Calculations For “A Note on Dhillon (1998)”

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Equation (4)

Equation (3) is equivalent to:

$$\hat{\lambda}_1 + \hat{\lambda}_4 = c + d \quad (1)$$
$$\hat{\lambda}_2 + \hat{\lambda}_4 = 2c + d \quad (2)$$
$$\hat{\lambda}_3 = d \quad (3)$$

Subtracting the first equation from the second, we obtain:

$$\hat{\lambda}_2 - \hat{\lambda}_1 = c \quad (4)$$

Thus, we need $\hat{\lambda}_2 - \hat{\lambda}_1 > 0$ to ensure that $c > 0$. Substituting from (4) for c, and from (3) for d into the first equation, we obtain:

$$\hat{\lambda}_1 + \hat{\lambda}_4 = \hat{\lambda}_2 - \hat{\lambda}_1 + \hat{\lambda}_3 \Leftrightarrow \quad (5)$$
$$\hat{\lambda}_2 + \hat{\lambda}_3 = 2\hat{\lambda}_1 + \hat{\lambda}_4 \quad (6)$$

Thus, any solution to (3) in the main text must satisfy (4) in the main text. On the other hand, given any solution that satisfies (4) in the main text, we can use (4) and (3) above to determine c and d such that (3) in the main text holds.

Equation (7)

The first three conditions of equation (6) are:

$$\hat{\lambda}_1 + \lambda_4 = 7c + d \quad (7)$$
$$\hat{\lambda}_2 + \lambda_4 = 10c + d \quad (8)$$
$$\hat{\lambda}_3 + \lambda_4 = 6c + d \quad (9)$$

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Subtracting (7) from (8) we obtain:

$$\hat{\lambda}_2 - \hat{\lambda}_1 = 3c \tag{10}$$

Subtracting (9) from (7), we get:

$$\hat{\lambda}_1 - \hat{\lambda}_3 = c \tag{11}$$

Using (11) to substitute for c in (10), we get:

$$\hat{\lambda}_2 - \hat{\lambda}_1 = 3(\hat{\lambda}_1 - \hat{\lambda}_3) \Leftrightarrow \tag{12}$$

$$4\hat{\lambda}_1 - \hat{\lambda}_2 - 3\hat{\lambda}_3 = 0 \tag{13}$$

Equation (10)

We can write out (9) as:

$$\hat{\lambda}_1 + \lambda_3 + \hat{\lambda}_4 = 12c + d \tag{14}$$

$$\hat{\lambda}_2 + \lambda_3 + \hat{\lambda}_4 = 15c + d \tag{15}$$

$$\lambda_3 = 5c + d \tag{16}$$

$$0 = d \tag{17}$$

Using (17) to drop d from (14)-(16), and then using (16) to replace λ_3 in (14) and (15), we get:

$$\hat{\lambda}_1 + 5c + \hat{\lambda}_4 = 12c \tag{18}$$

$$\hat{\lambda}_2 + 5c + \hat{\lambda}_4 = 15c \tag{19}$$

which is equivalent to:

$$\hat{\lambda}_1 + \hat{\lambda}_4 = 7c \tag{20}$$

$$\hat{\lambda}_2 + \hat{\lambda}_4 = 10c \tag{21}$$

We solve (21) for c, and substitute into (20) to get:

$$\hat{\lambda}_1 + \hat{\lambda}_4 = \frac{7}{10}(\hat{\lambda}_2 + \hat{\lambda}_4) \Leftrightarrow \tag{22}$$

$$10\hat{\lambda}_1 + 10\hat{\lambda}_4 = 7\hat{\lambda}_2 + 7\hat{\lambda}_4 \Leftrightarrow \tag{23}$$

$$10\hat{\lambda}_1 - 7\hat{\lambda}_2 + 3\hat{\lambda}_4 = 0 \tag{24}$$

Equation (13)

We can write out (12) as:

$$\begin{aligned} \hat{\lambda}_1 + \lambda_2 + \hat{\lambda}_4 &= 16c + d & (25) \\ \lambda_2 + \hat{\lambda}_4 &= 10c + d & (26) \\ \lambda_2 + \hat{\lambda}_3 &= 14c + d & (27) \\ \hat{\lambda}_3 &= 5c + d & (28) \end{aligned}$$

We use (28) to replace d by $\hat{\lambda}_3 - 5c$ in (25)-(27):

$$\begin{aligned} \hat{\lambda}_1 + \lambda_2 + \hat{\lambda}_4 &= 11c + \hat{\lambda}_3 & (29) \\ \lambda_2 + \hat{\lambda}_4 &= 5c + \hat{\lambda}_3 & (30) \\ \lambda_2 + \hat{\lambda}_3 &= 9c + \hat{\lambda}_3 & (31) \end{aligned}$$

Equation (32) tells us that λ_2 equals $9c$ which we substitute into (29) and (30) to get:

$$\begin{aligned} \hat{\lambda}_1 + 9c + \hat{\lambda}_4 &= 11c + \hat{\lambda}_3 & (32) \\ 9c + \hat{\lambda}_4 &= 5c + \hat{\lambda}_3 & (33) \end{aligned}$$

We solve (33) for c, and substitute into (32) to get:

$$\begin{aligned} \hat{\lambda}_1 + \hat{\lambda}_4 &= 2 \cdot \frac{\hat{\lambda}_3 - \hat{\lambda}_4}{4} + \hat{\lambda}_3 \Leftrightarrow & (34) \\ 2\hat{\lambda}_1 + 2\hat{\lambda}_4 &= \hat{\lambda}_3 - \hat{\lambda}_4 + 2\hat{\lambda}_3 \Leftrightarrow & (35) \\ 2\hat{\lambda}_1 - 3\hat{\lambda}_3 + 3\hat{\lambda}_4 &= 0 & (36) \end{aligned}$$

Equation (14)

Equations (7), (10) and (13) from the text are:

$$\begin{aligned} 4\hat{\lambda}_1 - \hat{\lambda}_2 - 3\hat{\lambda}_3 &= 0 & (37) \\ 10\hat{\lambda}_1 - 7\hat{\lambda}_2 + 3\hat{\lambda}_4 &= 0 & (38) \\ 2\hat{\lambda}_1 - 3\hat{\lambda}_3 + 3\hat{\lambda}_4 &= 0 & (39) \end{aligned}$$

We solve (39) for $3\hat{\lambda}_4$ and substitute into (38), to get:

$$\begin{aligned} 10\hat{\lambda}_1 - 7\hat{\lambda}_2 + 3\hat{\lambda}_3 - 2\hat{\lambda}_1 &= 0 \Leftrightarrow & (40) \\ 8\hat{\lambda}_1 - 7\hat{\lambda}_2 + 3\hat{\lambda}_3 &= 0 & (41) \end{aligned}$$

We add (37) and (41) and get:

$$\begin{aligned} 12\hat{\lambda}_1 - 8\hat{\lambda}_2 &= 0 \Leftrightarrow & (42) \\ \frac{\hat{\lambda}_1}{\hat{\lambda}_2} &= \frac{2}{3} & (43) \end{aligned}$$

We solve (43) for $\hat{\lambda}_1$ and then substitute into (37) to get:

$$\begin{aligned} \frac{8}{3}\hat{\lambda}_2 - \hat{\lambda}_2 - 3\hat{\lambda}_3 &= 0 \Leftrightarrow && (44) \\ \frac{\hat{\lambda}_2}{\hat{\lambda}_3} &= \frac{9}{5} && (45) \end{aligned}$$

We can combine (43) and (45) to obtain:

$$\hat{\lambda}_1 = \frac{2}{3} \cdot \frac{9}{5}\hat{\lambda}_3 = \frac{6}{5}\hat{\lambda}_3 \quad (46)$$

Plugging (46), and also (45) (solved for $\hat{\lambda}_2$) into (39), we get:

$$\begin{aligned} \frac{12}{5}\hat{\lambda}_3 - 3\hat{\lambda}_3 + 3\hat{\lambda}_4 &= 0 \Leftrightarrow && (47) \\ \frac{\hat{\lambda}_3}{\hat{\lambda}_4} &= \frac{5}{1} && (48) \end{aligned}$$

Equation (20)

We can write out (19) as:

$$\begin{aligned} \lambda_1 &= c + d && (49) \\ \lambda_1 + \lambda_2 &= 2c + d && (50) \\ \lambda_3 &= c + d && (51) \end{aligned}$$

We thus conclude from (49) and (51) that $\lambda_1 = \lambda_3$. Subtracting (49) from (50) we obtain:

$$\lambda_2 = c \quad (52)$$

If we choose $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 = \lambda_3$, then we can use (52) to determine c, and then (49) to determine d.