Expansions for Joint Laplace Transform of Stationary Waiting Times in (max,+)-Linear Systems with Poisson Input*

Hayriye Ayhan
School of Industrial and Systems Engineering
Georgia Institute of Technology, Atlanta, GA 30332-0205

François Baccelli
École Normale Supérieure
Département de Mathématiques et d'Informatique
LIENS, 45 Rue d'Ulm 75230 Paris Cedex 05

July 1999

Abstract

We give a Taylor series expansion for the joint Laplace transform of stationary waiting times in open (max,+)-linear stochastic systems with Poisson input. Probabilistic expressions are derived for coefficients of all orders. Combining this new result with the earlier expansion formula for the mean stationary waiting times, we also provide a Taylor series expansion for the covariance of stationary waiting times in such systems.

It is well known that (max,+) linear systems can be used to represent stochastic Petri nets belonging to the class of event graphs. This class contains various instances of queueing networks like acyclic or cyclic fork-and-join queueing networks, finite or infinite capacity tandem queueing networks with various types of blocking, synchronized queueing networks and so on. It also contains some basic manufacturing models such as kanban networks, assembly systems and so forth. The applicability of this expansion technique is discussed for several systems of this type.

Keywords: Queueing Networks, Stochastic Event Graphs, Differentiability of Functionals of Poisson Processes.

AMS subject classifications: 60K25, 60G55, 90B22.

*This research has been supported by the National Science Foundation Grant DMI-9713974 and by the TMR Grant RB-FMRX-CT-96-0074 (ALAPEDES PROJECT) of the European Commission

1 Introduction

Under the notion of an open (max,+)-linear stochastic system, one understands a sequence $\{X_n\}$ of random vectors satisfying the recursion $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$ where the addition $\oplus$ means maximization and the multiplication $\otimes$ means addition. Here $\{T_n\}$ is an increasing sequence of real-valued random variables, and $\{A_n\}$ and $\{B_n\}$ are stationary sequences of random matrices. Such systems allow one to represent the dynamics of stochastic Petri nets belonging to the class of event graphs (see [2]). In particular this class contains various instances of queueing networks like acyclic and cyclic fork-join queueing networks, finite or infinite capacity tandem queueing networks with various kinds of blocking (manufacturing and communication), synchronized queueing networks etc. It also contains some basic manufacturing systems like Kanban networks, assembly systems and so forth. In all these models, T_n is the arrival epoch of the nth customer to the network and the coordinates X_n^i of the state vector $X_n = (X_n^1, X_n^2, \cdots, X_n^\alpha)$ represent absolute times (like beginning of the nth service in the ith queue) which grow to ∞ when n increases unboundedly. For this reason one is actually more interested in the differences $W_n^i = X_n^i - T_n$ (like the waiting time of the nth customer until the beginning of his service in queue i), which are expected to admit a certain stationary state $W^i = \lim_{n\to\infty} W_n^i$ (in distribution) under certain rate conditions. In most cases, and particularly for systems α greater than 2, it is impossible to determine the characteristics of the random vector $W = (W^1, \cdots, W^\alpha)$ in closed form. This motivated the research to derive an expansion formula for the characteristics of W.

Assuming that $\{T_n\}$ is a stationary Poisson process with intensity λ and that the sequences $\{A_n\}$ and $\{B_n\}$ have certain independence properties, our objective in this paper is to derive a series expansion for $E[e^{-s_1 W^i - s_2 W^j}]$, where W^i, W^j $(i \neq j)$ are the ith and the jth components of the stationary waiting time vector and s_1 and s_2 are nonnegative real numbers. Namely, we want to derive an expansion formula with respect to the arrival rate λ for the joint Laplace transform of stationary waiting times in Poisson driven (max,+)-linear systems. In general it is impossible to get a closed form expression for such Laplace transforms for systems of dimension α larger than 2. Even in the case when all system data are exponential, analytical formulae only exist for specific models, like for instance two station tandem queues (see [14]). Using the expansion formula for $E[e^{-s_1 W^i - s_2 W^j}]$, it is straightforward to get an expansion formula for $E[W^i W^j]$ with respect to λ.

Under similar assumptions, Baccelli and Schmidt [8] derived a series expansion for EW^i for $i = 1, \cdots, \alpha$. Combining this with the expansion formula of $E[W^i W^j]$, we can easily derive a series expansion for $\mathrm{Cov}(W^i W^j)$ which is an important performance measure in queueing networks. This expansion approach was generalized to other characteristics (such as higher order moments, Laplace transform, tail probability and transient behavior) of stationary waiting times by Baccelli, Hasenfuss and Schmidt [4], [5]. Combining our series expansion for $Cov(W^i W^j)$ with the results of [5], we can also get a series expansion for $Cov(W^i (W^j - W^i))$.

The technique used in order to obtain the expansion formulae in [4],[5],[8] is a general method which consists of expanding the expectation of a functional of a marked point process using its factorial moment measures. The roots of this method can be traced back to the following papers: [3],[9],[10],[11],[15],[16] and [19]. In order to get an expansion formula for the joint Laplace transform of stationary waiting times, we will use a more direct approach based on a general theory on the differentiability of functionals of Poisson processes (see [6] and [13]).

The paper is organized as follows. In Section 2, some preliminaries are given. Section 3 contains the expansion formulae for the joint Laplace transform and the covariances, together with the technical conditions under which these formulae hold. Section 3 also describes an estimate of the analyticity region, namely the region where the Taylor expansion can practically be used. In particular, Section 3.1 presents the assumptions required for the existence of expansion formulae. Section 3.2 and Section 3.3 provide the Taylor series expansions and the explicit form of the coefficients of these expansions for the joint Laplace transform and covariances, respectively. Section 3.4 characterizes the region where the Taylor series of the joint Laplace transform is absolutely convergent in λ. In section 4 we present several examples to illustrate the main results of Section 3. Section 5 is devoted to the proof of the expansion formulae. In particular, Section 5.1 shows that under some moment conditions, functions of interest are differentiable in λ while Section 5.2 deals with the explicit computation of the coefficients of the expansion formulae. Clearly, if one can guarantee the existence of the more restrictive conditions in Section 3.4 (existence of exponential moments), differentiability of all orders is granted (as a result of the analyticity in λ) and results in Section 5.1 can then be omitted. Finally, some particularly technical steps required in Section 5.2 are presented in the appendix.

2 Preliminaries

The basic reference algebra throughout this paper is the so called (max,+) algebra on the real line $\mathbb{R}$, namely the semi-field with the two operations $(\oplus, \otimes)$, where $\oplus$ is the maximization in the conventional algebra and $\otimes$ is the addition in the conventional algebra (see [2] for more details on this formalism). As mentioned earlier the dynamics of (max,+) linear systems can be captured by the α-dimensional vectorial recurrence equations

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1} \tag{2.1}$$

with initial condition X_0 where

- $\{T_n\}$ is a non-decreasing sequence of real-valued random numbers (the epochs of the Poisson arrival process),
- $\{A_n\}$ is a stationary and ergodic sequence of $\alpha \times \alpha$ matrices with real-valued random entries,
- $\{B_n\}$ is a stationary and ergodic sequence of $\alpha \times 1$ matrices with real-valued random entries,
- $\{X_n\}$ is a sequence of $\alpha-$dimensional state vectors.

Various examples of discrete event systems with state variables satisfying an equation of type (2.1) are given in Section 4. As mentioned above, the components of the state vector represent absolute times which grow to ∞ when n increases unboundedly, and one is more interested in the differences

$$W_n^i = X_n^i - T_n$$

(like the waiting time of the nth customer until he joins server i). Let $\tau_n = T_{n+1} - T_n$, $n \geq 0$. By subtracting T_{n+1} from both sides of (2.1), the new state vector W_{n+1} can be written as

$$W_{n+1} = A_n \otimes C(\tau_n) \otimes W_n \oplus B_{n+1},$$

where $C(x)$ is the $\alpha \times \alpha$ matrix with all diagonal entries equal to $-x$ and all non-diagonal entries equal to $-\infty$.

Assume that the underlying probability space $(\Omega, \mathcal{F}, P)$ is equipped with a measurable (pointwise) shift $\theta : \Omega \to \Omega$, which leaves the probability measure P–invariant and is P–ergodic. The random variables $Z_n = (A_n, B_n)$ and the Poisson point process are assumed to be defined on this probability space and to be consistent with θ, i.e. $A_n = A \circ \theta^n$ and $B_n = B \circ \theta^n$ and $\tau_n = \tau \circ \theta^n$, for all integers n and some random variables $A \in \mathbb{R}^{\alpha \times \alpha}$, $B \in \mathbb{R}^{\alpha \times 1}$ and $\tau \in \mathbb{R}$. Under these conditions, plus integrability assumptions, it is shown in [2] that for all $\lambda < a^{-1}$ where a is the maximal (max,+) Lyapunov exponent of the sequence $\{A_n\}$, $\{W_n\}$ couples with a unique stationary sequence $\{W \circ \theta^n\}$, where W is the unique finite random variable solution of the functional equation

$$W \circ \theta = A \otimes C \otimes W \oplus Bo\theta$$

which is determined by the matrix-series

$$W \quad = \quad D_0 \oplus \bigoplus_{k \geq 1} C(T_{-k}) \otimes D_k$$

with $D_0 = B_0$ and

$$D_k \quad = \quad \Big(\bigotimes_{n=1}^{k} A_{-n} \Big) \otimes B_{-k} \tag{2.2}$$

for all $k \geq 1$.

For $A \in \mathbb{R}^{\alpha \times \alpha}$ and $B \in \mathbb{R}^{\alpha}$, let $\| A \|_\infty = \bigoplus_{i,j=1}^{\alpha} A_{i,j}$ and $\| B \|_\infty = \bigoplus_{i=1}^{\alpha} B^i$. It is straightforward to show that

$$D_n^i \leq \| D_n \|_\infty \leq \| \bigotimes_{k=1}^{n} A_{-k} \|_\infty + \| B_{-n} \|_\infty,$$

for any $i \in \{1, \cdots, \alpha\}$. It follows from Theorem 7.27 of [2] that

$$\lim_{n\to\infty} \frac{\| \bigotimes_{k=1}^{n} A_{-k} \|_\infty}{n} \quad = \quad a,$$

where a is again the maximal Lyapunov exponent of the sequence $\{A_n\}$. Therefore, with probability 1

$$\lim_{n\to\infty} \frac{D_n^i}{n} \leq a, \tag{2.3}$$

for all $i \in \{1, \cdots, \alpha\}$.

Note that whenever we write an equality or an inequality which involves random variables we mean that the equality or inequality holds P–almost surely (i.e. with probability 1.) However, in order to keep the statements as simple as possible we omit phrases such as "with probability 1" or "almost surely" .

3 Main Results

3.1 Assumptions

Before presenting our main result which states that $E[e^{-s_1 W^i - s_2 W^j}]$ $(i \neq j)$ can be expanded into a finite power series with respect to the arrival intensity λ, we will list the assumptions under which this result holds.

Support and Monotonicity Assumptions. We assume that each entry of the random matrix A_n is either a.s. non-negative or equal to $-\infty$, i.e.

$$(A_n)_{i,j} \geq 0 \text{ or } (A_n)_{i,j} = -\infty \text{ a.s.}$$

and that all entries on the diagonal of A_n are non-negative, i.e. $(A_n)_{i,i} \geq 0$. We also assume that there is an integer $0 < \alpha' \leq \alpha$ such that the first α' coordinates of the α-dimensional random vectors B_n are non-negative, i.e. $B_n^i \geq 0$ for all $1 \leq i \leq \alpha'$. Let D_n be defined as in (2.2) with $D_0 = B_0$. First α' coordinates of D_n are assumed to be nondecreasing in n i.e.

$$0 \leq D_0^i \leq D_1^i \leq \cdots, \text{ for all } i = 1, \cdots, \alpha'.$$

As it is shown in [8], the above assumptions hold whenever the recurrence equations (2.1) originate from a so-called open stochastic event graph.

We will also need the following additional assumption on D_n^i and D_n^j for $i < j$

$$
\begin{aligned}
D_0^i &\leq D_0^j && (3.4)\\
D_n^i - D_0^i &\leq D_n^j - D_0^j \text{ for all } n \geq 1 \text{ and } i, j = 1, \cdots, \alpha'. && (3.5)
\end{aligned}
$$

Note that putting (3.4) and (3.5) together we obtain

$$
D_n^i \leq D_n^j \text{ for all } n \geq 0,\ i < j \text{ and } i, j = 1, \cdots, \alpha' \qquad (3.6)
$$

Stochastic assumptions. Recall that in the previous section we have already mentioned that $\{T_n\}$ is a Poisson process with intensity λ and $\{A_n, B_n\}$ is a stationary and ergodic sequence of random matrices, independent of $\{T_n\}$. In addition to these stochastic assumptions we need the following. For all $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$, let

$$
H_{n,l} = \bigoplus_{i=1}^{\alpha'} \{(A_{-(nl+1)} \otimes A_{-(nl+2)} \otimes \cdots \otimes A_{-(n+1)l} \otimes (B_{-(n+1)l} \oplus O)^i\},
$$

where O is the α dimensional column vector with all its components equal to 0. We assume that

$$
\lim_{l \to \infty} H_{0,l} = \infty.
$$

Note that this condition is practically always fulfilled since for this to hold, it is sufficient to have $EA_{i,i} > 0$ for some i. We also assume that there exists a positive integer $r' \in \mathbb{N}$ large enough such that

$$
\lambda < l(EH_{0,l})^{-1}
$$

for all integers $l \geq r'$. This will always be the case if the stability condition $\lambda a < 1$ is satisfied.

Finally, we assume that $\{H_{n,r'}\}_{n \geq 0}$ is a sequence of 1–dependent random variables (for systems derived from stochastic event graphs with independent firing times, a general result states that the sequence Z_n is M–dependent, $M \leq \alpha$. Our methodology can be extended to this more general case). An immediate consequence of the stochastic assumptions is the following lemma which is also given in Hasenfuss [13].

Lemma 3.1 *There exists an integer $r \in \mathbb{N}$, $r \geq r'$, such that*

$$
\lambda < (r-1)(EH_{0,r})^{-1}
$$

and $\{H_{n,r}\}_{n \geq 0}$ is a sequence of 1–dependent random variables.

Throughout the rest of this paper we suppress the index r and simply write H_n instead of $H_{n,r}$ but implicitly assume that r is chosen according to Lemma 3.1.

3.2 Taylor Series Expansion for Laplace Transforms

We recall a class of polynomials which first appeared in the expansion formula of EW in [8] and which will also appear in the expansion formula for $E[e^{-s_1W^i-s_2W^j}]$. These polynomials enjoy several nice properties which can be found in [4] and [8]. They are given by the following formula

$$p_k(x_0,\cdots,x_{k-1}) = \sum_{(i_0,i_1,\cdots,i_{k-1})\in\mathcal{N}_k} (-1)^{\gamma_k(i_0,i_1,\cdots,i_{k-1})} \frac{x_0^{i_0}}{i_0!}\frac{x_1^{i_1}}{i_1!}\cdots\frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!},$$

where

$$\mathcal{N}_k = \{(i_0,i_1,\cdots,i_{k-1})\in\{0,1,\cdots,i_{k-1}\}^k : i_0+i_1+\cdots+i_{k-1}=k \text{ and if } i_s=l>1, \text{ then } i_{s-1 \bmod k}=\cdots=i_{s-l+1 \bmod k}=0\}$$

and

$$\gamma_k(i_0,i_1,\cdots,i_{k-1}) = 1+\sum_{n=0}^{k-1}\mathbb{I}(i_n>0),$$

for all $k\geq 1$, with $\mathbb{I}(i_n>0)=1$ whenever $i_n>0$ and $\mathbb{I}(i_n>0)=0$ otherwise.

In particular it is shown in [8] that if $E(H_n)^{m+3}<\infty$ and the above assumptions on $\{A_n,B_n\}$ hold, then for all $i=1,\cdots,\alpha'$

$$E(W^i)=\sum_{k=0}^{m}\lambda^k E[p_{k+1}(D_0^i,\cdots,D_k^i)]+\mathcal{O}(\lambda^{m+1})$$

for $\lambda\in[0,a^{-1})$. Recall that a is the maximal (max,+) Lyapunov exponent of the sequence $\{A_n\}$.

We are now ready to state our main result. For all pairs of integers l and m and all pairs of real numbers x and y, let

$$F^{[l,m]}(x,y)=(-1)^{l+m}\frac{e^{-s_1x-s_2y}}{(s_1+s_2)^l s_2^m}. \tag{3.7}$$

Let d_p^i and d_p^j, $p\in\mathbb{N}$, be two sequences of real variables. We will need the following functions of these variables, which depend on four integers l,m, u and k such that $0\leq l<m\leq k$, and $0\leq u\leq k-m$ for some k:

$$h_{l,m,u}(d_{l+1}^i,\ldots,d_{l+u}^i,d_m^i)=\begin{cases}1 & \text{for } u=0\\ p_u(d_m^i,d_{l+1}^i,\ldots,d_{l+u-1}^i)-p_u(d_{l+1}^i,\ldots,d_{l+u}^i) & \text{otherwise}\end{cases} \tag{3.8}$$

where $p_u(\ldots)$ are the polynomials defined above. Finally, for all $l < n \le m \le k$, let

$$g_{k,l,m,n}(d^i_l, \ldots, d^i_m, d^j_m, \ldots, d^j_k) = h_{l,m,m-n}(d^i_{l+1}, \ldots, d^i_{l+m-n}, d^i_m)$$

$$\left(\frac{(d^i_m - d^i_l)^{k-m}}{(k-m)!} - \sum_{u=1}^{k-m} \frac{(-1)^u (d^i_m - d^i_l)^{k-m-u}}{(k-m-u)!} \left[p_u(d^j_{m+1}, \ldots, d^j_{m+u}) - p_u(d^j_m, \ldots, d^j_{m+u-1}) \right] \right). \quad (3.9)$$

Theorem 3.1 *Suppose that the above assumptions on $\{A_n, B_n\}$ and $\{D_n\}$ hold and $E(H_n)^{m+2} < \infty$. Then for $i < j$ and $i, j \in \{1, \cdots, \alpha'\}$, $E[e^{-s_1 W^i - s_2 W^j}]$ is $(m+1)$ times differentiable in λ for all λ in a right neighborhood of the origin,*

$$\lim_{\lambda \downarrow 0} \frac{d^k}{d\lambda^k} E[e^{-s_1 W^i - s_2 W^j}] = k! E[q_{k+1}(D^i_0, \cdots, D^i_k, D^j_0, \cdots, D^j_k)],$$

for $k = 0, 1, \cdots, m$ and $E[e^{-s_1 W^i - s_2 W^j}]$ can be expanded as a Taylor series of order m with respect to λ i.e.

$$E[e^{-s_1 W^i - s_2 W^j}] = \sum_{k=0}^{m} \lambda^k E[q_{k+1}(D^i_0, \cdots, D^i_k, D^j_0, \cdots, D^j_k)] + \mathcal{O}(\lambda^{m+1}),$$

for all arrival intensities $\lambda \in [0, a^{-1})$ where

$$q_{k+1}(d^i_0, \cdots, d^i_k, d^j_0, \cdots, d^j_k) = \quad (3.10)$$

$$\sum_{n=0}^{k} (-1)^{k-n} \sum_{v=n}^{k} \binom{v-1}{n-1} F^{[v,k-v]}(d^i_n, d^j_n)$$

$$+ \sum_{l=0}^{k-1} \sum_{n=l+1}^{k} (-1)^{k-n} \sum_{v=n}^{k} \binom{v-l}{n-l} \binom{k-v+l}{l} \Big[F^{[k-v+l,v-l]}(d^i_l, d^j_n - d^i_n + d^i_l)$$

$$- F^{[k-v+l+1,v-l-1]}(d^i_l, d^j_n - d^i_n + d^i_l) \Big]$$

$$+ \sum_{b=0}^{k-1} \sum_{n=0}^{b} (-1)^{b-n+1} \sum_{v=n}^{l} \binom{v-1}{n-1} F^{[v,b-v]}(d^i_n, d^j_n) \left[p_{k-b}(d^j_{n+1}, \cdots, d^j_{k-b+n}) - p_{k-b}(d^j_n, \cdots, d^j_{n+k-b-1}) \right]$$

$$+ \sum_{b=1}^{k-1} \sum_{l=0}^{b-1} \sum_{n=l+1}^{b} \sum_{m=n}^{n+k-b} (-1)^{k-m} \sum_{v=n}^{b} \binom{v-l}{n-l} \binom{b-v+l}{l} \Big[F^{[b-v+l,v-l]}(d^i_l, d^j_m - d^i_m + d^i_l)$$

$$- F^{[b-v+l+1,v-l-1]}(d^i_l, d^j_m - d^i_m + d^i_l) \Big] g_{n+k-b,l,m,n}(d^i_l, \cdots, d^i_m, d^j_m, \cdots, d^j_{n+k-b}),$$

with the conventions that summation over an empty set is 0 and that $\binom{n_1}{n_2} = 0$, $\binom{n_2}{n_2} = 1$ whenever $n_2 < 0$.

The proof of the above theorem will be given in Section 5.

As is mentioned in Hasenfuss [13], H_n is nothing but the maximum of various sums of service times. Thus, for condition $E(H_n)^{m+2} < \infty$ to hold it is sufficient to have $(m+2)^{\text{th}}$ moment of firing (or service) times finite.

Note that when $s_1 = 0$, Theorem 3.1 yields the expansion formula for $E[e^{-s_2 W^j}]$ (Laplace transform of the jth component of stationary waiting time vector). In this case with straightforward algebra we obtain

$$\begin{aligned} q_{k+1}(d_0^j, \cdots, d_k^j) &= \frac{1}{s_2^k} \sum_{n=0}^{k} (-1)^n \binom{k}{n} e^{-s_2 d_n^j} + \sum_{b=0}^{k-1} \frac{1}{s_2^b} \sum_{n=0}^{b} (-1)^{b+1} \binom{b}{n} e^{-s_2 d_n^j} \\ &\quad \left[p_{k-b}(d_{n+1}^j, \cdots, d_{k-b+n}^j) - p_{k-b}(d_n^j, \cdots, d_{n+k-b-1}^j) \right]. \end{aligned}$$

This coefficient coincides with the one obtained in [4] in the expansion formula of the Laplace transform of stationary waiting times. Setting $s_2 = 0$, with some elementary (but tedious) computation, we can obtain the same coefficients for the expansion formula of $E[e^{-s_1 W^i}]$.

3.3 Taylor Series Expansion for Covariances

The following result follows from Theorem 3.1 and its proof.

Corollary 3.1 *Suppose that the above assumptions on $\{A_n, B_n\}$ and $\{D_n\}$ hold and $E(H_n)^{m+4} < \infty$. Then for $i < j$ and $i, j \in \{1, \cdots, \alpha'\}$, $E[W^i W^j]$ is $(m+1)$ times differentiable in λ, for all λ in a right neighborhood of the origin,*

$$\lim_{\lambda \downarrow 0} \frac{d^k}{d\lambda^k} E[W^i W^j] = k! E[q'_{k+1}(D_0^i, \cdots, D_k^i, D_0^j, \cdots, D_k^j)]$$

for $k = 0, 1, \cdots, m$ and $E[W^i W^j]$ can be expanded as a Taylor series of order m with respect to λ i.e.

$$E[W^i W^j] = \sum_{k=0}^{m} \lambda^k E[q'_{k+1}(D_0^i, \cdots, D_k^i, D_0^j, \cdots, D_k^j)] + \mathcal{O}(\lambda^{m+1})$$

for all arrival intensities $\lambda \in [0, a^{-1})$ where

$$q'_{k+1}(d_0^i, \cdots, d_k^i, d_0^j, \cdots, d_k^j) = \tag{3.11}$$

$$\sum_{n=0}^{k} (-1)^{k-n} \sum_{v=n}^{k} \binom{v-1}{n-1} \tilde{F}^{[v,k-v]}(d_n^i, d_n^j)$$

$$+ \sum_{l=0}^{k-1} \sum_{n=l+1}^{k} (-1)^{k-n} \sum_{v=n}^{k} \binom{v-l}{n-l} \binom{k-v+l}{l} [\tilde{F}^{[k-v+l, v-l]}(d_l^i, d_n^j - d_n^i + d_l^i)$$

$$-\tilde{F}^{[k-v+l+1,v-l-1]}(d_l^i, d_n^j - d_n^i + d_l^i)]$$

$$+ \sum_{b=0}^{k-1}\sum_{n=0}^{b}(-1)^{b-n+1}\sum_{v=n}^{l}\binom{v-1}{n-1}\tilde{F}^{[v,b-v]}(d_n^i, d_n^j)\left[p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j) - p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\right]$$

$$+ \sum_{b=1}^{k-1}\sum_{l=0}^{b-1}\sum_{n=l+1}^{b}\sum_{m=n}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}[\tilde{F}^{[b-v+l,v-l]}(d_l^i, d_m^j - d_m^i + d_l^i) -$$
$$\tilde{F}^{[b-v+l+1,v-l-1]}(d_l^i, d_m^j - d_m^i + d_l^i)]g_{n+k-b,l,m,n}(d_l^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b}^j),$$

with the same conventions as above and with

$$\tilde{F}^{[l,m]}(x,y) = \frac{xy^{l+m+1}}{(l+m)!} - \frac{lxy^{l+m+1} + mxy^{l+m+1} + ly^{l+m+2}}{(l+m+1)!} + \frac{(l^2+lm+l)y^{l+m+2}}{(l+m+2)!}.$$

An important performance measure in queueing network analysis is $\mathrm{Cov}(W^iW^j)$ $(i \neq j)$ which is in general impossible to compute in closed form. Combining the expansion formula of Corollary 3.1 with the expansion formula of [8], it is straightforward to get an expansion formula with respect to λ for $\mathrm{Cov}(W^iW^j)$. Under the assumptions of Corollary 3.1, we have

$$\mathrm{Cov}(W^iW^j) = \sum_{k=0}^{m}\lambda^k\tilde{q}_{k+1}(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j) + \mathcal{O}(\lambda^{m+1}) \tag{3.12}$$

where for all $k \geq 0$

$$\tilde{q}_{k+1}(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j) = E[q'_{k+1}(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j)]$$
$$-\sum_{l=0}^{k}E[p_{l+1}(D_0^i,\cdots,D_l^i)]E[p_{k-l+1}(D_0^j,\cdots,D_{k-l}^j)].$$

In several applications, one is also interested in the covariance between the random variables W^i and $W^j - W^i$. It is immediate to derive a Taylor series expansion for this last covariance from the above formula and the expansion which was derived for the variance of W^i in [5]. Then under the assumptions of Corollary 3.1, we have

$$\mathrm{Cov}(W^i(W^j - W^i)) = \sum_{k=0}^{m}\lambda^k q_{k+1}^*(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j) + \mathcal{O}(\lambda^{m+1}) \tag{3.13}$$

where for all $k \geq 0$

$$q_{k+1}^*(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j) = \tilde{q}_{k+1}(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j)] - E[\hat{q}_{k+1}(D_0^i,\cdots,D_k^i)]$$
$$+\sum_{l=0}^{k}E[p_{l+1}(D_0^i,\cdots,D_l^i)]E[p_{k-l+1}(D_0^i,\cdots,D_{k-l}^i)].$$

with $\hat{q}_1(d_0^i) = (d_0^i)^2$ and for $k \geq 1$

$$\begin{aligned} &\hat{q}_{k+1}(d_0^i, \cdots, d_k^i) \\ = \ & \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \frac{2(d_n^i)^{k+2}}{(k+2)!} - \sum_{n=0}^{k-1} \sum_{l=n}^{k-1} (-1)^{l-n} \binom{l}{n} \frac{2(d_n^i)^{l+2}}{(l+2)!} \left[p_{k-l}(d_{n+1}^i, \cdots, d_{n+k-l}^i) - p_{k-l}(d_n^i, \cdots, d_{n+k-l-1}^i) \right] \end{aligned}$$

3.4 Convergence of the Expansion

A natural question which arises in relation with Theorem 3.1 is that of the region where the infinite Taylor series expansion of the joint Laplace transform of interest seen as a power series in the parameter λ is absolutely convergent (even when H_n admits moments of all orders, this last property is not granted by the method of [6], and actually, there are examples where the radius of convergence of such expansions is strictly less than a^{-1}, see [7]). Indeed, it is only when the parameter λ is in this region that the Taylor series expansion has a remainder term which tends to zero when the order of the expansion grows large. The method that we use below is adapted from that of Baccelli and Hong [7].

Theorem 3.2 *Assume that for some $K \geq 1$,*

$$\alpha_K = ||A_1 \otimes \cdots \otimes A_K||_\infty$$

is such that $\lambda E \alpha_K < K$ and such that the function

$$\zeta \to E \exp(\zeta \alpha_K)$$

is finite in a right-neighborhood of 0. If in addition for some ζ in this neighborhood,

$$(E \exp(\zeta \alpha_K))^{\frac{1}{K}} < \frac{\zeta}{\lambda} - 1, \tag{3.14}$$

then the infinite Taylor series expansion in λ at point 0 of the function $E[e^{-s_1 W^i - s_2 W^j}]$ is absolutely convergent and

$$E[e^{-s_1 W^i - s_2 W^j}] = \sum_{k=0}^{\infty} \lambda^k E[q_{k+1}(D_0^i, \cdots, D_k^i, D_0^j, \cdots, D_k^j)]. \tag{3.15}$$

Remark 3.1

- *There always exists a K such that $\lambda E \alpha_K < K$ if $\lambda a < 1$, since $\frac{E(\alpha_K)}{K} \to a$.*
- *It is always the case that $E \exp(\zeta \alpha_K)$ is finite in a right-neighborhood of 0 when the firing (or service) times have exponential moments.*
- *Condition (3.14) is always satisfied when λ is small enough.*

Proof Consider the vector (W_n^i, W_n^j) of $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ as a function of the state vector $\mathcal{Z}_n = (T_n, X_n^1, \ldots, X_n^\alpha)$ of $\mathbb{R}^{\alpha+1}$: $W_n^i = X_n^i - T_n$, $W_n^j = X_n^j - T_n$. It follows from Lemma 2 in [7] that this function is 1-Lipschitz w.r.t. the projective norm $\mathcal{D}$ defined therein (this is a direct application of this lemma when taking $\mathcal{H} = Id$, G_1 a permutation such that $(G_1(\mathcal{Z}_n))_1 = X_n^i$ and $(G_1(\mathcal{Z}_n))_2 = X_n^j$, $G_2(\mathcal{Z}_n)$ the mapping with all its coordinates equal to T_n and f the projection on the first two coordinates).

In addition, it is easy to check that the mapping

$$(x, y) \to \phi(x, y) = \exp(-s_1 x - s_2 y),$$

where s_1 and s_2 are two fixed real positive parameters is such that

$$|\phi(x, y) - \phi(x', y')| \leq (s_1 + s_2)||(x, y) - (x', y')||,$$

where $||.||$ denotes the supremum norm.

Thus, the mapping from $\mathbb{R}^{\alpha+1}$ to $\mathbb{R}$ defined by the relation

$$\psi(\mathcal{Z}_n) = \exp(-s_1 W_n^i - s_2 W_n^j)$$

is $(s_1 + s_2)$–Lipschitz w.r.t. the projective norm $\mathcal{D}$.

Therefore we can apply Theorem 1 and the results of Section 3.3.3 in [7] to derive the domain of absolute convergence of the series expansion. □

4 Examples

In this section we provide several examples of discrete event systems with state variables satisfying a vectorial recurrence equation of type (2.1) to illustrate the expansion formulae of the previous section. The first example concerns a system for which the joint Laplace transform is already known. It is presented here for the sake of checking the correctness of the expansion formula.

4.1 Tandem Queue with Deterministic Service Times

Consider a two station tandem queueing network where service times at both stations are deterministic. Let σ_1 and σ_2 be the service time at station 1 and station 2 respectively. Suppose $\sigma_2 < \sigma_1 < \infty$. Let W^i be the stationary waiting time of a randomly chosen customer until departure from station i, $i = 1, 2$. Assume that the arrivals to the network form a Poisson process with rate $\lambda < 1/\sigma_1$. It is shown in [8] that this system satisfies the monotonicity, support and stochastic assumptions of Section 3. Then Theorem 3.1 can be employed to obtain an expression for $E[e^{-s_1 W^1 - s_2 W^2}]$. Note that in this case

$$D_n^1 = (n+1)\sigma_1 \text{ and } D_n^2 = (n+1)\sigma_1 + \sigma_2, \qquad n \geq 0.$$

Hence, it follows from Theorem 3.1 that

$$E[q_1(D_0^1, D_0^2)] = q_1(D_0^1, D_0^2) = e^{-s_1 D_0^1 - s_2 D_0^2} = e^{-s_1\sigma_1 - s_2(\sigma_1+\sigma_2)} = e^{-(s_1+s_2)\sigma_1 - s_2\sigma_2}$$

and for $k \geq 1$

$$q_{k+1}(D_0^1, \cdots, D_k^1, D_0^2, \cdots, D_k^2) =$$

$$(-1)^k \sum_{n=0}^{k} (-1)^{k-n} \sum_{j=n}^{k} \binom{j-1}{n-1} \frac{e^{-s_1(n+1)\sigma_1 - s_2((n+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^j s_2^{k-j}} \tag{4.16}$$

$$+ \quad (-1)^k \sum_{l=0}^{k-1} \sum_{n=l+1}^{k} \sum_{j=n}^{k} \binom{j-l}{n-l} \binom{k-j+l}{l} e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)} ((-1)^{k-n} \frac{1}{(s_1+s_2)^{k-j+l} s_2^{j-l}}$$
$$+(-1)^{k-n+1} \frac{1}{(s_1+s_2)^{k-j+l+1} s_2^{j-l-1}}) \tag{4.17}$$

$$+ \quad \sum_{i=0}^{k-1} \sum_{n=0}^{i} (-1)^{i-n+1} \sum_{j=n}^{i} \binom{j-1}{n-1} (-1)^i \frac{e^{-s_1(n+1)\sigma_1 - s_2((n+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^j s_2^{i-j}}$$
$$\Big[p_{k-i}(D_{n+1}^2, \cdots, D_{k-i+n}^2) - p_{k-i}(D_n^2, \cdots, D_{n+k-i-1}^2)\Big] \tag{4.18}$$

$$+ \quad \sum_{i=1}^{k-1} \sum_{l=0}^{i-1} \sum_{n=l+1}^{i} \sum_{m=n}^{n+k-i} (-1)^{i-n+k-m} \sum_{j=n}^{i} \binom{j-l}{n-l} \binom{i-j+l}{l} (-1)^i e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)} \tag{4.19}$$
$$\left((-1)^{i-n} \frac{1}{(s_1+s_2)^{i-j+l} s_2^{j-l}} + (-1)^{i-n+1} \frac{1}{(s_1+s_2)^{i-j+l+1} s_2^{j-l-1}} \right) g_{k+n-i,l,m,n}(D_l^1, \cdots, D_m^1, D_m^2, \cdots, D_{n+k-i}^2).$$

We first consider (4.17). Changing the order of summation and substituting $k-j+l \to j$ in (4.17) we obtain

$$-(-1)^k \sum_{l=0}^{k-1} \sum_{j=l}^{k-1} \binom{j}{l} e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)} ((-1)^{k-l} \frac{1}{(s_1+s_2)^j s_2^{k-j}} + (-1)^{k-l+1} \frac{1}{(s_1+s_2)^{j+1} s_2^{k-j-1}})$$

$$= \quad -(-1)^k \sum_{l=0}^{k-1} \sum_{j=l}^{k-1} \binom{j-1}{l-1} (-1)^{k-l} \frac{e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^j s_2^{k-j}}$$
$$+(-1)^k \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-l} \frac{e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^k}, \tag{4.20}$$

where the equality follows from collecting similar terms together and then substituting $j+1 \to j$. Adding (4.20) to (4.16) and using the fact that $\binom{k-1}{l-1} + \binom{k-1}{l} = \binom{k}{l}$, we obtain

$$(-1)^k e^{-(s_1+s_2)\sigma_1 - s_2\sigma_2} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{e^{-(s_1+s_2)l\sigma_1}}{(s_1+s_2)^k} = (-1)^k e^{-(s_1+s_2)\sigma_1 - s_2\sigma_2} \frac{(e^{-(s_1+s_2)\sigma_1} - 1)^k}{(s_1+s_2)^k}$$
$$= e^{-(s_1+s_2)\sigma_1 - s_2\sigma_2} \frac{(1 - e^{-(s_1+s_2)\sigma_1})^k}{(s_1+s_2)^k}. \qquad (4.21)$$

We now consider (4.18) and (4.19). By the 1-invariance property of the polynomials (see [8]), it follows that (4.18) is nonzero only when $i = k-1$. Then (4.18) is equal to

$$(-1)^{k-1} \sum_{n=0}^{k-1} (-1)^{k-n} \sum_{j=n}^{k-1} \binom{j-1}{n-1} \frac{e^{-s_1(n+1)\sigma_1 - s_2((n+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^j s_2^{k-1-j}} \sigma_1. \qquad (4.22)$$

It follows from Theorem 7 in [4] that for $u \geq 2$

$$p_u(D_m^1, D_{l+1}^1, \cdots, D_{l+u-1}^1) - p_u(D_{l+1}^1, \cdots, D_{l+u}^1) = \sigma_1^u \Big(\frac{(m-l)^u}{u!} - \frac{(m-l)^{u-1}}{(u-1)!} \Big).$$

The above equality and some algebra yield that if $k - i \geq 2$

$$\sum_{m=n}^{n+k-i} (-1)^{i-n+k-m} g_{k+n-i,l,m,n}(D_l^1, \cdots, D_m^1, D_m^2, \cdots, D_{n+k-i}^2) = 0.$$

Hence, we again only consider the case $i = k-1$. Note that in this case m can only attain the values n and $n+1$ and (4.19) is equal to

$$-(-1)^{k-1} \sum_{l=0}^{k-2} \sum_{n=l+1}^{k-1} \sum_{j=n}^{k-1} \binom{j-l}{n-l} \binom{i-j+l}{l} e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)}$$
$$\Bigg((-1)^{k-1-n} \frac{1}{(s_1+s_2)^{k-1-j+l} s_2^{j-l}} + (-1)^{k-n} \frac{1}{(s_1+s_2)^{k-j+l} s_2^{j-l-1}} \Bigg) \sigma_1. \qquad (4.23)$$

Using the analysis that we have employed in the simplification of (4.17), (4.23) reduces to

$$(-1)^{k-1} \sum_{l=0}^{k-2} \sum_{j=l}^{k-2} \binom{j-1}{l-1} (-1)^{k-1-l} \frac{e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^j s_2^{k-1-j}} \sigma_1$$
$$- \quad (-1)^{k-1} \sum_{l=0}^{k-2} \binom{k-2}{l} (-1)^{k-1-l} \frac{e^{-s_1(l+1)\sigma_1 - s_2((l+1)\sigma_1+\sigma_2)}}{(s_1+s_2)^{k-1}} \sigma_1. \qquad (4.24)$$

Adding (4.24) to (4.22), we obtain

$$-e^{-(s_1+s_2)\sigma_1 - s_2\sigma_2} \frac{(1 - e^{-(s_1+s_2)\sigma_1})^{k-1}}{(s_1+s_2)^{k-1}} \sigma_1. \qquad (4.25)$$

Summing (4.21) and (4.25) gives for $k \geq 1$

$$q_{k+1}(D_0^1, \cdots, D_k^1, D_0^2, \cdots, D_k^2) = e^{-(s_1+s_2)\sigma^1 - s_2\sigma^2} \left(\frac{(1-e^{-(s_1+s_2)\sigma^1})^k}{(s_1+s_2)^k} - \frac{(1-e^{-(s_1+s_2)\sigma^1})^{k-1}}{(s_1+s_2)^{k-1}} \sigma_1 \right). \quad (4.26)$$

Clearly, in this case, there are more efficient ways of obtaining such an expansion, such as a direct use of the Pollaczek-Khintchine formula (see for example Wolff [18], page 386). Since $\sigma_1 > \sigma_2$

$$E[e^{-s_1 W^1 - s_2 W^2}] = E[e^{-s_1 W^1 - s_2(\sigma_2 + W^1)}] = E[e^{-(s_1+s_2)W^1}]E[e^{-s_2\sigma_2}]$$

and from the Pollaczek-Khintchine formula we have

$$E[e^{-(s_1+s_2)W^1}]E[e^{-s_2\sigma_2}] = e^{-(s_1+s_2)\sigma_1 - s_2\sigma_2} \left(1 + \sum_{k=1}^{\infty} \lambda^k \left(\frac{1-e^{-(s_1+s_2)\sigma_1}}{s_1+s_2} \right)^{k-1} \left(\frac{1-e^{-(s_1+s_2)\sigma_1}}{s_1+s_2} - \sigma_1 \right) \right). \quad (4.27)$$

Note that the coefficients of the expansion in (4.27) are the same as those obtained from the expansion formula of Theorem 3.1 (i.e. those given in (4.26).)

4.2 Tandem Queue with Random Service Times

Consider a two station tandem queueing network where service times at both stations are discrete random variables. Let $\{\sigma_n^1\}$ and $\{\sigma_n^2\}$ be the sequence of service times at station 1 and station 2 respectively. Suppose $P(\sigma_n^1 = 1.0) = 0.3, P(\sigma_n^1 = 3.0) = 0.1, P(\sigma_n^1 = 4.0) = 0.6$ and $P(\sigma_n^2 = 2.0) = 0.3, P(\sigma_n^2 = 5.0) = 0.25, P(\sigma_n^2 = 7.0) = 0.45$ for all n. Let W^i be the stationary waiting time of a randomly chosen customer until the beginning of his service on station i, $i = 1, 2$. Assume that the arrivals to the network form a Poisson point process with rate $\lambda < 0.2$. Our objective with this example is to get approximations for $\mathrm{Cov}(W^1 W^2)$ and $\mathrm{Cov}(W^1(W^2 - W^1))$ using (3.12) and (3.13). It is shown in [8] that for $i = 1, 2$

$$D_n^i = \max_{1 \leq l_n \leq \ldots \leq l_1 \leq i} \{ \sum_{k=l_1}^{i-1} \sigma_0^k + \sum_{k=l_2}^{l_1} \sigma_{-1}^k + \cdots + \sum_{k=l_n}^{l_{n-1}} \sigma_{-n+1}^k + \sum_{k=1}^{l_n} \sigma_{-n}^k \}.$$

Note that in this case $D_0^1 = 0$ and $D_0^2 = \sigma_0^1$. Using the above expression, for $n \geq 1$, we have $D_n^2 = \max\{D_n^1 + \sigma_0^1, U_n\}$ where U_n is a random variable written as the maximum of various sums of the service times $\sigma_0^1, \cdots, \sigma_{-n+1}^1, \sigma_{-n}^1$, $\sigma_0^2, \cdots, \sigma_{-n+1}^2, \sigma_{-n}^2$. Thus, the monotonicity assumptions of Section 3.1 are satisfied. Seidel et al [17] provide an efficient algorithm for the computation of the coefficients that appear in the Taylor series expansion of mean stationary waiting times in tandem queueing networks when service times are discrete random variables. This algorithm facilitates the computation of the coefficients in (3.12) and (3.13). Figure 1 displays a comparison of approximations of various order for $\mathrm{Cov}(W^1 W^2)$ with those obtained from a simulation study with respect to the traffic intensity $\rho = \lambda a$. Similarly, Figure 2 displays a comparison of approximations of various order for $\mathrm{Cov}(W^1(W^2 - W^1))$ with those obtained from a simulation study with respect to the traffic intensity ρ.

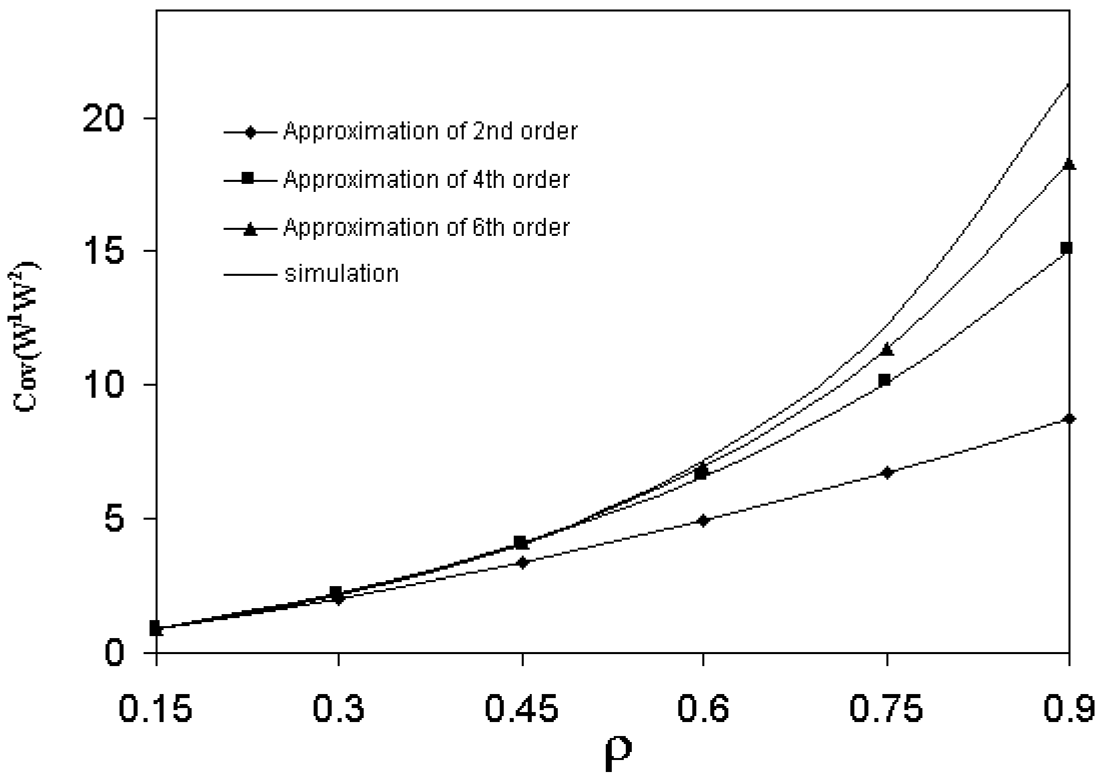

Figure 1: Approximations for $\mathrm{Cov}(W^1W^2)$ for the tandem queue example.

4.3 Kanban System with Deterministic Processing Times

Consider a two stage kanban system where processing times at both stations are deterministic. This example is studied in section 3.1 of [4] and section 4.2.4 of [8], where approximations for mean stationary waiting times are developed. The event graph representation of this kanban system is also given in [4] and [8]. In this simple example, each stage has a single machine and the number of jobs in the environment of each machine (i.e. in the input buffer, on the machine being processed, in the output buffer) is restricted to 2.

Let σ_1 and σ_2 be the service time at station 1 and station 2 respectively. Suppose $\sigma_1 = \sigma_2 = 4$. As is shown in [8], the stationary waiting time vector W has 7 components. The fourth component W^4 is the stationary time from arrival until a job leaves station 1 and the seventh component W^7 is the stationary time from arrival until a job leaves station 2. Assume that the arrivals to the network form a Poisson process with rate $\lambda < 0.25$. We are interested in computing $E[e^{-s_1W^4-s_2W^7}]$ and $\mathrm{Cov}(W^4W^7)$. Note that in this case $D_n^4 = 4+4n$ and $D_n^7 = 8+4n$. Exploiting this simple structure of $\{D_n^4\}$ and $\{D_n^7\}$, we can compute all the coefficients of the expansion formula in Theorem 3.1 and get an exact expression for $E[e^{-s_1W^4-s_2W^7}]$ as

$$e^{-4s_1-8s_2}\Big(1+\frac{\lambda(\frac{1-e^{-4s_1-4s_2}}{s_1+s_2}-4)(s_1+s_2)}{s_1+s_2-\lambda(1-e^{-4s_1-4s_2})}\Big).$$

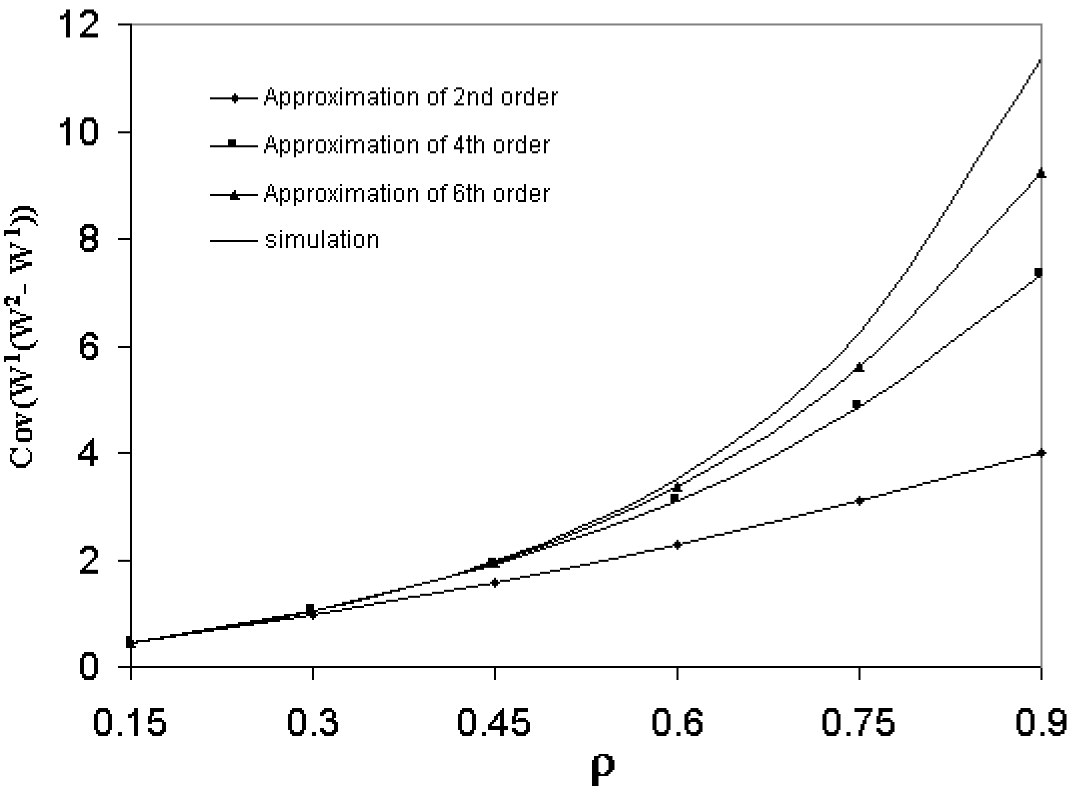

Figure 2: Approximations for $\mathrm{Cov}(W^1(W^2 - W^1))$ for the tandem queue example.

Moreover, the analysis of [4] yields that

$$E[W^4] = \frac{4 - 8\lambda}{1 - 4\lambda} \quad \text{and} \quad E[W^7] = \frac{8 - 24\lambda}{1 - 4\lambda}.$$

Hence, we can easily compute $\mathrm{Cov}(W^4 W^7)$ as

$$\mathrm{Cov}(W^4 W^7) = \frac{64}{3} \frac{\lambda(1-\lambda)}{(1-4\lambda)^2}.$$

4.4 Kanban System with Random Processing Times

Consider the two stage kanban system of the previous example. Assume that service times at both stations (unlike the previous example) are now random and uniformly distributed over the interval (0,4). Let $\{\sigma_n^1\}$ and $\{\sigma_n^2\}$ be the sequence of service times at station 1 and station 2 respectively. Then, $D_0^4 = \sigma_0^1$ and $D_0^7 = \sigma_0^1 + \sigma_0^2$. Moreover, using the expressions for A_n and B_n given on page 165 of [8], one can see that for $n \geq 1$, $D_n^7 = \max\{D_n^4 + \sigma_0^2, V_n\}$ where V_n is a random variable written as the maximum of various sums of the service times $\sigma_0^1, \cdots, \sigma_{-n+1}^1, \sigma_{-n}^1$, $\sigma_0^2, \cdots, \sigma_{-n+1}^2, \sigma_{-n}^2$. Thus, the monotonicity assumptions of Section 3.1 are satisfied. Clearly, the coefficients of the expansion formulae in Theorem 3.1 and in Corollary 3.1 are difficult to compute since they now involve multidimensional integrals. Note that $\{D_n^4\}$ and $\{D_n^7\}$ are no longer deterministic sequences and we need to compute $E[q_{k+1}(\cdots)]$ and $E[q'_{k+1}(\cdots)]$. However, we can (for example) use Monte Carlo simulation (see for example Fishman [12]) to evaluate these multi-dimensional integrals numerically. Using Monte Carlo simulation we obtain the coefficients of the expansion for $\text{Cov}(W^4W^7)$ as

$$\text{Cov}(W^4W^7) \quad = \quad 1.33 + 5.01\lambda + 22.41\lambda^2 + 75.36\lambda^3 + 205.82\lambda^4 + 578.74\lambda^5 + 1428.09\lambda^6 + \mathcal{O}(\lambda^7)$$

Figure 3 displays a comparison of approximations of various orders for $\text{Cov}(W^4W^7)$ with those obtained from a simulation study with respect to the traffic intensity $\rho = \lambda a$.

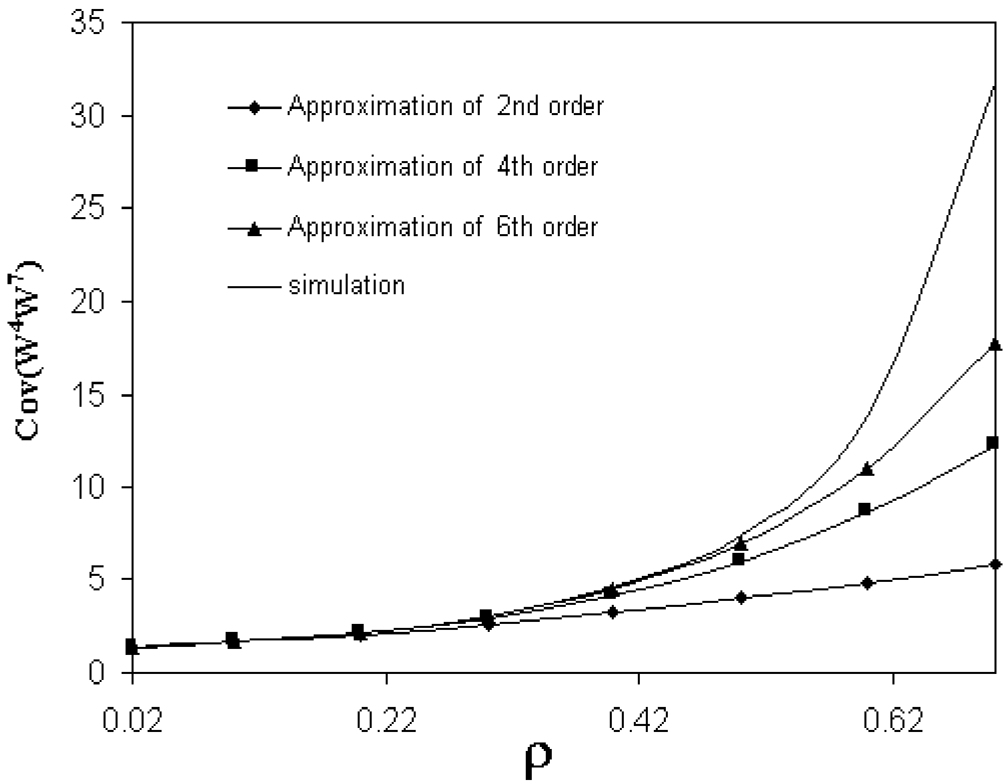

Figure 3: Approximations for $\text{Cov}(W^4W^7)$ for the kanban example.

4.5 Blocking Before Service

Consider a system of five stations in tandem with "blocking before service"; that is at each station, a customer can only start his service whenever the downstream station is empty (this is also called communication blocking). This example is studied in Section 4.2.3 of [8], where approximations for mean stationary waiting times are developed. Let σ^i be the deterministic service time at station i. Suppose $\sigma^1 \leq \sigma^2 \leq \cdots \leq \sigma^5$. Let W^4 be the stationary time from arrival until a random customer starts his service at station 4 and let W^5 be the stationary time from arrival until a random customer starts his service at station 5. Assume that the arrivals to the network form a Poisson process with rate $\lambda < \sigma^4 + \sigma^5$. Note that in this case $D_n^4 = \sigma_1 + \sigma_2 + \sigma_3 + n(\sigma_4 + \sigma_5)$ and $D_n^5 = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + n(\sigma_4 + \sigma_5)$ and using our expansion formula, we can get the following exact expression:

$$E[e^{-s_1 W^4 - s_2 W^5}] = e^{-(\sigma_1+\sigma_2+\sigma_3)(s_1+s_2)-\sigma_4 s_2}\Big(1 + \lambda \frac{(\frac{1-e^{-(\sigma_4+\sigma_5)(s_1+s_2)}}{s_1+s_2} - \sigma_4 - \sigma_5)(s_1+s_2)}{s_1+s_2-\lambda(1-e^{-(\sigma_4+\sigma_5)(s_1+s_2)})}\Big).$$

Moreover, the analysis of [4] yields that

$$\begin{aligned} E[W^4] &= \sigma_1 + \sigma_2 + \sigma_3 + \lambda \frac{(\sigma_4+\sigma_5)^2}{2-2\lambda(\sigma_4+\sigma_5)} \\ E[W^5] &= \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \lambda \frac{(\sigma_4+\sigma_5)^2}{2-2\lambda(\sigma_4+\sigma_5)}. \end{aligned}$$

Hence, we can easily compute $\mathrm{Cov}(W^4 W^5)$ as

$$\mathrm{Cov}(W^4W^5) = \frac{\lambda(12\sigma_5^2\sigma_4 - 6\lambda\sigma_4^2\sigma_5^2 - 4\lambda\sigma_4\sigma_5^3 - 4\lambda\sigma_4^3\sigma_5 - \lambda\sigma_4^4 + 4\sigma_5^3 + 12\sigma_5\sigma_4^2 + 4\sigma_4^3 - \lambda\sigma_5^4)}{12(1-\lambda(\sigma_5+\sigma_4))^2}.$$

4.6 Blocking After Service

In this final example, we consider a system of four stations in tandem with "blocking after service". Unlike communication blocking, in each station a customer can always start his service but when his service is completed he can only proceed to the downstream station if the downstream station is empty (this is also called manufacturing blocking). Let W^2 be the stationary time from arrival until a random customer leaves station 2 and let W^4 be the stationary time from arrival until a random customer leaves station 4 (hence the network). Let σ_1, σ_2, σ_3 and σ_4 be the service time at station 1, station 2, station 3 and station 4 respectively. Suppose $\sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 5$ and $\sigma_4 = 5$. Assume that the arrivals to the network form a Poisson process with rate $\lambda < 0.2$. We are interested in computing $E[e^{-s_1 W^2 - s_2 W^4}]$ and $\mathrm{Cov}(W^2 W^4)$. Note that in this case $D_n^2 = 4 + 5n$ and $D_n^4 = 14 + 5n$ and using Theorem 3.1, we can get an exact expression for $E[e^{-s_1 W^2 - s_2 W^4}]$ as

$$E[e^{-s_1 W^2 - s_2 W^4}] = e^{-4s_1 - 14 s_2}\Big(1 + \lambda \frac{(\frac{1-e^{-5s_1-5s_2}}{s_1+s_2} - 5)(s_1+s_2)}{s_1+s_2-\lambda(1-e^{-5s_1-5s_2})}\Big).$$

Moreover, the analysis of [4] yields that

$$E[W^2] = \frac{8 - 15\lambda}{2(1 - 5\lambda)}$$
$$E[W^4] = \frac{28 - 115\lambda}{2(1 - 5\lambda)}.$$

Hence, we can easily compute $\mathrm{Cov}(W^2W^4)$ as

$$\mathrm{Cov}(W^2W^4) = \frac{125}{12}\frac{\lambda(4 - 5\lambda)}{(1 - 5\lambda)^2}.$$

5 Proof of the Expansion Formula

This section is devoted to the proof of Theorem 3.1. The objective here is (a)to show that under some moment conditions, the functions of interest are differentiable in λ, and (b)to give a representation of the coefficients of the Taylor series expansion. Note that under the more restrictive assumptions of Section 3.4 (existence of exponential moments), we have analyticity in λ, so that the differentiability of all orders is then granted.

Recall the marked point process $(T, Z) = \{T_n, Z_n\}$ defined in Section 2. Let ψ be a real valued functional of the marked point process (T, Z) i.e. a measurable mapping $\psi : \mathcal{M} \times \mathcal{K}^\infty \to \mathbb{R}$ where $\mathcal{M}$ is the space of all realizations of $\{T_n\}$ and $\mathcal{K}^\infty$ is the space of all sequences of potential marks. Since the stochastic systems considered in this paper have the property that the future development of the input does not influence the present state of the system, we can and do assume that the values of ψ depend on the restriction of (T, Z) to the negative half line only. Let the counting measure $\sum_n \delta_{t_n}$ represent a sequence of points $\{t_n\}$. Moreover, let $t < 0$ be an arbitrary but fixed real number and let T^t denote the restriction of T to the interval $(t, 0)$, that is T^t can be represented by the random counting measure $\sum_n \delta_{T_n} 1(t < T_n < 0)$.

We will need the notions of *uniform coupling* and *uniform boundedness*. The following definitions are from [5] and [13].

Definition 5.1 *(Uniform coupling) Let* $\lambda_0 > 0$ *be fixed. Suppose there exists a (finite) random variable* $\tau_0 : \Omega \to (-\infty, 0]$ *such that*

$$\psi\Big(T^t(\omega), Z(\omega)\Big) = \psi\Big(T(\omega), Z(\omega)\Big) \qquad (5.28)$$

for all $t < \tau(\omega)$ *and such that* τ_0 *does not depend on* $\lambda \in [0, \lambda_0]$*. Then this random variable* τ_0 *is called a (uniform) coupling time of order 0. Moreover, assume that for* $k > 0$ *there exists a finite random variable* $\tau_k : \Omega \to (-\infty, 0]$ *such that*

$$\int_t^0 \cdots \int_t^0 \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \Big\{ \psi\Big(T + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) - \psi\Big(T^t + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) \Big\} dt_1 \cdots dt_k = 0 \tag{5.29}$$

whenever $t < \tau_k$ and τ_k does not depend on $\lambda \in [0, \lambda_0]$. Then we say that τ_k is a (uniform) coupling time of order k for ψ.

Note that condition (5.28) and (5.29) are automatically satisfied if

$$\psi\Big(T + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) \quad = \quad \psi\Big(T^t + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) \tag{5.30}$$

holds for all $t, t_v \leq 0$ and $l \leq k$ whenever $t < \tau_k$.

Definition 5.2 *(Uniform Boundedness) Assume that there exists a random variable $\eta_0 : \Omega \to [0, \infty)$ such that $E\eta_0 < \infty$ and*

$$| \psi(T^t, Z) | \leq \eta_0 \tag{5.31}$$

for all $t \leq 0$ and for each $\lambda \in [0, \lambda_0]$. Moreover, suppose that for $k > 0$ there is a random variable $\eta_k : \Omega \to [0, \infty)$ such that $E\eta_k < \infty$ and

$$\int_t^0 \cdots \int_t^0 \Big| \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \psi\Big(T^t + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) \Big| dt_1 \cdots dt_k \leq \eta_k \tag{5.32}$$

for all $t \leq 0$ and for each $\lambda \in [0, \lambda_0]$. Then we say that we have uniform boundedness of order k for ψ.

Finally, we define the so-called *expansion kernels* $\psi_{x_1, \cdots, x_k}$ which were first introduced in Reiman and Simon [16] (see also Błaszczyszyn [9]).

Definition 5.3 *(Expansion Kernels) For any $x < 0$ and $z \in \mathcal{K}^\infty$, let*

$$\psi_x(\mu, Z) \quad = \quad \psi(\mu^x + \delta_x, Z) - \psi(\mu^x, Z)$$

where μ^x denotes the restriction of the counting measure μ to the interval $(x, 0)$. For any $k \geq 1$ and $x_1, \cdots, x_k \in \mathbb{R}$, the functional $\psi_{x_1, \cdots, x_k}$ is defined by iteration of the mapping $\psi \to \psi_x$, that is

$$\psi_{x_1, \cdots, x_k}(\mu, Z) \quad = \quad (\cdots (\psi_{x_1})_{x_2} \cdots)_{x_k}(\mu, Z).$$

From the definition of the expansion kernels one can obtain the following relationship (see Błaszczyszyn [9])

$$\psi_{x_1,\cdots,x_k}(o,Z) = \begin{cases} \sum_{l=0}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l}}\psi(\sum_{v\in\pi}\delta_{x_v},Z) & \text{for } x_1<\cdots<x_k \\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

where o denotes the null measure (i.e. $o(\mathbb{R})=0$) and $K_{k,l}$ denotes the collection of all subsets of $\{1,\cdots,k\}$ containing l elements.

It is shown in [5] and [13] (see corollary 5.1) that if conditions (5.28), (5.29), (5.31) and (5.32) hold for all $0<k<m+1$ and if almost surely

$$\begin{aligned} &\lim_{\lambda\downarrow 0}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\psi\Big(T+\sum_{v=1}^{l}\delta_{t_v},Z\Big)dt_1\cdots dt_k \\ = &\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\psi\Big(\sum_{v=1}^{l}\delta_{t_v},Z\Big)dt_1\cdots dt_k, \end{aligned} \tag{5.34}$$

then

$$\begin{aligned} \lim_{\lambda\downarrow 0}\frac{d^k}{d\lambda^k}E\psi(T,Z) &= \int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\psi\Big(\sum_{v=1}^{l}\delta_{t_v},Z\Big)dt_1\cdots dt_k \\ &= k!\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}E\psi_{x_1,\cdots,x_k}(o,Z)dx_k\cdots dx_1. \end{aligned} \tag{5.35}$$

We now define for $i,j=1,\cdots,\alpha'$

$$\psi^{i,j}\Big(T,Z\Big) = G\Big(W^i(T,Z),W^j(T,Z)\Big) = G\Big(D_0^i\oplus\bigoplus_{n=1}^{\infty}(D_n^i\otimes T_{-n}),D_0^j\oplus\bigoplus_{n=1}^{\infty}(D_n^j\otimes T_{-n})\Big) \tag{5.36}$$

where $G:\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a nonnegative function such that $G(x,y)\le cx^{\nu_1}y^{\nu_2}$ for all $x,y\ge 0$, where ν_1,ν_2 are non-negative integers and c is a positive finite constant. Note that for $G(x,y)=e^{-s_1x-s_2y}$, $s_1,s_2\ge 0$, $E\psi^{i,j}(T,Z)$ is the joint Laplace transform of W^i,W^j. Similarly, for $G(x,y)=xy$, $E\psi^{i,j}(T,Z)$ is the expected value of the product of W^i and W^j.

5.1 Uniform Coupling and Uniform Boundedness

Let us start with a few preliminary results. It is shown in [13] that the following useful bound can be obtained on D_{nr}^i where r is determined by Lemma 3.1:

Lemma 5.1 *For $n,r\in\mathbb{N}$ and r is determined by Lemma 3.1*

$$D_{nr}^i\le H_0+H_1+\cdots+H_{n-1},$$

for all $1\le i\le\alpha'$.

We also need the following results:

Proposition 5.1 *If the assumptions of Section 3 hold and* $E[H_n^{\nu_1+\nu_2+m+2}] < \infty$ *then for* $0 \leq k < m+1$

$$\lim_{\lambda \downarrow 0} \frac{d^k}{d\lambda^k} E\psi^{i,j}(T,Z) \;=\; k! \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} E\psi^{i,j}_{x_1,\cdots,x_k}(o,Z)dx_k \cdots dx_1,$$

where $\psi^{i,j}(T,Z)$ *is as defined in (5.36).*

The proof of the above proposition follows from the following lemmas and Corollary 5.1 of [13].

Lemma 5.2 *If the intensity of* $\lambda = \lambda_0$ *of* T *is such that* $\lambda_0 a < 1$*, then for all* $k \in \mathbb{N}$*, there exists a random variable* τ_k *such that for all* $0 \leq l \leq k$*, and for all choices of real numbers* $t_1, \cdots, t_l$

$$\psi^{i,j}\Big(T^t + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) \;=\; \psi^{i,j}\Big(T + \sum_{v=1}^{l} \delta_{t_v}, Z\Big)$$

whenever $t < \tau_k$*.*

Proof Using the inequality in (2.3), we obtain that $D^i_{n+k} + T_{-n}$ and $D^j_{n+k} + T_{-n}$ tend to $-\infty$ for any $k \in \mathbb{N}$ as n tends to ∞. Then there exists an N_k such that $D^j_{n+k} + T_{-n} < 0 \; \forall n \geq N_k$. Since $D^i_{n+k} \leq D^j_{n+k}$, we also have $D^i_{n+k} + T_{-n} < 0 \; \forall n \geq N_k$. Rest of the proof is similar to the proof of Lemma 3 in Baccelli et al [6] and it is omitted. □

The above lemma gives the existence of a coupling time. We next show that for an appropriate choice of the reference probability space, the coupling time τ_k is uniform with respect to λ in the interval $[0, \lambda_0]$.

Lemma 5.3 *Let* λ_0 *be as in Lemma 5.2. There exists a probability space on which are defined* Z *and a family of point processes* $\{T(\lambda)\}$*, where* $T(\lambda)$ *is a stationary Poisson process of intensity* $\lambda \in [0, \lambda_0]$*. For all* $k \in \mathbb{N}$*, there exists a random variable* τ_k *on this space such that for all* $0 \leq l \leq k$*, for all choices of real numbers* $t_1, \cdots, t_l$*, and for all* $\lambda \in [0, \lambda_0]$

$$\psi^{i,j}\Big(T^t(\lambda) + \sum_{v=1}^{l} \delta_{t_v}, Z\Big) \;=\; \psi^{i,j}\Big(T(\lambda) + \sum_{v=1}^{l} \delta_{t_v}, Z\Big)$$

whenever $t < \tau_k$*.*

Proof Since $\{D^i_n\}$ and $\{D^j_n\}$ are monotone in n and $D^i_n \leq D^j_n$ for all n, proof is analogous to the proof of Lemma 4 in [6] and it is omitted. □

Hence, condition (5.30) and as an immediate consequence conditions (5.28) and (5.29) are satisfied. We now show that the uniform boundedness conditions in (5.31) and (5.32) hold. Before going into the details of this proof we state two simple inequalities which will be used in our analysis. For $x_1, \cdots, x_k \in \mathbb{R}_0^+$, $i_1, \cdots, i_k, k \in \mathbb{N}$ and $n \in \mathbb{N}_0$,

$$x_1^{i_1} \cdots x_k^{i_k} \leq x_1^{i_1+\cdots+i_k} + \cdots + x_k^{i_1+\cdots+i_k} \tag{5.37}$$

$$(x_1 + \cdots + x_k)^n \leq k^n(x_1^n + \cdots + x_k^n). \tag{5.38}$$

Proofs of the above inequalities are given in [6].

Lemma 5.4 *Suppose that the assumptions of Proposition 5.1 hold. If $\lambda_0 < a^{-1}$, then there exists a non-negative random variable η_0 with finite expectation such that $0 \leq \psi^{i,j}((T^t(\lambda), Z) \leq \eta_0$ and $E\psi^{i,j}(T(\lambda), Z) < \infty$ for all $t < 0$ and for all $\lambda \in [0, \lambda_0]$.*

Proof We first show that $E\psi^{i,j}(T, Z) = E[G(W^i(T, Z), W^j(T, Z))]$ is finite for $T = T(\lambda_0)$. Let $\lambda = \lambda_0$ and let r be chosen according to Lemma 3.1. Since $G(x, y) \leq cx^{\nu_1}y^{\nu_2}$, we have

$$\begin{aligned}
& \psi^{i,j}\Big(T, Z\Big) \\
= \;& G\Big(D_0^i \oplus \bigoplus_{n=1}^{\infty}(D_n^i \otimes T_{-n}), D_0^j \oplus \bigoplus_{n=1}^{\infty}(D_n^j \otimes T_{-n})\Big) \\
\leq \;& c\Big(D_0^i \oplus \bigoplus_{n=1}^{\infty}(D_n^i \otimes T_{-n})\Big)^{\nu_1}\Big(D_0^j \oplus \bigoplus_{n=1}^{\infty}(D_n^j \otimes T_{-n})\Big)^{\nu_2} \\
\leq \;& c\Big(D_0^i + \sup_{p\geq 0}\max_{1\leq q\leq r}\{D_{pr+q}^i + T_{-(pr+q)}\}_+\Big)^{\nu_1}\Big(D_0^j + \sup_{p\geq 0}\max_{1\leq q\leq v}\{D_{pr+q}^j + T_{-(pr+q)}\}_+\Big)^{\nu_2} \\
\leq \;& c\Big(D_0^j + \sup_{p\geq 0}\{D_{(p+1)r}^i + T_{-(pr+1)}\}_+\Big)^{\nu_1+\nu_2} \\
= \;& c\Big(D_0^j + \sup_{p\geq 0}\{H_0 + \cdots + H_p + T_{-(pr)}\}_+\Big)^{\nu_1+\nu_2} \\
\leq \;& c\Big(D_0^j + H_0 + \varphi^{(0)}(T, Z) + \varphi^{(1)}(T, Z)\Big)^{\nu_1+\nu_2}
\end{aligned} \tag{5.39}$$

($\{x\}_+ = \max(0, x)$) where we used the monotonicity properties of $\{D_n^i\}$, $\{D_n^j\}$, $D_0^i \leq D_0^j$ and Lemma 5.1. Note that $\varphi^{(b)}(T, Z)$, $b = 0, 1$, denotes the random variable

$$\varphi^{(b)}(T, Z) = \sup_{p\geq 1}\Big\{\sum_{k=1}^{p}\Big(H_{2k-b} + (T_{-(2k-b)r} - T_{-(2k-b-1)r})\Big)\Big\}_+.$$

Applying (5.38) to (5.39), we obtain

$$| \psi^{i,j}(T, Z) | \leq c4^{\nu_1+\nu_2}\Big((D_0^j)^{\nu_1+\nu_2} + (H_0)^{\nu_1+\nu_2} + (\varphi^{(0)}(T, Z))^{\nu_1+\nu_2} + (\varphi^{(1)}(T, Z))^{\nu_1+\nu_2}\Big).$$

Note that it follows from the stochastic assumptions of Section 3 that $\varphi^{(b)}(T, Z)$, $b = 0, 1$ is the supremum of a random walk with negative drift since from Lemma 3.1 we have

$$E[H_{2k-l}+(T_{-(2k-l)r}-T_{-(2k-l-1)r})]=EH_0-\frac{r}{\lambda}<EH_0-\frac{r-1}{\lambda}<0$$

for $b=0,1$. The finiteness of $E[\psi^{i,j}(T,Z)]$ now follows from the well known fact the $(\nu_1+\nu_2)^{\text{th}}$ moment of the maximum of a random walk with negative drift is finite if $(\nu_1+\nu_2+1)^{\text{th}}$ moment of its increments is finite (see for example Theorem 2.1 on page 184 of Asmussen [1]). This is true since for $b=0,1$

$$E\Big[\Big(H_{2k-b}+(T_{-(2k-b)r}-T_{-(2k-b-1)r})\Big)^{\nu_1+\nu_2+1}\Big]\leq 2^{\nu_1+\nu_2+1}\Big\{E\Big[(H_0)^{\nu_1+\nu_2+1}\Big]+(\nu_1+\nu_2+1)!\Big(\frac{r}{\lambda}\Big)^{\nu_1+\nu_2+1}\Big\}<\infty$$

whenever $E[(H_0)^{\nu_1+\nu_2+1}]<\infty$. We have actually done more than proving the finiteness of $E[\psi^{i,j}(T,Z)]$. We have shown that η_0 can be chosen as

$$\eta_0=c4^{\nu_1+\nu_2}\Big((D_0^j)^{\nu_1+\nu_2}+(H_0)^{\nu_1+\nu_2}+(\varphi^{(0)}(T,Z))^{\nu_1+\nu_2}+(\varphi^{(1)}(T,Z))^{\nu_1+\nu_2}\Big). \tag{5.40}$$

This is true since

$$\begin{aligned}\psi^{i,j}\Big(T^t,Z\Big) &= G\Big(D_0^i\oplus\bigoplus_{n:T_{-n}\geq t}(D_n^i\otimes T_{-n}),D_0^j\oplus\bigoplus_{n:T_{-n}\geq t}(D_n^j\otimes T_{-n})\Big)\\ &\leq c\Big(D_0^i\oplus\bigoplus_{n:T_{-n}\geq t}(D_n^i\otimes T_{-n})\Big)^{\nu_1}\Big(D_0^j\oplus\bigoplus_{n:T_{-n}\geq t}(D_n^j\otimes T_{-n})\Big)^{\nu_2}\\ &\leq c\Big(D_0^i\oplus\bigoplus_{n\geq 1}(D_n^i\otimes T_{-n})\Big)^{\nu_1}\Big(D_0^j\oplus\bigoplus_{n\geq 1}(D_n^j\otimes T_{-n})\Big)^{\nu_2}.\end{aligned}$$

Hence, η_0 as given in (5.40) can be used to show that there exists a random variable η_0 such that $0\leq\psi^{i,j}(T^t(\lambda),Z)\leq\eta_0$ for all $t<0$ and for all $\lambda\in[0,\lambda_0]$. □

Lemma 5.5 *Suppose that the assumptions of Proposition 5.1 hold. Then for $k>0$ there is a random variable $\eta_k:\Omega\to[0,\infty)$ such that $E\eta_k<\infty$ and*

$$\int_t^0\cdots\int_t^0\Big|\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\psi^{i,j}\Big(T^t+\sum_{v=1}^{l}\delta_{t_v},Z\Big)\Big|dt_1\cdots dt_k\leq\eta_k$$

for all $t\leq 0$ and for each $\lambda\in[0,\lambda_0]$.

Proof Recall that $K_{k,l}$ denotes the collection of all subsets π of $\{1,\cdots,k\}$ containing precisely l elements. Then

$$\begin{aligned}
&\int_t^0 \cdots \int_t^0 \Big|\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\psi^{i,j}\Big(T^t+\sum_{v=1}^{l}\delta_{t_v},Z\Big)\Big|dt_1\cdots dt_k \\
=\ &\int_t^0 \cdots \int_t^0 \Big|\sum_{l=0}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l}}\psi^{i,j}\Big(T^t+\sum_{v\in\pi}\delta_{t_v},Z\Big)\Big|dt_1\cdots dt_k \\
=\ & k!\int_t^0\int_{x_k}^0 \cdots \int_{x_2}^0 \Big|\sum_{l=0}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l}}\psi^{i,j}\Big(T^t+\sum_{v\in\pi}\delta_{x_v},Z\Big)\Big|dx_1\cdots dx_k \qquad (5.41)
\end{aligned}$$

where we split the range of integration $(t,0)^k$ into $k!$ subranges. Clearly, the value of $k!$ resulting integrals is the same. For each realization of the point process $T+\sum_{v\in\pi}\delta_{x_v}$, let the sequences $\{g_n^\pi\}$ and $\{h_v^\pi\}$ be the indices of these points originating from $T=\{T_{-n}\}$ and $\{x_v, v\in\pi\}$, respectively. Using this notation we see that for any $\pi\subseteq\{1,\cdots,k\}$

$$\psi^{i,j}\Big(T^t+\sum_{v\in\pi}\delta_{t_v},Z\Big)=G\Big(D_0^i\oplus\bigoplus_{n:T_{-n}\ge t}(D^i_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{v\in\pi}(D^i_{h_v^\pi}\otimes x_v),D_0^j\oplus\bigoplus_{n:T_{-n}\ge t}(D^j_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{v\in\pi}(D^j_{h_v^\pi}\otimes x_v)\Big).$$

Then

$$\begin{aligned}
&\sum_{l=0}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l}}\psi^{i,j}\Big(T^t+\sum_{v\in\pi}\delta_{x_v},Z\Big) \\
=\ &\sum_{l=1}^{k}(-1)^{k-l}\Big\{\sum_{\pi\in K_{k,l},k\in\pi}\psi^{i,j}\Big(T^t+\sum_{v\in\pi}\delta_{x_v},Z\Big)-\sum_{\pi\in K_{k,l},k\notin\pi}\psi^{i,j}\Big(T^t+\sum_{v\in\pi}\delta_{x_v},Z\Big)\Big\} \\
=\ &\sum_{l=1}^{k}\sum_{\pi\in K_{k,l},k\in\pi}\Big\{G\Big(D_0^i\oplus\bigoplus_{n:T_{-n}\ge t}(D^i_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{v\in\pi}(D^i_{h_v^\pi}\otimes x_v),D_0^j\oplus\bigoplus_{n:T_{-n}\ge t}(D^j_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{v\in\pi}(D^j_{h_v^\pi}\otimes x_v)\Big) \\
&\qquad -G\Big(D_0^i\oplus\bigoplus_{n=1}^{l(x_k)-1}(D^i_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{n\ge l(x_k)}^{T_{-n}<t}(D^i_{g_n^\pi-1}\otimes T_{-n})\bigoplus_{v\in\pi\backslash\{k\}}(D^i_{h_v^\pi}\otimes x_v), \\
&\qquad D_0^j\oplus\bigoplus_{n=1}^{l(x_k)-1}(D^j_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{n\ge l(x_k)}^{T_{-n}<t}(D^j_{g_n^\pi-1}\otimes T_{-n})\bigoplus_{v\in\pi\backslash\{k\}}(D^j_{h_v^\pi}\otimes x_v)\Big)\Big\} \qquad (5.42)
\end{aligned}$$

where $l(x_k)$ is defined by the relation $x\in[T_{-l(x)},T_{-(l(x)-1)}]$. The differences between the functions $G(\cdot)$ in (5.42) is zero if

$$D_0^i\oplus\bigoplus_{n=1}^{l(x_k)-1}(D^i_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{v\in\pi\backslash\{k\}}(D^i_{h_v^\pi}\otimes x_v)\ \ge\ \bigoplus_{n\ge l(x_k)}^{T_{-n}<t}(D^i_{g_n^\pi-1}\otimes T_{-n})\oplus(D^i_{h_k^\pi}\otimes x_k)\text{ and} \qquad (5.43)$$

$$D_0^j\oplus\bigoplus_{n=1}^{l(x_k)-1}(D^j_{g_n^\pi}\otimes T_{-n})\oplus\bigoplus_{v\in\pi\backslash\{k\}}(D^j_{h_v^\pi}\otimes x_v)\ \ge\ \bigoplus_{n\ge l(x_k)}^{T_{-n}<t}(D^j_{g_n^\pi-1}\otimes T_{-n})\oplus(D^j_{h_k^\pi}\otimes x_k). \qquad (5.44)$$

For simplicity let $g_n = g_n^{\{1,\cdots,k\}}$ and $h_v = h_v^{\{1,\cdots,k\}}$ where for each realization of the point process $T + \sum_{v=1}^{k} \delta_v$, $\{g_n^{\{1,\cdots,k\}}\}$ and $\{h_v^{\{1,\cdots,k\}}\}$ are the indices of the points originating from $T = \{T_{-n}\}$ and $\{x_v, v \in \{1, \cdots, k\}\}$ respectively. Clearly, $g_n^\pi \le g_n$ and $h_v^\pi \le h_v$ for all subsets $\pi \subseteq \{1, \cdots, k\}$. Since $D_0^i \ge 0$, $D_0^j \ge 0$, $\{D_n^i\}$ and $\{D_n^j\}$ are monotone sequences and $g_n \le n + k$, we can conclude that for the difference in (5.42) not to vanish it is necessary to have

$$
\begin{aligned}
0 < \bigoplus_{n \ge l(x_k)} (D^i_{n+k} \otimes T_{-n}) \oplus (D^i_{h_k} \otimes T_{-(l(x_k)-1)}) &= \bigoplus_{n \ge l(x_k)} (D^i_{n-1+k} \otimes T_{-(n-1)}) \text{ or} \\
0 < \bigoplus_{n \ge l(x_k)} (D^j_{n+k} \otimes T_{-n}) \oplus (D^j_{h_k} \otimes T_{-(l(x_k)-1)}) &= \bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)})
\end{aligned}
$$

since $h_k = l(x_k) - 1 + k$. The absolute value of (5.42) can be bounded by

$$
\begin{aligned}
& \sum_{l=1}^{k} \binom{k-1}{l-1} c \Big[\Big(\bigoplus_{n \ge l(x_k)} (D^i_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_1} \Big(\bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_2} \\
& \mathbb{1}\Big(x_k : \bigoplus_{n \ge l(x_k)} (D^i_{n-1+k} \otimes T_{-(n-1)}) > 0 \Big) \mathbb{1}\Big(x_k : \bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) > 0 \Big) \\
+ & \Big(D^i_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^i_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v \in \pi \setminus \{k\}} (D^i_{h^\pi_v} \otimes x_v) \Big)^{\nu_1} \Big(\bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_2} \\
& \mathbb{1}\Big(x_k : D^i_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^i_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v \in \pi \setminus \{k\}} (D^i_{h^\pi_v} \otimes x_v) \ge \bigoplus_{n \ge l(x_k)}^{T_{-n} < t} (D^i_{g^\pi_n - 1} \otimes T_{-n}) \oplus (D^i_{h^\pi_k} \otimes x_k) \Big) \\
& \mathbb{1}\Big(x_k : \bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) > 0 \Big) \\
+ & \Big(\bigoplus_{n \ge l(x_k)} (D^i_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_1} \Big(D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v \in \pi \setminus \{k\}} (D^j_{h^\pi_v} \otimes x_v) \Big)^{\nu_2} \\
& \mathbb{1}\Big(x_k : \bigoplus_{n \ge l(x_k)} (D^i_{n-1+k} \otimes T_{-(n-1)}) > 0 \Big) \\
& \mathbb{1}\Big(x_k : D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v \in \pi \setminus \{k\}} (D^j_{h^\pi_v} \otimes x_v) \ge \bigoplus_{n \ge l(x_k)}^{T_{-n} < t} (D^j_{g^\pi_n - 1} \otimes T_{-n}) \oplus (D^j_{h^\pi_k} \otimes x_k) \Big) \Big] \\
\le \; & 2^{k-1} c \Big[\Big(\bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_1 + \nu_2} \\
+ & \Big(D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v \in \pi \setminus \{k\}} (D^j_{h^\pi_v} \otimes x_v) \Big)^{\nu_1} \Big(\bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_2} \\
+ & \Big(\bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) \Big)^{\nu_1} \Big(D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v \in \pi \setminus \{k\}} (D^j_{h^\pi_v} \otimes x_v) \Big)^{\nu_2} \Big] \\
& \mathbb{1}\Big(x_k : \bigoplus_{n \ge l(x_k)} (D^j_{n-1+k} \otimes T_{-(n-1)}) > 0 \Big) \qquad (5.45)
\end{aligned}
$$

where we used the fact that $D^i_n \le D^j_n$ for all n. Note that

$$\begin{aligned}
& D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{g^\pi_n} \otimes T_{-n}) \oplus \bigoplus_{v\in\pi\setminus\{k\}} (D^j_{h^\pi_v} \otimes x_v) \\
\le\ & D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{n+k} \otimes T_{-n}) \oplus \bigoplus_{v\in\pi\setminus\{k\}} (D^j_{h_v} \otimes x_v) \\
=\ & D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{n+k} \otimes T_{-n}) \oplus \bigoplus_{v\in\pi\setminus\{k\}} (D^j_{l(x_v)-1+v} \otimes T_{-(l(x_v)-1)}) \\
\le\ & D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{n+k} \otimes T_{-n}) \oplus \bigoplus_{v\in\{1,\cdots,k-1\}} (D^j_{l(x_v)-1+v} \otimes T_{-(l(x_v)-1)}) \\
=\ & D^j_0 \oplus \bigoplus_{n=1}^{l(x_k)-1} (D^j_{n+k} \otimes T_{-n}) \le D^j_0 \oplus \bigoplus_{n\ge1} (D^j_{n+k} \otimes T_{-n}).
\end{aligned}$$

Then an upper bound on (5.45) is

$$\begin{aligned}
& 2^{k-1}c\Big[\Big(\sup_{n\ge l(x_k)} (D^j_{n-1+k} + T_{-(n-1)})\Big)^{\nu_1+\nu_2} + \Big(\max\{D^j_0, \sup_{n\ge1}(D^j_{n+k}+T_{-n})\}\Big)^{\nu_1}\Big(\sup_{n\ge l(x_k)} (D^j_{n-1+k} + T_{-(n-1)})\Big)^{\nu_2} \\
+ & \Big(\sup_{n\ge l(x_k)} (D^j_{n-1+k} + T_{-(n-1)})\Big)^{\nu_1}\Big(\max\{D^j_0, \sup_{n\ge1}(D^j_{n+k}+T_{-n})\}\Big)^{\nu_2}\Big]\mathbb{1}\Big(x_k : \sup_{n\ge l(x_k)} (D^j_{n-1+k} + T_{-(n-1)}) > 0\Big).
\end{aligned}$$

Let $l^* = (r-1)^2 + (r-1)(k-1)$ where r is the integer defined in Lemma 3.1. It is shown in [6] and [13] that if $x_k < T_{-l^*}$ then

$$\begin{aligned}
\sup_{n\ge l(x_k)} (D^j_{n-1+k} + T_{-(n-1)}) &\le \sup_{n\ge l(x_k)} (\xi^{(0)}_{p(n)} + \xi^{(1)}_{p(n)}) \\
&\le \phi^{(0)}_{x_k} + \phi^{(1)}_{x_k} \qquad (5.46)
\end{aligned}$$

where $p(n) \in \mathbb{N}$ is such that $p(n)(r-1) < n \le (p(n)+1)(r-1)$ and $\xi^{(b)}_{p(n)} = \xi^{(b)}_{p(n)}(T,Z)$, $b = 0,1$ is defined as

$$\xi^{(b)}_{p(n)}(T,Z) = \sum_{l:0\le 2l-b\le p(n)-1} \Big(H_{2l-b} + (T_{-(2l+1-b)(r-1)} - T_{-(2l-b)(r-1)})\Big)$$

and $\phi^{(b)}_{x_k} = \phi^{(b)}_{x_k}(T,Z) = \sup_{n\ge l(x_k)}(\xi^{(b)}_{p(n)}(T,Z))$. Let $\phi^{(b)}(T,Z) = \sup_{p\ge1}(\xi^{(b)}_p(T,Z))$. Notice that $\xi^{(b)}_{p(n)}$ is a random walk with a negative drift and $\phi^{(b)}(T,Z)$ can be interpreted as the supremum of a random walk with a negative drift. Then the $(\nu_1+\nu_2)$th moment of $\phi^{(b)}$ is finite whenever $(\nu_1+\nu_2+1)$th moment of its increments is finite. But for $b = 0,1$

$$\begin{aligned}
& E\Big[\Big(H_{2l-b} + (T_{-(2l+1-b)(r-1)} - T_{-(2l-b)(r-1)})\Big)^{\nu_1+\nu_2+1}\Big] \\
\le\ & 2^{\nu_1+\nu_2+1}\Big(E[(H_0)^{\nu_1+\nu_2+1}] + (\nu_1+\nu_2+1)!\Big(\frac{r-1}{\lambda}\Big)^{\nu_1+\nu_2+1}\Big) < \infty
\end{aligned}$$

where we used the inequalities (5.37) and (5.38). On the other hand, if $x_k \in [T_{-l^*}, 0)$ then

$$\begin{aligned}
\sup_{n\geq l(x_k)} (D^j_{n-1+k} + T_{-(n-1)}) &= \max\{\max_{l(x_k)\leq n\leq l^*} (D^j_{n-1+k} + T_{-(n-1)}), \sup_{n>l^*} (D^j_{n-1+k} + T_{-(n-1)})\} \\
&\leq \max\{H_0 + \cdots + H_{kr-1}, \phi^{(0)}_{x_k} + \phi^{(1)}_{x_k}\} \\
&\leq (H_0 + \cdots + H_{kr-1}) + \phi^{(0)}_{x_k} + \phi^{(1)}_{x_k}. \qquad (5.47)
\end{aligned}$$

As a next step we develop an upper bound on $\max\{D^j_0, \sup_{n\geq 1}(D^j_{n+k} + T_{-n})\}$.

$$\begin{aligned}
\max\{D^j_0, \sup_{n\geq 1}(D^j_{n+k} + T_{-n})\} &\leq D^j_0 + \sup_{p\geq 0} \max_{1\leq q\leq r} (D^j_{(pr+q)+k} + T_{-(pr+q)}) \\
&\leq D^j_0 + \sup_{p\geq 0}(D^j_{(p+1)r+k} + T_{-(pr+1)})_+ \\
&\leq D^j_0 + \sup_{p\geq 0}(D^j_{(p+k+1)r} + T_{-(pr+1)})_+ \\
&\leq D^j_0 + \sup_{p\geq 0}(H_0 + H_1 + \cdots + H_{p+k} + T_{-pr})_+ \\
&\leq D^j_0 + H_0 + (-T_{-kr}) + \beta^{(0)}_k + \beta^{(1)}_k \qquad (5.48)
\end{aligned}$$

where for $b = 0, 1$

$$\beta^{(b)}_k = \sup_{p\geq 0}\Big(\sum_{l:1\leq 2l-b\leq p+k} (H_{2l-b} + (T_{-(2l-k-b)r} - T_{-(2l-k-b-1)r})\Big)_+.$$

Again $\beta^{(b)}_k$ is the supremum of a random walk with a negative drift. The $(\nu_1+\nu_2)$th moment of this supremum is finite as long as $(\nu_1 + \nu_2 + 1)$th moment of its increments is finite. But for $b = 0, 1$

$$\begin{aligned}
&E\Big[\Big(H_{2l-b} + (T_{-(2l-k-b)r} - T_{-(2l-k-b-1)r})\Big)^{\nu_1+\nu_2+1}\Big] \\
\leq\ & 2^{\nu_1+\nu_2+1}\Big(E[(H_0)^{\nu_1+\nu_2+1}] + (\nu_1+\nu_2+1)!\Big(\frac{r}{\lambda}\Big)^{\nu_1+\nu_2+1}\Big) < \infty.
\end{aligned}$$

We are now ready to put all the pieces together. First notice that the right hand side of equation (5.41) can be bounded above by splitting the outer most integral into two parts:

$$\begin{aligned}
&\int_t^0 \int_{x_k}^0 \cdots \int_{x_2}^0 \Big|\sum_{l=0}^k (-1)^{k-l} \sum_{\pi\in K_{k,l}} \psi\Big(T^t + \sum_{v\in\pi} \delta_{x_v}, Z\Big)\Big| dx_1 \cdots dx_k \\
\leq\ & 2^{k-1} c\Big\{\int_{T_{-l^*}}^0 \int_{x_k}^0 \cdots \int_{x_2}^0 \Big[\Big(\sup_{n\geq l(x_k)} (D^j_{n-1+k} + T_{-(n-1)})\Big)^{\nu_1+\nu_2} \\
&+\Big(\max\{D^j_0, \sup_{n\geq 1}(D^j_{n+k} + T_{-n})\}\Big)^{\nu_1}\Big(\sup_{n\geq l(x_k)} (D^j_{n-1+k} + T_{-(n-1)})\Big)^{\nu_2}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\sup_{n\geq l(x_k)}(D^j_{n-1+k}+T_{-(n-1)})\Big)^{\nu_1}\Big(\max\{D^j_0,\sup_{n\geq 1}(D^j_{n+k}+T_{-n})\}\Big)^{\nu_2}\Big]dx_1\cdots dx_k\\
+\;&\int_{-\infty}^{T_{-l^*}}\int_{x_k}^0\cdots\int_{x_2}^0\Big[\Big(\sup_{n\geq l(x_k)}(D^j_{n-1+k}+T_{-(n-1)})\Big)^{\nu_1+\nu_2}\\
&+\Big(\max\{D^j_0,\sup_{n\geq 1}(D^j_{n+k}+T_{-n})\}\Big)^{\nu_1}\Big(\sup_{n\geq l(x_k)}(D^j_{n-1+k}+T_{-(n-1)})\Big)^{\nu_2}\\
&+\Big(\sup_{n\geq l(x_k)}(D^j_{n-1+k}+T_{-(n-1)})\Big)^{\nu_1}\Big(\max\{D^j_0,\sup_{n\geq 1}(D^j_{n+k}+T_{-n})\}\Big)^{\nu_2}\Big]\\
&\mathbb{1}\Big(x_k:\sup_{n\geq l(x_k)}(D^j_{n-1+k}+T_{-(n-1)})>0\Big)dx_1\cdots dx_k\Big\}
\end{aligned}
\tag{5.49}
$$

Using (5.47), (5.48) and the inequalities (5.37) and (5.38), the first integral in (5.49) can be bounded by

$$
\frac{3(kr+2)^{\nu_1+\nu_2+1}+2\cdot 5^{\nu_1+\nu_2+1}}{k!}\Big(2(H_0)^{\nu_1+\nu_2+k}+\sum_{l=0}^{kr-1}(H_l)^{\nu_1+\nu_2+k}+\sum_{l=0}^{1}(\phi^{(l)})^{\nu_1+\nu_2+k}+\sum_{l=0}^{1}(\beta_k^{(l)})^{\nu_1+\nu_2+k}+(-T_{-kr})^{\nu_1+\nu_2+k}+(-T_{-l^*})^{\nu_1+\nu_2+k}\Big).
$$

Let

$$
\vartheta^{(b)}=\vartheta^{(b)}(T,Z)=\sup_{p\geq 1}\{-T_{-(p+1)(r-1)}:\xi_p^{(b)}(T,Z)>0\}
$$

for $b=0,1$. Then the second integral in (5.49) can be bounded by

$$
\begin{aligned}
&\Big\{\Big(\phi^{(0)}+\phi^{(1)}\Big)^{\nu_1+\nu_2}+\Big(D^j_0+H_0+\beta^{(0)}_k+\beta^{(1)}_k+(-T_{-kr})\Big)^{\nu_1}\Big(\phi^{(0)}+\phi^{(1)}\Big)^{\nu_2}+\\
&\Big(\phi^{(0)}+\phi^{(1)}\Big)^{\nu_1}\Big(D^j_0+H_0+\beta^{(0)}_k+\beta^{(1)}_k+(-T_{-kr})\Big)^{\nu_2}\Big\}\int_{-\infty}^0\int_{x_k}^0\cdots\int_{x_2}^0\mathbb{1}\Big(x_k:-x_k<\vartheta^{(0)}+\vartheta^{(1)}\Big)dx_1\cdots dx_k\\
=\;&\Big\{\Big(\phi^{(0)}+\phi^{(1)}\Big)^{\nu_1+\nu_2}+\Big(D^j_0+H_0+\beta^{(0)}_k+\beta^{(1)}_k+(-T_{-kr})\Big)^{\nu_1}\Big(\phi^{(0)}+\phi^{(1)}\Big)^{\nu_2}+\\
&\Big(\phi^{(0)}+\phi^{(1)}\Big)^{\nu_1}\Big(D^j_0+H_0+\beta^{(0)}_k+\beta^{(1)}_k+(-T_{-kr})\Big)^{\nu_2}\Big\}\frac{(\vartheta^{(0)}+\vartheta^{(1)})^k}{k!}\\
\leq\;&\frac{(3\cdot 2^{\nu_1+\nu_2+k}+2\cdot 5^{\nu_1+\nu_2+k})}{k!}\Big(\sum_{l=0}^{1}(\phi^{(l)})^{\nu_1+\nu_2+k}+\sum_{l=0}^{1}(\beta^{(l)}_k)^{\nu_1+\nu_2+k}+2(H_0)^{\nu_1+\nu_2+k}+\\
&(-T_{-kr})^{\nu_1+\nu_2+k}+\sum_{l=0}^{1}(\vartheta^{(l)})^{\nu_1+\nu_2+k}\Big)
\end{aligned}
$$

where the first bound follows from (5.46) and Lemma 7.6 of [13] and the inequality again follows from (5.37) and (5.38). Thus, we can conclude that

$$
\int_t^0\cdots\int_t^0\Big|\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\psi^{i,j}\Big(T^t+\sum_{j=1}^{l}\delta_{t_j},Z\Big)\Big|dt_1\cdots dt_k\leq\eta_k,
$$

where

$$\eta_k = \tilde{c}_k\Big(2(H_0)^{\nu_1+\nu_2+k} + \sum_{l=0}^{kr-1}(H_l)^{\nu_1+\nu_2+k} + \sum_{l=0}^{1}(\phi^{(l)})^{\nu_1+\nu_2+k} + \sum_{l=0}^{1}(\beta_k^{(l)})^{\nu_1+\nu_2+k} + (-T_{-kr})^{\nu_1+\nu_2+k} + (T_{-l^*})^{\nu_1+\nu_2+k} + \sum_{l=0}^{1}(\vartheta^{(l)})^{\nu_1+\nu_2+k}\Big)$$

with $\tilde{c}_k = 2^{k-1}c(3(kr+2)^{\nu_1+\nu_2+1} + 2\cdot 5^{\nu_1+\nu_2+k+1} + 3\cdot 2^{\nu_1+\nu_2+k})$. It follows from Lemma 9 of [6] that $(\nu_1+\nu_2+k)$th moment of $\vartheta^{(u)}(T,Z)$, $u=0,1$ is finite provided that $E(H_n)^{\nu_1+\nu_2+k+1} < \infty$. Hence $E\eta_k < \infty$, and η_k is a uniformly integrable bound for all $k \in \{1,\cdots,m+1\}$ as long as $E(H_n)^{\nu_1+\nu_2+m+2} < \infty$, which completes the proof of Lemma 5.5. □

Finally, the proof of continuity property follows from an argument similar to the one given in section 3.1.7 of [6]. Since conditions (5.28), (5.29), (5.31), (5.32) and (5.34) are satisfied it follows from Corollary 5.1 of [13] that the result of Proposition 5.1 holds.

5.2 Computation of the coefficients

Since we have shown that Proposition 5.1 can be applied to derive a Taylor series expansion for $EG(W^i, W^j)$, we now explicitly compute the coefficients of the expansion when

$$G(W^i, W^j) = e^{-s_1W^i - s_2W^j} \text{ for } i < j \text{ and } s_1, s_2 \geq 0. \tag{5.50}$$

We first provide a recursive representation of these coefficients and then use this result to provide their explicit form in a second step. Recall that

$$\psi^{i,j}\Big(T,Z\Big) = G\Big(W^i(T,Z), W^j(T,Z)\Big).$$

Let $q_1(D_0^i, D_0^j) = \psi^{i,j}(o,Z)$ and for $k \geq 1$

$$q_{k+1}(D_0^i,\cdots,D_k^i,D_0^j,\cdots,D_k^j) = \int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\psi^{i,j}_{x_1,\cdots,x_k}(o,Z)dx_k\cdots dx_1 \tag{5.51}$$

for $G(W^i, W^j)$ as defined in (5.50). Let

$$S_k = \{(i_1, i_2, \cdots, i_k) \in \{1,2,\cdots,k\}^k : i_1 \leq i_2 \leq \cdots \leq i_k, i_l \geq l, j = l, \cdots k-1, i_k = k\}$$

and let

$$\mathcal{X}(i_1, i_2, \cdots, i_k) = \begin{cases} 1 & \text{if } i_p = \max\{l : D_l^i - D_0^i \leq D_p^j - D_0^j\} \text{ for all } p = 1,\cdots,k \\ 0 & \text{otherwise} \end{cases}$$

Throughout our developments we set $i_p + 1 = i_{p+1}$ if $i_p = k$ for $p \leq k-1$.

Theorem 5.1 *Suppose that the monotonicity assumptions of Section 3 hold, then for $i, j \in \{1, \cdots, \alpha'\}$ and $i < j$, $q_1(D_0^i, D_0^j) = e^{-s_1 D_0^i - s_2 D_0^j}$ and for all $k \geq 1$*

$$q_{k+1}(D_0^i, D_1^i, \cdots, D_k^i, D_0^j, D_1^j, \cdots, D_k^j) = \sum_{(i_1,\cdots,i_k)\in S_k} \mathcal{X}(i_1, \cdots, i_k)$$

$$\Big(\sum_{p=0}^{k-1}\Big[\int_{D_p^j - D_0^j}^{D_{i_p+1}^i - D_0^i} \{q_k(\underbrace{D_0^i, \cdots, D_0^i}_{i_p}, D_{i_p+1}^i - u, \cdots, D_k^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u)$$

$$-q_k(\underbrace{D_0^i, \cdots, D_0^i}_{i_p+1}, D_{i_p+1}^i - u, \cdots, D_{k-1}^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\}du$$

$$+ \sum_{v=1}^{i_{p+1}-i_p-1} \int_{D_{i_p+v}^i - D_0^i}^{D_{i_p+v+1}^i - D_0^i} \{q_k(\underbrace{D_0^i, \cdots, D_0^i}_{i_p+v}, D_{i_p+v+1}^i - u, \cdots, D_k^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u)$$

$$-q_k(\underbrace{D_0^i, \cdots, D_0^i}_{i_p+v+1}, D_{i_p+v+1}^i - u, \cdots, D_{k-1}^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\}du$$

$$+ \int_{D_{i_{p+1}}^i - D_0^i}^{D_{p+1}^j - D_0^j} \{q_k(\underbrace{D_0^i, \cdots, D_0^i}_{i_{p+1}}, D_{i_{p+1}+1}^i - u, \cdots, D_k^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u)$$

$$-q_k(\underbrace{D_0^i, \cdots, D_0^i}_{i_{p+1}+1}, D_{i_{p+1}+1}^i - u, \cdots, D_{k-1}^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\}du\Big]\Big) \tag{5.52}$$

with the convention that summation over an empty set is zero and $i_0 = 0$.

Proof By definition

$$q_1(D_0^i, D_0^j) = e^{-s_1 D_0^i - s_2 D_0^j}.$$

Using (5.33), (5.36), (5.50) and (5.51) we have for $k \geq 1$

$$q_{k+1}(D_0^i, D_1^i, \cdots, D_k^i, D_0^j, D_1^j, \cdots, D_k^j) = \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty \sum_{l=0}^{k} (-1)^{k-l}$$

$$\sum_{\pi \in K_{k,l}} e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{l}(D_n^i - t_{\pi(n)})\right) - s_2\left(D_0^j \oplus \bigoplus_{n=1}^{l}(D_n^j - t_{\pi(n)})\right)} dt_k \cdots dt_1 \tag{5.53}$$

where $K_{k,l}$ again denotes the collection of subsets of $\{1, \cdots, k\}$ containing l elements and $\pi_{(n)}$ denotes the n^{th} order statistic of π, i.e., the n^{th} smallest element of the $l - tupel$ $\pi \in K_{k,l}$. Pick $(i_1, \cdots, i_k) \in S_k$ and suppose

$$i_p = \max\{n : D_n^i - D_0^i \leq D_p^j - D_0^j\}. \tag{5.54}$$

As a next step using the order in (5.54) we decompose the outer integral in (5.53) in the following way

$$\begin{aligned}\int_0^\infty \cdots &= \sum_{p=0}^{k-1}\Big(\int_{D_p^j - D_0^j}^{D_{i_p+1}^i - D_0^i} \cdots + \sum_{v=1}^{i_{p+1}-i_p-1} \int_{D_{i_p+v}^i - D_0^i}^{D_{i_p+v+1}^i - D_0^i} \cdots + \int_{D_{i_{p+1}}^i - D_0^i}^{D_{p+1}^j - D_0^j} \cdots\Big) + \int_{D_k^j - D_0^j}^{\infty} \cdots \\ &= \sum_{p=0}^{k-1}\Big(\int_{D_p^j - D_0^j}^{D_{i_p+1}^i - D_0^i} \cdots + \sum_{v=1}^{i_{p+1}-i_p-1} \int_{D_{i_p+v}^i - D_0^i}^{D_{i_p+v+1}^i - D_0^i} \cdots + \int_{D_{i_{p+1}}^i - D_0^i}^{D_{p+1}^j - D_0^j} \cdots\Big)\end{aligned} \tag{5.55}$$

with the convention that the summation over an empty set is zero. Since $t_1 \le t_2 \le \cdots \le t_k$ and $D_0^j \le \cdots \le D_k^j$ and $t_1 \ge D_k^j - D_0^j$, it follows that $t_m \ge t_1 \ge D_k^j - D_0^j \ge D_n^j - D_0^j \ge D_n^i - D_0^i$ and therefore $D_n^i - t_m \le D_0^i$ and $D_n^j - t_m \le D_0^j$ for all $1 \le m, n \le k$ and the integrand of the integral from $D_k^j - D_0^j$ to ∞ is

$$\sum_{l=0}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l}} e^{-s_1 D_0^i - s_2 D_0^j} = e^{-s_1 D_0^i - s_2 D_0^j}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l} = 0.$$

For each remaining integral in (5.55) we decompose the inner summation according to

$$\sum_{l=0}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l}}\cdots = \sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi}\cdots + \sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\notin\pi}\cdots \tag{5.56}$$

We first consider first of these sums for $0 \le p \le k-1$ for each integral in (5.55),

$$\int_{D_p^j - D_0^j}^{D_{i_p+1}^i - D_0^i}\cdots\int_{t_{k-1}}^{\infty}\sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi} e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{l}(D_n^i - t_{\pi(n)})\right) - s_2\left(D_0^j \oplus \bigoplus_{n=1}^{l}(D_n^j - t_{\pi(n)})\right)} dt_k\cdots dt_1 =$$

$$\int_{D_p^j - D_0^j}^{D_{i_p+1}^i - D_0^i}\cdots\int_{t_{k-1}}^{\infty}\sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi} e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{\min\{l,i_p\}}(D_n^i - t_{\pi(n)}) \oplus \bigoplus_{n=\min\{l,i_p\}+1}^{l}(D_n^i - t_{\pi(n)})\right)}$$

$$e^{-s_2\left(D_0^j \oplus \bigoplus_{n=1}^{\min\{l,p\}}(D_n^j - t_{\pi(n)}) \oplus \bigoplus_{n=\min\{l,p\}+1}^{l}(D_n^j - t_{\pi(n)})\right)} dt_k\cdots dt_1. \tag{5.57}$$

Since $t_1 \ge D_p^j - D_0^j \ge D_{i_p}^i - D_0^i$, it follows that $D_n^j - t_{\pi(n)} \le D_n^j - t_1 \le D_p^j - t_1 \le D_0^j$ for $n \le \min\{p, l\}$ and $D_n^i - t_{\pi(n)} \le D_n^i - t_1 \le D_{i_p}^i - t_1 \le D_0^i$ for $n \le \min\{i_p, l\}$. Substituting $t_{\pi(n)} \to t_{\pi(n)} - t_1$ in (5.57) yields the plus term in the first integral in (5.52) for the particular ordering considered in (5.54). For $v \ge 1$ we have $t_1 \ge D_p^j - D_0^j$ and hence $D_n^j - t_{\pi(n)} \le D_n^j - t_1 \le D_p^j - t_1 \le D_0^j$ for $n \le \min\{p, l\}$ still holds and since $t_1 \ge D_{i_p+v}^1 - D_0^i$, we have $D_n^i - t_{\pi(n)} \le D_n^i - t_1 \le D_{i_p+v}^i - t_1 \le D_0^i$ for $n \le \min\{l, i_p + v\}$. Thus, for $v \ge 1$

$$\int_{D_{i_p+v}^i - D_0^i}^{D_{i_p+v+1}^i - D_0^i}\cdots\int_{t_{k-1}}^{\infty}\sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi} e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{l}(D_n^i - t_{\pi(n)})\right) - s_2\left(D_0^j \oplus \bigoplus_{n=1}^{l}(D_n^j - t_{\pi(n)})\right)} dt_k\cdots dt_1 =$$

$$\int_{D_{i_p+v}^i - D_0^i}^{D_{i_p+v+1}^i - D_0^i}\cdots\int_{t_{k-1}}^{\infty}\sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi}\Big(e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{\min\{l,i_p+v\}}(D_n^i - t_{\pi(n)}) \oplus \bigoplus_{n=\min\{l,i_p+v\}+1}^{l}(D_n^i - t_{\pi(n)})\right)}$$

$$e^{-s_2\left(D_0^j \oplus \bigoplus_{n=1}^{\min\{l,p\}}(D_n^j - t_{\pi(n)}) \oplus \bigoplus_{n=\min\{l,p\}+1}^{l}(D_n^j - t_{\pi(n)})\right)}\Big) dt_k\cdots dt_1 \tag{5.58}$$

Again substituting $t_{\pi(n)} \to t_{\pi(n)} - t_1$ in (5.58) yields the plus term of the second integral in (5.52) for the particular ordering considered in (5.54). Similarly, for the last integral we have

$$\int_{D_{i_{p+1}}^i - D_0^i}^{D_{p+1}^j - D_0^j}\cdots\int_{t_{k-1}}^{\infty}\sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi} e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{l}(D_n^i - t_{\pi(n)})\right) - s_2\left(D_0^j \oplus \bigoplus_{n=1}^{l}(D_n^j - t_{\pi(n)})\right)} dt_k\cdots dt_1 =$$

$$\int_{D_{i_{p+1}}^i - D_0^i}^{D_{p+1}^j - D_0^j}\cdots\int_{t_{k-1}}^{\infty}\sum_{l=1}^{k}(-1)^{k-l}\sum_{\pi\in K_{k,l},1\in\pi}\Big(e^{-s_1\left(D_0^i \oplus \bigoplus_{n=1}^{\min\{l,i_{p+1}\}}(D_n^i - t_{\pi(n)}) \oplus \bigoplus_{n=\min\{l,i_{p+1}\}+1}^{l}(D_n^i - t_{\pi(n)})\right)}$$

$$e^{-s_2\left(D_0^j \oplus \bigoplus_{n=1}^{\min\{l,p\}}(D_n^j - t_{\pi(n)}) \oplus \bigoplus_{n=\min\{l,p\}+1}^{l}(D_n^j - t_{\pi(n)})\right)}\Big) dt_k\cdots dt_1 \tag{5.59}$$

and substituting $t_{\pi_{(n)}} \to t_{\pi_{(n)}} - t_1$ in (5.59) gives the plus term of the last integral in (5.52) for the particular ordering considered in (5.54). The minus terms in (5.52) follow analogously from the second summation in (5.56). $\square$

We now show that the expression for $q_{k+1}(D_0^i, \cdots, D_k^i, D_0^j, \cdots, D_k^j)$ does not depend on $(i_1, \cdots, i_k) \in S_k$, $\forall k \geq 1$ as long as the monotonicity assumptions of Section 3 are satisfied. Let $T_k^{(i_1,\cdots,i_k)}$ be the linear transformation given in (5.52) for $(i_1, \cdots, i_k) \in S_k$. Hence, for $f : \mathbb{R}^{2k} \to \mathbb{R}$

$$\begin{aligned}
&T_k^{(i_1,\cdots,i_k)}(f(D_0^i, \cdots, D_{k-1}^i, D_0^j, \cdots, D_{k-1}^j)) = \\
&\sum_{p=0}^{k-1}\Big[\int_{D_p^j - D_0^j}^{D_{i_p+1}^i - D_0^i} \{f(\underbrace{D_0^i, \cdots, D_0^i}_{i_p}, D_{i_p+1}^i - u, \cdots, D_k^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u) \\
&\qquad - f(\underbrace{D_0^i, \cdots, D_0^i}_{i_p+1}, D_{i_p+1}^i - u, \cdots, D_{k-1}^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\}du \\
&\quad + \sum_{l=1}^{i_{p+1}-i_p-1} \int_{D_{i_p+l}^i - D_0^i}^{D_{i_p+l+1}^i - D_0^i} \{f(\underbrace{D_0^i, \cdots, D_0^i}_{i_p+l}, D_{i_p+l+1}^i - u, \cdots, D_k^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u) \\
&\qquad - f(\underbrace{D_0^i, \cdots, D_0^i}_{i_p+l+1}, D_{i_p+l+1}^i - u, \cdots, D_{k-1}^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\}du \\
&\quad + \int_{D_{i_{p+1}}^i - D_0^i}^{D_{p+1}^j - D_0^j} \{f(\underbrace{D_0^i, \cdots, D_0^i}_{i_{p+1}}, D_{i_{p+1}+1}^i - u, \cdots, D_k^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u) \\
&\qquad - f(\underbrace{D_0^i, \cdots, D_0^i}_{i_{p+1}+1}, D_{i_{p+1}+1}^i - u, \cdots, D_{k-1}^i - u, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\}du\Big]. \quad (5.60)
\end{aligned}$$

The following lemma says that $T_k^{(i_1,\cdots,i_k)}$ does not depend on $(i_1, \cdots, i_k)$ under the monotonicity assumptions.

Lemma 5.6 *Let $f : \mathbb{R}^{2k} \to \mathbb{R}$ be such that*

$$f(D_0^i, \cdots, D_{k-1}^i, D_0^j, \cdots, D_{k-1}^j) = h(D_{v_1}^i, D_{v_2}^i, \cdots, D_{v_m}^i, D_{w_1}^j, \cdots, D_{w_n}^j) \quad (5.61)$$

where $h : \mathbb{R}^{n+m} \to \mathbb{R}$, $n + m \leq 2k$ and $0 \leq v_1 < v_2 < \cdots < v_m \leq w_1 < w_2 < \cdots < w_n \leq k-1$. Then

$$\begin{aligned}
T_k^{(i_1,\cdots,i_k)}(f(D_0^i, \cdots, D_{k-1}^i, D_0^j, \cdots, D_{k-1}^j)) &= T_k^{(i'_1,\cdots,i'_k)}(f(D_0^i, \cdots, D_{k-1}^i, D_0^j, \cdots, D_{k-1}^j)) \\
&\textit{for any } (i_1, \cdots, i_k), (i'_1, \cdots, i'_k) \in S_k
\end{aligned}$$

Proof Without loss of generality, we assume that $v_1 \neq 0$ and $w_1 \neq 0$. If $v_1 = 0$, we have $D_{v_1}^i - u = D_0^i$ in the integrals below. Similarly, if $v_1 = 0$ and $m = 1$ and $w_1 = v_m$, we have $D_{w_1}^j - u = D_0^j$. It follows from (5.60) and (5.61) that with $v_0 = 0$ and with the convention that the summation over an empty set is zero

$$\begin{aligned}
&T_k^{(i_1,\cdots,i_k)}(f(D_0^i, \cdots, D_{k-1}^i, D_0^j, \cdots, D_{k-1}^j)) = \\
&\int_0^{D_{v_1+1}^i - D_0^i} h(D_{v_1+1}^i - u, D_{v_2+1}^i - u, \cdots, D_{v_m+1}^i - u, D_{w_1+1}^j - u, \cdots, D_{w_n+1}^j - u)du +
\end{aligned}$$

$$\sum_{p=1}^{m-1}\int_{D^i_{v_{p+1}}-D^i_0}^{D^i_{v_{p+1}+1}-D^i_0} h(\underbrace{D^i_0,\cdots,D^i_0}_{p},D^i_{v_{p+1}+1}-u,\cdots,D^i_{v_m+1}-u,D^j_{w_1+1}-u,\cdots,D^j_{w_n+1}-u)du+$$
$$\int_{D^i_{v_m+1}-D^i_0}^{D^j_{w_1+1}-D^j_0} h(D^i_0,\cdots,D^i_0,D^j_{w_1+1}-u,\cdots,D^j_{w_n+1}-u)du+$$
$$\sum_{p=0}^{n-2}\int_{D^j_{w_{p+1}+1}-D^j_0}^{D^j_{w_{p+2}+1}-D^j_0} h(D^i_0,\cdots,D^i_0,\underbrace{D^j_0,\cdots,D^j_0}_{p+1},D^j_{w_{p+2}+1}-u,\cdots,D^j_{w_n+1}-u)du+$$
$$\int_{D^j_{w_n+1}-D^j_0}^{D^j_k-D^j_0} h(D^i_0,\cdots,D^i_0,D^j_0,\cdots,D^j_0)du$$
$$-\quad\Big\{\sum_{p=0}^{m-1}\int_{D^i_{v_p}-D^i_0}^{D^i_{v_{p+1}}-D^i_0} h(\underbrace{D^i_0,\cdots,D^i_0}_{p},D^i_{v_{p+1}}-u,\cdots,D^i_{v_m}-u,D^j_{w_1}-u,\cdots,D^j_{w_n}-u)du+$$
$$\int_{D^i_{v_m}-D^i_0}^{D^j_{w_1}-D^j_0} h(D^i_0,\cdots,D^i_0,D^j_{w_1}-u,\cdots,D^j_{w_n}-u)du+$$
$$\sum_{p=0}^{n-2}\int_{D^j_{w_{p+1}}-D^j_0}^{D^j_{w_{p+2}}-D^j_0} h(D^i_0,\cdots,D^i_0,\underbrace{D^j_0,\cdots,D^j_0}_{p+1},D^j_{w_{p+2}}-u,\cdots,D^j_{w_n}-u)du+$$
$$\int_{D^j_{w_n}-D^j_0}^{D^j_k-D^j_0} h(D^i_0,\cdots,D^i_0,D^j_0,\cdots,D^j_0)du\Big\}$$
$$=\quad \mathcal{H}_{0,1}(D^i_0,D^i_{v_1+1},D^i_{v_2+1},\cdots,D^i_{v_m+1},D^j_{w_1+1},\cdots,D^j_{w_n+1})-\mathcal{H}_{0,2}(D^i_{v_1+1},D^i_{v_2+1},\cdots,D^i_{v_m+1},D^j_{w_1+1},\cdots,D^j_{w_n+1})+$$
$$\sum_{p=1}^{m-1}(\mathcal{H}_{p,1}(D^i_0,D^i_{v_{p+1}+1},\cdots,D^i_{v_m+1},D^j_{w_1+1},\cdots,D^j_{w_n+1})-\mathcal{H}_{p,2}(D^i_0,D^i_{v_p+1},\cdots,D^i_{v_m+1},D^j_{w_1+1},\cdots,D^j_{w_n+1}))+$$
$$\mathcal{H}_{m,1}(D^i_0,D^j_0,D^j_{w_1+1},\cdots,D^j_{w_n+1})-\mathcal{H}_{m,2}(D^i_0,D^i_{v_m+1},D^j_{w_1+1},\cdots,D^j_{w_n+1})+$$
$$\sum_{p=0}^{n-l-1}(\mathcal{H}_{m+1+p,1}(D^i_0,D^j_0,D^j_{w_{p+2}+1},\cdots,D^j_{w_n+1})-\mathcal{H}_{m+1+p,2}(D^i_0,D^j_0,D^j_{w_{p+1}+1},\cdots,D^j_{w_n+1}))+$$
$$\mathcal{H}_{m+n-l+1,1}(D^i_0,D^j_0,D^j_k)-\mathcal{H}_{m+n-l+1,2}(D^i_0,D^j_0,D^j_{w_n+1})$$
$$-\quad\Big\{\mathcal{H}_{0,1}(D^i_0,D^i_{v_1},D^i_{v_2},\cdots,D^i_{v_m},D^j_{w_1},\cdots,D^j_{w_n})-\mathcal{H}_{0,2}(D^i_{v_1},D^i_{v_2},\cdots,D^i_{v_m},D^j_{w_1},\cdots,D^j_{w_n})+$$
$$\sum_{p=1}^{m-1}(\mathcal{H}_{p,1}(D^i_0,D^i_{v_{p+1}},\cdots,D^i_{v_m},D^j_{w_1},\cdots,D^j_{w_n})-\mathcal{H}_{p,2}(D^i_0,D^i_{v_p},\cdots,D^i_{v_m},D^j_{w_1},\cdots,D^j_{w_n}))+$$
$$\mathcal{H}_{m,1}(D^i_0,D^j_0,D^j_{w_1},\cdots,D^j_{w_n})-\mathcal{H}_{m,2}(D^i_0,D^i_{v_m},D^j_{w_1},\cdots,D^j_{w_n})+$$
$$\sum_{p=0}^{n-l-1}(\mathcal{H}_{m+1+p,1}(D^i_0,D^j_0,D^j_{w_{p+2}},\cdots,D^j_{w_n})-\mathcal{H}_{m+1+p,2}(D^i_0,D^j_0,D^j_{w_{p+1}},\cdots,D^j_{w_n}))+$$
$$\mathcal{H}_{m+n-l+1,1}(D^i_0,D^j_0,D^j_k)-\mathcal{H}_{m+n-l+1,j}(D^i_0,D^j_0,D^j_{w_n})\Big\} \tag{5.62}$$

where $\mathcal{H}_{p,r}(\cdots)$ correspond to the value of the integrals evaluated at the end points. Since (5.62) does not depend on $(i_1,\cdots,i_k)$, we get the desired the result. Moreover, it can be seen from (5.62) that $T_k^{(i_1,\cdots,i_k)}(f(D^i_0,\cdots,D^i_{k-1},D^j_0,\cdots,D^j_{k-1}))$ is written as the sum of functions of the form given in (5.61) (i.e. sum of functions whose parameters are ordered whenever the monotonicity assumptions hold). □

We can now state our next theorem.

Theorem 5.2 *Suppose that the monotonicity assumptions of Section 3 hold. Then for each $i, j \in \{1, \cdots, \alpha'\}$ and $i < j$, we have*

$$
\begin{aligned}
&q_{k+1}(D_0^i, \cdots, D_k^i, D_0^j, \cdots, D_k^j) = \\
&\sum_{p=0}^{k-1} \int_{D_p^i - D_0^i}^{D_{p+1}^i - D_0^i} \{q_k(\underbrace{D_0^i, \cdots, D_0^i}_{p}, D_{p+1}^i - u, \cdots, D_k^i - u, D_1^j - u \cdots, D_k^j - u) \\
&\qquad - q_k(\underbrace{D_0^i, \cdots, D_0^i}_{p+1}, D_{p+1}^i - u, \cdots, D_{k-1}^i - u, D_0^j, D_1^j - u, \cdots, D_{k-1}^j - u)\} du \\
&+ \int_{D_k^i - D_0^i}^{D_1^j - D_0^j} \{q_k(\underbrace{D_0^i, \cdots, D_0^i}_{k}, D_1^j - u, \cdots, D_k^j - u) - q_k(\underbrace{D_0^i, \cdots, D_0^i}_{k}, D_0^j, D_1^j - u, \cdots, D_{k-1}^j - u)\} du \\
&+ \sum_{p=1}^{k-1} \int_{D_p^j - D_0^j}^{D_{p+1}^j - D_0^j} \{q_k(\underbrace{D_0^i, \cdots, D_0^i}_{k}, \underbrace{D_0^j, \cdots, D_0^j}_{p}, D_{p+1}^j - u, \cdots, D_k^j - u) \\
&\qquad - q_k(\underbrace{D_0^i, \cdots, D_0^i}_{k}, \underbrace{D_0^j, \cdots, D_0^j}_{p+1}, D_{p+1}^j - u, \cdots, D_{k-1}^j - u)\} du
\end{aligned}
\tag{5.63}
$$

for all $k \geq 1$.

Proof Since $q_1(D_0^i, D_0^j) = e^{-s_1 D_0^i - s_2 D_0^j}$, it follows from Lemma 5.6 and its proof with an induction argument that for all $k \geq 1$ the expression for $q_{k+1}(D_0^i, \cdots, D_k^i, D_0^j, \cdots, D_k^j)$ does not depend on $(i_1, \cdots, i_k) \in S_k$. For the sake of simplicity we can choose $i_1 = i_2 = \cdots = i_k = k$, for all $k \geq 1$ and obtain the recursive relationship in (5.63) from (5.52). □

Finally, we provide an explicit representation of the coefficients $q_k(\cdots)$, $k \geq 1$ which show up in the Taylor series expansion of $E[e^{-s_1 W^i - s_2 W^j}]$ in Theorem 3.1.

Theorem 5.3 *For $k \geq 1$ and $d_0^i \leq d_0^j$, $0 \leq d_n^i - d_0^i \leq d_n^j - d_0^j$ for $i < j$, a solution to the integral recursive equation (5.63) is given by the functions $q_k(\cdots)$ defined in (3.10).*

Proof is given in the Appendix.

Note that the existence of a Taylor series expansion for $E[W^i W^j]$ under the assumptions of Corollary 3.1 follows from section 5.1. The explicit expression of the coefficients $q_k'(\cdots)$ for $k \geq 1$ is straightforward to obtain from $q_k(\cdots)$.

6 Appendix–Proof of Theorem 5.3

We use induction. Recall that $F^{[l,m]}(x, y) = (-1)^{l+m} \frac{e^{-s_1 x - s_2 y}}{(s_1+s_2)^l s_2^m}$. First suppose $k = 1$. Then

$$
q_2(d_0^i, d_1^i, d_0^j, d_1^j) = \int_0^{d_1^i - d_0^i} \{q_1(d_1^i - u, d_1^j - u) - q_1(d_0^i, d_0^j)\} du + \int_{d_1^i - d_0^i}^{d_1^j - d_0^j} \{q_1(d_0^i, d_1^j - u) - q_1(d_0^i, d_0^j)\} du
$$

$$= \int_0^{d_1^i - d_0^i} \{F^{[0,0]}(d_1^i - u, d_1^j - u) - F^{[0,0]}(d_0^i, d_0^j)\} du + \int_{d_1^i - d_0^i}^{d_1^j - d_0^j} \{F^{[0,0]}(d_0^i, d_1^j - u) - F^{[0,0]}(d_0^i, d_0^j)\} du$$
$$= -F^{[0,1]}(d_0^i, d_0^j) + F^{[1,0]}(d_1^i, d_1^j) - (F^{[1,0]} - F^{[0,1]})(d_0^i, d_1^j - d_1^i + d_0^i) - F^{[0,0]}(d_0^i, d_0^j)(p_1(d_1^j) - p_1(d_0^j))$$

which satisfies the expression in (3.10) for $k = 1$. Similarly,

$$\begin{aligned} q_3(d_0^i, d_1^i, d_2^i, d_0^j, d_1^j, d_2^j) \;=\; & F^{[0,2]}(d_0^i, d_0^j) - (F^{[1,1]} + F^{[2,0]})(d_1^i, d_1^j) + F^{[2,0]}(d_2^i, d_2^j) + \\ & (F^{[0,2]} - F^{[1,1]})(d_0^i, d_2^j - d_2^i + d_0^i) + (F^{[1,1]} - F^{[2,0]})(d_1^i, d_2^j - d_2^i + d_0^i) + \\ & (F^{[2,0]} + F^{[1,1]} - 2F^{[0,2]})(d_0^i, d_1^j - d_1^i + d_0^i) - F^{[0,0]}(d_0^i, d_0^j)(p_2(d_1^j, d_2^j) - p_2(d_0^j, d_1^j)) + \\ & F^{[0,1]}(d_0^i, d_0^j)(p_1(d_1^j) - p_1(d_0^j)) - F^{[1,0]}(d_1^i, d_1^j)(p_1(d_2^j) - p_1(d_1^j)) + \\ & (F^{[0,1]} - F^{[1,0]})(d_0^i, d_2^j - d_2^i + d_0^i)(p_1(d_2^i) - p_1(d_1^i)) - \\ & (F^{[0,1]} - F^{[1,0]})(d_0^i, d_1^j - d_1^i + d_0^i)(p_1(d_1^i) - p_1(d_0^i) + p_1(d_2^j) - p_1(d_1^j)) \end{aligned}$$

which again satisfies the expression in (3.10) for $k = 2$. In the following k will always be assumed to be greater than two, i.e. $k \geq 3$. It follows from Theorem 5.2 that

$$q_{k+1}(d_0^i, d_1^i, \cdots, d_k^i, d_0^j, d_1^j, \cdots, d_k^j) =$$
$$\sum_{p=0}^{k-1} \int_{d_p^i - d_0^i}^{d_{p+1}^i - d_0^i} q_k(\underbrace{d_0^i, \cdots, d_0^i}_{p}, d_{p+1}^i - u, \cdots, d_k^i - u, d_1^j - u, d_2^j - u, \cdots, d_k^j - u) du \quad (6.64)$$
$$- \sum_{p=0}^{k-1} \int_{d_p^i - d_0^i}^{d_{p+1}^i - d_0^i} q_k(\underbrace{d_0^i, \cdots, d_0^i}_{p+1}, d_{p+1}^i - u, \cdots, d_{k-1}^i - u, d_0^j, d_1^j - u, \cdots, d_{k-1}^j - u) du + \quad (6.65)$$
$$\int_{d_k^i - d_0^i}^{d_1^j - d_0^j} q_k(\underbrace{d_0^i, \cdots, d_0^i}_{k}, d_1^j - u, d_2^j - u, \cdots, d_k^j - u) du \quad (6.66)$$
$$- \int_{d_k^i - d_0^i}^{d_1^j - d_0^j} q_k(\underbrace{d_0^i, \cdots, d_0^i}_{k}, d_0^j, d_1^j - u, \cdots, d_{k-1}^j - u) du + \quad (6.67)$$
$$\sum_{p=1}^{k-1} \int_{d_p^j - d_0^j}^{d_{p+1}^j - d_0^j} q_k(\underbrace{d_0^i, \cdots, d_0^i}_{k}, \underbrace{d_0^j, \cdots, d_0^j}_{p}, d_{p+1}^j - u, \cdots, d_k^j - u) du \quad (6.68)$$
$$- \sum_{p=1}^{k-1} \int_{d_p^j - d_0^j}^{d_{p+1}^j - d_0^j} q_k(\underbrace{d_0^i, \cdots, d_0^i}_{k}, \underbrace{d_0^j, \cdots, d_0^j}_{p+1}, d_{p+1}^j - u, \cdots, d_{k-1}^j - u) du. \quad (6.69)$$

In what follows we will evaluate (6.64) to (6.69) separately. Now assume that the expression in (3.10) is correct for some $k \geq 3$, i.e. $q_k(\cdots)$ satisfies the expression in (3.10). Then changing the order of summation and using the 1-invariance property (see Baccelli and Schmidt [8]) of $p_k(\cdots)$ and $g_{k,l,m,n}(\cdots)$ and the fact that if $d_l^i = d_{l+1}^i = \cdots = d_m^i$ then

$$g_{k,l,m,n}(d_l^i, \cdots, d_m^i, d_m^j, \cdots, d_k) = \begin{cases} 0 & \text{if } m > n \\ (-1)^{k-m+1}\{p_k(d_{m+1}^j, \cdots, d_{k+m}^j) - \\ p_k(d_m^j, \cdots, d_{k+m-1}^j)\} & \text{if } m = n \end{cases}$$

with some algebra we obtain (6.64) as

$$\sum_{p=0}^{k-1}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i} q_k(\underbrace{d_0^i,\cdots,d_0^i}_{p},d_{p+1}^i-u,\cdots,d_k^i-u,d_1^j-u,d_2^j-u,\cdots,d_k^j-u)du \tag{6.70}$$

$$= \sum_{n=0}^{k-2}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)-F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_k^i+d_0^i)\Big)+ \tag{6.71}$$

$$\sum_{n=0}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v+1,k-v-1]}(d_{n+1}^i,d_{n+1}^j)-F^{[v+1,k-v-1]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i))\Big)+ \tag{6.72}$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+ (-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)- [(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_{n+1}^j-d_k^i+d_0^i)\Big)+ \tag{6.73}$$

$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big[(-1)^{k-1-n}F^{[k-1-v+l,v-l]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)+ (-1)^{k-n}F^{[k-v+l,v-l-1]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)\Big]\Big(p_1(d_{n+1}^i)-p_1(d_{l+1}^i)\Big)+ \tag{6.74}$$

$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-v+l,v-l]}+ (-1)^{k-n}F^{[k-v+l+1,v-l-1]}](d_{l+1}^i,d_{n+1}^j-d_{n+1}^i+d_{l+1}^i)- [(-1)^{k-1-n}F^{[k-v+l,v-l]}+(-1)^{k-n}F^{[k-v+l+1,v-l-1]}](d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)\Big)+ \tag{6.75}$$

$$\sum_{n=0}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v,b-v+1]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)-F^{[v,b-v+1]}(d_0^i,d_{n+1}^j-d_k^i+d_0^i)\Big) \Big\{p_{k-b-1}(d_{n+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(d_{n+1}^j,\cdots,d_{n+k-b-1}^j)\Big\}+ \tag{6.76}$$

$$\sum_{n=0}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v+1,b-v]}(d_{n+1}^i,d_{n+1}^j)-F^{[v+1,b-v]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)\Big) \Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(d_{n+1}^j,\cdots,d_{n+k-b-1}^j)\Big\}+$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)- [F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_{n+1}^j-d_k^i+d_0^i)\Big) \Big\{p_{k-b-1}(d_{n+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(d_{n+1}^j,\cdots,d_{n+k-b-1}^j)\Big\}+ \tag{6.77}$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b-1}(-1)^{k-m-1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}(d_0^i,d_{m+1}^j-d_{m+1}^i+d_0^i)- F^{[b-v+l+1,v-l-1]}(d_0^i,d_{m+1}^j-d_{m+1}^i+d_0^i)\Big]$$

$$\Big[\sum_{p=n+1}^{m}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\{p_{m-n}(d_{m+1}^i,\underbrace{d_0^i,\cdots,d_0^i}_{p-l-1},d_{p+1}^i-u,\cdots,d_{l+m-n}^i-u)-$$

$$p_{m-n}(\underbrace{d_0^i,\cdots,d_0^i}_{p-l-1},d_{p+1}^i-u,\cdots,d_{l+m-n-1}^i-u)\}\Big(\frac{(d_{m+1}^i-d_0^i-u)^{k-m+n-b-1}}{(k-m+n-b-1)!}-$$

$$\sum_{r=1}^{k-m+n-b-1}\frac{(-1)^r(d_{m+1}^i-d_0^i-u)^{k-m+n-b-r-1}}{(k-m+n-b-r-1)!}\{p_r(d_{m+2}^j,\cdots,d_{m+r+1}^j)-p_r(d_{m+1}^j,\cdots,d_{m+r}^j)\}\Big)du\Big] \quad (6.78)$$

$$+\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b-1}(-1)^{k-m-1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big(F^{[b-v+l,v-l]}(d_0^i,d_{m+1}^j-d_{m+1}^i+d_0^i)-$$

$$F^{[b-v+l+1,v-l-1]}(d_0^i,d_{m+1}^j-d_{m+1}^i+d_0^i)\Big)$$

$$\Big[\sum_{p=l+1}^{n}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\{p_{m-n}(d_{m+1}^i,\underbrace{d_0^i,\cdots,d_0^i}_{p-l-1},d_{p+1}^i-u,\cdots,d_{l+m-n}^i-u)-$$

$$p_{m-n}(\underbrace{d_0^i,\cdots,d_0^i}_{p-l-1},d_{p+1}^i-u,\cdots,d_{l+m-n-1}^i-u)\}\Big(\frac{(d_m^i-d_0^i-u)^{k-m+n-b-1}}{(k-m+n-b-1)!}-$$

$$\sum_{r=1}^{k-m+n-b-1}\frac{(-1)^r(d_{m+1}^i-d_0^i-u)^{k-m+n-b-r-1}}{(k-m+n-b-r-1)!}\{p_r(d_{m+2}^j,\cdots,d_{m+r+1}^j)-p_r(d_{m+1}^j,\cdots,d_{m+r}^j)\}\Big)du\Big] \quad (6.79)$$

$$+\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n}^{n+k-b-1}(-1)^{k-m-1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}$$

$$\Big([F^{[b-v+l+1,v-l]}-F^{[b-v+l+2,v-l-1]}](d_0^i,d_{m+1}^j-d_{m+1}^i+d_0^i)-$$

$$[F^{[b-v+l+1,v-l]}-F^{[b-v+l+2,v-l-1]}](d_{l+1}^i,d_{m+1}^j-d_{m+1}^i+d_{l+1}^i)\Big)$$

$$g_{k-b+n,l+1,m+1,n+1}(d_{l+1}^i,\cdots,d_{m+1}^i,d_{m+1}^j,\cdots,d_{n+k-b}^j). \quad (6.80)$$

Similarly, (6.66) can be obtained as

$$\int_{d_k^i-d_0^i}^{d_1^j-d_0^j}q_k(\underbrace{d_0^i,\cdots,d_0^i}_{k},d_1^j-u,d_2^j-u,\cdots,d_k^j-u)du=$$

$$\sum_{n=0}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_k^i+d_0^i)-F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_1^j+d_0^j)\Big)+ \quad (6.81)$$

$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-v+l-1,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_{n+1}^j-d_k^i+d_0^i)$$

$$-[(-1)^{k-1-n}F^{[k-v+l-1,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_{n+1}^j-d_1^j+d_0^j)\Big)+ \quad (6.82)$$

$$\sum_{n=0}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v,b-v+1]}(d_0^i,d_{n+1}^j-d_k^i+d_0^i)-F^{[v,b-v+1]}(d_0^i,d_{n+1}^j-d_1^j+d_0^j)\Big)$$

$$\Big\{p_{k-b-1}(d_{n+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(d_{n+1}^j,\cdots,d_{n+k-b-1}^j)\Big\}+ \quad (6.83)$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_{n+1}^j-d_k^i+d_0^i)$$

$$-[F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_{n+1}^j-d_1^j+d_0^j)\Big)$$

$$\Big\{p_{k-b-1}(d_{n+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(d_{n+1}^j,\cdots,d_{n+k-b-1}^j)\Big\}. \quad (6.84)$$

Finally, to complete the plus terms in (5.63) we consider (6.68). Note that using the recursive integral relationship of the polynomials (see Baccelli and Schmidt [8]) we can easily show that the following equality holds

$$\begin{aligned} & \sum_{p=n}^{k-2}\int_{d_p^j}^{d_{p+1}^j}\{p_{k-b-1}(\underbrace{u,\cdots,u}_{p-n},d_{p+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(\underbrace{u,\cdots,u}_{p-n+1},d_{p+2}^j,\cdots,d_{n+k-b-1}^j)\}du \\ = & \sum_{p=0}^{k-2-n}\int_{d_{p+n}^j}^{d_{p+n+1}^j}\{p_{k-b-1}(\underbrace{u,\cdots,u}_{p},d_{p+n+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(\underbrace{u,\cdots,u}_{p+1},d_{p+n+2}^j,\cdots,d_{k+n-b-1}^j)\}du \\ = & p_{k-b}(d_{n+1}^j,\cdots,d_{k+n-b}^j). \end{aligned}$$

Using the above relationship with some algebra we obtain (6.68) as

$$\sum_{p=1}^{k-1}\int_{d_p^j-d_0^j}^{d_{p+1}^j-d_0^j} q_k(\underbrace{d_0^i,\cdots,d_0^i}_{k},\underbrace{d_0^j,\cdots,d_0^j}_{p},d_{p+1}^j-u,\cdots,d_k^j-u)du \tag{6.85}$$

$$= \sum_{n=0}^{k-2}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}F^{[v,k-v-1]}(d_0^i,d_0^j)\Big(p_1(d_k^j)-p_1(d_{n+1}^j)\Big)+ \tag{6.86}$$

$$\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_1^j+d_0^j)-F^{[v,k-v]}(d_0^i,d_0^j)\Big)+ \tag{6.87}$$

$$\begin{aligned} & \sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big[(-1)^{k-1-n}F^{[k-1-v+l,v-l]}+ \\ & \qquad (-1)^{k-n}F^{[k-v+l,v-l-1]}\Big](d_0^i,d_0^j)\Big(p_1(d_k^j)-p_1(d_{n+1}^j)\Big)+ \end{aligned} \tag{6.88}$$

$$\begin{aligned} & \sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}(d_0^i,d_{n+1}^j-d_{l+1}^j+d_0^j)+ \\ & (-1)^{k-n}F^{[k-v+l,v-l]}(d_0^i,d_{n+1}^j-d_{l+1}^j+d_0^j)]-[(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}(d_0^i,d_0^j)+ \\ & \qquad (-1)^{k-n}F^{[k-v+l,v-l]}(d_0^i,d_0^j)]\Big)+ \end{aligned} \tag{6.89}$$

$$\begin{aligned} & \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}(d_0^i,d_{n+1}^j-d_1^j+d_0^j) \\ & \qquad +(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_{n+1}^j-d_1^j+d_0^j)]- \\ & [(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_{n+1}^j-d_{l+1}^j+d_0^j)\Big)+ \end{aligned} \tag{6.90}$$

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d_0^i,d_0^j)p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)+ \tag{6.91}$$

$$\begin{aligned} & \sum_{n=0}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v,b-v+1]}(d_0^i,d_{n+1}^j-d_1^j+d_0^j)-F^{[v,b-v+1]}(d_0^i,d_0^j)\Big) \\ & \qquad \Big\{p_{k-b-1}(d_{n+2}^j,\cdots,d_{k-b+n}^j)-p_{k-b-1}(d_{n+1}^j,\cdots,d_{n+k-b-1}^j)\Big\}+ \end{aligned} \tag{6.92}$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}\Big](d_0^i,d_0^j)$$

$$p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n})+ \tag{6.93}$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d^i_0,d^j_{n+1}-d^j_{l+1}+d^j_0)-$$
$$[F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d^i_0,d^j_0)\Big)$$
$$\Big\{p_{k-b-1}(d^j_{n+2},\cdots,d^j_{k-b+n})-p_{k-b-1}(d^j_{n+1},\cdots,d^j_{n+k-b-1})\Big\}+ \tag{6.94}$$

$$\sum_{l=1}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d^i_0,d^j_{n+1}-d^j_1+d^j_0)-$$
$$[F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d^i_0,d^j_{n+1}-d^j_{l+1}+d^j_0)\Big)$$
$$\Big\{p_{k-b-1}(d^j_{n+2},\cdots,d^j_{k-b+n})-p_{k-b-1}(d^j_{n+1},\cdots,d^j_{n+k-b-1})\Big\}. \tag{6.95}$$

With similar analysis and with some algebra we have (6.65) as

$$\sum_{p=0}^{k-1}\int_{d^i_p-d^i_0}^{d^i_{p+1}-d^i_0} q_k(\underbrace{d^i_0,\cdots,d^i_0}_{p+1},d^i_{p+1}-u,\cdots,d^i_{k-1}-u,d^j_0,d^j_1-u,\cdots,d^j_{k-1}-u)du \tag{6.96}$$

$$= (-1)^{k-1}F^{[0,k-1]}(d^i_0,d^j_0)(d^i_k-d^i_0)+$$
$$\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d^i_0,d^j_n-d^i_n+d^i_0)-F^{[v,k-v]}(d^i_0,d^j_n-d^i_k+d^i_0)\Big)+ \tag{6.97}$$
$$\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v+1,k-v-1]}(d^i_n,d^j_n)-F^{[v+1,k-v-1]}(d^i_0,d^j_n-d^i_n+d^i_0)\Big)+ \tag{6.98}$$
$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d^i_0,d^j_n-d^i_n+d^i_0)$$
$$-[(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d^i_0,d^j_n-d^i_k+d^i_0)\Big)+ \tag{6.99}$$
$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big[(-1)^{k-1-n}F^{[k-1-v+l,v-l]}+$$
$$(-1)^{k-n}F^{[k-v+l+,v-l-1]}\Big](d^i_0,d^j_n-d^i_n+d^i_0)\Big(p_1(d^i_n)-p_1(d^i_l)\Big)+ \tag{6.100}$$
$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-v+l,v-l]}+$$
$$(-1)^{k-n}F^{[k-v+l+1,v-l-1]}](d^i_l,d^j_n-d^i_n+d^i_l)-[(-1)^{k-1-n}F^{[k-v+l,v-l]}+$$
$$(-1)^{k-n}F^{[k-v+l+1,v-l-1]}](d^i_0,d^j_n-d^i_n+d^i_0)\Big)+ \tag{6.101}$$
$$\sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d^i_0,d^j_0)\int_0^{d^i_k-d^i_0}\{p_{k-b-1}(d^j_1,\cdots,d^j_{k-b-1})-p_{k-b-1}(d^j_0,d^j_1-u,\cdots,d^j_{k-b-2}-u)\}du+$$
$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v,b-v+1]}(d^i_0,d^j_n-d^i_n+d^i_0)-F^{[v,b-v+1]}(d^i_0,d^j_n-d^i_k+d^i_0)\Big)$$
$$\Big\{p_{k-b-1}(d^j_{n+1},\cdots,d^j_{k-b+n-1})-p_{k-b-1}(d^j_n,\cdots,d^j_{n+k-b-2})\Big\}+ \tag{6.102}$$

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v+1,b-v]}(d_n^i,d_n^j)-F^{[v+1,b-v]}(d_0^i,d_n^j-d_n^i+d_0^i)\Big)$$
$$\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}+ \tag{6.103}$$
$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_n^i+d_0^i)-$$
$$[F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_k^i+d_0^i)\Big)$$
$$\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}+ \tag{6.104}$$
$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b-1}(-1)^{k-m-1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$
$$\Big[\sum_{p=n}^{m}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\{p_{m-n}(d_m^i,\underbrace{d_0^i,\cdots,d_0^i}_{p-l},d_{p+1}^i-u,\cdots,d_{l+m-n-1}^i-u)-$$
$$p_{m-n}(\underbrace{d_0^i,\cdots,d_0^i}_{p-l},d_{p+1}^i-u,\cdots,d_{l+m-n}^i-u)\}\Big(\frac{(d_m^i-d_0^i-u)^{k-m+n-b-1}}{(k-m+n-b-1)!}-$$
$$\sum_{r=1}^{k-m+n-b-1}\frac{(-1)^r(d_m^i-d_0^i-u)^{k-m+n-b-r-1}}{(k-m+n-b-r-1)!}\{p_r(d_{m+1}^j,\cdots,d_{m+u}^j)-p_r(d_m^j,\cdots,d_{m+r}^j)\}\Big)du\Big]+ \tag{6.105}$$
$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b-1}(-1)^{k-m-1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$
$$\Big[\sum_{p=l}^{n-1}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\{p_{m-n}(d_m^i,\underbrace{d_0^i,\cdots,d_0^i}_{p-l},d_{p+1}^i-u,\cdots,d_{l+m-n-1}^i-u)-$$
$$p_{m-n}(\underbrace{d_0^i,\cdots,d_0^i}_{p-l},d_{p+1}^i-u,\cdots,d_{l+m-n}^i-u)\}\Big(\frac{(d_m^i-d_0^i-u)^{k-m+n-b-1}}{(k-m+n-b-1)!}-$$
$$\sum_{r=1}^{k-m+n-b-1}\frac{(-1)^r(d_m^i-d_0^i-u)^{k-m+n-b-r-1}}{(k-m+n-b-r-1)!}\{p_r(d_{m+1}^j,\cdots,d_{m+r}^j)-p_r(d_m^j,\cdots,d_{m+r-1}^j)\}\Big)du\Big]+ \tag{6.106}$$
$$\sum_{l=1}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n}^{n+k-b-1}(-1)^{k-m-1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l+1,v-l]}-$$
$$F^{[b-v+l+2,v-l-1]}](d_0^i,d_m^j-d_m^i+d_0^i)-[F^{[b-v+l+1,v-l]}-$$
$$F^{[b-v+l+2,v-l-1]}](d_l^i,d_m^j-d_m^i+d_l^i)\Big)g_{n+k-b-1,l,m,n}(d_l^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b-1}^j). \tag{6.107}$$

We now consider (6.67)

$$\int_{d_k^i-d_0^i}^{d_1^j-d_0^j}q_k(\underbrace{d_0^i,\cdots,d_0^i}_{k},d_0^j,d_1^j-u,\cdots,d_{k-1}^j-u)=$$
$$(-1)^{k-1}F^{[0,k-1]}(d_0^i,d_0^j)\Big(p_1(d_1^j)-p_1(d_0^j)-p_1(d_k^i)+p_1(d_0^i)\Big)+$$
$$\sum_{n=0}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_n^j-d_k^i+d_0^i)-F^{[v,k-v]}(d_0^i,d_n^j-d_1^j+d_0^j)\Big)+ \tag{6.108}$$

$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-v+l-1,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_n^j-d_k^i+d_0^i\Big)$$
$$-[(-1)^{k-1-n}F^{[k-v+l-1,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_n^j-d_1^j+d_0^j)\Big)+ \quad (6.109)$$
$$\sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d_0^i,d_0^j)\int_{d_k^i-d_0^i}^{d_1^j-d_0^j}\{p_{k-b-1}(d_1^j,\cdots,d_{k-b-1}^j)-p_{k-b-1}(d_0^j,d_1^j-u,\cdots,d_{k-b-2}^j-u)\}du+$$
$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v,b-v+1]}(d_0^i,d_n^j-d_k^i+d_0^i)-F^{[v,b-v+1]}(d_0^i,d_n^j-d_1^j+d_0^j)\Big)$$
$$\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}+ \quad (6.110)$$
$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_k^i+d_0^i\Big)$$
$$-[F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_1^j+d_0^j)\Big)$$
$$\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}. \quad (6.111)$$

Finally, we compute (6.69) as

$$\sum_{p=1}^{k-1}\int_{d_p^j-d_0^j}^{d_{p+1}^j-d_0^j}q_k(\underbrace{d_0^i,\cdots,d_0^i}_{k},\underbrace{d_0^j,\cdots,d_0^j}_{p+1},d_{p+1}^j-u,\cdots,d_{k-1}^j-u)du= \quad (6.112)$$
$$\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}F^{[v,k-v-1]}(d_0^i,d_0^j)\Big(p_1(d_k^j)-p_1(d_n^j)\Big)+$$
$$(-1)^{k-1}F^{[0,k-1]}(d_0^i,d_0^j)\Big(p_1(d_k^j)-p_1(d_1^j)\Big)+ \quad (6.113)$$
$$\sum_{n=2}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_n^j-d_1^j+d_0^j)-F^{[v,k-v]}(d_0^i,d_0^j)\Big)+ \quad (6.114)$$
$$\sum_{l=0}^{k-2}\sum_{n=l+1}^{k-2}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big[(-1)^{k-1-n}F^{[k-1-v+l,v-l]}+$$
$$(-1)^{k-n}F^{[k-v+l,v-l-1]}\Big](d_0^i,d_0^j)\Big(p_1(d_k^j)-p_1(d_n^j)\Big)+ \quad (6.115)$$
$$\sum_{n=2}^{k-1}\sum_{v=n}^{k-1}\binom{v}{n}\Big([(-1)^{k-1-n}F^{[k-1-v,v+1]}+(-1)^{k-n}F^{[k-v,v]}](d_0^i,d_n^j-d_1^j+d_0^j)-$$
$$[(-1)^{k-1-n}F^{[k-1-v,v+1]}+(-1)^{k-n}F^{[k-v,v]}](d_0^i,d_0^j)\Big)+$$
$$\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+$$
$$(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_n^j-d_l^j+d_0^j)-[(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_0^j)\Big)+ \quad (6.116)$$
$$\sum_{l=2}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big([(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+$$
$$(-1)^{k-n}F^{[k-v+l+1,v-l-1]}](d_0^i,d_n^j-d_1^j+d_0^j)-[(-1)^{k-1-n}F^{[k-1-v+l,v-l+1]}+$$
$$(-1)^{k-n}F^{[k-v+l,v-l]}](d_0^i,d_n^j-d_l^j+d_0^j)\Big)+ \quad (6.117)$$

$$
\begin{aligned}
&\sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d_0^i,d_0^j)p_{k-b}(d_0^j,\cdots,d_{k-b-1}^j)- \\
&\sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d_0^i,d_0^j)\int_{d_0^j}^{d_1^j}\{p_{k-b-1}(d_0^j,\cdots,d_{k-b-1}^j)-p_{k-b-1}(u,d_1^j,\cdots,d_{k-b-2}^j)\}du+ \\
&\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d_0^i,d_0^j)p_{k-b}(d_n^j,\cdots,d_{k-b+n-1}^j)+ \qquad (6.118)\\
&\sum_{n=2}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}\Big(F^{[v,b-v+1]}(d_0^i,d_n^j-d_1^j+d_0^j)-F^{[v,b-v+1]}(d_0^i,d_0^j)\Big)\\
&\qquad\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}+ \qquad (6.119)\\
-&\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}](d_0^i,d_0^j)\\
&\qquad p_{k-b}(d_n^j,\cdots,d_{k-b+n-1}^j)+ \qquad (6.120)\\
&\sum_{n=2}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v}{n}\Big([F^{[b-v,v+1]}-F^{[b-v+1,v]}](d_0^i,d_n^j-d_1^j+d_0^j)-\\
&\qquad [F^{[b-v,v+1]}-F^{[b-v+1,v]}](d_0^i,d_0^j)\Big)\\
&\qquad\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}+\\
&\sum_{l=1}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_l^j+d_0^j)-\\
&\qquad [F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_0^j)\Big)\\
&\qquad\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}+ \qquad (6.121)\\
&\sum_{l=2}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big([F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_1^j+d_0^j)-\\
&\qquad [F^{[b-v+l,v-l+1]}-F^{[b-v+l+1,v-l]}](d_0^i,d_n^j-d_l^j+d_0^j)\Big)\\
&\qquad\Big\{p_{k-b-1}(d_{n+1}^j,\cdots,d_{k-b+n-1}^j)-p_{k-b-1}(d_n^j,\cdots,d_{n+k-b-2}^j)\Big\}. \qquad (6.122)
\end{aligned}
$$

Since $q_{k+1}(\cdots)$ is given by the sum of (6.65),(6.67),(6.69) subtracted from the sum of (6.64),(6.66) and (6.68), we continue by the computation of the differences. We start by subtracting (6.97) from (6.71)

$$
\begin{aligned}
&\sum_{n=0}^{k-2}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)-F^{[v,k-v]}(d_0^i,d_{n+1}^j-d_k^i+d_0^i)\Big)\\
&-(-1)^{k-1}F^{[0,k-1]}(d_0^i,d_0^j)\Big(d_k^i-d_0^i\Big)\\
&-\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-1}{n-1}\Big(F^{[v,k-v]}(d_0^i,d_n^j-d_n^i+d_0^i)-F^{[v,k-v]}(d_0^i,d_n^j-d_k^i+d_0^i)\Big)\\
=\;&(-1)^kF^{[0,k-1]}(d_0^i,d_0^j)\Big(d_k^i-d_0^i\Big)+\sum_{n=1}^{k-1}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v-1,k-v+1]}(d_0^i,d_n^j-d_n^i+d_0^i)
\end{aligned}
$$

$$-\sum_{n=1}^{k-1}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v-1,k-v+1]}(d_0^i, d_n^j - d_k^i + d_0^i) \tag{6.123}$$

where we substituted $n+1 \to n$ in (6.71), $v+1 \to v$ in (6.71) and (6.97), used the fact that $\binom{v-2}{n-2} + \binom{v-2}{n-1} = \binom{v-1}{n-1}$ to obtain the equality in (6.123). Similarly, subtracting (6.98) from (6.72) we have

$$\sum_{n=1}^{k}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v,k-v]}(d_n^i, d_n^j) - \sum_{n=1}^{k}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v,k-v]}(d_0^i, d_n^j - d_n^i + d_0^i). \tag{6.124}$$

The difference of (6.99) and (6.73) is equal to

$$\begin{aligned}
&\sum_{n=1}^{k-1}\sum_{v=n}^{k-1}\binom{v}{n}\Big([(-1)^{k-n}F^{[k-v-1,v+1]} + (-1)^{k-n+1}F^{[k-v,v]}](d_0^i, d_n^j - d_n^i + d_0^i) \\
&-[(-1)^{k-n}F^{[k-v-1,v+1]} + (-1)^{k-n+1}F^{[k-v,v]}](d_0^i, d_n^j - d_k^i + d_0^i)\Big) + \\
&+\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k}\binom{v-l}{n-l}\binom{k-v+l}{l}\Big([(-1)^{k-n}F^{[k-v+l-1,v-l+1]} + (-1)^{k-n+1}F^{[k-v+l,v-l]}](d_0^i, d_n^j - d_n^i + d_0^i) \\
&\qquad -[(-1)^{k-n}F^{[k-v+l-1,v-l+1]} + (-1)^{k-n+1}F^{[k-v+l,v-l]}](d_0^i, d_n^j - d_k^i + d_0^i)\Big)
\end{aligned} \tag{6.125}$$

which we obtained by substituting $n+1 \to n$, $v+1 \to v$ and $l+1 \to l$ in (6.73). Similarly, the difference of (6.100) and (6.74) and the difference of (6.101) and (6.75) are equal to (6.126) and (6.127) respectively.

$$\begin{aligned}
&\sum_{n=1}^{k-1}\sum_{v=n}^{k-1}\binom{v}{n}[(-1)^{k-n}F^{[k-v-1,v]} + (-1)^{k-n+1}F^{[k-v,v-1]}](d_0^i, d_n^j - d_n^i + d_0^i)\Big(p_1(d_n^i) - p_1(d_0^i)\Big) + \\
&\sum_{l=1}^{k-1}\sum_{n=l+1}^{k}\sum_{v=n}^{k}\binom{v-l}{n-l}\binom{k-v+l}{l}\Big[(-1)^{k-n}F^{[k-v+l-1,v-l]} + \\
&\qquad (-1)^{k-n+1}F^{[k-v+l,v-l-1]}\Big](d_0^i, d_n^j - d_n^i + d_0^i)\Big(p_1(d_n^i) - p_1(d_l^i)\Big)
\end{aligned} \tag{6.126}$$

$$\begin{aligned}
&\sum_{l=1}^{k-1}\sum_{n=l+1}^{k}\sum_{v=n}^{k}\binom{v-l}{n-l}\binom{k-v+l}{l}\Big([(-1)^{k-n}F^{[k-v+l,v-l]} + (-1)^{k-n+1}F^{[k-v+l+1,v-l-1]}](d_l^i, d_n^j - d_n^i + d_l^i) \\
&\qquad -[(-1)^{k-n}F^{[k-v+l,v-l]} + (-1)^{k-n+1}F^{[k-v+l+1,v-l-1]}](d_0^i, d_n^j - d_n^i + d_0^i)\Big)
\end{aligned} \tag{6.127}$$

Similar analysis yield (6.128) and (6.129) for the differences of (6.102) and (6.76) and (6.103) and (6.77) respectively.

$$\begin{aligned}
&\sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v-1,b-v+1]}(d_0^i, d_n^j - d_n^i + d_0^i)\Big\{p_{k-b}(d_{n+1}^j, \cdots, d_{k-b+n}^j) - \\
&\qquad p_{k-b}(d_n^j, \cdots, d_{n+k-b-1}^j)\Big\} \\
-\;&\sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v-1,b-v+1]}(d_0^i, d_n^j - d_k^i + d_0^i)\Big\{p_{k-b}(d_{n+1}^j, \cdots, d_{k-b+n}^j) - \\
&\qquad p_{k-b}(d_n^j, \cdots, d_{n+k-b-1}^j)\Big\}
\end{aligned}$$

$$- \sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d_0^i,d_0^j)\int_0^{d_k^i-d_0^i}\{p_{k-b-1}(d_1^j,\cdots,d_{k-b-1}^j)-p_{k-b-1}(d_0^j,d_1^j-u,\cdots,d_{k-b-2}^j-u)\}du \quad (6.128)$$

$$\sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d_n^i,d_n^j)\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}$$

$$- \sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d_0^i,d_n^j-d_n^i+d_0^i)$$
$$\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\} \quad (6.129)$$

Substituting $b+1\to b$ in (6.104) and (6.77) and substituting $n+1\to n$, $v+1\to v$ and $l+1\to l$ in (6.77) and taking the difference we obtain

$$\sum_{n=1}^{k-2}\sum_{b=n+1}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v}{n}[F^{[b-v-1,v+1]}-F^{[b-v,v]}](d_0^i,d_n^j-d_n^i+d_0^i)$$
$$\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}$$
$$- \sum_{n=1}^{k-2}\sum_{b=n+1}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v}{n}[F^{[b-v-1,v+1]}-F^{[b-v,v]}](d_0^i,d_n^j-d_k^i+d_0^i)$$
$$\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}$$
$$+ \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l-1,v-l+1]}-F^{[b-v+l,v-l]}\Big](d_0^i,d_n^j-d_n^i+d_0^i)$$
$$\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}$$
$$- \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l-1,v-l+1]}-F^{[b-v+l,v-l]}\Big](d_0^i,d_n^j-d_k^i+d_0^i))$$
$$\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}. \quad (6.130)$$

Taking the sum of (6.105) and (6.106) and subtracting from the sum of (6.78) and (6.79) with some algebra we have

$$\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{k-n}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l-1,v-l]}-F^{[b-v+l,v-l-1]}\Big](d_0^i,d_n^j-d_n^i+d_0^i)$$
$$\Big[\sum_{p=l}^{n-1}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\Big(\frac{(d_n^i-d_0^i-u)^{k-b}}{(k-b)!}-\sum_{r=1}^{k-b-1}\frac{(-1)^r(d_m^i-d_0^i-u)^{k-b-r}}{(k-b-r)!}\{p_r(d_{m+1}^j,\cdots,d_{m+r}^j)-p_r(d_m^j,\cdots,d_{m+r-1}^j)\}$$
$$-(-1)^{k-b}\{p_{k-b}(d_{n+1}^j,\cdots,d_{n+k-b}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\}\Big)du\Big]+$$
$$\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}\sum_{m=n+1}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l-1,v-l]}-F^{[b-v+l,v-l-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$
$$\Big[\sum_{p=l}^{m-1}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\{p_{m-n}(d_m^i,\underbrace{d_0^i,\cdots,d_0^i}_{p-l},d_{p+1}^i-u,\cdots,d_{l+m-n-1}^i-u)-$$

$$p_{m-n}(\underbrace{d_0^i,\cdots,d_0^i}_{p-l},d_{p+1}^i-u,\cdots,d_{l+m-n}^i-u)\}\Big(\frac{(d_m^i-d_0^i-u)^{k-m+n-b}}{(k-m+n-b)!}-$$

$$\sum_{r=1}^{k-m+n-b-1}\frac{(-1)^r(d_n^i-d_0^i-u)^{k-m+n-b-r}}{(k-m+n-b-r)!}\{p_r(d_{m+1}^j,\cdots,d_{m+r}^j)-p_r(d_m^j,\cdots,d_{m+r-1}^j)\}$$

$$-(-1)^{k-b-m+n}\{p_{k-b-m+n}(d_{m+1}^j,\cdots,d_{m+k-b}^j)-p_{k-b-m+n}(d_m^j,\cdots,d_{m+k-b-1}^j)\}\Big)du\Big]+$$

$$\sum_{n=1}^{k-2}\sum_{b=n+1}^{k-1}(-1)^{k-n}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v-1,v]}-F^{[b-v,v-1]}\Big](d_0^i,d_n^j-d_n^i+d_0^i)$$

$$\Big[\sum_{p=0}^{n-1}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\Big(\frac{(d_n^i-d_0^i-u)^{k-b}}{(k-b)!}-\sum_{r=1}^{k-b-1}\frac{(-1)^r(d_m^i-d_0^i-u)^{k-b-r}}{(k-b-r)!}\{p_r(d_{m+1}^j,\cdots,d_{m+r}^j)-p_r(d_m^j,\cdots,d_{m+r-1}^j)\}$$

$$-(-1)^{k-b}\{p_{k-b}(d_{n+1}^j,\cdots,d_{n+k-b}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\}\Big)du\Big]+$$

$$\sum_{n=1}^{k-2}\sum_{b=n+1}^{k-1}\sum_{m=n+1}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v-1,v]}-F^{[b-v,v-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$

$$\Big[\sum_{p=0}^{m-1}\int_{d_p^i-d_0^i}^{d_{p+1}^i-d_0^i}\{p_{m-n}(d_m^i,\underbrace{d_0^i,\cdots,d_0^i}_{p},d_{p+1}^i-u,\cdots,d_{l+m-n-1}^i-u)-$$

$$p_{m-n}(\underbrace{d_0^i,\cdots,d_0^i}_{p},d_{p+1}^i-u,\cdots,d_{l+m-n}^i-u)\}\Big(\frac{(d_m^i-d_0^i-u)^{k-m+n-b}}{(k-m+n-b)!}-$$

$$\sum_{r=1}^{k-m+n-b-1}\frac{(-1)^r(d_m^i-d_0^i-u)^{k-m+n-b-r}}{(k-m+n-b-r)!}\{p_r(d_{m+1}^j,\cdots,d_{m+r}^j)-p_r(d_m^j,\cdots,d_{m+r-1}^j)\}$$

$$-(-1)^{k-b-m+n}\{p_{k-b-m+n}(d_{m+1}^j,\cdots,d_{n+k-b}^j)-p_{k-b-m+n}(d_m^j,\cdots,d_{n+k-b-1}^j)\}\Big)du\Big]. \quad (6.131)$$

Subtracting (6.107) from (6.80) we have

$$\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}\sum_{m=n}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_l^i,d_m^j-d_m^i+d_l^i)$$
$$g_{k-b+n,l,m,n}(d_l^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b}^j)$$
$$-\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}\sum_{m=n}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$
$$g_{k-b+n,l,m,n}(d_l^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b}^j) \quad (6.132)$$

where we substituted $b+1\to b$ in (6.107) and (6.80) and $m+1\to m$, $n+1\to n$, $v+1\to v$ and $l+1\to l$ in (6.80) to obtain (6.132). For convenience, we split the expression in (6.132) into three parts in the following way

$$\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}\sum_{m=n}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_l^i,d_m^j-d_m^i+d_l^i)$$
$$g_{k-b+n,l,m,n}(d_l^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b}^j) \quad (6.133)$$
$$-\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}\sum_{m=n+1}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$

$$g_{k-b+n,l,m,n}(d^i_l,\cdots,d^i_m,d^j_m,\cdots,d^j_{n+k-b})$$

$$- \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{k-n}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d^i_0,d^j_n-d^i_n+d^i_0)$$
$$\Big(g_{k-b+n,l,n,n}(d^i_l,\cdots,d^i_n,d^j_n,\cdots,d^j_{n+k-b})+(-1)^{k-b}\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n})-p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\}\Big) \tag{6.134}$$

$$- \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d^i_0,d^j_n-d^i_n+d^i_0)$$
$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n})-p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}. \tag{6.135}$$

Instead of subtracting (6.81) directly from (6.108), for convenience we add (6.114) to (6.108) and subtract this sum from the sum of (6.87) and (6.81) which yields

$$(-1)^kF^{[0,k-1]}(d^i_0,d^j_0)\Big(p_1(d^j_1)-p_1(d^j_0)\Big)+ \tag{6.136}$$

$$(-1)^kF^{[0,k]}(d^i_0,d^j_0)-(-1)^kF^{[0,k-1]}(d^i_0,d^j_0)\Big(p_1(d^i_k)-p_1(d^i_0)\Big)+$$
$$\sum_{n=1}^{k}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v-1,k-v+1]}(d^i_0,d^j_n-d^i_k+d^i_0). \tag{6.137}$$

Similarly we subtract the sum of (6.116), (6.117), (6.109) from the sum of (6.89), (6.90), (6.82) and obtain

$$\sum_{n=1}^{k-1}\sum_{v=n}^{k-1}\binom{v}{n}\Big[(-1)^{k-n}F^{[k-v-1,v+1]}+(-1)^{k-n+1}F^{[k-v,v]}\Big](d^i_0,d^j_n-d^i_k+d^i_0)$$
$$+ \sum_{l=1}^{k-1}\sum_{n=l+1}^{k}\sum_{v=n}^{k}\binom{v-l}{n-l}\binom{k-v+l}{l}\Big[(-1)^{k-n}F^{[k-v+l-1,v-l+1]}+$$
$$(-1)^{k-n+1}F^{[k-v+l,v-l]}\Big](d^i_0,d^j_n-d^i_n+d^i_k). \tag{6.138}$$

Taking the sum of (6.119) and (6.110) and subtracting from the sum of (6.92) and (6.83), we have

$$\sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v-1,b-v+1]}(d^i_0,d^j_n-d^i_k+d^i_0)$$
$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n})-p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}$$
$$- \sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v}{n}F^{[v-1,b-v+1]}(d^i_0,d^j_0)\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n})-p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}$$
$$- \sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d^i_0,d^j_0)\int_{d^i_k-d^i_0}^{d^j_1-d^j_0}\{p_{k-b-1}(d^j_1-u,\cdots,d^j_{k-b-1}-u)-$$
$$p_{k-b-1}(d^j_0,d^j_1-u,\cdots,d^j_{k-b-2}-u)\}du. \tag{6.139}$$

Finally, taking the sum of (6.121),(6.122),(6.111) and subtracting from the sum of (6.94),(6.95) and (6.84) we have

$$\sum_{n=1}^{k-2}\sum_{b=n+1}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v-1,v+1]}-F^{[b-v,v]}\Big](d^i_0,d^j_n-d^i_k+d^i_0)$$

$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}$$

$$+\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l-1,v-l+1]} - F^{[b-v+l,v-l]}\Big](d^i_0, d^j_n - d^i_k + d^i_0)$$

$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}$$

$$+\sum_{n=1}^{k-2}\sum_{b=n+1}^{k-1}(-1)^{b-n}\sum_{v=n}^{b-1}\binom{v}{n}\Big[F^{[b-v-1,v+1]} - F^{[b-v,v]}\Big](d^i_0, d^j_0)$$

$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}$$

$$+\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l-1,v-l+1]} - F^{[b-v+l,v-l]}\Big](d^i_0, d^j_0)$$

$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\}. \tag{6.140}$$

We now consider the remaining terms in the difference of (6.68) and (6.69). We start by the difference of (6.113) and (6.86). Analysis similar to the ones above yields the following

$$\sum_{n=1}^{k-1}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v,k-v-1]}(d^i_0,d^j_0)\Big(p_1(d^j_k) - p_1(d^j_n)\Big) +$$
$$(-1)^k F^{[0,k-1]}(d^i_0,d^j_0)\Big(p_1(d^j_k) - p_1(d^j_1)\Big). \tag{6.141}$$

The difference of (6.115) and (6.88) yields

$$\sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}\sum_{v=n}^{k}\binom{v-l}{n-l}\binom{k-v+l}{l}\Big[(-1)^{k-n}F^{[k-1-v+l,v-l]} +$$
$$(-1)^{k-n+1}F^{[k-v+l,v-l-1]}\Big](d^i_0,d^j_0)\Big(p_1(d^j_k) - p_1(d^j_n)\Big)$$
$$+\sum_{n=1}^{k-1}\sum_{v=n}^{k-1}\binom{v}{n}\Big[(-1)^{k-n}F^{[k-1-v,v]} + (-1)^{k-n+1}F^{[k-v,v-1]}\Big](d^i_0,d^j_0)\Big(p_1(d^j_k) - p_1(d^j_n)\Big). \tag{6.142}$$

Finally, the difference of (6.118) and (6.91) and the difference of (6.120) and (6.93) are equal to (6.142) and (6.143) respectively.

$$\sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d^i_0,d^j_0)\Big\{p_{k-b}(d^j_1,\cdots,d^j_{k-b}) - p_{k-b}(d^j_0,\cdots,d^j_{k-b-1})\Big\} +$$
$$\sum_{b=0}^{k-2}(-1)^{b+1}F^{[0,b]}(d^i_0,d^j_0)\int_{d^j_0}^{d^j_1}\{p_{k-b-1}(d^j_0,\cdots,d^j_{k-b-1}) - p_{k-b-1}(u,d^j_1,\cdots,d^j_{k-b-2})\}du +$$
$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d^i_0,d^j_0)\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{k-b+n-1})\Big\}, \tag{6.143}$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]} - F^{[b-v+l+1,v-l-1]}\Big](d^i_0,d^j_0)$$
$$\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{k-b+n-1})\Big\}. \tag{6.144}$$

We are now ready to sum up the differences given in (6.123) to (6.144) in order to obtain the expression for $q_{k+1}(\cdots)$. Note that with the convention that $\binom{n_1}{n_2} = 0$ when $n_1 < n_2$ we have $\sum_{l=1}^{n-1}\binom{v-1}{n-l}\binom{k-v+1}{l} - \sum_{l=1}^{n-1}\binom{v}{n-l}\binom{k-v}{l} = \binom{v-1}{n-1} - \binom{k-v}{n-1}$ and $\binom{v-1}{n-1} + \binom{v-1}{n} = \binom{v}{n}$. Then the sum of (6.123), (6.124), (6.125), (6.127), (6.137), (6.138) is equal to

$$\sum_{n=0}^{k}(-1)^{k-n}\sum_{v=n}^{k}\binom{v-1}{n-1}F^{[v,k-v]}(d_n^i, d_n^j) + \sum_{l=0}^{k-1}\sum_{n=l+1}^{k}\sum_{v=n}^{k}\binom{v-l}{n-l}\binom{k-v+l}{l}\Big[(-1)^{k-n}F^{[k-v+l,v-l]} + (-1)^{k-n+1}F^{[k-v+l+1,v-l-1]}\Big](d_l^i, d_n^j - d_n^i + d_l^i). \tag{6.145}$$

In a similar fashion with some elementary (but tedious) algebra, the sum of (6.128),(6.129),(6.130),(6.133),(6.135), (6.136), (6.139), (6.140), (6.141), (6.142), (6.143) and (6.144) can be obtained as

$$\begin{aligned}&\sum_{n=0}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d_n^i, d_n^j)\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j) - p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}\\ &+ \sum_{n=1}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]} - F^{[b-v+1,v-1]}\Big](d_0^i, d_n^j - d_n^i + d_0^i)\\ &\qquad\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j) - p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}. \end{aligned}\tag{6.146}$$

We now consider (6.131). Using Theorem 8 in Baccelli et al [4] and substituting $b-1 \to b$, (6.131) can be written as

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}(-1)^{k-n+1}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l+1}{l+1}\Big[F^{[b-v+l,v-l]} - F^{[b-v+l+1,v-l-1]}\Big](d_0^i, d_{n+1}^j - d_{n+1}^i + d_0^i)\, f(n, n+1, l, b) + \tag{6.147}$$

$$\sum_{l=0}^{k-3}\sum_{n=l+1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+2}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l+1}{l+1}\Big[F^{[b-v+l,v-l]} - F^{[b-v+l+1,v-l-1]}\Big](d_0^i, d_m^j - d_m^i + d_0^i)\, f(n, m, l, b) + \tag{6.148}$$

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{k-n}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]} - F^{[b-v+1,v-1]}\Big](d_0^i, d_n^j - d_n^i + d_0^i) f(n-1, n, -1, b) + \tag{6.149}$$

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b-1}(-1)^{k-m}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]} - F^{[b-v+1,v-1]}\Big](d_0^i, d_m^j - d_m^i + d_0^i) f(n-1, m, -1, b) \tag{6.150}$$

where we also substituted $l-1 \to l$, $v-1 \to v$ and $n-1 \to n$ in the first two summations and defined $f(\cdot,\cdot,\cdot,\cdot)$ as

$$\begin{aligned} f(n,m,l,b) &= (-1)^{k-b+m-n-1}\Big(\frac{(d_{m-n+l}^i - d_m^i)^{k-b}}{(k-b)!} - \sum_{r=l+1}^{m-n+l-1}\{p_{m-n+l-r}(d_{r+1}^i,\cdots,d_{m-n+l}^i) - \\ &\qquad p_{m-n+l-r}(d_m^i, d_{r+1}^i,\cdots,d_{m-n+l-1}^i)\}\frac{(d_r^i - d_m^i)^{k-b-m+n-l+r}}{(k-b-m+n-l+r)!}\Big) \\ &+ (-1)^{k-n+m-b}\sum_{v=1}^{k-m+n-b}\Big\{p_v(d_{m+1}^j,\cdots,d_{m+v}^j) - p_v(d_m^j,\cdots,d_{m+v-1}^j)\Big\}\Big(\frac{(d_{m-n+l}^i - d_m^i)^{k-b-v}}{(k-b-v)!} \end{aligned}$$

$$- \sum_{r=l+1}^{m-n+l-1} \{p_{m-n+l-r}(d^i_{r+1},\cdots,d^i_{m-n+l}) - p_{m-n+l-r}(d^i_m,d^i_{r+1},\cdots,d^i_{m-n+l-1})\} \frac{(d^i_r - d^i_m)^{k-b-m+n-l+r-v)}}{(k-b-m+n-l+r-v)!}\Big).$$

Note that (6.149) is equal to

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{k-n}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]} - F^{[b-v+1,v-1]}\Big](d^i_0, d^j_n - d^i_n + d^i_0)$$
$$\Big(g_{k-b+n,0,n,n}(d^i_0,\cdots,d^i_n,d^j_n,\cdots,d^j_{n+k-b}) + (-1)^{k-b}\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\}\Big).$$

Adding this to (6.146), we get

$$\sum_{n=0}^{k-1}\sum_{b=n}^{k-1}(-1)^{b-n+1}\sum_{v=n}^{b}\binom{v-1}{n-1}F^{[v,b-v]}(d^i_n,d^j_n)\Big\{p_{k-b}(d^j_{n+1},\cdots,d^j_{k-b+n}) - p_{k-b}(d^j_n,\cdots,d^j_{n+k-b-1})\Big\} +$$
$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}(-1)^{k-n}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]}(d^i_0,d^j_n - d^i_n + d^i_0) - F^{[b-v+1,v-1]}(d^i_0,d^j_n - d^i_n + d^i_0)\Big]$$
$$g_{k-b+n,0,n,n}(d^i_0,\cdots,d^i_n,d^j_n,\cdots,d^j_{n+k-b}) \tag{6.151}$$
$$+ \sum_{n=1}^{k-1}(-1)^{k-n}\sum_{v=n}^{k-1}\binom{v}{n}\Big[F^{[k-1-v,v]} - F^{[k-v,v-1]}\Big](d^i_0,d^j_n - d^i_n + d^i_0)\Big\{p_1(d^j_{n+1}) - p_1(d^j_n)\Big\}. \tag{6.152}$$

We also have

$$\begin{aligned} f(n,m,l,b) + f(n-1,m,l-1,b) &= g_{k-b+n,l,m,n}(d^i_l,\cdots,d^i_m,d^j_m,\cdots,d^j_{n+k-b}) \quad \text{and} \\ f(m-1,m,l-1,b) &= g_{k-b+m,l,m,m}(d^i_l,\cdots,d^i_m,d^j_m,\cdots,d^j_{m+k-b}) + \\ &\quad (-1)^{k-b}\Big\{p_{k-b}(d^j_{m+1},\cdots,d^j_{k-b+m}) - p_{k-b}(d^j_m,\cdots,d^j_{k-b+m-1})\Big\}. \end{aligned}$$

Using the above relationship together with some algebra, we obtain the sum of (6.147),(6.148) and (6.150) and (6.134) as

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]} - F^{[b-v+1,v-1]}\Big](d^i_0,d^j_m - d^i_m + d^i_0)$$
$$g_{k-b+n,0,m,n}(d^i_0,\cdots,d^i_m,d^j_m,\cdots,d^j_{n+k-b})$$
$$- \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big[F^{[k-1-v+l,v-l]} -$$
$$F^{[k-v+l,v-l-1]}\Big](d^i_0,d^j_{n+1} - d^i_{n+1} + d^i_0)\Big(d^i_{n+1} - d^i_{l+1}\Big)$$
$$- \sum_{l=1}^{k-2}\sum_{n=l+1}^{k-1}(-1)^{k-n}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{k-1-v+l}{l}\Big[F^{[k-1-v+l,v-l]} -$$
$$F^{[k-v+l,v-l-1]}\Big](d^i_0,d^j_n - d^i_n + d^i_0)\Big(d^i_n - d^i_l\Big). \tag{6.153}$$

Adding (6.126) to (6.153), we have

$$\sum_{n=1}^{k-2}\sum_{b=n}^{k-2}\sum_{m=n+1}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v}{n}\Big[F^{[b-v,v]}-F^{[b-v+1,v-1]}\Big](d_0^i,d_m^j-d_m^i+d_0^i)$$
$$g_{k-b+n,0,m,n}(d_0^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b}^j)$$
$$+\quad\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v}{n}\Big[F^{[k-1-v,v]}-F^{[k-v,v-1]}\Big](d_0^i,d_{n+1}^j-d_{n+1}^i+d_0^i)\Big(d_{n+1}^i-d_1^i\Big)$$
$$-\quad\sum_{n=1}^{k-1}(-1)^{k-n-1}\sum_{v=n}^{k-1}\binom{v}{n}\Big[F^{[k-1-v,v]}-F^{[k-v,v-1]}\Big](d_0^i,d_n^j-d_n^i+d_0^i)\Big(d_n^i-d_0^i\Big).\qquad(6.154)$$

Finally, summing (6.152), (6.154) and (6.133) and changing the order of summation yields

$$\sum_{b=0}^{k-1}\sum_{n=0}^{b}(-1)^{b-n+1}\sum_{v=n}^{l}\binom{v-1}{n-1}F^{[v,b-v]}(d_n^i,d_n^j)\Big\{p_{k-b}(d_{n+1}^j,\cdots,d_{k-b+n}^j)-p_{k-b}(d_n^j,\cdots,d_{n+k-b-1}^j)\Big\}$$
$$+\quad\sum_{b=1}^{k-1}\sum_{l=0}^{b-1}\sum_{n=l+1}^{b}\sum_{m=n}^{n+k-b}(-1)^{k-m}\sum_{v=n}^{b}\binom{v-l}{n-l}\binom{b-v+l}{l}\Big[F^{[b-v+l,v-l]}-F^{[b-v+l+1,v-l-1]}\Big](d_l^i,d_m^j-d_m^i+d_l^i)$$
$$g_{k-b+n,l,m,n}(d_l^i,\cdots,d_m^i,d_m^j,\cdots,d_{n+k-b}^j).\qquad(6.155)$$

Putting (6.155) and (6.145) together gives the expression for $q_{k+1}(\cdots)$ which completes the proof of Theorem 5.3.

□

References

[1] Asmussen, S. (1987) *Applied Probability and Queues*, John Wiley and Sons, Chichester.

[2] Baccelli, F., Cohen, G., Olsder G.J. and Quadrat J-P.(1992) *Synchronization and Linearity: An Algebra for Discrete Event Systems*, John Wiley and Sons, Chichester.

[3] Baccelli, F. and Brémaud, P. (1993) "Virtual Customers in Sensitivity and Light Traffic Analysis via Campbell's Formula for Point Processes", *Adv. in Appl. Probab.* **25** 221-224.

[4] Baccelli, F., Hasenfuss, S. and Schmidt V. (1997) "Transient and Stationary Waiting Times in (max, +) Linear Systems with Poisson Input", *Queueing Systems*, **26** 301-342.

[5] Baccelli, F., Hasenfuss, S. and Schmidt V. (1998) "Expansions for Steady State Characteristics in (max, +) Linear Systems", *Commun. Statist.-Stochastic Models*, to appear.

[6] Baccelli, F., Hasenfuss, S. and Schmidt V. (1998) "Differentiability of Poisson Driven Stochastic Systems" *Stochastic Process. Appl.*, to appear.

[7] Baccelli, F. and Hong D. (1998) "Analyticity of iterates of non-expansive maps", *Advances in Applied Probability*, under review.

[8] Baccelli, F. and Schmidt V. (1996) "Taylor Series Expansions for Poisson-Driven (max,+)-Linear Systems", *Annals of Applied Probability*, **6** 138-185.

[9] Błaszczyszyn, B. (1995) "Factorial Moment Expansion for Stochastic Systems", *Stochastic Process. Appl.*, **56** 321-335.

[10] Błaszczyszyn, B., Frey, A. and Schmidt, V. (1995) "Light-traffic Approximations for Markov modulated Multi-server Queues", *Stochastic Models*, **11**, 423-445.

[11] Błaszczyszyn, B., Rolski, T. and Schmidt, V.(1995) "Light-traffic Approximations in Queues and Related Stochastic Models", In *Advances in Queueing: Theory, Methods and Open Problems* (J.H. Dshalalow, ed.) 379-406. CRC Press, Boca Raton, FL.

[12] Fishman, G.S. (1996), *Monte Carlo*. Springer-Verlag, New York.

[13] Hasenfuss, S. (1998), *Performance Analysis of (max,+)-Linear Systems via Taylor Series Expansions*, Ph.D. Dissertation, University of Ulm.

[14] Karpelevitch, F.I. and Kreinin, A. Ya. (1992) "Joint Distributions in Poissonian Tandem Queues",*Queueing Systems*, **12** 273-286.

[15] Kroese, D.P. and Schmidt, V. (1996) "Light-traffic Analysis for Queues with Spatially Distributed Arrivals", *Math. Oper. Res.*, **21** 135-157.

[16] Reiman, B. and Simon, B. (1989) "Open Queueing Systems in Light Traffic", *Math. Oper. Res.*, **14**, 26-59.

[17] Seidel, W., Kocemba, K.v. and Mitreiter, K. (1997) "On a Taylor Series Expansion for waiting Times in Tandem Queues: An Algorithm for Calculating the Coefficients and an Investigation of the Approximation Error", *Working paper*.

[18] Wolff, R. (1989) *Stochastic Modeling and the Theory of Queues*. Prentice Hall, Englewood Cliffs.

[19] Zazanis, M.A. (1992) "Analyticity of Poisson Driven Stochastic Systems", *Adv. in Appl. Probab.*, **24**, 532-541.