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On the cardinality and sum of reciprocals of primitive sequences

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Abstract. Let $\mathcal{A}(2n)$ denote the set of primitive sequences $A(2n)$ with cardinality n. In this paper, we consider the upper bound of reciprocal sum of $A \in \mathcal{A}(2n)$ and obtain

$$\max_{A\in\mathcal{A}(2n)} \sum_{i=1}^{n} \frac{1}{a_i} = \log 3 + O\left(\frac{1}{n^{\log_3 2}}\right)$$

as $n \to \infty$. We also find some interesting properties of $|\mathcal{A}(2n)|$.

Keywords. Primitive sequence, reciprocal sum, cardinality.

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1. Introduction

We first introduce some notations. Let $A = \{a_i\}_{i\geq 1}$ be a subset of $\mathbb{N}$. For convenience we set $a_1 < a_2 < \ldots$. Note that A can also be viewed as an increasing sequence $(a_i)_{i\geq 1}$. For $x \leq y$, $(x, y]$ and $[x, y]$ equal the set of integers n such that $x < n \leq y$ and $x \leq n \leq y$, respectively. We use the abbreviation $A(x) = A \cap [1, x]$. A sequence $A = (a_i)_{i\geq 1}$ is primitive, if $a_i \nmid a_j$ for $i \neq j$. In 1935, Paul Erdös [3] proved that for every primitive sequence $A = (a_i)_{i\geq 1}$,

$$\frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} = o(1) \text{ as } n \to \infty. \tag{1.1}$$

In the same year, Felix Behrend [2] showed that there exists a constant γ such that for every primitive sequence $A = (a_i)_{i\geq 1}$,

$$\frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} \leq \gamma \frac{1}{(\log\log n)^{1/2}} \text{ for } n \geq 3. \tag{1.2}$$

Later in 1967 Paul Erdös, András Sárközy and Endre Szemerédi [4] proved that for every infinite primitive sequence $A = (a_i)_{i\geq 1}$,

$$\sum_{a_i \leq x} \frac{1}{a_i} = o\left(\frac{\log x}{(\log\log x)^{1/2}}\right), \tag{1.3}$$

and that this bound is best possible. In fact, one may refer to the paper by Rudolf Ahlswede and Levon H. Khachatrian [1] for more details on relevant results. On the other hand, it is easy to verify by the pigeonhole principle that the cardinality of each primitive sequence $A(2n)$ is less than $n + 1$. As Professor Qi Sun told us, Paul Erdös wrote to Chao Ko in 1960s and suggested to find some properties of primitive sequences $A(2n)$ with cardinality n where $n \in \mathbb{N}$. Denote by $\mathcal{A}(2n)$ the set of such primitive sequences, and by $s(n)$ the cardinality of $\mathcal{A}(2n)$. Chao Ko and

Qi Sun [5] showed that $a_1 \geq 2^{\lfloor \log_3 2n \rfloor}$ holds for all $A \in \mathcal{A}(2n)$. It is easy to see that $\{n+1, n+2, \ldots, 2n\} \in \mathcal{A}(2n)$, hence we have

$$\min_{A \in \mathcal{A}(2n)} \sum_{i=1}^{n} \frac{1}{a_i} = \log 2 + O\left(\frac{1}{n}\right) \tag{1.4}$$

as $n \to \infty$. In this paper, we consider the upper bound of reciprocal sum of $A \in \mathcal{A}(2n)$ and some other interesting properties of $s(n)$. We have the following theorems.

Theorem 1.1. *If n goes to infinity, then*

$$\max_{A \in \mathcal{A}(2n)} \sum_{i=1}^{n} \frac{1}{a_i} = \log 3 + O\left(\frac{1}{n^{\log_3 2}}\right). \tag{1.5}$$

Theorem 1.2. *Let t be a positive integer. If n satisfies* (i) $n = 6t$, *or* (ii) $n = 12t+9$ *where* $t \not\equiv 0 \pmod 5$, *then* $s(n+1) = 2s(n)$.

Theorem 1.3. *Let t be a positive integer. If n satisfies* (i) $n = 6t+4$ *where* $t \not\equiv 0 \pmod 3$, *or* (ii) $n = 12t+1$ *where* $t \not\equiv 1 \pmod 3$ *and* $t \not\equiv 4 \pmod 5$, *then* $s(n+1) = s(n)$.

2. Proofs of the theorems

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1. *Let $A^{(1)}(2n) = \{2^{k_i} i : i = 1, 3, \ldots, 2n-1\}$ where each k_i satisfies $\frac{2n}{3^{k_i+1}} < i \leq \frac{2n}{3^{k_i}}$, then $A^{(1)}(2n) \in \mathcal{A}(2n)$. Moreover, all $A \in \mathcal{A}(2n)$ have the form $\{2^{\alpha_i} i : i = 1, 3, \ldots, 2n-1\}$ with each $\alpha_i \geq k_i$.*

Proof. It is easy to show that $2^{k_i} i \leq 2n$ for each odd i. We now divide $\{1, 2, \ldots, 2n\}$ into the following $k+1$ subsets:

$$\left(\frac{2n}{3^1}, \frac{2n}{3^0}\right], \left(\frac{2n}{3^2}, \frac{2n}{3^1}\right], \ldots, \left(\frac{2n}{3^{k+1}}, \frac{2n}{3^k}\right],$$

where $k = \lfloor \log_3 2n \rfloor$. Given $2i-1$ and $2j-1$, where $2i-1 < 2j-1$, in the same subset, say $\left(\frac{2n}{3^{\alpha+1}}, \frac{2n}{3^\alpha}\right]$, we have $k_{2i-1} = k_{2j-1} = \alpha$. If $2^{k_{2i-1}}(2i-1) \mid 2^{k_{2j-1}}(2j-1)$, then $2i-1 | 2j-1$. Thus $2i-1 \leq (2j-1)/3 \leq 2n/3^{\alpha+1}$, which contradicts to the assumption $2i-1 \in \left(\frac{2n}{3^{\alpha+1}}, \frac{2n}{3^\alpha}\right]$. If we pick $2i-1 < 2j-1$ from two different subsets, then $k_{2i-1} > k_{2j-1}$. It readily follows that neither of $2^{k_{2i-1}}(2i-1)$ and $2^{k_{2j-1}}(2j-1)$ divides another. Thus $A^{(1)}(2n) \in \mathcal{A}(2n)$.

We next show that each $A \in \mathcal{A}(2n)$ has the form $\{2^{\alpha_i} i : i = 1, 3, \ldots, 2n-1\}$. Otherwise, by the pigeonhole principle, there exists an odd i and two integers $0 \leq \beta_1 < \beta_2$ such that both $2^{\beta_1} i$ and $2^{\beta_2} i$ are in A. Thus $2^{\beta_1} i \mid 2^{\beta_2} i$, which leads to a contradiction. Note that for odd i, if both $2^{\alpha_i} i$ and $2^{\alpha_{3i}} 3i$ are in A, then $\alpha_i \geq \alpha_{3i} + 1$. Now if odd $i \in \left(\frac{2n}{3^{k_i+1}}, \frac{2n}{3^{k_i}}\right]$, we have $3^j i \leq 2n$ for $j = 0, 1, \ldots, k_i$. Thus, $\alpha_i \geq k_i$ by induction. □

Example 2.1. If $n = 12$, then $A^{(1)}(24) = \{4, 6, 9, 10, 11, 13, 14, 15, 17, 19, 21, 23\}$.

Remark 2.1. One readily verifies $a_1^{(1)} = 2^{\lfloor \log_3 2n \rfloor}$. In this case, the inequality of Ko and Sun mentioned above is in fact an equality.

Proof of Theorem 1.1. Since

$$\sum_{1\le n\le x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right)$$

and

$$\left|\log\frac{x}{2}-\log\left\lfloor\frac{x}{2}\right\rfloor\right|\le\log\frac{x}{2}-\log\left(\frac{x}{2}-1\right)=O\left(\frac{1}{x}\right)$$

as $x\to\infty$, it follows that

$$\begin{aligned}S(x):=\sum_{\substack{1\le n\le x\\ n\text{ odd}}}\frac{1}{n}&=\sum_{1\le n\le x}\frac{1}{n}-\frac{1}{2}\sum_{1\le n\le\lfloor x/2\rfloor}\frac{1}{n}\\&=\left(\log x+\gamma+O\left(\frac{1}{x}\right)\right)-\frac{1}{2}\left(\log\left\lfloor\frac{x}{2}\right\rfloor+\gamma+O\left(\frac{1}{x}\right)\right)\\&=\frac{\log x}{2}+\frac{\gamma+\log 2}{2}+O\left(\frac{1}{x}\right).\end{aligned}$$

Using this estimate, it follows from Lemma 2.1 that

$$\begin{aligned}\max_{A\in\mathcal{A}(2n)}\sum_{i=1}^{n}\frac{1}{a_i}&=\sum_{j=0}^{\lfloor\log_3 2n\rfloor}\frac{1}{2^j}\sum_{\frac{2n}{3^{j+1}}<2i-1\le\frac{2n}{3^j}}\frac{1}{2i-1}\\&=\sum_{j=0}^{\lfloor\log_3 2n\rfloor}\frac{1}{2^j}\left(S\left(\frac{2n}{3^j}\right)-S\left(\frac{2n}{3^{j+1}}\right)\right)\\&=\sum_{j=0}^{\lfloor\log_3 2n\rfloor}\frac{\log 3}{2^{j+1}}+O\left(\sum_{j=0}^{\lfloor\log_3 2n\rfloor}\frac{(3/2)^j}{2n}\right)\\&=\log 3+O\left(\frac{1}{n^{\log_3 2}}\right)\end{aligned}$$

as $n\to\infty$. □

Remark 2.2. For $x\ge 1$, let $\mathcal{A}'(x)$ be the set of primitive sequences $A(x)$ with cardinality $\lfloor(x+1)/2\rfloor$, we also have

$$\max_{A\in\mathcal{A}'(x)}\sum_{a\in A}\frac{1}{a}=\log 3+O\left(\frac{1}{x^{\log_3 2}}\right)\tag{2.1}$$

as $x\to\infty$.

Letting $\{2^{\alpha_i}i : i=1,3,\dots,2n-1\}=A\in\mathcal{A}(2n)$ and $\{2^{\alpha'_i}i : i=1,3,\dots,2n+1\}=B\in\mathcal{A}(2n+2)$, we next prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We first prove case (i). Note that $2n+1=12t+1$ and $2n+2=2(6t+1)$. Since $3\nmid 12t+1$, for any proper divisor d of $12t+1$, we have $d\le\frac{12t+1}{5}\le\frac{12t+2}{3}=\frac{2n+2}{3}$. Thus, $\alpha'_d\ge 1$, which leads to $2^{\alpha'_d}d\nmid 12t+1$. We therefore have

$$\#\{B\in\mathcal{A}(2n+2) : 6t+1\in B\}=\#\{A\in\mathcal{A}(2n) : 6t+1\in A\}=s(n).$$

Since $3 \nmid 6t+1$, for any proper divisor d of $6t+1$, we have $d \le \frac{6t+1}{5} \le \frac{12t+2}{9} = \frac{2n+2}{3^2}$. Thus, $\alpha'_d \ge 2$, which leads to $2^{\alpha'_d} d \nmid 2(6t+1)$. We therefore have

$$\#\{B \in \mathcal{A}(2n+2) : 6t+1 \in B\} = \#\{B \in \mathcal{A}(2n+2) : 2(6t+1) \in B\} = \frac{s(n+1)}{2}.$$

This readily implies $s(n+1) = 2s(n)$.

We next prove case (ii). Note that $2n+1 = 24t+19$ and $2n+2 = 4(6t+5)$. Since $3(6t+5) \in A, B$, we have $\alpha_{6t+5} = 1$ and $\alpha'_{6t+5} = 1, 2$. Note also since that $3 \nmid 24t+19$, for any proper divisor d of $24t+19$, we have $d \le \frac{24t+19}{5} \le \frac{24t+20}{3} = \frac{2n+2}{3}$. Thus, $\alpha'_d \ge 1$, which leads to $2^{\alpha'_d} d \nmid 24t+19$. We therefore have

$$\#\{B \in \mathcal{A}(2n+2) : 2(6t+5) \in B\} = \#\{A \in \mathcal{A}(2n) : 2(6t+5) \in A\} = s(n).$$

Since $t \not\equiv 0 \pmod 5$, it follows that $15 \nmid 6t+5$. For any proper divisor d of $6t+5$, we have $d \le \frac{6t+5}{7} \le \frac{24t+20}{27} = \frac{2n+2}{3^3}$. Thus, $\alpha'_d \ge 3$, which leads to $2^{\alpha'_d} d \nmid 4(6t+5)$. We therefore have

$$\#\{B \in \mathcal{A}(2n+2) : 2(6t+5) \in B\} = \#\{B \in \mathcal{A}(2n+2) : 4(6t+5) \in B\} = \frac{s(n+1)}{2}.$$

This readily implies $s(n+1) = 2s(n)$. □

Proof of Theorem 1.3. We first prove case (i). Note that $2n+1 = 3(4t+3)$ and $2n+2 = 2(6t+5)$, so that

$$\#\{B \in \mathcal{A}(2n+2) : 6t+5 \in B\} = \#\{A \in \mathcal{A}(2n) : 2(4t+3) \in A\}.$$

Since $t \not\equiv 0 \pmod 3$, it follows that $3 \nmid 4t+3$. For any proper divisor d of $4t+3$, we have $d \le \frac{4t+3}{5} \le \frac{12t+8}{9} = \frac{2n}{3^2}$. Thus, $\alpha_d \ge 2$, which leads to $2^{\alpha_d} d \nmid 2(4t+3)$. We therefore have

$$\#\{A \in \mathcal{A}(2n) : 2(4t+3) \in A\} = \#\{A \in \mathcal{A}(2n) : 4t+3 \in A\} = \frac{s(n)}{2}.$$

Since $3 \nmid 6t+5$, for any proper divisor d of $6t+5$, we have $d \le \frac{6t+5}{5} \le \frac{2(6t+5)}{9} = \frac{2n+2}{3^2}$. Thus, $\alpha'_d \ge 2$, which leads to $2^{\alpha'_d} d \nmid 2(6t+5)$. We therefore have

$$\#\{B \in \mathcal{A}(2n+2) : 6t+5 \in B\} = \#\{B \in \mathcal{A}(2n+2) : 2(6t+5) \in B\} = \frac{s(n+1)}{2}.$$

This readily implies $s(n+1) = s(n)$.

We next prove case (ii). Note that $2n+1 = 3(8t+1)$ and $2n+2 = 4(6t+1)$. Since $3(6t+1) \in A, B$, we have $\alpha_{6t+1} = 1$ and $\alpha'_{6t+1} = 1, 2$. It also follows that

$$\#\{B \in \mathcal{A}(2n+2) : 2(6t+1) \in B\} = \#\{A \in \mathcal{A}(2n) : 2(8t+1) \in A\}.$$

Since $t \not\equiv 1 \pmod 3$, it follows that $3 \nmid 8t+1$. For any proper divisor d of $8t+1$, we have $d \le \frac{8t+1}{5} \le \frac{24t+2}{9} = \frac{2n}{3^2}$. Thus, $\alpha_d \ge 2$, which leads to $2^{\alpha_d} d \nmid 2(8t+1)$. We therefore have

$$\#\{A \in \mathcal{A}(2n) : 2(8t+1) \in A\} = \#\{A \in \mathcal{A}(2n) : 8t+1 \in A\} = \frac{s(n)}{2}.$$

Since $t \not\equiv 4 \pmod 5$, it follows that $15 \nmid 6t+1$. For any proper divisor d of $6t+1$, we have $d \le \frac{6t+1}{7} \le \frac{4(6t+1)}{27} = \frac{2n+2}{3^3}$. Thus, $\alpha'_d \ge 3$, which leads to $2^{\alpha'_d} d \nmid 4(6t+1)$. We therefore have

$$\#\{B \in \mathcal{A}(2n+2) : 2(6t+1) \in B\} = \#\{B \in \mathcal{A}(2n+2) : 4(6t+1) \in B\} = \frac{s(n+1)}{2}.$$

This readily implies $s(n+1) = s(n)$. □

3. Final remarks

The first 46 members of $s(n)$ are listed in sequence A174094 of OEIS [6]. Since $46 = 6\times 7+4$ and $7 \not\equiv 0 \pmod 3$, by Theorem 1.3, we have $s(47) = s(46) = 529920$. Moreover, denote by $\hat{s}(n)$ the number of members in $\mathcal{A}(2n)$ with $a_1 = 2^{\lfloor \log_3 2n \rfloor}$. It is not difficult to see that both $s(n)$ and $\hat{s}(n)$ go to infinity as n goes to infinity. Naturally we have the following question: can we find the formula or the order of $s(n)$ or $\hat{s}(n)$?

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