1}^{L} U(C_{n-2N-N_{A,i}}^{K_i,i}) - \sum_{\substack{i=1 \\ i\neq l}}^{L} U(C_{n-3N-N_{A,i}}^{K_i,i}) - U(C_{n-N-N_{A,l}}^{K_l,l}) \tag{27}$$

with the convention that the maximum of an empty set is $-\infty$ and the summation over an empty set is 0.

Proof We first consider $W_n^{1,l}$. We have from (5) that

$$\begin{aligned} W_n^{1,l} &\geq X_{n-1}^{1,l} - X_{n-N_{1,l}}^A \\ &= X_{n-1}^{1,l} - X_{n-N}^{1,l} + X_{n-N}^{1,l} - X_{n-N_{1,l}}^A \\ &\geq X_{n-1}^{1,l} - X_{n-N}^{1,l} - (X_{n-N_{1,l}}^A - X_{n-N-N_{1,l}}^A) \end{aligned}$$

where the last inequality follows from (1) since $X_{n-N}^{1,l} \geq X_{n-N-N_{1,l}}^A$. Moreover, in the time interval from $X_{n-N}^{1,l}$ to $X_{n-1}^{1,l}$, customers $p_{n-N+1}^{1,l}, p_{n-N+2}^{1,l}, \ldots, p_{n-1}^{1,l}$ are served at station 1 in line l. Thus,

$$W_n^{1,l} \geq \sum_{r=n-N+1}^{n-1} B_r^{1,l} - U(C_{n-N-N_{1,l}}^A).$$

Next consider $k \in \{2, \ldots, K_l\}$. From (5),

$$\begin{aligned} W_n^{k,l} &\geq X_{n-1}^{k,l} - X_{n-N_{k,l}}^{k-1,l} \\ &= X_{n-1}^{k,l} - X_{n-N}^{k,l} + X_{n-N}^{k,l} - X_{n-N_{k,l}}^{k-1,l} \\ &\geq X_{n-1}^{k,l} - X_{n-N}^{k,l} - (X_{n-N_{k,l}}^{k-1,l} - X_{n-N-N_{k,l}}^{k-1,l}) \end{aligned}$$

where the last inequality follows from (1) since $X_{n-N}^{k,l} \geq X_{n-N-N_{k,l}}^{k-1,l}$. Since in the time interval from $X_{n-N}^{k,l}$ to $X_{n-1}^{k,l}$, customers $p_{n-N+1}^{k,l}, p_{n-N+2}^{k,l}, \ldots, p_{n-1}^{k,l}$ are served at station k in line l, we have

$$W_n^{k,l} \geq \sum_{r=n-N+1}^{n-1} B_r^{k,l} - U(C_{n-N-N_{k,l}}^{k-1,l}).$$

We next consider $W_n^{A,l}$. It follows from (6) that

$$\begin{aligned} W_n^{A,l} &\geq X_{n-1}^A - X_{n-N_{A,l}}^{K_l,l} \\ &= X_{n-1}^A - X_{n-N}^A + X_{n-N}^A - X_{n-N_{A,l}}^{K_l,l} \\ &\geq X_{n-1}^A - X_{n-N}^A - (X_{n-N_{A,l}}^{K_l,l} - X_{n-N-N_{A,l}}^{K_l,l}) \end{aligned}$$

where the last inequality follows from (2), since we know that $X^A_{n-N} \geq X^{K_l,l}_{n-N-N_{A,l}}$. Moreover, customers $p^A_{n-N+1}, p^A_{n-N+2}, \ldots, p^A_{n-1}$ are served at the assembly station in the time interval from X^A_{n-N} to X^A_{n-1}. Then

$$W_n^{A,l} \geq \sum_{r=n-N+1}^{n-1} B_r^A - U(C^{K_l,l}_{n-N-N_{A,l}}). \tag{28}$$

It again follows from (6) that for all $i \in \{1, \ldots, L\} \setminus \{l\}$

$$\begin{aligned}
W_n^{A,l} &\geq X^{K_i,i}_{n-N_{A,i}} - X^{K_l,l}_{n-N_{A,l}} \\
&= X^{K_i,i}_{n-N_{A,i}} - X^{K_i,i}_{n-N-N_{A,i}} + X^{K_i,i}_{n-N-N_{A,i}} - X^{K_l,l}_{n-N_{A,l}} \\
&\geq X^{K_i,i}_{n-N_{A,i}} - X^{K_i,i}_{n-N-N_{A,i}} - (X^{K_l,l}_{n-N_{A,l}} - X^{K_l,l}_{n-2N-N_{A,l}}) \\
&= X^{K_i,i}_{n-N_{A,i}} - X^{K_i,i}_{n-N-N_{A,i}} - (X^{K_l,l}_{n-N_{A,l}} - X^{K_l,l}_{n-N-N_{A,l}} + X^{K_l,l}_{n-N-N_{A,l}} - X^{K_l,l}_{n-2N-N_{A,l}})
\end{aligned}$$

where the last inequality follows since we know from (1) and (2) that $X^{K_i,i}_{n-N-N_{A,i}} \geq X^{K_l,l}_{n-2N-N_{A,l}}$ for all $i \in \{1, \ldots, L\} \setminus \{l\}$. But from (17), for all $i \in \{1, \ldots, L\} \setminus \{l\}$

$$\begin{aligned}
X^{K_i,i}_{n-N_{A,i}} - X^{K_i,i}_{n-N-N_{A,i}} &\geq \max_{k=1,\ldots,K_i} \sum_{r=n-N+1-N_{A,i}-\sum_{q=k+1}^{K_i} N_{q,i}}^{n-N_{A,i}-\sum_{q=k+1}^{K_i} N_{q,i}} B_r^{k,i} - U(C^{K_i,i}_{n-2N-N_{A,i}}) - U(C^{K_i,i}_{n-3N-N_{A,i}}) \\
&= \max_{k=1,\ldots,K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i} - U(C^{K_i,i}_{n-2N-N_{A,i}}) - U(C^{K_i,i}_{n-3N-N_{A,i}}).
\end{aligned}$$

Thus, for all $i \in \{1, \ldots, L\} \setminus \{l\}$

$$\begin{aligned}
W_n^{A,l} \geq &\max_{k=1,\ldots,K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i} - U(C^{K_i,i}_{n-2N-N_{A,i}}) - U(C^{K_i,i}_{n-3N-N_{A,i}}) \\
&-U(C^{K_l,l}_{n-N-N_{A,l}}) - U(C^{K_l,l}_{n-2N-N_{A,l}}).
\end{aligned}$$

Taking the maximum over all $i \in \{1, \ldots, L\} \setminus \{l\}$ yields

$$\begin{aligned}
W_n^{A,l} \geq &\max_{\substack{i=1,\ldots,L \\ i\neq l}} \max_{k=1,\ldots,K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i} - \sum_{i=1}^{L} U(C^{K_i,i}_{n-2N-N_{A,i}}) - \sum_{\substack{i=1 \\ i\neq l}}^{L} U(C^{K_i,i}_{n-3N-N_{A,i}}) \\
&-U(C^{K_l,l}_{n-N-N_{A,l}}).
\end{aligned} \tag{29}$$

and the maximum of (28) and (29) yields (27).

4 Tail Asymptotics of Cycle Times and Waiting Times

Our objective in this section is to obtain the tail asymptotics of the n^{th} cycle time and waiting time at each station. Results follow immediately from the bounds in Section 3 and the properties of the subexponential distributions given in the Appendix. The following proposition provides the tail asymptotics for the n^{th} cycle time.

Proposition 4.1 *For* $l \in \{1,\ldots,L\}$, $k \in \{1,\ldots,K_l\}$ *and* $n \geq \max\{N - N_{[k+1],l}, \max_{\substack{i=1,\ldots,L \\ i\neq l}}\{\sum_{q=1}^{k} N_{q,l} + N - N_{1,i}\}\}$

$$\mathbb{P}(C_n^{k,l} > x) \sim N(\sum_{k=1}^{K_l}\sum_{l=1}^{L} c_{k,l} + c_A)\overline{F}(x) \tag{30}$$

and for $n \geq \max_{l=1,\ldots,L}\{N - N_{1,l}\}$

$$\mathbb{P}(C_n^{A} > x) \sim N(\sum_{k=1}^{K_l}\sum_{l=1}^{L} c_{k,l} + c_A)\overline{F}(x). \tag{31}$$

Proof First consider the tail asymptotics of $C_n^{k,l}$ for $l \in \{1,\ldots,L\}$, $k \in \{1,\ldots,K_l\}$. From (11) and Proposition 6.2, for $n \geq \max\{N - N_{[k+1],l}, \max_{\substack{i=1,\ldots,L \\ i\neq l}}\{\sum_{q=1}^{k} N_{q,l} + N - N_{1,i}\}\}$,

$$\limsup_{x\to\infty}\frac{\mathbb{P}(C_n^{k,l} > x)}{\overline{F}(x)} \leq \limsup_{x\to\infty}\frac{\mathbb{P}(U(C_n^{k,l}) > x)}{\overline{F}(x)} = \limsup_{x\to\infty}\frac{\mathbb{P}(U(C_{2N}^{k,l}) > x)}{\overline{F}(x)} = N(\sum_{k=1}^{K_l}\sum_{l=1}^{L} c_{k,l} + c_A). \tag{32}$$

Let

$$L(C_n^{k,l}) = \max\{\max_{j=1,\ldots,k}\sum_{r=n+1-\sum_{q=j+1}^{k} N_{q,l}}^{n+N-\sum_{q=j+1}^{k} N_{q,l}} B_r^{j,l}, \sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}}^{n+N-\sum_{q=1}^{k} N_{q,l}} B_r^{A},$$
$$\max_{j=k+1,\ldots,K_l}\sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_l} N_{[q+1],l}} B_r^{j,l}, \max_{\substack{i=1,\ldots,L \\ i\neq l}}\max_{j=1,\ldots,K_i}\sum_{r=n+1-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}}^{n+N-\sum_{q=1}^{k} N_{q,l}-\sum_{q=j}^{K_i} N_{[q+1],i}} B_r^{j,i}\}.$$

Then (17) is equivalent to

$$C_n^{k,l} \geq L(C_n^{k,l}) - U(C_{n-N}^{k,l}) - U(C_{n-2N}^{k,l}).$$

Note that $L(C_n^{k,l})$ is independent of $U(C_{n-N}^{k,l})$ and $U(C_{n-2N}^{k,l})$ since the service time terms that appear in the expression of $L(C_n^{k,l})$ do not appear in the expressions of $U(C_{n-N}^{k,l})$ and $U(C_{n-2N}^{k,l})$. Then from Propositions 6.1, 6.2 and 6.3, we have for $n \geq \max\{N - N_{[k+1],l}, \max_{\substack{i=1,\ldots,L \\ i\neq l}}\{\sum_{q=1}^{k} N_{q,l} + N - N_{1,i}\}\}$,

$$\begin{aligned}\liminf_{x\to\infty}\frac{\mathbb{P}(C_n^{k,l} > x)}{\overline{F}(x)} &\geq \liminf_{x\to\infty}\frac{\mathbb{P}(L(C_n^{k,l}) - U(C_{n-N}^{k,l}) - U(C_{n-2N}^{k,l}) > x)}{\overline{F}(x)} \\ &= \liminf_{x\to\infty}\frac{\mathbb{P}(L(C_{3N}^{k,l}) - U(C_{2N}^{k,l}) - U(C_{N}^{k,l}) > x)}{\overline{F}(x)} = N(\sum_{l=1}^{L}\sum_{k=1}^{K_l} c_{k,l} + c_A)\end{aligned} \tag{33}$$

and (32) together with (33) gives the tail asymptotics of $C_n^{k,l}$. Next we consider C_n^A. It follows from (12) and Proposition 6.2 that for $n \geq \max_{l=1,\dots,L}\{N - N_{1,l}\}$

$$\limsup_{x\to\infty} \frac{\mathbb{P}(C_n^A > x)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\mathbb{P}(U(C_n^A) > x)}{\overline{F}(x)} = \limsup_{x\to\infty} \frac{\mathbb{P}(U(C_N^A) > x)}{\overline{F}(x)} = N(\sum_{k=1}^{K_l}\sum_{l=1}^{L} c_{k,l} + c_A). \quad (34)$$

Let

$$L(C_n^A) = \max\{\sum_{r=n+1}^{n+N} B_r^A, \max_{l=1,\dots,L}\max_{k=1,\dots,K_l} \sum_{r=n+1-\sum_{q=k}^{K_l} N_{[q+1],l}}^{n+N-\sum_{q=k}^{K_l} N_{[q+1],l}} B_r^{k,l}\}.$$

Then (18) is equivalent to

$$C_n^A \geq L(C_n^A) - U(C_{n-N}^A).$$

Then since $L(C_n^A)$ is independent of $U(C_{n-N}^A)$, it follows from Propositions 6.1, 6.2 and 6.3 that for $n \geq \max_{l=1,\dots,L}\{N - N_{1,l}\}$

$$\begin{aligned}\liminf_{x\to\infty} \frac{\mathbb{P}(C_n^{k,l} > x)}{\overline{F}(x)} &\geq \liminf_{x\to\infty} \frac{\mathbb{P}(L(C_n^A) - U(C_{n-N}^A) > x)}{\overline{F}(x)} \\ &= \liminf_{x\to\infty} \frac{\mathbb{P}(L(C_{2N}^A) - U(C_N^A) > x)}{\overline{F}(x)} = N(\sum_{l=1}^{L}\sum_{k=1}^{K_l} c_{k,l} + c_A)\end{aligned}$$

which together with (34) completes the proof. □

Note that since the convergence in (32) and (33) is uniform in n, one can conclude the uniformity of convergence in n in (30). Similarly, the convergence in (31) is uniform in n.

The tail asymptotics of cycle times is the same for all stations (including the assembly station). Hence, the cycle time asymptotics does not depend on at which station the cycle starts. Moreover, as expected when $L = 1$, tail asymptotics of cycle times reduces to the one obtained by Ayhan, Palmowski and Schlegel [5] for the cyclic tandem queue. The next proposition will demonstrate that tail asymptotics of waiting times at the assembly station is different from that of station $k \in \{1, \dots, K_l\}$ in line $l \in \{1, \dots, L\}$.

Proposition 4.2 *For $l \in \{1, \dots, L\}$, $k \in \{1, \dots, K_l\}$ with $B_{k,l} \in \mathcal{S}$ and $n \geq N$*

$$\mathbb{P}(W_n^{k,l} > x) \sim (N-1)c_{k,l}\overline{F}(x) \quad (35)$$

and if $B_A \in \mathcal{S}$ or $B_{k,i} \in \mathcal{S}$ for some $i \in \{1, \dots, L\}\backslash\{l\}$ and $k \in \{1, \dots, K_i\}$ for $n \geq N + \max_{\substack{i=1,\dots,L\\ i\neq l}}\{N - N_{1,i}\}$,

$$\mathbb{P}(W_n^{A,l} > x) \sim \Big((N-1)c_A + N(\sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{k=1}^{K_i} c_{k,i})\Big)\overline{F}(x). \quad (36)$$

Proof We start with $W_n^{k,l}$ for $l \in \{1, \dots, L\}$, $k \in \{1, \dots, K_l\}$. From (7) and Proposition 6.2 for $n \geq N$

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{W_n^{k,l} > x\}}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=n-N+1}^{n-1} B_r^{k,l} > x\}}{\overline{F}(x)}$$
$$= \limsup_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=1}^{N-1} B_r^{k,l} > x\}}{\overline{F}(x)} = (N-1)c_{k,l}.$$

Note that $\sum_{r=n-N+1}^{n-1} B_r^{1,l}$ is independent of $U(C^A_{n-N-N_{1,l}})$ since $U(C^A_{n-N-N_{1,l}})$ is written in terms of $B^{1,l}_{n-2N+1}, \dots, B^{1,l}_{n-N}$. Then from (25) and Propositions 6.1 and 6.2, we obtain for $n \geq N$,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{W_n^{1,l} > x\}}{\overline{F}(x)} \geq \liminf_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=n-N+1}^{n-1} B_r^{1,l} - U(C^A_{n-N-N_{1,l}}) > x\}}{\overline{F}(x)}$$
$$= \liminf_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=N+1}^{2N-1} B_r^{1,l} - U(C^A_{N-N_{1,l}}) > x\}}{\overline{F}(x)} = (N-1)c_{1,l}.$$

Similarly, $\sum_{r=n-N+1}^{n-1} B_r^{k,l}$ is independent of $U(C^{k-1,l}_{n-N-N_{k,l}})$ for $k \in \{2, \dots, K_l\}$ and $l \in \{1, \dots, K_l\}$ since $U(C^{k-1,l}_{n-N-N_{k,l}})$ has only the $B^{k,l}_{n-2N+1}, \dots, B^{k,l}_{n-N}$ terms. Thus, from (25) and Propositions 6.1 and 6.2, for $n \geq N$

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{W_n^{k,l} > x\}}{\overline{F}(x)} \geq \liminf_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=n-N+1}^{n-1} B_r^{k,l} - U(C^{k-1,l}_{n-N-N_{k,l}}) > x\}}{\overline{F}(x)}$$
$$= \liminf_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=N+1}^{2N-1} B_r^{k,l} - U(C^{k-1,l}_{N-N_{k,l}}) > x\}}{\overline{F}(x)} = (N-1)c_{k,l}.$$

This completes the proof for the asymptotics of $W_n^{k,l}$. Next consider $W_n^{A,l}$. From Proposition 6.2 and (8), for $n \geq N + \max_{\substack{i=1,\dots,L \\ i\neq l}} \{N - N_{1,i}\}$

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{W_n^{A,l} > x\}}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=n-N+1}^{n-1} B_r^A + \sum_{\substack{i=1 \\ i\neq l}}^{L} \sum_{k=1}^{K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i} > x\}}{\overline{F}(x)}$$

$$= \limsup_{x\to\infty} \frac{\mathbb{P}\{\sum_{r=1}^{N-1} B_r^A + \sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{k=1}^{K_i}\sum_{r=1}^{N} B_r^{k,i} > x\}}{\overline{F}(x)}$$

$$= (N-1)c_A + N(\sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{k=1}^{K_i} c_{k,i}).$$

in order to get a lower bound on the tail asymptotics of $W_n^{A,l}$, we start with (27). Note that $\sum_{r=n-N+1}^{n-1} B_r^A$ and $\sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i}$ for $i \in \{1,\ldots,L\}\setminus\{l\}$, $k \in \{1,\ldots,K_i\}$ are independent of $U(C_{n-2N-N_{A,i}}^{K_i,i})$, $i \in \{1,\ldots,L\}$, $U(C_{n-3N-N_{A,i}}^{K_i,i})$, $i \in \{1,\ldots,L\}\setminus\{l\}$ and $U(C_{n-N-N_{A,l}}^{K_l,l})$. In order to see this, note that the largest index for the $B_\cdot^{k,i}$ terms that appear in $U(C_{n-2N-N_{A,i}}^{K_i,i})$ is $n-N-\sum_{q=k}^{K_i} N_{[q+1],i}$ and the largest index for the $B_\cdot^A$ terms that appear in $U(C_{n-2N-N_{A,i}}^{K_i,i})$ is $n-2N$. Moreover, one can see that $U(C_{n-N-N_{A,l}}^{K_l,l})$ is written in terms of $B_{n+1-2N-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i},\ldots,$ $B_{n-N-\sum_{q=k}^{K_i} N_{[q+1],i}}^{k,i}$ for $i \in \{1,\ldots,L\}\setminus\{l\}$, $k \in \{1,\ldots,K_i\}$ and $B_{n+1-2N}^A,\ldots,B_{n-N}^A$. For notational convenience, set

$$L(W_n^{A,l}) = \max\{\sum_{r=n-N+1}^{n-1} B_r^A, \max_{\substack{i=1,\ldots,L\\ i\neq l}} \max_{k=1,\ldots,K_i} \sum_{r=n-N+1-\sum_{q=k}^{K_i} N_{[q+1],i}}^{n-\sum_{q=k}^{K_i} N_{[q+1],i}} B_r^{k,i}\}.$$

Then from (27) and Propositions 6.1, 6.2 and 6.3, we have for all $n \geq N + \max_{\substack{i=1,\ldots,L\\ i\neq l}}\{N - N_{1,i}\}$

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{W_n^{A,l} > x\}}{\overline{F}(x)}$$

$$\geq \liminf_{x\to\infty} \frac{\mathbb{P}\{L(W_n^{A,l}) - \sum_{i=1}^{L} U(C_{n-2N-N_{A,i}}^{K_i,i}) - \sum_{\substack{i=1\\ i\neq l}}^{L} U(C_{n-3N-N_{A,i}}^{K_i,i}) - U(C_{n-N-N_{A,l}}^{K_l,l}) > x\}}{\overline{F}(x)}$$

$$= \liminf_{x\to\infty} \frac{\mathbb{P}\{L(W_{4N}^{A,l}) - \sum_{i=1}^{L} U(C_{2N-N_{A,i}}^{K_i,i}) - \sum_{\substack{i=1\\ i\neq l}}^{L} U(C_{N-N_{A,i}}^{K_i,i}) - U(C_{3N-N_{A,l}}^{K_l,l}) > x\}}{\overline{F}(x)}$$

$$= (N-1)c_A + N(\sum_{\substack{i=1\\ i\neq l}}^{L}\sum_{k=1}^{K_i} c_{k,i})$$

which completes the proof. □

From the proof of Proposition 4.2, one can see that the convergence in (35) and (36) is uniform in n.

Tail asymptotics of the waiting time of the n^{th} customer at station $k \in \{1,\ldots,K_l\}$ in line $l \in \{1,\ldots,L\}$ is the same as the tail asymptotics of the waiting times in a closed tandem queue which is given in Ayhan, Palmowski and Schlegel [5]. However, the tail asymptotics of the waiting time of the n^{th} arriving customer from line $l \in \{1,\dots,L\}$ at the assembly station is different and it depends not only on the service time at the assembly station but also on the service times of all stations in lines $\{1,\dots,L\} \setminus \{l\}$. However, as expected, if $L=1$, tail asymptotics of the waiting times at the assembly station also reduces to the one given in [5].

5 Stationary Results

In this section, our objective is to analyze the asymptotic tail behavior of stationary cycle times and waiting times. Using the analysis in Section 7.5 of Baccelli, Cohen, Olsder and Quadrat [7], we first derive a sufficient condition under which these stationary characteristics exist and then employ Propositions 4.1 and 4.2 to determine the tail asymptotics of stationary cycle times and waiting times. Let $C^{k,l}$ denote the stationary cycle time at station $k \in \{1,\dots,K_l\}$ in line $l \in \{1,\dots,L\}$ and C^A denote the stationary cycle time at the assembly station. Similarly, $W^{k,l}$ is the stationary waiting time at station $k \in \{1,\dots,K_l\}$ in line $l \in \{1,\dots,L\}$ and $W^{A,l}$ is the stationary waiting time of an arbitrary customer coming from line l at the assembly station.

Assume that there exists $l \in \{1,\dots,L\}$ and $k \in \{1,\dots,K_l\}$ such that $N_{k,l} > 0$ and $B_{k,l}(\cdot)$ has infinite support or if $N_{A,l} > 0$ for all $l \in \{1,\dots,L\}$ and $B_A(\cdot)$ has infinite support. Thus, we assume that among the stations which are ready to start serving a customer at time 0, at least one of them has service time distribution with infinite support. Clearly, this condition is satisfied if service time distributions at all stations have infinite support. Note that the subexponential distributions have infinite support. Under this condition it follows from Theorem 7.94 of Baccelli, Cohen, Olsder and Quadrat [7] that

$$X_n^{k,l} - X_{n-1}^{j,i} \quad \text{for all } l,i \in \{1,\dots,L\} \text{ and for all } k \in \{1,\dots,K_l\},\ j \in \{1,\dots,K_i\}, \tag{37}$$

$$X_n^{k,l} - X_{n-1}^{A} \quad \text{for all } l \in \{1,\dots,L\} \text{ and for all } k \in \{1,\dots,K_l\}, \tag{38}$$

$$X_n^{A} - X_{n-1}^{k,l} \quad \text{for all } l \in \{1,\dots,L\} \text{ and for all } k \in \{1,\dots,K_l\}, \tag{39}$$

$$X_n^{A} - X_{n-1}^{A} \tag{40}$$

admit a stationary regime. This stationary regime is unique, integrable, directly reachable, independent of the initial condition and (37) to (40) couple with it in finite time. In order to see this, note that the condition of Theorem 7.94 of [7] is satisfied if there exists $l \in \{1,\dots,L\}$ and $k \in \{1,\dots,K_l\}$ such that $N_{k,l} > 0$ and $B_{k,l}(\cdot)$ has infinite support or if $N_{A,l} > 0$ for all $l \in \{1,\dots,L\}$, $B_A(\cdot)$ has infinite support (see Remark 7.91 of [7] and note that the matrix E of Theorem 7.94 of [7] has non-negative entries). At this point one might notice that the state variables in Theorem 7.94 of [7] are not necessarily the departure times. In particular, let $Y_n^{k,l}$ be the time that the n^{th} customer starts his service at station $k \in \{1,\dots,K_l\}$ in line $l \in \{1,\dots,L\}$. Similarly, Y_n^A denotes the beginning of the n^{th} service at the assembly station. Hence, $X_n^{k,l} = Y_n^{k,l} + B_n^{k,l}$ and $X_n^A = Y_n^A + B_n^A$. Then the state variables consist of $Y_n^{k,l}$ and $X_n^{k,l}$ if $N_{k,l} > 0$ and $N_{[k+1],l} > 0$, $Y_n^{k,l}$ if $N_{k,l} > 0$ and $N_{[k+1],l} = 0$,

$X_n^{k,l}$ if $N_{k,l} = 0$, for $l \in \{1, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$, Y_n^A and X_n^A if $N_{A,l} > 0$ for all $l \in \{1, \ldots, L\}$ and if there exists $l \in \{1, \ldots, L\}$ such that $N_{1,l} > 0$, Y_n^A if $N_{A,l} > 0$ for all $l \in \{1, \ldots, L\}$ and $N_{1,l} = 0$ for all $l \in \{1, \ldots, L\}$ and X_n^A if there exists $l \in \{1, \ldots, L\}$ with $N_{A,l} = 0$. Hence, for example if $Y_n^{k,l}$ for some $l \in \{1, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$, $X_n^{i,j}$ for $(i, j) \neq (k, l)$ and X_n^A belong to the state vector then Theorem 7.94 of [7] would imply that the differences of the form $Y_n^{k,l} - X_{n-1}^{i,j}$, $Y_n^{k,l} - Y_{n-1}^{k,l}$, $Y_n^{k,l} - X_{n-1}^A$, ... admit a stationary regime with which they couple in finite time. Using the definition of coupling on page 87 of Baccelli and Brémaud [6] and the relationships between $Y_n^{k,l}$ and $X_n^{k,l}$ and Y_n^A and X_n^A, we can conclude that the quantities in (37) to (40) also admit a unique stationary regime and they couple with this stationary regime in finite time.

We now know that if there exists $l \in \{1, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$ such that $N_{k,l} > 0$ and $B_{k,l}(\cdot)$ has infinite support or if $N_{A,l} > 0$ for all $l \in \{1, \ldots, L\}$ and $B_A(\cdot)$ has infinite support then the process $\{X_n^{k,l} - X_{n-1}^{k,l}\}_{n \geq 1}$ couples with a stationary process $\{Z_n\}_{n \geq 1}$ in finite time. Thus, there exists a finite random variable T such that $X_n^{k,l} - X_{n-1}^{k,l} = Z_n$ for all $n \geq T$ (see the definition of coupling on page 87 of Baccelli and Brémaud [6]). Then for all $x \geq 0$

$$
\begin{aligned}
&\Big|\mathbb{P}\{\sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \leq x\} - \mathbb{P}\{\sum_{i=n+1}^{n+N} Z_i \leq x\}\Big| \\
= &\Big|\mathbb{P}\{\sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \leq x, \sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) = \sum_{i=n+1}^{n+N} Z_i\} \\
&+\mathbb{P}\{\sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \leq x, \sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \neq \sum_{i=n+1}^{n+N} Z_i\} \\
&-\mathbb{P}\{\sum_{i=n+1}^{n+N} Z_i \leq x, \sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) = \sum_{i=n+1}^{n+N} Z_i\} \\
&-\mathbb{P}\{\sum_{i=n+1}^{n+N} Z_i \leq x, \sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \neq \sum_{i=n+1}^{n+N} Z_i\}\Big| \\
= &\Big|\mathbb{P}\{\sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \leq x, \sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \neq \sum_{i=n+1}^{n+N} Z_i\} \\
&-\mathbb{P}\{\sum_{i=n+1}^{n+N} Z_i \leq x, \sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \neq \sum_{i=n+1}^{n+N} Z_i\}\Big| \\
\leq &\ 2\mathbb{P}\{\sum_{i=n+1}^{n+N} (X_i^{k,l} - X_{i-1}^{k,l}) \neq \sum_{i=n+1}^{n+N} Z_i\} \leq 2P(T > n).
\end{aligned}
\tag{41}
$$

Since T is a finite random variable, it follows from (41) that

$$\lim_{n \to \infty} \Big|\mathbb{P}\{C_n^{k,l} \leq x\} - \mathbb{P}\{C^{k,l} \leq x\}\Big| = 0 \tag{42}$$

for $l \in \{1, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$. Similarly, one can show that

$$\lim_{n \to \infty} \Big|\mathbb{P}\{C_n^A \leq x\} - \mathbb{P}\{C^A \leq x\}\Big| = 0. \tag{43}$$

Since the convergence in (30) and (31) is uniform in n, combining (42) and (43) with Proposition 4.1, we have the following result.

Proposition 5.1 *If there exists $l \in \{1, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$ such that $N_{k,l} > 0$ and $B_{k,l}(\cdot)$ has infinite support or if $N_{A,l} > 0$ for all $l \in \{1, \ldots, L\}$ and $B_A(\cdot)$ has infinite support, then for $l \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K_l\}$*

$$\mathbb{P}(C^{k,l} > x) \sim N(\sum_{k=1}^{K_l} \sum_{l=1}^{L} c_{k,l} + c_A)\overline{F}(x)$$

and

$$\mathbb{P}(C^{A} > x) \sim N(\sum_{k=1}^{K_l} \sum_{l=1}^{L} c_{k,l} + c_A)\overline{F}(x).$$

Since $W_n^{k,l}$ for $k \in \{1, \ldots, K_l\}$, $l \in \{1, \ldots, L\}$ and $W_n^{A,l}$ can also be expressed in terms of the differences in (37) to (40) and since the convergence in (35) and (36) is uniform in n, we also have the following proposition.

Proposition 5.2 *If there exists $l \in \{1, \ldots, L\}$ and $k \in \{1, \ldots, K_l\}$ such that $N_{k,l} > 0$ and $B_{k,l}(\cdot)$ has infinite support or if $N_{A,l} > 0$ for all $l \in \{1, \ldots, L\}$ and $B_A(\cdot)$ has infinite support, then for $l \in \{1, \ldots, L\}$, $k \in \{1, \ldots, K_l\}$ with $B_{k,l} \in \mathcal{S}$*

$$\mathbb{P}(W^{k,l} > x) \sim (N-1)c_{k,l}\overline{F}(x)$$

and if $B_A \in \mathcal{S}$ or $B_{k,i} \in \mathcal{S}$ for some $i \in \{1, \ldots, L\} \setminus \{l\}$ and $k \in \{1, \ldots, K_i\}$

$$\mathbb{P}(W^{A,l} > x) \sim \Big((N-1)c_A + N(\sum_{\substack{i=1 \\ i \neq l}}^{L} \sum_{k=1}^{K_i} c_{k,i})\Big)\overline{F}(x).$$

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6 Appendix

In this section, we go over some properties of subexponential distributions which are used in our analysis. The interested reader is referred to Sigman [22] for a more detailed review of these distributions.

A distribution function F on $\mathbb{R}_+ = [0, \infty)$ with $F(x) < 1$ for all $x > 0$ is called subexponential ($F \in \mathcal{S}$) if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x)\,,$$

where F^{*2} denotes the convolution $F * F$.

The following proposition is the same as Proposition 1.1 of Sigman [22].

Proposition 6.1 *Let X and $Y \geq 0$ be independent random variables with distribution functions $F_X \in \mathcal{S}$ and F_Y, respectively. Then*

$$\mathbb{P}(X - Y > x) \sim \mathbb{P}(X > x) \quad \text{as } x \to \infty.$$

The next proposition follows from Propositions 2.7 and 2.8 of Sigman [22] by induction.

Proposition 6.2 *Let $F \in \mathcal{S}$ and let $F_1, \ldots, F_n$, $n \geq 1$, and $G_1, \ldots, G_m$, $m \geq 1$, be distributions on $\mathbb{R}_+$ such that $\overline{F}_i(x) \sim c_i \overline{F}(x)$ with $c_i > 0$, $1 \leq i \leq n$, and $\overline{G}_i(x) = o(\overline{F}(x))$ for $1 \leq i \leq m$. Then,*

$$\overline{F_1 * \ldots * F_n * G_1 * \ldots * G_m}(x) \sim \sum_{i=1}^{n} c_i \, \overline{F}(x) \, .$$

Finally, we present the following standard result.

Proposition 6.3 *Let F and $G_1, \ldots, G_n$, $n \geq 1$, be distributions on $\mathbb{R}_+$ such that $\overline{G}_i(x) \sim c_i \overline{F}(x)$ as $x \to \infty$; $c_i \geq 0$. Then,*

$$1 - \prod_{i=1}^{n} G_i(x) \sim \sum_{i=1}^{n} c_i \, \overline{F}(x) \, .$$

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