Interval Estimation for Reinforcement-Learning Algorithms in Continuous-State Domains

Martha White
Department of Computing Science
University of Alberta
whitem@cs.ualberta.ca

Adam White
Department of Computing Science
University of Alberta
awhite@cs.ualberta.ca

Abstract

The reinforcement learning community has explored many approaches to obtaining value estimates and models to guide decision making; these approaches, however, do not usually provide a measure of confidence in the estimate. Accurate estimates of an agent's confidence are useful for many applications, such as biasing exploration and automatically adjusting parameters to reduce dependence on parameter-tuning. Computing confidence intervals on reinforcement learning value estimates, however, is challenging because data generated by the agent-environment interaction rarely satisfies traditional assumptions. Samples of value-estimates are dependent, likely non-normally distributed and often limited, particularly in early learning when confidence estimates are pivotal. In this work, we investigate how to compute robust confidences for value estimates in *continuous* Markov decision processes. We illustrate how to use *bootstrapping* to compute confidence intervals online under a changing policy (previously not possible) and prove validity under a few reasonable assumptions. We demonstrate the applicability of our confidence estimation algorithms with experiments on exploration, parameter estimation and tracking.

1 Introduction

In reinforcement learning, an agent interacts with the environment, learning through trial-and-error based on scalar reward signals. Many reinforcement learning algorithms estimate values for states to enable selection of maximally rewarding actions. Obtaining confidence intervals on these estimates has been shown to be useful in practice, including directing exploration [17, 19] and deciding when to exploit learned models of the environment [3]. Moreover, there are several potential applications using confidence estimates, such as teaching interactive agents (using confidence estimates as feedback), adjusting behaviour in non-stationary environments and controlling behaviour in a parallel multi-task reinforcement learning setting.

Computing confidence intervals was first studied by Kaelbling for finite-state Markov decision processes (MDPs) [11]. Since this preliminary work, many model-based algorithms have been proposed for evaluating confidences for discrete-state MDPs. The extension to continuous-state spaces with model-free learning algorithms, however, has yet to be undertaken. In this work we focus on constructing confidence intervals for online model-free reinforcement learning agents.

The agent-environment interaction in reinforcement learning does not satisfy classical assumptions typically used for computing confidence intervals, making accurate confidence estimation challenging. In the discrete case, certain simplifying assumptions make classical normal intervals more appropriate; in the continuous setting, we will need a different approach.

The main contribution of this work is a method to robustly construct confidence intervals for approximated value functions in continuous-state reinforcement learning setting. We first describe *boot-*

strapping, a non-parametric approach to estimating confidence intervals from data. We then prove that bootstrapping can be applied to our setting, addressing challenges due to sample dependencies, changing policies and non-stationarity (because of learning). Then, we discuss how to address complications in computing confidence intervals for sparse or local linear representations, common in reinforcement learning, such as tile coding, radial basis functions, tree-based representations and sparse distributed memories. Finally, we propose several potential applications of confidence intervals in reinforcement learning and conclude with an empirical investigation of the practicality of our confidence estimation algorithm for exploration, tuning the temporal credit parameter and tracking.

2 Related Work

Kaelbling was the first to employ confidence interval estimation method for exploration in finite-state MDPs [11]. The agent estimates the probability of receiving a reward of 1.0 for a given state-action pair and constructs an upper confidence bound on this estimate using a Bernoulli confidence interval. Exploration is directed by selecting the action with the highest upper confidence bound, which corresponds to actions for which it has high uncertainty or high value estimates [11].

Interval estimation for model-based reinforcement learning with discrete state spaces has been quite extensively studied. Mannor *et al.* (2004) investigated confidence estimates for the parameters of the learned transition and reward models, assuming Gaussian rewards [5, 16]. The Model Based Interval Estimation Algorithm (MBIE) uses upper confidence bounds on the model transition probabilities to select the model that gives the maximal reward [22]. The Rmax algorithm uses a heuristic notion of confidence (state visitiation counts) to determine when to explore, or exploit the learned model [3]. Both Rmax and MBIE are guaranteed to converge to the optimal policy in polynomially many steps. These guarantees, however, become difficult for continuous state spaces.

A recently proposed framework, KWIK ("Knows What It Knows"), is a formal framework for algorithms that explore efficiently by minimizing the number of times an agent must return the response "I do not know" [23]. For example, for reinforcement learning domains, KWIK-RMAX biases exploration toward states that the algorithm currently does not "know" an accurate estimate of the value [23]. KWIK-RMAX provides an uncertainty estimate (not a confidence interval) on a linear model by evaluating if the current feature vector is contained in the span of previously observed feature vectors. Though quite general, the algorithm remains theoretical due to the requirement of a solution to the model.

Bayesian methods (e.g., GPTD [6]) provide a natural measure of confidence: one can use the posterior distribution to form credible intervals for the mean value of a state-action pair. However, if one wants to use non-Gaussian priors and likelihoods, then the Bayesian approach is intractable without appropriate approximations. Although this approach is promising, we are interested in computing classical frequentist confidence intervals for agents, while not restricting the underlying learning algorithm to use a model or particular update mechanism.

Several papers have demonstrated the empirical benefits of using heuristic confidence estimates to bias exploration [14, 17, 19] and guide data collection in model learning [9, 18]. For example, Nouri *et al.* [19] discretize the state space with a KD-tree and mark the state as "known" after reaching a visitation count threshold.

In the remainder of this work, we provide the first study of estimating confidence intervals for model-free, online reinforcement learning value estimates in the continuous-state setting.

3 Background

In this section, we will introduce the reinforcement learning model of sequential decision making and bootstrapping, a family of techniques used to compute confidence intervals for means of dependent data from an unknown (likely non-normal) underlying distribution.

3.1 Reinforcement Learning

In reinforcement learning, an agent interacts with its environment, receiving observations and selecting actions to maximize a scalar reward signal provided by the environment. This interaction is

usually modeled by a Markov decision process (MDP). An MDP consists of (S, A, P, R) where S is the set of states; A is a finite set of actions; P, the transition function, which describes the probability of reaching a state s' from a given state and action (s, a); and finally the reward function $R(s, a, s')$, which returns a scalar value for transitioning from state-action (s, a) to state s'. The state of the environment is said to be *Markov* if $Pr(s_{t+1}, r_{t+1}|s_t, a_t) = Pr(s_{t+1}, r_{t+1}|s_t, a_t, \ldots, s_0, a_0)$. The agent's objective is to learn a *policy*, $\pi : S \rightarrow A$, such that R is maximized for all $s \in S$.

Many reinforcement learning algorithms maintain an *state-action value function*, $Q^\pi(s, a)$, equal to the expected discounted sum of future rewards for a given state-action pair: $Q^\pi(s, a) = E_\pi \left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | s_t = s, a_t = a\right]$, where $\gamma \in [0\ 1]$ discounts the contribution of future rewards. The optimal state-action value function, $Q^*(s, a)$, is the maximum achievable value given the agent starts in state s and selects action a. The optimal policy, π^*, is greedy with respect to the optimal value function: $\pi^*(s) = \text{argmax}_{a \in A} Q^*(s, a)$ for all $s \in S$. During learning the agent must balance selecting actions to achieve high reward (according to $\hat{Q}(s, a)$) or selecting actions to gain more information about the environment. This is called the exploration-exploitation trade-off.

In many practical applications, the state space is too large to store in a table. In this case, a function approximator is used to estimate the value of a state-action pair. A linear function approximator produces a value prediction using a linear combination of basis units: $\hat{Q}(s, a) = \boldsymbol{\theta}^T \boldsymbol{\phi}(s, a)$. We refer the reader to the introductory text [25] for a more detailed discussion on reinforcement learning.

3.2 Bootstrapping a confidence interval for dependent data

Bootstrapping is a statistical procedure for estimating the distribution of a statistic (such as the sample mean), particularly when the underlying distribution is complicated or unknown, samples are dependent and power calculations (e.g. variance) are estimated with limited sample sizes [21]. This estimate can then be used to approximate a $1 - \alpha$ confidence interval around the statistic: an interval for which the probability of seeing the statistic outside of the interval is low (probability α). For example, for potentially dependent data sampled from an unknown distribution $P(X_1, X_2, \ldots)$, we can use bootstrapping to compute a confidence interval around the mean, $T_n = n^{-1} \sum_{i=1}^{n} x_n$.

The key idea behind bootstrapping is that the data is an appropriate approximation, P_n, of the true distribution: resampling from the data represents sampling from P_n. Samples are "drawn" from P_n to produce a bootstrap sample, $x_1^*, \ldots, x_n^* \subset \{x_1, \ldots, x_n\}$, and an estimate, T_n^*, of the statistic. This process is repeated B times, giving B samples of the statistic, $T_{n,1}^*, \ldots, T_{n,B}^*$. These, for example, can be used to estimate $\text{Var}_P(T_n) \approx \text{Var}_{P_n}(T_n) = \sum (T_{n,b}^* - \overline{T}_n^*)^2/(B-1)$.

Bootstrapped intervals have been shown to have a lower coverage error than normal intervals for dependent, non-normal data. A normal interval has a coverage error of $O(1/\sqrt{n})$, whereas bootstrapping has a coverage error of $O(n^{-3/2})$ [29]. The *coverage error* represents how quickly the estimated interval converges to the true interval: higher order coverage error indicates faster convergence[1]. Though the theoretical conditions for these guarantees are somewhat restrictive [29], bootstrapping has nevertheless proved very useful in practice for more general data [4, 21].

With the bootstrapped samples, a *percentile-t (studentized) interval* is constructed by

$$P(T \in (2T_n - T_{1-\alpha/2}^*, 2T_n - T_{\alpha/2}^*)) \geq 1 - \alpha$$

where T_β^* is the β sample quantile of $T_{n,1}^*, \ldots, T_{n,B}^*$. Usually, the β-quantile of an ordered population of size n is the continuous sample quantile:

$$(1 - r)T_{n,j}^* + rT_{n,j+1}^* \qquad \text{where} \quad j = \lfloor n\beta \rfloor + m,\ r = n\beta - j + m$$

where m is dependent on quantile type, with $m = \frac{\beta+1}{3}$ common for non-normal distributions.

The remaining question is how to bootstrap from the sequence of samples. In the next section, we describe the block bootstrap, applicable to Markov processes, which we will show represents the structure of data for value estimates in reinforcement learning.

[1] More theoretically, coverage error is the approximation error in the Edgeworth expansions used to approximate the distribution in bootstrap proofs.

3.2.1 Moving Block Bootstrap

In the *moving block bootstrap* method, blocks of consecutive samples are drawn with replacement from a set of overlapping blocks, making the k-th block $\{x_{k-1+t} : t = 1 \ldots, l\}$. The bootstrap resample is the concatenation of n/l blocks chosen randomly with replacement, making a time series of length n; B of these concatenated resamples are used in the bootstrap estimate. The block bootstrap is appropriate for sequential processes because the blocks implicitly maintain a time-dependent structure. An common heuristic for the block length, l, is $n^{1/3}$ [8].

The moving block bootstrap was designed for stationary, dependent data; however, our scenario involves *nonstationary* data. Lahiri [12] proved a coverage error of $o(n^{-1/2})$ when applying the moving block bootstrap to nonstationary, dependent data, better than the normal coverage error. Fortunately, the conditions are not restrictive for our scenario, described further in the next section.

Note that there are other bootstrapping techniques applicable to sequential, dependent data with lower coverage error, such as the double bootstrap [13], block-block bootstrap [1] and Markov or Sieve bootstrap [28]. In particular, the Markov bootstrap has been shown to have a lower coverage error for Markov data than the block bootstrap under certain restricted conditions [10]. These techniques, however, have not been shown to be valid for nonstationary data.

4 Confidence intervals for continuous-state Markov decision processes

In this section, we present a theoretically sound approach to constructing confidence intervals for parametrized $Q(s, a)$ using bootstrapping for dependent data. We then discuss how to address sparse representations, such as tile coding, which make confidence estimation more complicated.

4.1 Bootstrapped Confidence Intervals for Global Representations

The goal is to compute a confidence estimate for $Q(s_t, a_t)$ on time step t. Assume that we are learning a parametrized value function $Q(s, a) = f(\boldsymbol{\theta}, s, a)$, with $\boldsymbol{\theta} \in \mathbb{R}^d$ and a smooth function $f : \mathbb{R}^d \times S \times A \to \mathbb{R}$. A common example is a linear value function $Q(s, a) = \boldsymbol{\theta}^T \boldsymbol{\phi}(s, a)$, with $\boldsymbol{\phi} : S \times A \to \mathbb{R}^d$. During learning, we have a sequence of changing weights, $\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_n\}$ up to time step n, corresponding to the random process $\{\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_n\}$. If this process were stationary, then we could compute an interval around the mean of the process. In almost all cases, however, the process will be nonstationary with means $\{\mu_1, \ldots, \mu_n\}$. Instead, our goal is to estimate

$$\bar{f}_n(s, a) = n^{-1} \sum_{t=1}^{n} E[f(\boldsymbol{\Theta}_t, s, a)]$$

which represents the variability in the current estimation of the function $\hat{Q}$ for any given state-action pair, $(s, a) \in S \times A$. Because Q is parametrized, the sequence of weights, $\{\boldsymbol{\Theta}_t\}$, represents the variability for the uncountably many state-action pairs.

Assume that the weight vector on time step $t + 1$ is drawn from the unknown distribution $P_a[(\boldsymbol{\Theta}_{t+1}, s_{t+1})|(\boldsymbol{\theta}_t, s_t), \ldots, (\boldsymbol{\theta}_{t-k}, s_{t-k})]$, giving a k-order Markov dependence on previous states and weight vectors. Notice that P_a incorporates P and R, using s_t, $\boldsymbol{\theta}_t$ (giving the policy π) and R to determine the reward passed to the algorithm to then obtain $\boldsymbol{\theta}_{t+1}$. This allows the learning algorithm to select actions using confidence estimates based on the history of the k most recent $\boldsymbol{\theta}$, without invalidating that the sequence of weights are drawn from P_a. In practice, the length of the dependence, k, can be estimated using *auto-correlation* [2].

Applying the Moving Block Bootstrap method to a non-stationary sequence of θ's requires several assumptions on the underlying MDP and the learning algorithm. We require two assumptions on the underlying MDP: a bounded density function and a strong mixing requirement. The assumptions on the algorithm are less strict, only requiring that the algorithm be non-divergent and produce a sequence of $\{Q_t(s, a)\}$ that 1) satisfy a smoothness condition (a dependent Cramer condition), 2) have a bounded twelfth moment and 3) satisfy an m-dependence relation where sufficiently separated $Q_i(s, a), Q_j(s, a)$ are independent. Based on these assumptions (stated formally in the supplement), we can prove that the moving block bootstrap produces an interval with a coverage error of $o(n^{-1/2}$ for the studentized interval on $f_n(s, a)$.

Theorem 1 *Given that Assumption 1-7 are satisfied and there exists constants $C_1, C_2 > 0$, $0 < \alpha \leq \beta < 1/4$ such that $C_1 n^{\alpha} < l < C_2 n^{\beta}$ (i.e. l increases with n), then the moving block bootstrap produces a one-sided confidence interval that is consistent and has a coverage error of $o(n^{-1/2})$ for the studentization of the mean of the process $\{f(\boldsymbol{\theta}_t, s, a)\}$, where $Q_t(s,a) = f(\boldsymbol{\theta}_t, s, a)$.*

The proof for the above theorem follows Lahiri's proof [12] for the coverage error of the moving block bootstrap for nonstationary data. The general approach for coverage error proofs involve approximating the unknown distribution with an Edgeworth expansion (see [7]), with the coverage error dependent on the order of the expansion, similar to the the idea of a Taylor series expansion.

Assuming P_a is k-order Markov results in two important practical implications on the learning algorithm: 1) inability to use eligibility traces and 2) restrictions on updates to parameters (such as the learning rate). These potential issues, however, are actually not restrictive. First, the tail of eligibility traces has little effect, particularly for larger k; the most recent k weights incorporate the most important information for the eligibility traces. Second, the learning rate, for example, cannot be updated based on time. The learning rate, however, can still be adapted based on changes between weight vectors, a more principled approach taken, by the meta-learning algorithm, IDBD [24].

The final algorithm is summarized in the pseudocode below. In practice, a window of data of length w is stored due to memory restrictions; other data selection techniques are possible. Corresponding to the notation in Section 3.2, Q_i represents the data samples (of $\hat{Q}(s,a)$), $(Q^*_{i,1}, \ldots, Q^* i, M)$ the dependently sampled blocks for the ith resample and T^*_i the mean of the i resample.

Algorithm 1 GetUpperConfidence$(f(\cdot, s, a), \{\boldsymbol{\theta}_{n-w}, \ldots \boldsymbol{\theta}_n\}, \alpha)$

$l =$ block length, $B =$ num bootstrap resamples
last w weights and confidence level α (= 0.05)

$Q_N \leftarrow \{f(\boldsymbol{\theta}_{n-w}, s, a), \ldots f(\boldsymbol{\theta}_n, s, a)\}$
Blocks = $\{[Q_{n-w}, \ldots, Q_{n-w+l-1}], [Q_{n-w+1}, \ldots, Q_{n-w+l}], \ldots, [Q_{n-l+1}, \ldots, Q_n]\}$
$M \leftarrow \lfloor w/l \rfloor$ the number of length $l$blocks to sample with replacement and concatenate
for all $i = 1$ to B **do**
  $(Q^*_1, Q^*_2, \ldots, Q^*_{M*l}) \leftarrow$ concatMRandomBlocks(Blocks, M)
  $T^*_i = \frac{1}{M*l} \sum Q^*_j$
end for
sort($\{T^*_1, \ldots, T^*_B\}$)
$j \leftarrow \lfloor \frac{B\alpha}{2} + \frac{\alpha+2}{6} \rfloor, r \leftarrow \frac{B\alpha}{2} + \frac{\alpha+2}{6} - j$
$T^*_{\alpha/2} \leftarrow (1-r)T^*_j + rT^*_{j+1}$
Return 2mean(Q_N) $- T^*_{\alpha/2}$

4.2 Bootstrapped Confidence Intervals for Sparse Representations

We have shown that bootstrapping is a principled approach for computing intervals for global representations; sparse representations, however, complicate the solution. In an extreme case, for example, for linear representations, features active on time step t may have never been active before. Samples $Q_1(s_t, a_t), \ldots, Q_t(s_t, a_t)$ would therefore all equal $Q_0(s_t, a_t)$, because the weights would have never been updated for those features. Consequently, the samples erroneously indicate low variance for $Q(s_t, a_t)$.

We propose that, for sparse linear representations, the samples for the weights can be treated independently and still produce a reasonable, though currently unproven, bootstrap interval. Notice that for $\theta(i)$ the ith feature

$$P_a[(\boldsymbol{\theta}_t, s_t)|(\boldsymbol{\theta}_{t-1}, s_{t-1}), \ldots, (\boldsymbol{\theta}_{t-k}, s_{t-k})] = \Pi^d_{i=1} P_a[(\theta_t(i), s_t)|(\boldsymbol{\theta}_{t-1}, s_{t-1}), \ldots, (\boldsymbol{\theta}_{t-k}, s_{t-k})]$$

because updates to weights $\theta(i), \theta(j)$ are independent given the previous states and weights vectors for all $i, j \in \{1, \ldots, d\}$. We could, therefore, estimate upper confidence bounds on the individual weights, $ucb_i(s,a)$, and then combine them, via $ucb(s,a) = \sum^d_{i=1} ucb_i(s,a) * \phi_i(s,a)$, to produce an upper confidence bound on $Q(s_t, a_t)$. To approximate the variance of $\theta(i)$ on time step t, we can use the last w samples of $\theta(i)$ where $\theta(i)$ changed.

Proving coverage error results for sparse representations will require analyzing the covariance between components of $\boldsymbol{\theta}$ over time. The above approach for sparse representations does not capture this covariance; due to sparsity, however, the dependence between many of the samples for $\theta(i)$ and $\theta(j)$ will likely be weak. We could potentially extend the theoretical results by bounding the covariance between the samples and exploiting independencies. The means for individual weights could likely be estimated separately, therefore, and still enable a valid confidence interval. In future work, a potential extension is to estimate the covariances between the individual weights to improve the interval estimate.

5 Applications of confidence intervals for reinforcement learning

The most obvious application of interval estimation is to bias exploration to select actions with high uncertainty. Confidence-based exploration should be comparable to optimistic initialization in domains where exhaustive search is required and find better policies in domains where noisy rewards and noisy dynamics can cause the optimistic initialization to be prematurely decreased and inhibit exploration. Furthermore, confidence-based exploration reduces parameter tuning because the policy does not require knowledge of the reward range, as in softmax and optimistic initialization.

Confidence-based exploration could be beneficial in domains where the problem dynamics and reward function change over time. In an extreme case, the agent may converge to a near-optimal policy before the goal is teleported to another portion of the space. If the agent continues to act greedily with respect to its action-value estimates without re-exploring, it may act sub-optimally indefinitely. These tracking domains require that the agent "notice" that its predictions are incorrect and begin searching for a better policy. AN example of a changing reward signals arises in interactive teaching. In this scenario, the a human teaching shapes the agent by providing a drifting reward signal. Even in stationary domains, tracking the optimal policy may be more effective than converging due to the non-stationarity introduced by imperfect function approximation [26].

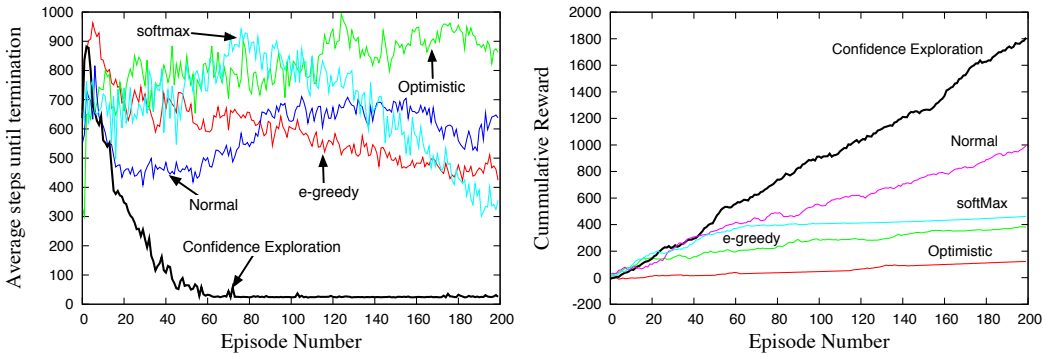
Another potential application of confidence estimation is to automate parameter tuning online. For example, many TD-based reinforcement learning algorithms use an eligibility parameter (λ) to address the credit assignment problem. Learning performance can be sensitive to γ. There has been little work, however, exploring the effects of different decay functions for λ; using different λ values for each state/feature; or for meta-learning λ. Confidence estimates could be used to increase λ when the agent is uncertain, reflecting and decrease λ for confident value estimates [25].

Confidence estimates could also be used to guide the behaviour policy for a parallel multi-task reinforcement learning system. Due to recent theoretical developments [15], several target value functions can be learned in parallel, off-policy, based on a single stream of data from a behaviour policy. The behaviour policy should explore to provide samples that generalize well between the various target policies, speeding overall convergence. For example, if one-sided intervals are maintained for each target value functions, the behaviour policy could select an action corresponding to the maximal sum of those intervals. Exploration is then biased to highly uncertain areas where more samples are required.

Finally, confidence estimates could be used to determine when features should be evaluated in a feature construction algorithm. Many feature construction algorithms, such as cascade correlation networks, interleave proposing candidate features and evaluation. In an online reinforcement learning setting, these methods freeze the representation for a fixed window of time to accurately evaluate the candidate [20]. Instead of using a fixed window, a more principled approach is to evaluate the features after the confidence on the weights of the candidate features reached some threshold.

6 Experimental Results

In this section, we provide a preliminary experimental investigation into the practicality of confidence estimation in continuous-state MDPs. We evaluate a naive implementation of the block bootstrap method for (1) exploration in a noisy reward domain, (2) automatically tuning λ in the Cartpole domain and (3) tracking a moving goal in a navigation task. In all tests we used the Sarsa(λ) learning algorithm with tile coding function approximation (see Sutton and Barto [25]). All experiments were evaluated using RL-Glue [27] and averaged over 30 independent runs.



(a) Exploration: convergence

(b) Exploration: comparison

Figure 1: Results showing (a) convergence of various exploration techniques in the navigation task and (b) average cumulative reward of various exploration techniques on the navigation task.

6.1 Exploration

To evaluate the effectiveness of confidence-based exploration, we use a simple two-goal continuous navigation task. The *small goal* yields a reward of 1.0 on every visit. The *flashing goal* yields a reward selected uniformly from $\{100, -100, 5, -5, 50\}$. The reward on all other steps is zero and $\gamma = 0.99$ (similar results for -1 per step and $\gamma = 1.0$). The agent's observation is a continuous (x, y) position and actions move the agent $\{$N,S,E,W$\}$ perturbed by uniform noise 10% of the time. We present only the first 200 episodes to highlight early learning performance.

Similar to Kaelbling, we select the action with the highest upper confidence in each state. We compare our confidence exploration algorithm to three baselines commonly used in continuous state MDPs: (1) ϵ-greedy (selecting the highest-value action with probability $1 - \epsilon$, random otherwise), (2) optimistic initialization (initializing all weights to a high fixed value to encourage exploration) and (3) softmax (choosing actions probabilistically according to their values). We also compare our algorithm to an exploration policy using normal (instead of bootstrapped) intervals to investigate the effectiveness of making simplifying assumptions on the data distribution. We present the results for the best parameter setting for each exploration policy for clarity. Figure 1 summarizes the results.

The ϵ-greedy policy convergences slowly to the small goal. The optimistic policy slowly converges to the small goal for lower initializations and does not favour either goal for higher initializations. The softmax policy navigates to the small goal on most runs and also convergences slowly. The normal-interval exploration policy does prefer the flashing goal but not as quickly as the bootstrap policy. Finally, the bootstrap-interval exploration policy achieves highest cumulative reward and is the only policy that converges to the flashing goal, despite the large variance in the reward signal.

6.2 Adjusting Lambda

To illustrate the effect of adjusting λ based on confidence intervals, we study the Cartpole problem. We selected Cartpole because the performance of Sarsa is particularly sensitive to λ in this domain. The objective in Cartpole is to apply forces to a cart on a track to keep a pole from falling over. An episode ends when the pole falls past a given angle or the cart reaches the end of the track. The reward is +1 for each step of the episode. The agent's observations are the cart position and velocity and the poles' angle and angular velocity. The Cartpole environment is based on Sutton and Barto's [25] pole-balancing task and is available in RL-library [27].

To adjust the λ value, we reset λ on every time step: $\lambda = \text{normalized}(ucb)$ where $ucb = 0.9 * ucb + 0.1 * \text{getUpperConfidence}(\boldsymbol{\phi}(s, a), \theta, \alpha)$. The confidence estimates were only used to adjust λ for clarity: exploration was performed using optimistic initialization. Figure 2 presents the average balancing time on the last episode for various values of λ. The flat line depicts the average balancing time for Sarsa with λ tuned via confidence estimates. Setting λ via confidence estimates achieves performance near the best value of λ. We also tested adjusting λ using normal confidence intervals, however, the normal confidence intervals resulted in worse performance then any fixed value of λ.

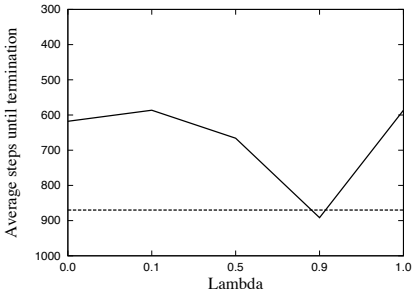


Figure 2: Performance of Sarsa(λ) on Cartpole for various values of λ. The straight line depicts the performance of Sarsa with λ adjusted using the confidence estimation algorithm.

6.3 Non-stationary Navigation Task

One natural source of non-stationarity is introduced by *shaping* a robot through successive approximations to a goal task (e.g., changing the reward function). We studied the effects of this form of non-stationarity, where the agent learns to go to a goal and then another, better goal becomes available (near the first goal to better guide it to the next goal). In our domain, the agent receives -1 reward per step and +10 at termination in a goal region. After 150 episodes, the goal region is teleported to a new location within 50 steps of the previous goal. The agent receives +10 in the new goal and now 0 in the old goal. We used $\epsilon = 0$ to enable exploration only with optimistic initialization.

We recorded the number of times the agent converged to the new goal with the change after an initial learning period of 150 episodes. The bootstrap-based explorer found the new goal 70% of the time. It did not always find the new goal because the -1 structure biased it to stay with the safe 0 goal. Interestingly, optimistic initialization was unable to find the new goal because of this bias, illustrating that the confidence-based explorer detected the increase in variance and promoted re-exploration automatically.

7 Conclusion

In this work, we investigated constructing confidence intervals on value estimates in the continuous-state reinforcement learning setting. We presented a robust approach to computing confidence estimates for function approximation using bootstrapping, a nonparametric estimation technique. We proved that our confidence estimate has low coverage error under mild assumptions on the learning algorithm. In particular, we did so even for a changing policy that uses the confidence estimates. We illustrated the usefulness of our estimates for three applications: exploration, tuning λ and tracking.

We are currently exploring several directions for future work. We have begun testing the confidence-based exploration on a mobile robot platform. Despite the results presented in this work, many traditional deterministic, negative cost-to-goal problems (e.g., Mountain Car, Acrobot and Puddle World) are efficiently solved using optimistic exploration. Robotic tasks, however, are often more naturally formulated as continual learning tasks with a sparse reward signal, such as negative reward for bumping into objects, or a positive reward for reaching some goal. We expect confidence based techniques to perform better in these settings where the reward range may be truly unknown (e.g. generated dynamically by a human teacher) and under natural variability in the environment (noisy sensors and imperfect motion control). We have also begun evaluating confidence-interval driven behaviour for large-scale, parallel off-policy learning on the same robot platform.

There are several potential algorithmic directions, in addition to those mentioned throughout this work. We could potentially improve coverage error by extending other bootstrapping techniques, such as the Markov bootstrap, to non-stationary data. We could also explore the theoretical work on exponential bounds, such as the Azuma-Hoeffding inequality, to obtain different confidence estimates with low coverage error. Finally, it would be interesting to extend the theoretical results in the paper to sparse representations.

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Interval Estimation for Reinforcement-Learning Algorithms in Continuous-State Domains: Supplementary Material

Martha White
Department of Computing Science
University of Alberta
whitem@cs.ualberta.ca

Adam White
Department of Computing Science
University of Alberta
awhite@cs.ualberta.ca

Applicability Proofs for Block Bootstrapping in Reinforcement Learning

We prove the consistency and coverage error for the bootstrapped studentized interval around our the sample mean of the sequence of parameters for global function approximation. Each parameter vector $\boldsymbol{\theta}_t$ on time step t corresponds to the action-value function Q_t on that time step, with $Q(s,a) = f(\boldsymbol{\theta}, s, a)$ for some bounded function f. A common example of f is a linear function $f(\boldsymbol{\theta}, s, a) = \boldsymbol{\theta}^T \boldsymbol{\phi}(s,a)$ with features given by the function $\boldsymbol{\phi} : S \times A \rightarrow \mathbb{R}^d$.

Let $\{\boldsymbol{\theta}_t\}$ be the sequence of weight vectors, drawn from probability distributions $P_a(< \boldsymbol{\Theta}_t, s_t > | < \boldsymbol{\theta}_{t-1}, s_{t-1} >, \ldots, < \theta_{t-k-1}, s_{t-k-1})$ with means μ_t. For a given state-action pair $(s,a) \in S \times A$, with $g(\boldsymbol{\theta}) = f(\boldsymbol{\theta}, s, a)$, we are estimating

$$\bar{g}_n = n^{-1} \sum_{i=1}^{n} g_i$$

where $g_i = E[(g(\boldsymbol{\Theta}_i)]$.

In order to prove a coverage error of $o(n^{-1/2})$ for the studentized interval on $f(\mu_n, s, a)$ for any given $(s,a) \in S \times A$, we will need the following assumptions, simplified from Lahiri's Theorem 4.1 [5] for our scenario. The proof will be for any (s,a), so we fix an $(s,a) \in S \times A$ and let $g(\theta) = f(\theta, s, a)$.

Let $Y_n =< S_n, A_n, R_n >$, the triplet obtained from acting in the given MDP, $G = (S, A, P, R)$. The triplets are drawn from the implicit from the probability distribution $P_R(s,a,r)$ computed using $P(s,a,s')$ and $R(s,a,s')$, giving the probability of receiving reward r after taking action a in state s. Let $D_j = \sigma(Y_j)$, the σ-fields of the random variables Y_n.

Assumption 1 *For any $(s,a) \in S \times A$, the density function, $P_R(s,a,\cdot) : R \rightarrow \mathbb{R}$ is continuous and bounded.*

This assumption is required so that we can approximate this (continuous) density function with infinitely many samples from Y_n. This in turn enables us to define the σ-fields in terms of the Y_n, placing the strong mixing assumptions instead on Y_n, rather than on our sequence, $\{\boldsymbol{\theta}_t\}$. The next two assumptions are the typical assumptions placed on the sequence of σ-fields for bootstrapping proofs. Essentially, Assumptions 2 and 3 both place a strong mixing assumptions on Y_n. Mixing assumptions are a common restriction in reinforcement learning, related to ergodicity of an MDP [2, 7]. Any MDP satisfying the above mixing assumptions is ergodic [3]. Any stationary positive recurrent Markov chain (essentially ergodic) with trivial tail field is strongly mixing [6](Page 553).

Assumption 2 *There exists $d > 0$ such that for all $m, n \in \mathbb{N}$, $A \in D_{-\infty}^{n}$ and $B \in D_{n+m}^{\infty}$,*

$$|P(A \cap B) - P(A)P(B)| \leq d^{-1} e^{-dm}$$

Assumption 3 *There exists $d > 0$ such that for all $m, n, q \in \mathbb{N}$, $A \in D_{m-q}^{m+q}$,*

$$E|P(A|D_j : j \neq n) - P(A|D_j : 0 < |n - j| \leq m + q)| \leq d^{-1}e^{-dm}$$

The remaining assumptions are on the sequence of function values, $\{g(\boldsymbol{\theta}_t)\}$ (implicitly assumptions on the sequence of weights $\{\boldsymbol{\theta}_t\}$). The assumptions include boundedness assumptions on certain moments of the sequence, a smoothness condition and an m-dependence requirement.

Assumption 4 *g is a continuos bounded function and $\sup_{j\geq 1} E||g(\boldsymbol{\theta}_j)||^{12} < \infty$*

Assumption 5 *(Conditional Cramer condition) There exists $d > 0$ such that for all $m, n \in \mathbb{N}$, $d^{-1} < m < n$ and all $t \in \mathbb{R}$ with $t \geq d$,*

$$E|E[e^{it(g(\boldsymbol{\theta}_{n-m})+\ldots+g(\boldsymbol{\theta}_{n+m})}|D_j : j \neq n]| \leq e^{-d}$$

Assumption 6 *For $|s - t| > m$, $\mathrm{Cov}(g(\boldsymbol{\theta}_s), g(\boldsymbol{\theta}_t)) = 0$*

We expect our Q-values to change smoothly and to remain reasonably bounded, so Assumptions 4 and 5 are unrestrictive assumptions. The m-dependence assumption is slightly stronger, though m can be quite large, so we can still have quite long-range dependence.

Finally, we want to say something about the properties of the algorithm, essentially restricting to non-divergent algorithms and those that have a measurable update function.

Assumption 7 *$M_n = \mathrm{Var}(n^{-1/2}(g(\boldsymbol{\theta}_1)+\ldots+g(\boldsymbol{\theta}_n)))$ is non-singular $\forall\, n$ and $M = \lim_{n\to\infty} M_n$ exists and is non-singular.*

This assumptions is necessary to ensure that we reach a normal distribution in the limit and can therefore use an Edgeworth expansion to approximate the true distribution. Notice therefore that for non-convergent (but not divergent) sequences of $\boldsymbol{\theta}_n$, we can still apply the bootstrap, as long as the conditions on M_n are met. For convergent sequences, $M_n \to 0$, because the weights stop changing; however, in practice, the weights will always oscillate around the true $\boldsymbol{\theta}_0$. Therefore, assuming the above does not practically restrict convergence. We expect that we can drop this condition and in fact expect many of the conditions to simplify assuming $\boldsymbol{\theta}_t \to \boldsymbol{\theta}_0$, but we leave this to future work.

Assumption 8 *For the algorithm update function $U : R \times \mathbb{R}^{k\times d} \to \mathbb{R}^d$ which takes the reward and last k weight vectors as input to produce the next weight vector, for given k weight vectors, $U(\cdot, \boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1}, \ldots, \boldsymbol{\theta}_{t-k})$ is a measurable function.*

This assumption is not restrictive, as most algorithm will smoothly update $\boldsymbol{\theta}$ based on the reward.

With just these assumptions, we can proof the main result.

Theorem 1 *Given that Assumption 1-8 are satisfied and there exists constants $C_1, C_2 > 0$, $0 < \alpha \leq \beta < 1/4$ such that $C_1 n^\alpha < l < C_2 n^\beta$ (i.e. l increases with n), then the moving block bootstrap produces a one-sided confidence interval that is consistent and has a coverage error of $o(n^{-1/2})$ for the studentization of the mean of the process $\{Q_1(s, a) = f(\boldsymbol{\theta}_1, s, a), Q_2(s, a) = f(\boldsymbol{\theta}_2, s, a), \ldots\}$.*

Proof: For the proof, we need to satisfy the seven assumptions in Lahiri's Theorem 4.1 [5]. We will call these assumptions *requirements* to distinguish them from our assumptions. The proof will be organized based on these requirements (which will be stated below). The statements of the requirements will be italicized with justification of how that requirement is satisfied following the italicized statement.

Requirement 1 (C.1'): *$\sup_{j\geq 1} E||g(\boldsymbol{\theta}_j)||^4 < \infty$ and $M = \lim_{n->\infty} M_n$ exists and is non-singular* Satisfied by Assumptions 4 and 8

Requirement 2 (C.2): *There exists a $d > 0$ such that for $n, m \in \mathbb{N}$ with $m > d^{-1}$, there exists a D_{n-m}^{n+m} measurable p-variate random vector $\bar{Y}_{n,m}$ for which*

$$E||g(\boldsymbol{\theta}_n) - \bar{Y}_{n,m}|| \leq d^{-1}e^{-dm}$$

C.2 is satisfied by Assumption 1 and by the construction of Y_n, which we justify in the following. Essentially, this requirement says that for infinitely many Y_n, we can accurately approximate $g(\boldsymbol{\theta}_n)$. In Theorem 1 [4], an exponential decrease rate is proved for a kernel density estimator. Choose a kernel K satisfying:

1. The kernel function K is a probability density of bounded variation such that $\int K^2(u)du < \infty$; further, the derivative K' exists and is integrable.

2. For the parameters h_n, p_n defined in [4], $nh_n/p_n \to \infty$.

Now, by the theorem, because P_R is bounded and continuous and Y_n is strongly mixing, the error in the approximation of P_R using samples from Y_n decreases exponentially with n for some constants c_1, c_2 (based on norms on K, some constants, etc.), at a rate of

$$err \leq c_1 e^{-c_2 n} \tag{1}$$

With an accurate P_R distribution for a given (s, a), we can exactly compute the mean for the distribution $P_a(< \boldsymbol{\Theta}_t, s_t > \mid < \boldsymbol{\theta}_{t-1}, s_{t-1} >, \ldots, < \boldsymbol{\theta}_{t-k}, s_{t-k} >)$. Let $\bar{Y}_{n,m}$ be the function g applied to the approximation of this distribution using the current approximation of P_R with the $2m$ samples, $Y_{n-m}, \ldots, Y_{n+m}$. The construction of $\bar{Y}_{n,m}$ is possible because we have a measurable function (Borel function) between the σ-fields (on $Y_{n-m}, \ldots, Y_{n+m}$) and the σ-field on $g(\boldsymbol{\Theta})$ drawn from $P_a(< g(\boldsymbol{\Theta}_t), s_t > \mid < \boldsymbol{\theta}_{t-1}, s_{t-1} >, \ldots, < \boldsymbol{\theta}_{t-k}, s_{t-k} >)$. We have this measurable function because the kernel estimator is continuous, the update from a given reward to weight is measurable and g is continuous, making the composition measurable. Therefore, the error between this approximated mean and the L_2 norm of the random variable will decrease with the some C times the above rate (Equation 1), for some large enough C (which exists because P_R and g are bounded).

Setting $d = \min\{(C \times c_1)^{-1}, c_2\}$, we obtain the desired result.

Requirement 2 (C.2 (ii)): $\sup_{j \geq 1} E||g(\boldsymbol{\theta}_j)||^{12} < \infty$
The twelfth moment is bounded by Assumption 4.

Requirements 3 and **5** (C.3 and C.5) correspond exactly to our Assumptions 2 and 3.

Requirement 4 (C.4) corresponds exactly to our smoothness Cramer condition, Assumption 5.

Finally, there are two more requirements that restrict the heterogeneity of the means μ_t asymptotically. Let

$$\begin{aligned} m_t &= EU_t = l^{-1} \sum_{j=1}^{l} g_{t+j-1} \\ M_{nt} &= \mathrm{Var}(\sqrt{l} U_t) \\ U_t &= l^{-1} \sum_{j=1}^{l} g(\boldsymbol{\Theta}_{t+j-1}) \end{aligned}$$

where l is the block length and $1 \leq t \leq b$. Recall $M_n = \mathrm{Var}(n^{-1/2}(g(\boldsymbol{\Theta}_1) + \ldots + g(\boldsymbol{\Theta}_n)))$ and $\bar{g}(\mu_n) = n^{-1}(g_1, \ldots, g_n)$, where $g_i = E[g(\boldsymbol{\Theta}_i]$.

Since the sequence of μ_t eventually reaches it's limiting distribution (stationary) with mean μ and variance σ, for $n_0 \in \mathbb{N}$, $\mu = \mu_t = \mu_{t+1} = \ldots$ for all $t > n_0$. Note that this Markov chain becomes stationary because the $\boldsymbol{\theta}_t$ are drawn from a time-homogenous Markov chain, and time-homogenous Markov chains always reach a limiting distribution [1]. This fact enables us to satisfy the asymptotic heterogeneity conditions.

Requirement 6 (C.6):

$$\lim_{n \to \infty} \max\{l^2 ||m_t - \bar{\mu}_n|| : 1 \leq t \leq b\} = 0$$

Requirement 7 (C.7):[1]

$$\lim_{n\to\infty} \max\{l^2||M_{nt} - M_n|| : 1 \leq t \leq b\} = 0$$

Note that as $n \to \infty$, then by the assumption that $C_1 n^\alpha < l < C_2 n^\beta$, l also goes to ∞. As $l \to \infty$, the finite initial number of samples with means different from μ (the stationary distribution) will be dominated by the infinite tail of samples in the stationary distribution with mean μ. Therefore, clearly both $m_t \to g(\mu)$ and $\bar{\mu}_n \to g(\mu)$as $n \to \infty$. Similarly, the tail has the same variances; therefore, the difference between $M_{n,t}$ and M_n goes to zero.

Therefore, because our data meets the assumptions in [5], we know that the bootstrap is consistent and the coverage error for the studentized confidence interval on $\{Q_n\}$ is $o(n^{-1/2})$.

■

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[1] Note that C.7 is slightly different in the theorem, but Lahiri mentions that it can be simplified to what we have here because of requirement C.1