Supplemental Materials: Preliminary test and Stein-type shrinkage ridge estimators in robust regression

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ABSTRACT

This supplement contains the proofs of the theorems. It also provides more comparisons between the estimators and the simulation study nd the real data application.

Key words: M-estimation; Multicollinearity; Non-sample information; Outliers; Ridge regression; Shrinkage methods.

The supplement is divided as follows. Section S1 contains the detailed proofs of theorems. In Section S2 we analyze the performance of proposed ridge Huberized estimators in detail. Sections S3 and S4 show omitted information on the simulation study and real data application, respectively. To prove the theorems, we need the ADB, ADQB, and ADR of M-estimators.

Under the foregoing regularity conditions and the local alternatives $\{\mathcal{K}_{(n)}\}$, the ADB, ADQB, and ADR of M-estimators are given by

(i)
$$\mathrm{ADB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}) = \mathbf{0}, \tag{S1}$$
$$\mathrm{ADQB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}) = 0, \tag{S2}$$
$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}; W) = \eta^2 \operatorname{tr}(WC^{-1}), \tag{S3}$$

(ii)
$$\mathrm{ADB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}) = -\boldsymbol{\delta} H_{p+2}\left(\chi_p^2(\alpha); \Delta^2\right), \quad \text{where} \quad \Delta^2 = \eta^{-2}\boldsymbol{\delta}^T C \boldsymbol{\delta} \tag{S4}$$
$$\mathrm{ADQB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}) = \Delta^2\{H_{p+2}(\chi_p^2(\alpha); \Delta^2)\}^2, \tag{S5}$$
$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}; W) = \eta^2 \operatorname{tr}(WC^{-1})(1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) + \boldsymbol{\delta}^T W \boldsymbol{\delta} h_2(\alpha; \Delta^2), \tag{S6}$$

(iii)
$$\mathrm{ADB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}) = -c\boldsymbol{\delta} E\left[\chi_{p+2}^{-2}(\Delta^2)\right], \tag{S7}$$
$$\mathrm{ADQB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}) = c^2\Delta^2\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\}^2, \tag{S8}$$
$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}; W) = \eta^2 \operatorname{tr}(WC^{-1}) - c\eta^2 \operatorname{tr}(WC^{-1}) h_3(\Delta^2) + c(c+4)\boldsymbol{\delta}^T W \boldsymbol{\delta} E\left[\chi_{p+4}^{-4}(\Delta^2)\right], \tag{S9}$$

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(iv) $$\text{ADB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})\text{S}+}) = -\boldsymbol{\delta}\left\{cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] + E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))I(\chi_{p+2}^2(\Delta^2)\le c)\right]\right\}, \quad \text{(S10)}$$

$$\text{ADQB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})\text{S}+}) = \Delta^2\left\{cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] - E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))I(\chi_{p+2}^2(\Delta^2)\le c)\right]\right\}^2 \quad \text{(S11)}$$

$$\begin{aligned}\text{ADR}(\hat{\boldsymbol{\beta}}_n^{(\text{M})\text{S}+}; W) &= \text{ADR}(\hat{\boldsymbol{\beta}}_n^{(\text{M})\text{S}}; W) + \boldsymbol{\delta}^T W\boldsymbol{\delta} h_4(\Delta^2)\\ &\quad -\eta^2\,\text{tr}(WC^{-1})E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right], \end{aligned}\quad \text{(S12)}$$

S1 Proof of The Theorems

Proof of Theorem 3.1: First, let us prove the following relation.

$$kC^{-1}(k) = I_p - R(k) \quad \text{(S13)}$$

Since we know that for all $k > 0$, $R(k) = (I_p + kC^{-1})^{-1}$. Then,

$$\begin{aligned} & R^{-1}(k) = I_p + kC^{-1}\\ \iff\ & kC^{-1} = R^{-1}(k) - I_p\\ \implies\ & kR(k)C^{-1} = I_p - R(k)\\ \implies\ & k(I_p + kC^{-1})^{-1}C^{-1} = I_p - R(k)\\ \implies\ & k(C(I_p + kC^{-1})^{-1} = I_p - R(k)\\ \implies\ & k(C + kI_p)^{-1} = I_p - R(k)\\ \implies\ & -kC^{-1}(k) = R(k) - I_p \end{aligned}$$

Now, suppose that $\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}(k)$ be an improved ridge M estimator. By the definition (23), the ADB of this estimator is given by

$$\begin{aligned}\text{ADB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}(k)) &= \lim_{n\to\infty} E\left[\sqrt{n}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}(k) - \boldsymbol{\beta}\right]\\ &= \lim_{n\to\infty} E\left[\sqrt{n}(R_n(k)\hat{\boldsymbol{\beta}}_n^{(\text{M})^*} - \boldsymbol{\beta})\right]\\ &= \lim_{n\to\infty} E\left[\sqrt{n}(R_n(k)\hat{\boldsymbol{\beta}}_n^{(\text{M})^*} - R_n(k)\boldsymbol{\beta} + R_n(k)\boldsymbol{\beta} - \boldsymbol{\beta})\right]\\ &= \lim_{n\to\infty} E\left[R_n(k)\sqrt{n}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*} - \boldsymbol{\beta})\right] + \lim_{n\to\infty} E\left[(R_n(k) - I_p)\sqrt{n}\boldsymbol{\beta}\right]\\ &= R(k)\text{ADB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}) - kC^{-1}(k)\boldsymbol{\delta}\end{aligned}$$

The lase line is resulted by Slustkey's theorem and Eq. (S13). We leave it to reader to substituting one of Eqs. (S1),(S4),(S7), and (S10) in the above expression and obtain the ADB of the ridge-type estimators.

The ADQB of $\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}(k)$ follows easily from the Eq. (24) as

$$\begin{aligned}\text{ADQB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}(k)) &= R(k)^T[\text{ADB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*})]^T R(k)\text{ADB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}) + k^2\boldsymbol{\delta}^T C^{-2}(k)\boldsymbol{\delta}\\ &\quad -2k\boldsymbol{\delta}^T C^{-1}(k)R(k)\text{ADB}(\hat{\boldsymbol{\beta}}_n^{(\text{M})^*}).\end{aligned}\quad \text{(S14)}$$

By Eqs. (S1),(S4),(S7), and (S10) and substituting them in (S14), the proof finishes.

Proof of Theorem 3.2: If $\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}(k)$ be an improved ridge M estimator, its ADR is

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}; W) &= \lim_{n\to\infty} E\left[n(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}(k)-\boldsymbol{\beta})^T W(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}(k)-\boldsymbol{\beta})\right] \\
&= \lim_{n\to\infty} E\left[n(R_n(k)\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-\boldsymbol{\beta})^T W(R_n(k)\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-\boldsymbol{\beta})\right] \\
&= \lim_{n\to\infty} E\left[n(R_n(k)\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-R_n(k)\boldsymbol{\beta}+R_n(k)\boldsymbol{\beta}-\boldsymbol{\beta})^T W(R_n(k)\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}\right. \\
&\qquad \left. -R_n(k)\boldsymbol{\beta}+R_n(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right] \\
&= \lim_{n\to\infty} E\left[n(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-\boldsymbol{\beta})^T[R_n(k)]^T W R_n(k)(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-\boldsymbol{\beta})\right] \\
&\quad + \lim_{n\to\infty} E\left[n\boldsymbol{\beta}^T(R_n(k)-I_p)^T W R_n(k)(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-\boldsymbol{\beta})\right] \\
&\quad + \lim_{n\to\infty} E\left[n(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}-\boldsymbol{\beta})^T[R_n(k)]^T W(R_n(k)-I_p)\boldsymbol{\beta}\right] \\
&\quad + \lim_{n\to\infty} E\left[n\boldsymbol{\beta}^T(R_n(k)-I_p)^T W(R_n(k)-I_p)\boldsymbol{\beta}\right]
\end{aligned}$$

By the definition of asymptotic distribution bias and risk and also, Eq. (S13), we have

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}; W) &= \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*}; W^*) - k\left[\boldsymbol{\delta}^T C^{-1}(k) W R(k)\mathrm{ADB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*})\right. \\
&\qquad \left. +[\mathrm{ADB}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})*})]^T[R_n(k)]^T W C^{-1}(k)\boldsymbol{\delta}\right] \qquad \text{(S15)}
\end{aligned}$$

We leave it to the reader to obtain the expressions presented in Theorem 3.2 by using one of Eqs. (S3),(S6),(S9), and (S12) in the Eq. (S15).

S2 Comparison between estimators

Proof of Theorem 4.1 (Comparison between PRRME and PRME):
Case 1. Null hypothesis, $\mathcal{H}_o : \boldsymbol{\beta} = \mathbf{0}$: In this case, the risk difference of two estimators is given by

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}(k)) &= \eta^2\,\mathrm{tr}\left(C^{-1}-[R(k)]^T C^{-1}R(k)\right) \\
&\quad \times\left(1-c\,h_3(0)+E\left[(1-c\chi_{p+2}^{-2}(0))^2 I(\chi_{p+2}^2(0)\le c)\right]\right)
\end{aligned}$$

The r.h.s of this expression is nonnegative when $1-ch_3(0)+E\left[(1-c\chi_{p+2}^{-2}(0))^2 I(\chi_{p+2}^2(0)\le c)\right]\ge 0$. On the other hand, we have

$$\begin{aligned}
& ch_3(0)-1 \le E\left[(1-c\chi_{p+2}^{-2}(0))^2 I(\chi_{p+2}^2(0)\le c)\right] \le E\left[(1-c\chi_{p+2}^{-2}(0))^2\right] \\
\Longrightarrow\quad & 2cE\left[\chi_{p+2}^{-2}(0)\right]-c^2E\left[\chi_{p+2}^{-4}(0)\right]-1 \le 1-2cE\left[\chi_{p+2}^{-2}(0)\right]+c^2E\left[\chi_{p+2}^{-4}(0)\right] \\
\Longrightarrow\quad & 2\left(c^2E\left[\chi_{p+2}^{-4}(0)\right]-2cE\left[\chi_{p+2}^{-2}(0)\right]+1\right)\ge 0 \\
\Longrightarrow\quad & 2\left(\frac{c^2}{p(p-2)}-\frac{2c}{p}+1\right)\ge 0 \\
\Longrightarrow\quad & 2\left(\frac{c^2-2c(p-2)+p(p-2)}{p(p-2)}\right)\ge 0 \qquad \Longrightarrow c^2-2c(p-2)+p(p-2)\ge 0
\end{aligned}$$

The last line always true for $p \geq 3$. Thus, with this condition under $\mathcal{H}_o$, PRRME uniformly dominates PRME.

Case 2. Alternative hypothesis, $\mathcal{H}_A : \boldsymbol{\beta} \neq \mathbf{0}$: We consider the risk difference between PRRME and PRME estimators as

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}; I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}(k); I_p) &= \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}; I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p) \\
&+\eta^2 \,\mathrm{tr}\left(C^{-1} - [R(k)]^T C^{-1} R(k)\right) E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right] \\
&+\boldsymbol{\delta}^T (I_p - [R(k)]^T R(k))\boldsymbol{\delta} Q(\Delta^2) - k\boldsymbol{\delta}^T (C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))\boldsymbol{\delta} \\
&\times E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right] \qquad \text{(S16)}
\end{aligned}$$

Substituting Eq. (S48) in Eq. (S16) results in the below equation.

$$\begin{aligned}
&\eta^2 \,\mathrm{tr}\left(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - c\,h_3(\Delta^2) + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\right) \\
&+c(c+4)\boldsymbol{\delta}^T (I_p - [R(k)]^T R(k))\boldsymbol{\delta}\left(E\left[\chi_{p+4}^{-4}(\Delta^2)\right] + h_4(\Delta^2)\right) - k^2 \boldsymbol{\delta}^T C^{-2}(k)\boldsymbol{\delta} \\
&-k\boldsymbol{\delta}^T \left[C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)\right] \boldsymbol{\delta} \left(E\left[\chi_{p+2}^{-2}(\Delta^2)\right] + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\right) \qquad \text{(S17)}
\end{aligned}$$

Eq. (S17) will be nonnegative if

$$\begin{aligned}
&\boldsymbol{\delta}^T \left[k^2 C^{-2}(k) + k\left[C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)\right] f_1(\Delta^2) - (I_p - [R(k)]^T R(k)) f_2(\Delta^2)\right] \boldsymbol{\delta} \\
&\quad \leq \eta^2 \,\mathrm{tr}\left(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - c\,h_3(\Delta^2) + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\right)
\end{aligned}$$

where

$$\begin{aligned}
f_1(\Delta^2) &= E\left[\chi_{p+2}^{-2}(\Delta^2)\right] + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right] \\
f_2(\Delta^2) &= c(c+4)\left(E\left[\chi_{p+4}^{-4}(\Delta^2)\right] + 2E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\right. \\
&\quad \left. -E\left[(1 - c\chi_{p+4}^{-2}(\Delta^2))^2 I(\chi_{p+4}^2(\Delta^2) \leq c)\right]\right)
\end{aligned}$$

Using Courant's theorem, PRRME dominates PRME when

$$\Delta^2 \leq \frac{\mathrm{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - c\,h_3(\Delta^2) + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right])}{Ch_{1.\min}} = \Delta_1^2(k).$$

and PRME dominates PRRME when

$$\Delta^2 \geq \frac{\mathrm{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - c\,h_3(\Delta^2) + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right])}{Ch_{1.\max}} = \Delta_1'^2(k).$$

where

$$
\begin{aligned}
Ch_{1.\min} &= Ch_{\min}\Big(\Big(k^2C^{-2}(k) + k\left[C^{-1}(k)R(k) + [R(k)]^TC^{-1}(k)\right] f_1(\Delta^2) \\
&\quad - \left(I_p - [R(k)]^TR(k)\right) f_2(\Delta^2)\Big)C^{-1}\Big) \\
Ch_{1.\max} &= Ch_{\max}\Big(\Big(k^2C^{-2}(k) + k\left[C^{-1}(k)R(k) + [R(k)]^TC^{-1}(k)\right] f_1(\Delta^2) \\
&\quad - \left(I_p - [R(k)]^TR(k)\right) f_2(\Delta^2)\Big)C^{-1}\Big)
\end{aligned}
$$

To obtain a condition on k, we consider the risk function of PRRME in terms of eigenvalues as

$$
\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}(k); I_p) = \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p) \\
-\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}\Big\{\eta^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
-\lambda_i\theta_i^2 h_4(\Delta^2) - 2k\theta_i^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2)\le c)\right]\Big\}
\end{aligned} \tag{S18}
$$

where $\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p)$ is given in

$$
\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p) &= \sum_{i=1}^{p}\frac{1}{(\lambda_i+k)^2}\Big\{\lambda_i\eta^2(1-cX(\Delta^2)) + c(c+4)\lambda_i^2\theta_i^2 E\left[\chi_{p+4}^{-4}(\Delta^2)\right] \\
&\quad + k^2\theta_i^2 + 2ck\lambda_i\theta_i^2 E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\Big\}.
\end{aligned} \tag{S19}
$$

Differentiating (S18) w.r.t. k, we obtain

$$
\begin{aligned}
\frac{\partial\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}(k); I_p)}{\partial k} &= \frac{\partial\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p)}{\partial k} \\
&\quad +2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^3}\Big\{\eta^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
&\quad -\lambda_i\theta_i^2 h_4(\Delta^2) - k\theta_i^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
&\quad +\lambda_i\theta_i^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2)\le c)\right]\Big\}
\end{aligned} \tag{S20}
$$

where

$$
\begin{aligned}
\frac{\partial\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(M)S}(k); I_p)}{\partial k} &= -2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^3}\Big\{\eta^2(1-cX(\Delta^2)) + c(c+4)\lambda_i\theta_i^2 E\left[\chi_{p+4}^{-4}(\Delta^2)\right] \\
&\quad -c\lambda_i\theta_i^2 E\left[\chi_{p+2}^{-2}(\Delta^2)\right] - k\theta_i^2\left(1 - cE\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right)\Big\}.
\end{aligned} \tag{S21}
$$

We define

$$
k_1(\Delta^2) = \frac{f_3(\Delta^2)}{\max_{1\le i\le p}\left\{\theta_i^2\left(1 - cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] - E\left[(1-c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2)\le c)\right]\right)\right\}}, \tag{S22}
$$

where

$$f_3(\Delta^2) = \min_{1\le i\le p}\Big\{\eta^2\Big(1 - ch_3(\Delta^2) - E\Big[(1-c\chi^{-2}_{p+2}(\Delta^2))^2 I(\chi^2_{p+2}(\Delta^2)\le c)\Big]\Big) + \lambda_i\theta_i^2\Big(c(c+4)E\Big[\chi^{-4}_{p+4}(\Delta^2)\Big] - cE\Big[\chi^{-2}_{p+2}(\Delta^2)\Big] - E\Big[(1-c\chi^{-2}_{p+2}(\Delta^2))I(\chi^2_{p+2}(\Delta^2)\le c)\Big]\Big)\Big\}.$$

So, for each positive k with $k \le k_1(\Delta^2)$, PRRME has a risk value less than that of PRME.

Proof of Theorem 4.2 (Comparsion between PRRME and URME):
Case 1. Null hypothesis, $\mathcal{H}_o : \boldsymbol{\beta} = \mathbf{0}$: The risk difference between two estimators is

$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}(k); I_p) = \eta^2\,\mathrm{tr}\Big(ch_3(0) + E\Big[(1-c\chi^{-2}_{p+2}(0))^2 I(\chi^2_{p+2}(0)\le c)\Big]\Big) \ge 0 \tag{S23}$$

since

$$-c\,h_3(0) \le E\Big[(1-c\chi^{-2}_{p+2}(0))^2 I(\chi^2_{p+2}(0)\le c)\Big] \le E\Big[(1-c\chi^{-2}_{p+2}(0))^2\Big].$$

Case 2. Alternative hypothesis, $\mathcal{H}_A : \boldsymbol{\beta} \neq \mathbf{0}$: The risk difference is

$$\begin{aligned}\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}+}(k); I_p) &= \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p)\\ &+\eta^2\,\mathrm{tr}\left([R(k)]^T C^{-1} R(k)\right) E\Big[(1-c\chi^{-2}_{p+2}(\Delta^2))^2 I(\chi^2_{p+2}(\Delta^2)\le c)\Big]\\ &-\boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta} h_4(\Delta^2) - k\left(\boldsymbol{\delta}^T C^{-1}(k)R(k)\boldsymbol{\delta} + \boldsymbol{\delta}^T[R(k)]^T C^{-1}(k)\boldsymbol{\delta}\right)\\ &\times E\left[(1-c\chi^2_{p+2}(\Delta^2))I(\chi^2_{p+2}(\Delta^2)\le c)\right]\\ &= \eta^2\,\mathrm{tr}([R(k)]^T C^{-1} R(k)) f_4(\Delta^2) - \boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta} f_5(\Delta^2)\\ &-k(\boldsymbol{\delta}^T C^{-1}(k)R(k)\boldsymbol{\delta} + \boldsymbol{\delta}^T[R(k)]^T C^{-1}(k)\boldsymbol{\delta}) f_6(\Delta^2)\end{aligned} \tag{S24}$$

where

$$\begin{aligned} f_4(\Delta^2) &= c\,h_3(\Delta^2) + E\Big[(1-c\chi^{-2}_{p+2}(\Delta^2))^2 I(\chi^2_{p+2}(\Delta^2)\le c)\Big]\\ f_5(\Delta^2) &= c(c+4)E\Big[\chi^{-4}_{p+4}(\Delta^2)\Big] + h_4(\Delta^2)\\ f_6(\Delta^2) &= cE\Big[\chi^{-2}_{p+2}(\Delta^2)\Big] + E\left[(1-c\chi^2_{p+2}(\Delta^2))I(\chi^2_{p+2}(\Delta^2)\le c)\right].\end{aligned}$$

The r.h.s of (S24) is non-negative when

$$\boldsymbol{\delta}^T\left[[R(k)]^T R(k) f_5(\Delta^2) + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) f_6(\Delta^2)\right]\boldsymbol{\delta} \le \eta^2\,\mathrm{tr}([R(k)]^T C^{-1} R(k)) f_4(\Delta^2); \tag{S25}$$

Using Courant's theorem, PRRME dominates URME when

$$\Delta^2 \le \frac{\mathrm{tr}([R(k)]^T C^{-1} R(k)) f_4(\Delta^2)}{Ch_{\min}\left(([R(k)]^T R(k) f_5(\Delta^2) + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) f_6(\Delta^2))C^{-1}\right)} = \Delta_2^2(k),$$

and URME is superior to PRRME when

$$\Delta^2 \geq \frac{\mathrm{tr}([R(k)]^T C^{-1} R(k)) f_4(\Delta^2)}{Ch_{\max}\left(([R(k)]^T R(k) f_5(\Delta^2) + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) f_6(\Delta^2)) C^{-1}\right)} = \Delta_2'^2(k).$$

To obtain a condition on k, we consider the risk difference between PRRME and URME in terms of eigenvalues as

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\mathrm{(M)U}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\mathrm{(M)S+}}(k); I_p) &= \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\mathrm{(M)U}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\mathrm{(M)S}}(k); I_p) \\
&\quad - \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2} \Big\{ \eta^2 E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c) \right] \\
&\quad - \lambda_i \theta_i^2 h_4(\Delta^2) - 2k\theta_i^2 E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c) \right] \Big\} \\
&= \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2} \Big\{ \eta^2 \left(c\, h_3(\Delta^2) - E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c) \right] \right) \\
&\quad - \lambda_i \theta_i^2 \left(c(c+4) E\left[\chi_{p+4}^{-4}(\Delta^2) \right] - h_4(\Delta^2) \right) \\
&\quad - 2k\theta_i^2 \left(cE\left[\chi_{p+2}^{-2}(\Delta^2) \right] - E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c) \right] \right) \Big\}.
\end{aligned}$$

The r.h.s. of the above expression will be non-negative if

$$k \leq \frac{f_7(\Delta^2)}{\max_{1 \leq i \leq p} \left\{ \theta_i^2 \left(cE\left[\chi_{p+2}^{-2}(\Delta^2) \right] - E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c) \right] \right) \right\}} = k_2(\Delta^2) \quad \text{(S26)}$$

where

$$\begin{aligned}
f_7(\Delta^2) \;=\; \min_{1 \leq i \leq p} & \Big\{ \eta^2 \left(ch_3(\Delta^2) - E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c) \right] \right) \\
& - \lambda_i \theta_i^2 \left(c(c+4) E\left[\chi_{p+4}^{-4}(\Delta^2) \right] - h_4(\Delta^2) \right) \Big\}.
\end{aligned}$$

Thus, for each positive k with $k < k_2(\Delta^2)$, PRRME has a risk less than URME.

Proof of Theorem 4.3 (Comparison between PRRME and PTRME):
Case 1. Null hypothesis, $\mathcal{H}_o : \boldsymbol{\beta} = \mathbf{0}$: The risk difference is

$$\eta^2\, \mathrm{tr}\left([R(k)]^T C^{-1} R(k) \right) \left(c\, h_3(0) - H_{p+2}(\chi_p^2(\alpha); 0) + E\left[(1 - c\chi_{p+2}^2(0))^2 I(\chi_{p+2}^2(0) \leq c) \right] \right) \geq 0,$$

for all α satisfy the condition

$$\left\{ \alpha : \chi_p^2(\alpha) \leq H_{p+2}^{-1} \left(\frac{2c(p-2) - c^2}{p(p-2)} + E\left[(1 - c\chi_{p+2}^{-2}(0))^2 I(\chi_{p+2}^2(0) \leq c) \right] \right) \right\}. \quad \text{(S27)}$$

Thus, the asymptotic distribution risk of PRRME is smaller than that of PTRME when the critical value $\chi_p^2(\alpha)$ satisfies the relation in (S27). However, the PTRME dominates PRRME when the critical value $\chi_p^2(\alpha)$ satisfies the opposite relation to (S27).

Case 2. Alternative hypothesis, $\mathcal{H}_A : \boldsymbol{\beta} \neq \mathbf{0}$: The risk differences is

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S+}}(k); I_p) &= \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k); I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k); I_p) \\
&\quad + \eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right) E\left[(1 - c\chi_{p+2}^{2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right] \\
&\quad - \boldsymbol{\delta}^T [R(k)]^T R(k) \boldsymbol{\delta} h_4(\Delta^2) - k\boldsymbol{\delta}^T \left(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)\right)\boldsymbol{\delta} \\
&\quad \times E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right] \\
&= \eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\Big(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2) \\
&\quad + E\left[(1 - c\chi_{p+2}^{2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\Big) \\
&\quad - \boldsymbol{\delta}^T [R(k)]^T R(k) \boldsymbol{\delta}\left(h_2(\alpha; \Delta^2) - c(c+4)E\left[\chi_{p+4}^{-4}(\Delta^2)\right] + h_4(\Delta^2)\right) \\
&\quad - k(\boldsymbol{\delta}^T C^{-1}(k) R(k)\boldsymbol{\delta} + \boldsymbol{\delta}^T [R(k)]^T C^{-1}(k)\boldsymbol{\delta})\Big(cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] \\
&\quad - H_{p+2}(\chi_p^2(\alpha); \Delta^2) + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\Big) \qquad \text{(S28)}
\end{aligned}$$

The r.h.s. of (S28) will be non-negative when

$$\begin{aligned}
&\boldsymbol{\delta}^T \Big([R(k)]^T R(k) f_8(\Delta^2) - k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) f_9(\Delta^2)\Big)\boldsymbol{\delta} \\
&\leq \eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\Big(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2) + E\left[(1 - c\chi_{p+2}^{2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\Big)
\end{aligned} \qquad \text{(S29)}$$

where

$$\begin{aligned}
f_8(\Delta^2) &= h_2(\alpha; \Delta^2) - c(c+4)E\left[\chi_{p+4}^{-4}(\Delta^2)\right] + h_4(\Delta^2) \\
f_9(\Delta^2) &= cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] - H_{p+2}(\chi_p^2(\alpha); \Delta^2) + E\left[(1 - c\chi_{p+2}^{-2}(\Delta^2)) I(\chi_{p+2}^2(\Delta^2) \leq c)\right] \qquad \text{(S30)}
\end{aligned}$$

Again with using Courant's theorem for Eq. (S29), PRRME dominates PTRME when

$$\begin{aligned}
\Delta^2 &\leq \frac{\operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\Big(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2) + E\left[(1 - c\chi_{p+2}^{2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]}{Ch_{\min}\left(([R(k)]^T R(k) f_8(\Delta^2) - k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) f_9(\Delta^2))C^{-1}\right)} \\
&= \Delta_3^2(k; \alpha)
\end{aligned}$$

and PTRME is superior to PRRME when

$$\begin{aligned}
\Delta^2 &\geq \frac{\operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\Big(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2) + E\left[(1 - c\chi_{p+2}^{2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]}{Ch_{\max}\left(([R(k)]^T R(k) f_8(\Delta^2) - k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) f_9(\Delta^2))C^{-1}\right)} \\
&= \Delta_3'^2(k; \alpha)
\end{aligned}$$

Now, we consider the risk difference of PRRME and PTRME as a function of eigenvalues as

$$\begin{aligned}
\sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2}\Big\{&\eta^2\left(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2) + E\left[(1 - c\chi_{p+2}^{2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2) \leq c)\right]\right) \\
&- \lambda_i \theta_i^2 f_8(\Delta^2) - 2k\theta_i^2 f_9(\Delta^2)\Big\}.
\end{aligned}$$

On the other hand, the risk difference is non-negative whenever there exists a $k \in (0, k_3(\Delta^2))$, where

$$k_3(\Delta^2) = \frac{f_{10}(\Delta^2)}{\max_{1\le i\le p}\left\{2\theta_i^2 f_9(\Delta^2)\right\}}.$$

where

$$f_{10}(\Delta^2) = \min_{1\le i\le p}\left\{\eta^2\left(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha);\Delta^2) + E\left[(1-c\chi_{p+2}^2(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right]\right) - \lambda_i\theta_i^2 f_8(\Delta^2)\right\}$$

Proof of Theorem 4.4 (Comparison between PRRME and SRME):
The risk difference of two estimators is given by

$$\begin{aligned}
\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)S}}(k);I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)S+}}(k);I_p) = \\
&\eta^2\,\mathrm{tr}([R(k)]^T C^{-1}R(k))E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
&+2k\boldsymbol{\delta}^T[R(k)]^T C^{-1}(k)\boldsymbol{\delta}E\left[(c\chi_{p+2}^{-2}(\Delta^2)-1)I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
&-\boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta}h_4(\Delta^2) \\
&= \eta^2\,\mathrm{tr}([R(k)]^T C^{-1}R(k))E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
&+\boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta}E\left[(1-c\chi_{p+4}^{2}(\Delta^2))^2 I(\chi_{p+4}^2(\Delta^2)\le c)\right] \\
&+2\boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta}E\left[(c\chi_{p+2}^{-2}(\Delta^2)-1)I(\chi_{p+2}^2(\Delta^2)\le c)\right] \\
&+2k\boldsymbol{\delta}^T[R(k)]^T C^{-1}(k)\boldsymbol{\delta}E\left[(c\chi_{p+2}^{-2}(\Delta^2)-1)I(\chi_{p+2}^2(\Delta^2)\le c)\right]
\end{aligned} \tag{S31}$$

The r.h.s. is positive, since the expectation of a positive random variable is positive and by the definition of an indicator function,

$$0 < \chi_{p+2}^2(\Delta^2) \le c \Longleftrightarrow c\chi_{p+2}^{-2}(\Delta^2) - 1 \ge 0.$$

We get $E\left[\left(c\chi_{p+2}^{-2}(\Delta^2)-1\right)I(\chi_{p+2}^2(\Delta^2)\le c)\right] \ge 0$. Thus, for all Δ^2, $\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)S+}}(k);I_p) \le \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)S}}(k);I_p)$. The PRRME not only confirms inadmissibility of SRME but also provides a simple superior estimator.
The expression (S31) in terms of eigenvalues is given by

$$\begin{aligned}
\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}&\left\{\eta^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right] - \lambda_i\theta_i^2 h_4(\Delta^2)\right. \\
&\left.-2k\theta_i^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))I(\chi_{p+2}^2(\Delta^2)\le c)\right]\right\}.
\end{aligned} \tag{S32}$$

We define

$$k_4(\Delta^2) = \frac{\min_{1\le i\le p}\left\{\eta^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))^2 I(\chi_{p+2}^2(\Delta^2)\le c)\right] - \lambda_i\theta_i^2 h_4(\Delta^2)\right\}}{\max_{1\le i\le p}\left\{2\theta_i^2 E\left[(1-c\chi_{p+2}^{-2}(\Delta^2))I(\chi_{p+2}^2(\Delta^2)\le c)\right]\right\}},$$

for $k \in [0, k_4(\Delta^2)]$, PRRME dominates SRME.

Comparison between URME and UME
The asymptotic risk function of $\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k)$ is given by

$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k)) = \eta^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2} + k^2 \sum_{i=1}^{p} \frac{\theta_i^2}{(\lambda_i + k)^2}. \tag{S33}$$

Differentiating (S33) w.r.t.k, we get

$$\frac{\partial \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k))}{\partial k} = 2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^3} (k\theta_i^2 - \eta^2). \tag{S34}$$

Next, we define

$$k_5 = \frac{\eta^2}{\max_{1 \le i \le p} \theta_i^2} = \frac{\eta^2}{\theta_{\max}^2}, \tag{S35}$$

where $\theta_{\max}$ is the largest element of $\boldsymbol{\theta}$. We see that a sufficient condition for (S34) to be negative is that there exists a $k \in (0, k_5)$ such that URME has smaller risk that UME.

Also, the risk difference is

$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}; I_p) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k); I_p) = \eta^2 \operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k)) - k^2 \boldsymbol{\delta}^T C^{-2}(k) \boldsymbol{\delta}. \tag{S36}$$

A sufficient condition for $\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k); I_p)$ to be smaller than $\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}; I_p) = \eta^2 \operatorname{tr}(C^{-1})$ is that Eq. (S36) be non-negative. i.e.

$$\boldsymbol{\delta}^T C^{-2}(k) \boldsymbol{\delta} \le \frac{\eta^2 \operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))}{k^2}. \tag{S37}$$

By Courant-Fisher theorem

$$Ch_{\min}(C^{-2}(k)C^{-1}) \le \frac{\boldsymbol{\delta}^T C^{-2}(k) \boldsymbol{\delta}^T}{\boldsymbol{\delta}^T \boldsymbol{\delta}^T} = \frac{\boldsymbol{\delta}^T C^{\frac{1}{2}} C^{-2}(k) C^{-1} C^{\frac{1}{2}} \boldsymbol{\delta}^T}{\boldsymbol{\delta}^T C^{\frac{1}{2}} C^{\frac{1}{2}} \boldsymbol{\delta}^T} \le Ch_{\max}(C^{-2}(k)C^{-1}),$$

and then

$$\frac{\boldsymbol{\delta}^T C^{-2}(k) \boldsymbol{\delta}^T}{Ch_{\max}(C^{-2}(k)C^{-1})} \le \boldsymbol{\delta}^T C \boldsymbol{\delta}^T \le \frac{\boldsymbol{\delta}^T C^{-2}(k) \boldsymbol{\delta}^T}{Ch_{\min}(C^{-2}(k)C^{-1})} \tag{S38}$$

Substituting Eq. (S37) in Eq. (S38) results in

$$\Delta^2 \le \frac{\operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))}{k^2 Ch_{\min}(C^{-2}(k)C^{-1})} = \Delta_4(k).$$

Comparison between PTRME and PTME
Case 1. Null hypothesis, $\mathcal{H}_o : \boldsymbol{\beta} = \mathbf{0}$: The risk difference in this case is given by

$$\eta^2 \operatorname{tr}(C^{-1} - R^T(k) C^{-1} R(k))(1 - H_{p+2}(\chi_p^2(\alpha); 0)) \ge 0.$$

Hence, $\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}$ uniformly for all k and α.
Case 2. Alternative hypothesis, $\mathcal{H}_A : \boldsymbol{\beta} \neq \mathbf{0}$: The risk difference is given by

$$\begin{aligned}
\text{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}; I_p) - \text{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}(k); I_p) = \\
&\eta^2 \operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) \\
&-k \left\{ \boldsymbol{\delta}^T C^{-1}(k) R(k) \boldsymbol{\delta} + \boldsymbol{\delta}[R(k)]^T C^{-1}(k) \boldsymbol{\delta} \right\} H_{p+2}(\chi_p^2(\alpha); \Delta^2) \\
&+\boldsymbol{\delta}^T (I_p - [R(k)]^T R(k)) \boldsymbol{\delta} h_2(\alpha; \Delta^2) - k^2 \boldsymbol{\delta}^T C^{-2}(k) \boldsymbol{\delta}
\end{aligned} \tag{S39}$$

The expression (S39) is non-negative if and only if

$$\begin{aligned}
&\eta^2 \operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) \geq \\
&\boldsymbol{\delta}^T \left[k^2 C^{-2}(k) + k(C^{-1}(k) R(k) + [R(k)]^T C^{-1}(k)) H_{p+2}(\chi_p^2(\alpha); \Delta^2) - (I_p - [R(k)]^T R(k)) h_2(\alpha; \Delta^2) \right] \boldsymbol{\delta},
\end{aligned}$$

By standard calculation using Courant's theorem, we obtain

$$\Delta^2 \leq \frac{\operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2))}{Ch_{\min}(f_{11}(\alpha, \Delta^2) C^{-1})} = \Delta_5^2(k, \alpha),$$

where

$$\begin{aligned}
f_{11}(\alpha, \Delta^2) &= k^2 C^{-2}(k) + k(C^{-1}(k) R(k) + [R(k)]^T C^{-1}(k)) H_{p+2}(\chi_p^2(\alpha); \Delta^2) \\
&\quad -(I_p - [R(k)]^T R(k)) h_2(\alpha; \Delta^2)
\end{aligned}$$

And, PTRME dominates PTME when

$$\Delta^2 \geq \frac{\operatorname{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2))}{Ch_{\max}(f_{11}(\alpha, \Delta^2) C^{-1})} = \Delta_5'^2(k, \alpha),$$

If we rewrite the expression of $\text{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}(k))$ in terms of eigenvalues and k, we have

$$\begin{aligned}
\text{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}(k)) &= \sum_{i=1}^{p} \frac{1}{(\lambda_i + k)^2} \left\{ \eta^2 \lambda_i (1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) + \lambda_i^2 \theta_i^2 h_2(\alpha; \Delta^2) \right. \\
&\quad \left. + k^2 \theta_i^2 + 2k \lambda_i \theta_i^2 H_{p+2}(\chi_p^2(\alpha); \Delta^2) \right\}
\end{aligned}$$

Differentiating the risk of PTRME w.r.t. k, we obtain

$$\begin{aligned}
\frac{\partial \text{ADR}(\hat{\boldsymbol{\beta}}_n^{\text{(M)PT}}(k))}{\partial k} &= -2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^3} \left\{ \eta^2 (1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) - k \theta_i^2 (1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) \right. \\
&\quad \left. + \lambda_i \theta_i^2 \left(H_{p+2}(\chi_p^2(\alpha); \Delta^2) - H_{p+4}(\chi_p^2(\alpha); \Delta^2) \right) \right\}
\end{aligned} \tag{S40}$$

A sufficient condition for (S40) to be negative is that there exists a $k \in (0, k_6(\alpha, \Delta^2))$, whenever

$$k_6(\alpha, \Delta^2) = \frac{f_{12}(\Delta^2)}{\max_{1 \leq i \leq p} \left\{ \theta_i^2 (1 - H_{p+2}(\chi_p^2(\alpha); \Delta^2)) \right\}} \tag{S41}$$

where

$$f_{12}(\Delta^2) = \min_{1\le i\le p}\left\{\eta^2(1-H_{p+2}(\chi_p^2(\alpha);\Delta^2)) + \lambda_i\theta_i^2\left(H_{p+2}(\chi_p^2(\alpha);\Delta^2) - H_{p+4}(\chi_p^2(\alpha);\Delta^2)\right)\right\} \quad \text{(S42)}$$

Thus, $\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k)$ is superior to $\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}$ whenever $k \in (0, k_6(\alpha,\Delta^2))$ under $\mathcal{H}_A$.

Comparison between PTRME and URME
The risk difference between PTRME and URME is

$$\begin{aligned}\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k)) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k)) &= \eta^2\,\mathrm{tr}([R(k)]^T C^{-1}R(k))H_{p+2}(\chi_p^2(\alpha);\Delta^2)\\ &\quad -\boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta} h_2(\alpha;\Delta^2) - k(\boldsymbol{\delta}^T C^{-1}(k)R(k)\boldsymbol{\delta}\\ &\quad +\boldsymbol{\delta}^T[R(k)]^T C^{-1}(k)\boldsymbol{\delta})H_{p+2}(\chi_p(\alpha);\Delta^2) \qquad \text{(S43)}\end{aligned}$$

The expression is non-negative whenever

$$\begin{aligned}\boldsymbol{\delta}^T\left[[R(k)]^T R(k)h_2(\alpha;\Delta^2) + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))H_{p+2}(\chi_p(\alpha);\Delta^2)\right]\boldsymbol{\delta}\\ \le \eta^2\,\mathrm{tr}([R(k)]^T C^{-1}R(k))H_{p+2}(\chi_p^2(\alpha);\Delta^2) \qquad \text{(S44)}\end{aligned}$$

Again, using the Courant's theorem, we obtain Eq. (S43)≥ 0 according to

$$\Delta^2 \le \frac{\mathrm{tr}([R(k)]^T C^{-1}R(k))H_{p+2}(\chi_p^2(\alpha);\Delta^2)}{CH_{\min}(f_{13}(\alpha,\Delta^2)C^{-1})} = \Delta_6^2(k,\alpha), \quad \text{(S45)}$$

where

$$f_{13}(\alpha,\Delta^2) = [R(k)]^T R(k)h_2(\alpha;\Delta^2) + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))H_{p+2}(\chi_p(\alpha);\Delta^2)$$

Thus, $\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k)$ dominates $\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k)$ whenever $\Delta^2 \in (0, \Delta_3^2(k,\alpha))$.

Rewriting the expression (S43) in terms of eigenvalues and k, we obtain

$$\begin{aligned}&\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k)) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k)) =\\ &\quad \sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}\left\{\eta^2 H_{p+2}(\chi_p^2(\alpha);\Delta^2) - \lambda_i\theta_i^2 h_2(\alpha;\Delta^2) - 2k\theta_i^2 H_{p+2}(\chi_p^2(\alpha);\Delta^2)\right\} \quad \text{(S46)}\end{aligned}$$

The r.h.s. is non-negative when $k \le k_7(\alpha,\Delta^2)$, where

$$k_7(\alpha,\Delta^2) = \frac{\min_{1\le i\le p}\left\{\eta^2 H_{p+2}(\chi_p^2(\alpha);\Delta^2) - \lambda_i\theta_i^2 h_2(\alpha;\Delta^2)\right\}}{\max_{1\le i\le p}\left\{2\theta_i^2 H_{p+2}(\chi_p^2(\alpha);\Delta^2)\right\}} \quad \text{(S47)}$$

Thus, PTRME dominates URME when $k \in (0, k_7(\alpha,\Delta^2))$ and URME will dominate PTRME when $k > k_7(\alpha,\Delta^2)$.

Comparison of SRME and SME
Case 1. Null hypothesis, $\mathcal{H}_o : \boldsymbol{\beta} = \mathbf{0}$: In this case, the risk difference is given by

$$\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k)) = \eta^2\,\mathrm{tr}(C^{-1} - [R(k)]^T C^{-1}R(k))(1 - ch_3(0))$$

which is non-negative for all k if $0 < c < 2(p-2)$, Since $h_3(0) = (2(p-2)-c)/(p(p-2))$.
Case 2. Alternative hypothesis, $\mathcal{H}_A : \boldsymbol{\beta} \neq 0$: In this case, the risk difference is given by

$$\begin{aligned}\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k)) &= \eta^2\,\mathrm{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - ch_3(\Delta^2)) \\ &+c(c+4)\boldsymbol{\delta}^T(I_p - [R(k)]^T R(k))\boldsymbol{\delta} E\left[\chi_{p+4}^{-4}(\Delta^2)\right] - k^2\boldsymbol{\delta}^T C^{-2}(k)\boldsymbol{\delta} \\ &-k\boldsymbol{\delta}^T\left[C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)\right]\boldsymbol{\delta} E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \end{aligned} \quad \text{(S48)}$$

Using Courant's theorem, the SRME is superior to SME whenever $\Delta^2 \leq \Delta_7^2(k)$ which is defined by

$$\Delta_7^2(k) = \frac{\mathrm{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - c\,h_3(\Delta^2))}{Ch_{\min}(f_{14}(\Delta^2)C^{-1})}, \quad \text{(S49)}$$

where

$$\begin{aligned} f_{14}(\Delta^2) &= k^2 C^{-1}(k) + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \\ &\quad -c(c+4)(I_p - [R(k)]^T R(k))E\left[\chi_{p+4}^{-4}(\Delta^2)\right] \end{aligned} \quad \text{(S50)}$$

However, SME is superior to SRME when $\Delta^2 > \Delta_7'^2(k)$ is defined by

$$\Delta_7'^2(k) = \frac{\mathrm{tr}(C^{-1} - [R(k)]^T C^{-1} R(k))(1 - ch_3(\Delta^2))}{Ch_{\max}(f_{14}(\Delta^2)C^{-1})}. \quad \text{(S51)}$$

In this case, the risk function in terms of eigenvalues is given by Eq. (S19). Differentiating it w.r.t. k, we obtain Eq. (S21). Hence, a sufficient condition for (S21) to be negative is that there exists a k such that $0 < k < k_8(\Delta^2)$ where

$$k_8(\Delta^2) = \frac{\min_{1\leq i\leq p}\left\{\eta^2(1 - ch_3(\Delta^2)) + c(c+4)\lambda_i\theta_i^2 E\left[\chi_{p+4}^{-4}(\Delta^2)\right] - c\lambda_i\theta_i^2 E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\}}{\max_{1\leq i\leq p}\left\{\theta_i^2\left(1 - cE\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right)\right\}}$$

Hence, $\hat{\boldsymbol{\beta}}_n^{(M)S}(k)$ has smaller risk than $\hat{\boldsymbol{\beta}}_n^{(M)S}$ for $k \in [0, k_8(\Delta^2)]$.

Comparison of SRME with URME
We consider the risk difference of SRME and URME as

$$\begin{aligned}\mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{U}}(k)) - \mathrm{ADR}(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k)) &= c\eta^2\,\mathrm{tr}([R(k)]^T C^{-1} R(k))\,h_3(\Delta^2) \\ &-c(c+4)\boldsymbol{\delta}^T[R(k)]^T R(k)\boldsymbol{\delta} E\left[\chi_{p+4}^{-4}(\Delta^2)\right] - ck(\boldsymbol{\delta}^T C^{-1}(k)R(k)\boldsymbol{\delta} \\ &+\boldsymbol{\delta}^T[R(k)]^T C^{-1}(k)\boldsymbol{\delta})E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \end{aligned} \quad \text{(S52)}$$

The above difference will be non-negative when

$$\boldsymbol{\delta}^T\left[(c+4)[R(k)]^T R(k) E\left[\chi_{p+4}^{-4}(\Delta^2)\right] + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right]\boldsymbol{\delta} \leq \eta^2\,\mathrm{tr}\left([R(k)]^T C^{-1} R(k)\right) h_3(\Delta^2)$$

Using Courant's theorem, the SRME dominates URME when

$$\Delta^2 \leq \frac{\operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right) h_3(\Delta^2)}{Ch_{\min}\left(f_{15}(\Delta^2) C^{-1}\right)} = \Delta_8^2(k),$$

and URME is superior to SRME when

$$\Delta^2 \geq \frac{\operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right) h_3(\Delta^2)}{Ch_{\max}\left(f_{15}(\Delta^2) C^{-1}\right)} = \Delta_8^{\prime 2}(k),$$

where

$$f_{15}(\Delta^2) = (c+4)[R(k)]^T R(k) E\left[\chi_{p+4}^{-4}(\Delta^2)\right] + k(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k)) E\left[\chi_{p+2}^{-2}(\Delta^2)\right].$$

To find a sufficient condition on k, we consider the expression (S52) according to eigenvalues as

$$\sum_{i=1}^{p} \frac{c\lambda_i}{(\lambda_i + k)^2} \left\{\eta^2 h_3(\Delta^2) - (c+4)\lambda_i \theta_i^2 E\left[\chi_{p+4}^{-4}(\Delta^2)\right] - 2k\theta_i^2 E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\}$$

It is non-negative when there exists a value of $k \in (0, k_9(\Delta^2))$, where

$$k_9(\Delta^2) = \frac{\min_{1\leq i\leq p}\left\{\eta^2 h_3(\Delta^2) - (c+4)\lambda_i \theta_i^2 E\left[\chi_{p+4}^{-4}(\Delta^2)\right]\right\}}{\max_{1\leq i\leq p}\left\{2\theta_i^2 E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\}}$$

Note that the risk difference (S52) under $\mathcal{H}_o : \boldsymbol{\beta} = \mathbf{0}$ is

$$c\eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right) h_3(0) \geq 0, \quad \text{if} \quad c \leq 2(p-2).$$

Thus, SRME dominates URME under the null hypothesis for $c \leq 2(p-2)$.

Comparison between SRME and PTRME

The risk difference in this case is

$$\begin{aligned}
\operatorname{ADR}\left(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{PT}}(k)\right) - \operatorname{ADR}\left(\hat{\boldsymbol{\beta}}_n^{(\mathrm{M})\mathrm{S}}(k)\right) = \\
&\eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\left(c\, h_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2)\right) \\
&-\boldsymbol{\delta}^T [R(k)]^T R(k) \boldsymbol{\delta}\left(h_2(\alpha; \Delta^2) - c(c+4) E\left[\chi_{p+4}^{-4}(\Delta^2)\right]\right) \\
&-k(\boldsymbol{\delta}^T C^{-1}(k) R(k) \boldsymbol{\delta} + \boldsymbol{\delta}^T [R(k)]^T C^{-1}(k) \boldsymbol{\delta}) \\
&\quad \times \left(cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] - H_{p+2}(\chi_p^2(\alpha); \Delta^2)\right)
\end{aligned} \tag{S53}$$

The right side of (S53) will be non-negative if

$$\begin{aligned}
\boldsymbol{\delta}^T &\left[f_{16}(\alpha, \Delta^2)[R(k)]^T R(k) - k f_{17}(\alpha, \Delta^2)(C^{-1}(k) R(k) + [R(k)]^T C^{-1}(k))\right] \boldsymbol{\delta} \leq \\
&\eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\left(c\, h_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha); \Delta^2)\right),
\end{aligned}$$

where

$$f_{16}(\alpha,\Delta^2) = h_2(\alpha;\Delta^2) - c(c+4)E\left[\chi_{p+4}^{-4}(\Delta^2)\right]$$
$$f_{17}(\alpha,\Delta^2) = cE\left[\chi_{p+2}^{-2}(\Delta^2)\right] - H_{p+2}(\chi_p^2(\alpha);\Delta^2). \quad \text{(S54)}$$

Using Courant's theorem, SRME dominates PTRME when

$$\Delta^2 \le \frac{\eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\left(c\,h_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha);\Delta^2)\right)}{Ch_{\min}\left(f_{16}(\alpha,\Delta^2)[R(k)]^T R(k) - k f_{17}(\alpha,\Delta^2)(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))C^{-1}\right)} = \Delta_9^2(k).$$

However, PTRME is superior to SRME when

$$\Delta^2 > \frac{\eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\left(c\,h_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha);\Delta^2)\right)}{Ch_{\max}\left(f_{16}(\alpha,\Delta^2)[R(k)]^T R(k) - f_{17}(\alpha,\Delta^2)(C^{-1}(k)R(k) + [R(k)]^T C^{-1}(k))C^{-1}\right)} = \Delta_9'^2(k).$$

Now, we consider the risk difference in (S53) as a function of eigenvalues as

$$\sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i+k)^2}\left\{\eta^2\left(c\,h_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha);\Delta^2)\right) - \lambda_i\theta_i^2 f_{16}(\alpha,\Delta^2) - 2k\theta_i^2 f_{17}(\alpha,\Delta^2)\right\}.$$

The above expression is non-negative whenever there exists a value $k \in (0, k_{10}(\Delta^2)]$, where

$$k_{10}(\Delta^2) = \frac{\min_{1\le i\le p}\left\{\eta^2\left(ch_3(\Delta^2) - H_{p+2}(\chi_p^2(\alpha);\Delta^2)\right) - \lambda_i\theta_i^2 f_{16}(\alpha,\Delta^2)\right\}}{\max_{1\le i\le p}\left\{2\theta_i^2 f_{17}(\alpha,\Delta^2)\right\}}.$$

Note that under $\mathcal{H}_o$, the risk difference (S53) reduces to

$$\eta^2 \operatorname{tr}\left([R(k)]^T C^{-1} R(k)\right)\left(ch_3(0) - H_{p+2}(\chi_p^2(\alpha);0)\right).$$

Thus, SRME dominates PTRME when

$$\left(ch_3(0) - H_{p+2}(\chi_p^2(\alpha);0)\right) \ge 0 \iff \frac{2c(p-2)-c^2}{p(p-2)} \ge H_{p+2}(\chi_p^2(\alpha);0)$$
$$\iff \chi_p^2(\alpha) \le H_{p+2}^{-1}\left(\frac{2c(p-2)-c^2}{p(p-2)}\right).$$

Otherwise, PTRME is superior to SRME.

In the end of this section, to have a clear imagine and better understanding, we consider $C = I_p$ and for different values of p, k, η^2 and α some graphs are presented. Figures S1 and S2 confirms that superiority of improved Huberized ridge estimators depends to k and Δ^2.

S3 More Details on Graphical Representations

In this section, more plots (Figures S3-S7) are presented to illustrate the properties of the proposed estimators, clearly.

Also, Table S1 is produced to find the effect of number of independent variables on choosing k. We supposed that some different values for k and in each case and compared the behavior of estimators with increasing p. The larger p, the better performance of all proposed estimators. However, observing the different behavior of them for $p = 30$ confirms that it depends to choice of k, η^2 and α for any value of p.

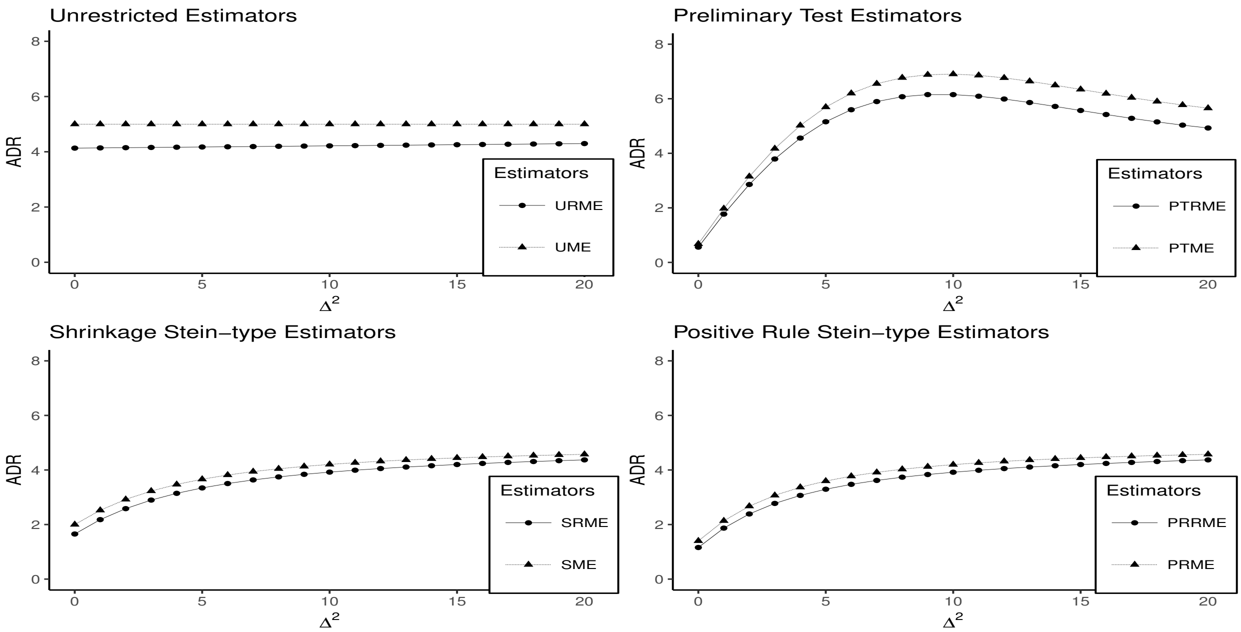

Figure S1: Comparison of Estimators for $p = 5$, $\alpha = 0.05$ and $k = 0.1$

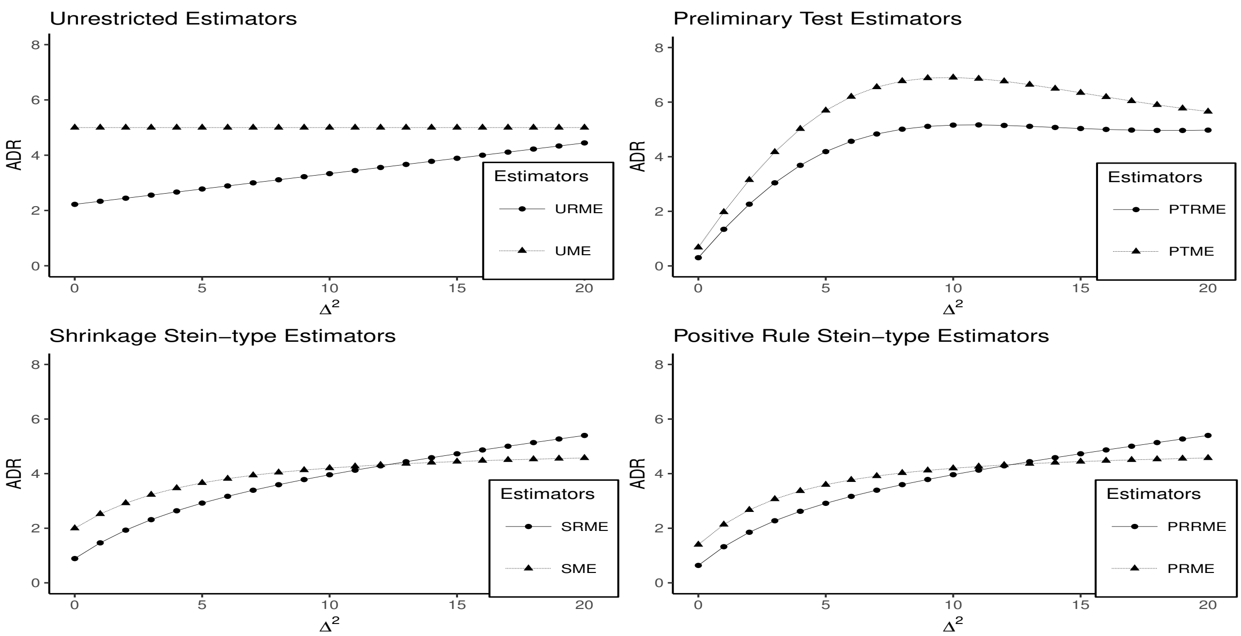

Figure S2: Comparison of Estimators for $p = 5$, $\alpha = 0.05$ and $k = 0.5$

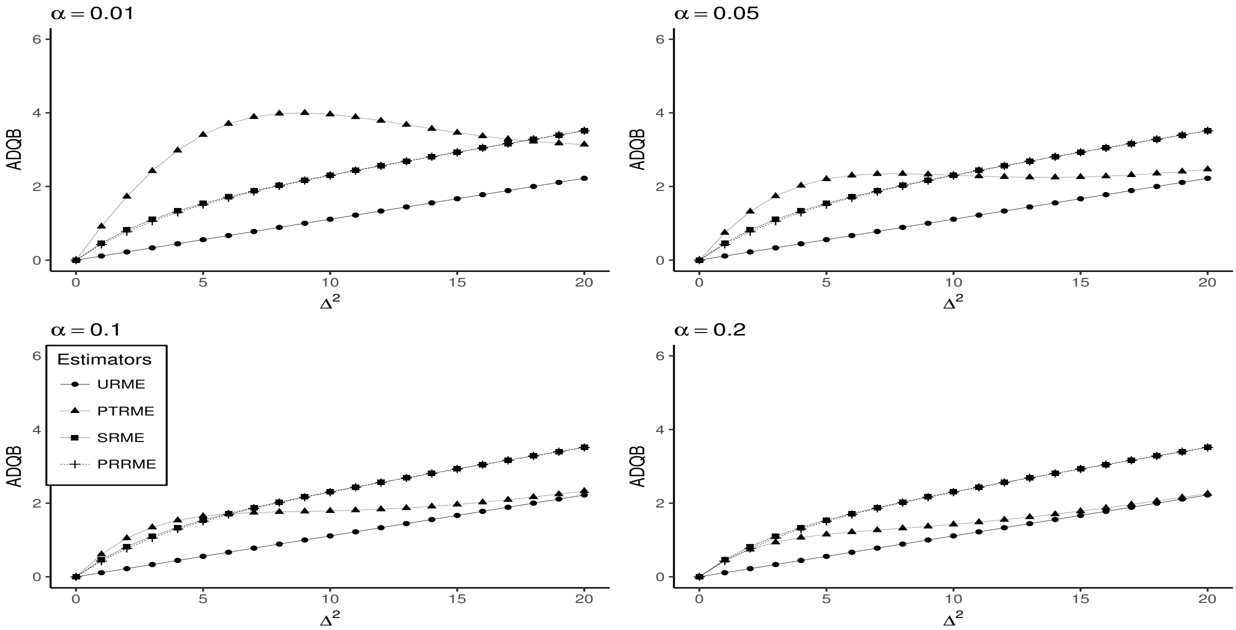

Figure S3: Asymptotic quadratic bias behavior of the estimators for $p = 5$, $k = 0.5$, $\eta^2 = 1$, and different α.

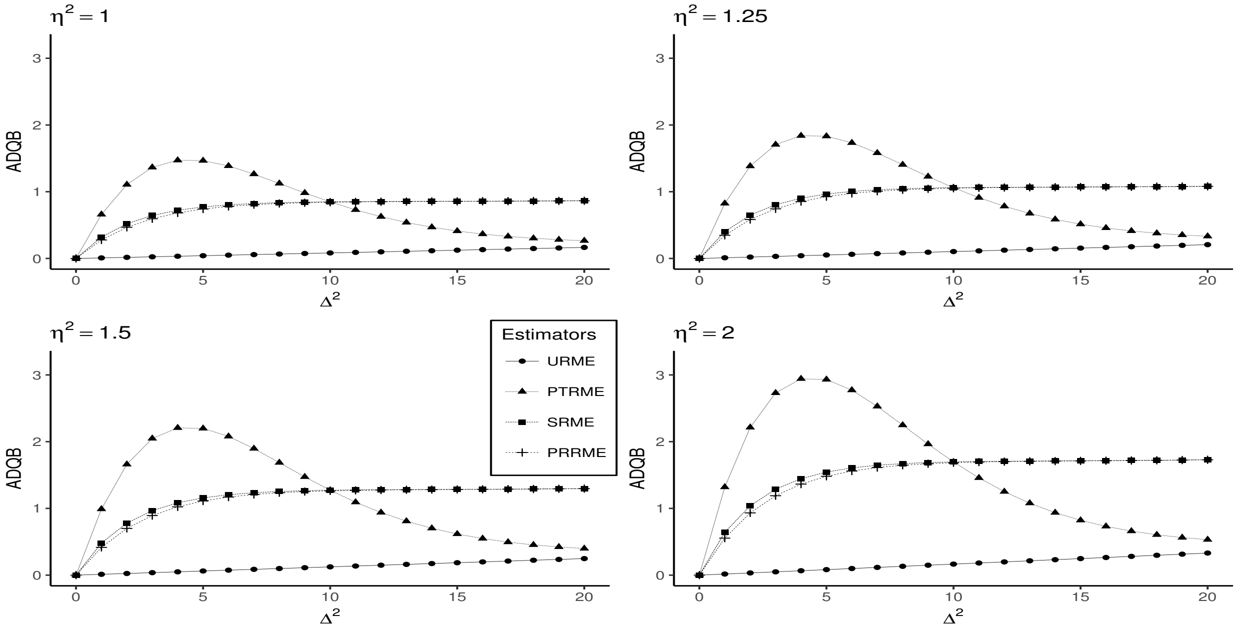

Figure S4: Asymptotic quadratic bias behavior of the estimators for $p = 5$, $k = 0.1$, $\alpha = 0.05$, and different η^2.

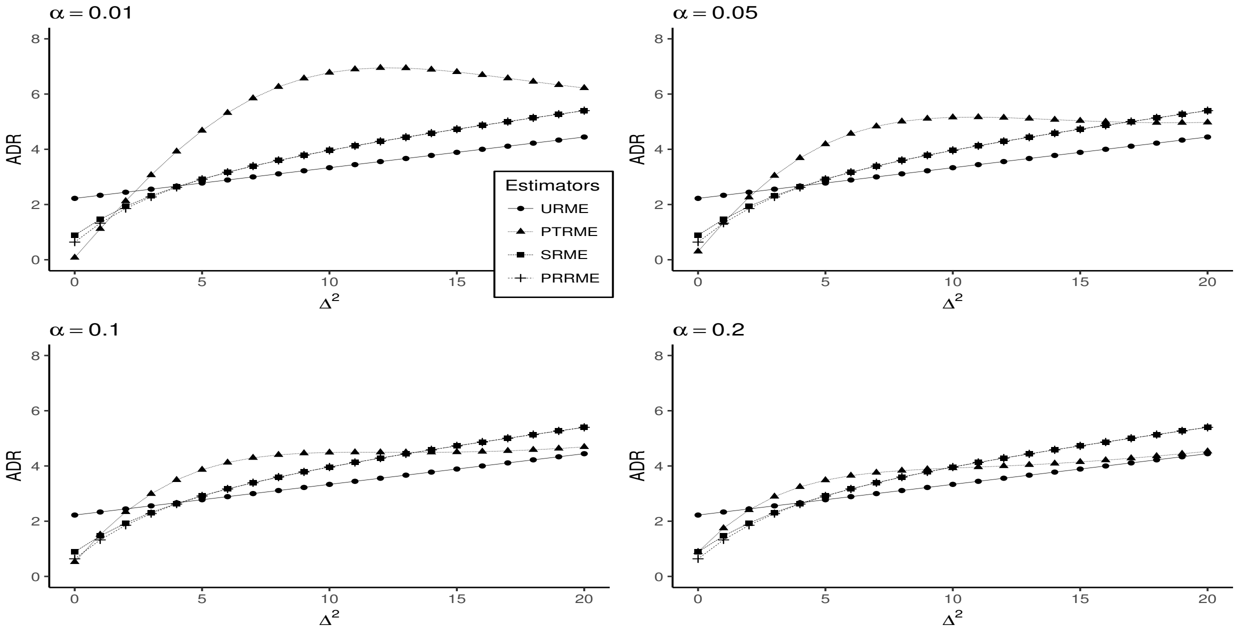

Figure S5: Asymptotic risk behavior of the estimators for $p = 5$, $k = 0.5$, $\eta^2 = 1$, and different α.

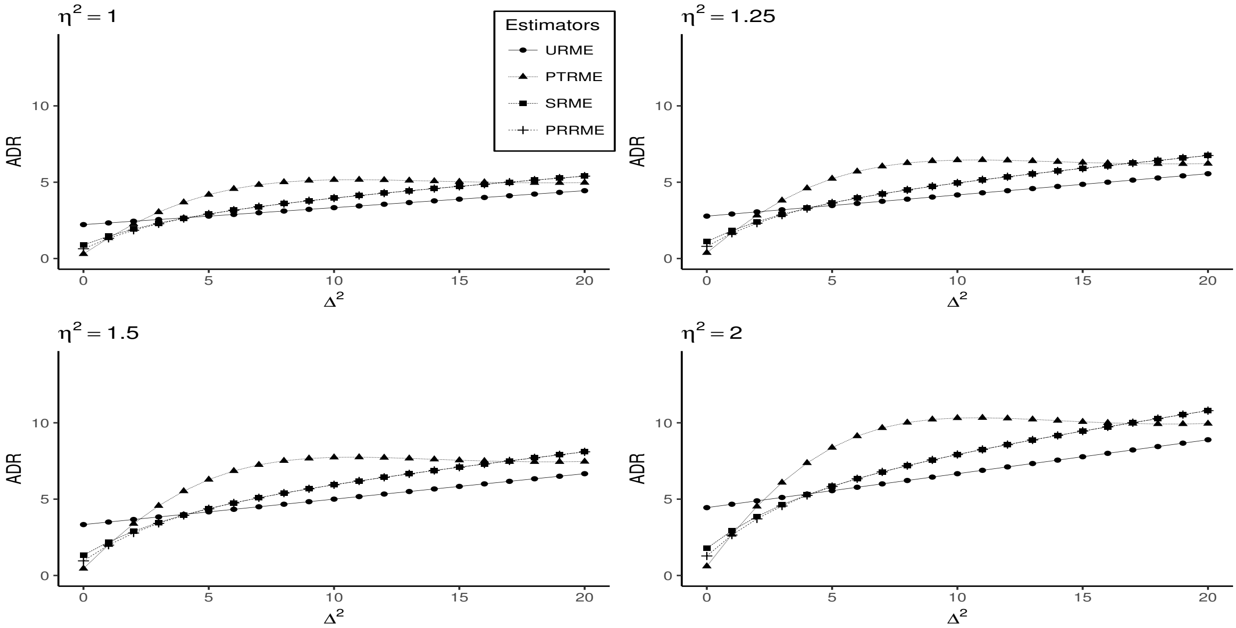

Figure S6: Asymptotic risk behavior of the estimators for $p = 5$, $k = 0.1$, $\alpha = 0.05$, and different η^2.

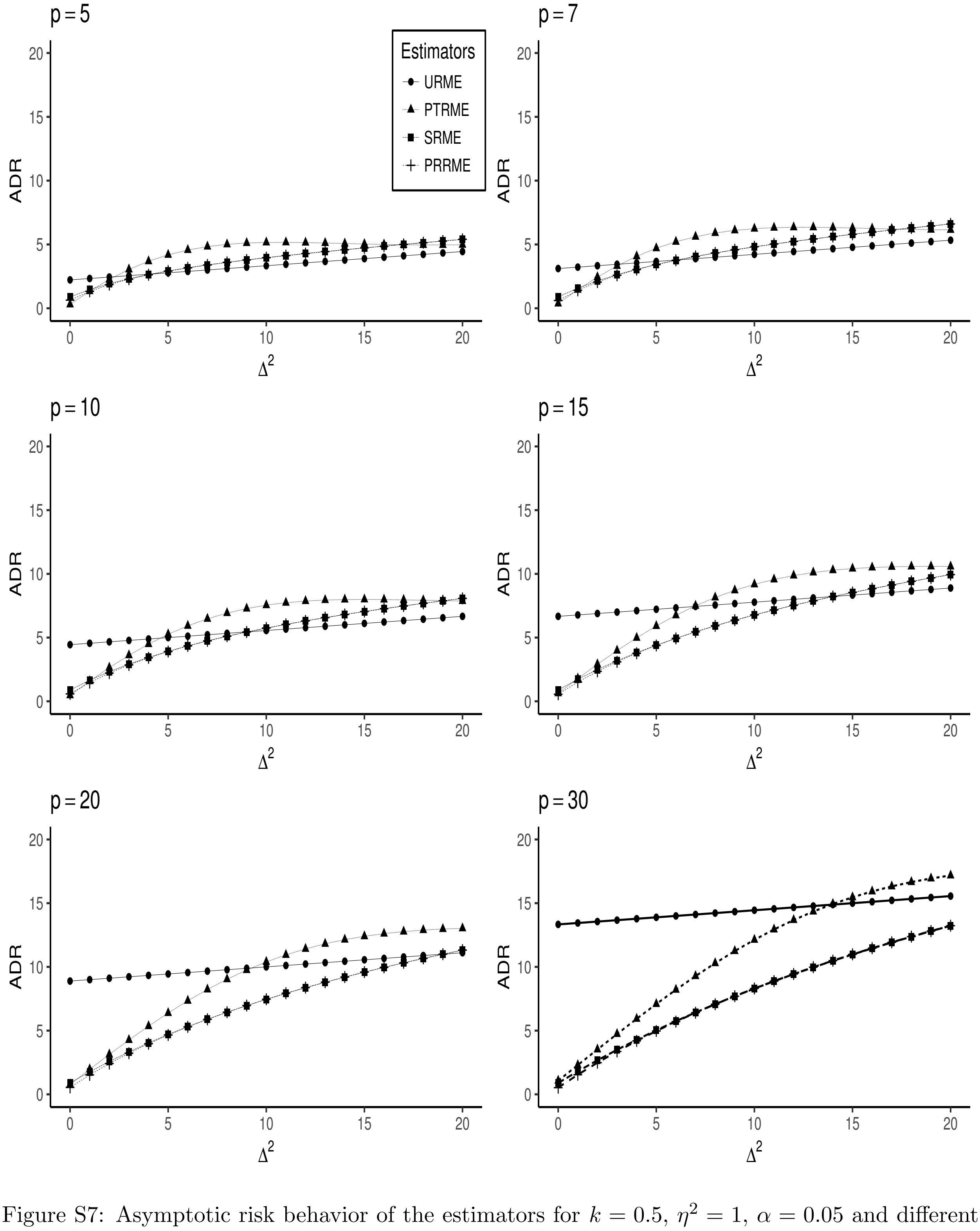

Figure S7: Asymptotic risk behavior of the estimators for $k = 0.5$, $\eta^2 = 1$, $\alpha = 0.05$ and different p.

Table S1: Relative efficiencies of the estimators compared with URME for $\eta^2 = 1$, $\alpha = 0.05$ and different p and k.

			Δ^2									
k	p	Estimators	0.000	3.333	6.667	10.000	13.333	16.667	20.000	23.333	26.667	30.000
		PTRME	7.376	1.107	0.769	0.720	0.755	0.819	0.883	0.932	0.964	0.982
	5	SRME	2.500	1.496	1.269	1.177	1.129	1.100	1.081	1.067	1.057	1.049
		PRRME	3.571	1.647	1.324	1.196	1.135	1.102	1.082	1.067	1.057	1.049
0.01		PTRME	9.374	1.778	1.103	0.909	0.855	0.859	0.886	0.919	0.947	0.968
	10	SRME	5.000	2.431	1.844	1.589	1.448	1.359	1.299	1.255	1.221	1.195
		PRRME	7.951	3.134	2.100	1.685	1.484	1.373	1.304	1.257	1.222	1.196
		PTRME	12.458	3.754	2.191	1.586	1.291	1.134	1.049	1.005	0.985	0.978
	30	SRME	15.000	6.157	4.141	3.251	2.749	2.428	2.204	2.040	1.915	1.816
		PRRME	26.296	17.434	7.760	4.542	3.270	2.653	2.305	2.086	1.935	1.825
		PTRME	7.376	1.023	0.720	0.685	0.729	0.801	0.872	0.926	0.961	0.981
	5	SRME	2.500	1.393	1.165	1.075	1.029	1.001	0.983	0.971	0.961	0.954
		PRRME	3.571	1.528	1.214	1.092	1.034	1.003	0.983	0.971	0.961	0.954
0.1		PTRME	9.374	1.647	1.031	0.861	0.820	0.833	0.868	0.907	0.940	0.965
	10	SRME	5.000	2.220	1.652	1.412	1.282	1.201	1.145	1.106	1.076	1.052
		PRRME	7.951	2.807	1.865	1.492	1.312	1.212	1.150	1.107	1.076	1.052
		PTRME	12.458	3.523	2.056	1.498	1.230	1.090	1.016	0.981	0.967	0.965
	30	SRME	15.000	5.549	3.651	2.839	2.388	2.102	1.904	1.760	1.650	1.563
		PRRME	26.296	13.479	6.307	3.821	2.791	2.278	1.983	1.796	1.666	1.570
		PTRME	7.376	0.699	0.638	0.704	0.796	0.878	0.935	0.968	0.986	0.994
	5	SRME	2.500	0.898	0.812	0.806	0.816	0.828	0.841	0.853	0.863	0.872
		PRRME	3.571	0.941	0.828	0.811	0.817	0.829	0.841	0.853	0.863	0.872
1		PTRME	9.373	0.972	0.734	0.714	0.756	0.818	0.878	0.926	0.958	0.978
	10	SRME	5.000	1.155	0.890	0.812	0.784	0.775	0.774	0.777	0.782	0.789
		PRRME	7.951	1.285	0.942	0.833	0.792	0.778	0.775	0.778	0.782	0.789
		PTRME	12.485	1.980	1.226	0.981	0.884	0.850	0.850	0.867	0.892	0.918
	30	SRME	15.000	2.419	1.522	1.203	1.043	0.950	0.891	0.851	0.823	0.803
		PRRME	26.296	3.245	1.859	1.355	1.114	0.984	0.910	0.858	0.826	0.805
		PTRME	7.376	0.893	0.927	0.956	0.976	0.988	0.994	0.997	0.999	0.999
	5	SRME	2.500	0.932	0.947	0.959	0.966	0.971	0.975	0.978	0.980	0.982
		PRRME	3.571	0.936	0.950	0.959	0.966	0.971	0.975	0.978	0.980	0.982
10		PTRME	9.374	0.887	0.907	0.934	0.957	0.974	0.985	0.992	0.995	0.998
	10	SRME	5.000	0.906	0.915	0.926	0.935	0.942	0.948	0.953	0.957	0.961
		PRRME	7.951	0.915	0.921	0.929	0.936	0.943	0.948	0.953	0.957	0.961
		PTRME	12.458	0.915	0.897	0.905	0.920	0.936	0.952	0.966	0.976	0.984
	30	SRME	15.000	0.921	0.897	0.895	0.898	0.901	0.906	0.910	0.914	0.918
		PRRME	26.296	0.937	0.913	0.906	0.904	0.905	0.908	0.911	0.914	0.918

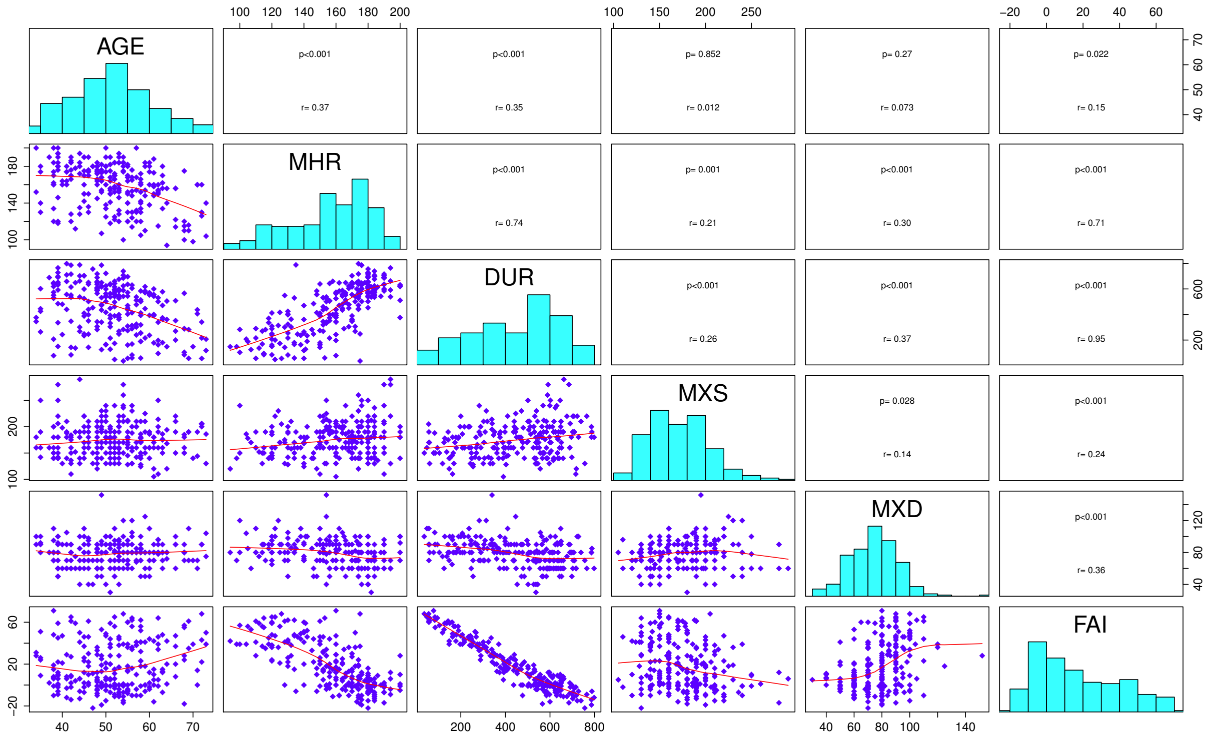

Figure S8: The scatter plot for independent variables of V02 data sets

S4 More Details on Simulation Studies

Some other tables (Tables S2-S7) for showing the effect of ρ^2 and Δ^2 on the number of parameters and sample size.

S5 More Details on Applications

In this section, we show that the used datasets violate two assumptions of multicollinearity and non-normal errors which means there are some outliers.

S5.1 V02 Dataset

Table S8 shows data summary while visual correlation matrix for the predictors is shown in Figure S8.

From Figure S10, we may guess the existence of multicollinearity. VIF values are shown in Table S11.

The values of Table S9 and $\kappa = 3451.728$ shows that there is a multicollinearity between independent variables.

Table S2: The relative efficiencies for the proposed estimators in case of $n = 20$, $p = 5$, and $\alpha = 0.05$

		No Outliers						10% Outliers						20% Outliers					
ρ^2	Δ^2	OLS	M	RRME	PTRME	SRME	RPRME	OLS	M	RRME	PTRME	SRME	PRRME	OLS	M	RRME	PTRME	SRME	PRRME
0	0	1.07	1.00	∞	3.36	2.03	2.72	0.21	1.00	∞	6.88	2.10	3.75	0.26	1.00	Inf	16.54	2.36	4.97
	0.01	1.07	1.00	100.78	3.28	2.10	2.64	0.19	1.00	200.90	5.18	2.18	3.36	0.27	1.00	527.77	15.32	2.15	4.79
	0.1	1.06	1.00	9.72	1.98	1.90	2.12	0.20	1.00	20.38	3.78	2.30	2.92	0.28	1.00	55.25	9.42	2.29	4.27
	0.5	1.07	1.00	1.96	0.98	1.43	1.45	0.21	1.00	4.32	1.67	1.91	2.00	0.28	1.00	10.85	3.38	2.17	2.73
	1	1.07	1.00	1.00	0.87	1.28	1.28	0.21	1.00	2.16	1.20	1.63	1.67	0.28	1.00	5.67	2.30	1.96	2.30
	2	1.07	1.00	0.50	0.94	1.19	1.19	0.20	1.00	1.00	0.98	1.40	1.40	0.27	1.00	2.68	1.48	1.83	1.83
	5	1.07	1.00	0.20	1.00	1.13	1.13	0.20	1.00	0.42	0.96	1.27	1.27	0.27	1.00	1.10	1.14	1.55	1.55
	10	1.07	1.00	0.10	1.00	1.08	1.08	0.20	1.00	0.21	0.96	1.18	1.18	0.27	1.00	0.54	0.95	1.38	1.38
	20	1.06	1.00	0.05	1.00	1.03	1.03	0.20	1.00	0.10	0.96	1.11	1.11	0.26	1.00	0.25	0.89	1.26	1.26
	30	1.07	1.00	0.03	1.00	0.99	0.99	0.19	1.00	0.07	0.94	1.04	1.04	0.28	1.00	0.18	0.87	1.21	1.21
0.5	0	0.01	0.01	∞	1.75	1.86	1.87	0.01	0.05	∞	7.30	1.74	4.00	0.02	0.09	Inf	12.19	2.68	4.20
	0.01	0.02	0.02	2.91	1.34	1.45	1.46	0.01	0.05	17.54	3.41	2.47	2.60	0.03	0.10	83.30	14.01	2.92	4.27
	0.1	0.07	0.06	0.99	0.94	1.02	1.02	0.01	0.07	2.20	1.80	1.54	1.73	0.03	0.11	9.85	5.99	2.90	3.28
	0.5	0.25	0.24	0.76	0.91	0.95	0.95	0.03	0.15	1.07	1.08	1.15	1.15	0.04	0.14	2.47	2.01	1.77	1.82
	1	0.44	0.42	0.68	0.95	0.95	0.95	0.05	0.23	0.82	0.99	0.92	1.04	0.05	0.17	1.60	1.53	1.47	1.49
	2	0.75	0.71	0.56	0.99	0.94	0.94	0.08	0.40	0.64	0.94	0.97	0.97	0.06	0.23	0.99	1.09	1.19	1.19
	5	1.24	1.16	0.38	1.00	0.92	0.92	0.13	0.66	0.45	0.97	0.93	0.93	0.10	0.35	0.63	1.00	1.05	1.05
	10	1.61	1.50	0.24	1.00	0.89	0.89	0.18	0.89	0.30	0.97	0.90	0.90	0.13	0.50	0.42	0.92	0.96	0.96
	20	1.81	1.69	0.14	1.00	0.85	0.85	0.21	1.06	0.18	0.97	0.84	0.84	0.17	0.65	0.27	0.89	0.89	0.89
	30	1.80	1.69	0.09	1.00	0.82	0.82	0.23	1.17	0.13	0.98	0.81	0.81	0.19	0.70	0.20	0.88	0.86	0.86
0.8	0	0.01	0.01	∞	1.81	1.89	1.91	0.01	0.04	∞	7.78	2.79	4.05	0.02	0.08	∞	12.02	2.91	4.11
	0.01	0.02	0.01	5.70	1.52	1.64	1.66	0.01	0.05	40.68	2.86	2.57	2.74	0.03	0.09	196.86	16.77	2.73	4.29
	0.1	0.03	0.03	1.21	0.99	1.11	1.11	0.01	0.05	4.05	2.51	1.80	2.21	0.03	0.10	22.08	8.30	3.25	3.75
	0.5	0.11	0.10	0.82	0.90	0.97	0.97	0.02	0.09	1.58	1.27	1.36	1.36	0.03	0.11	4.82	3.15	2.20	2.30
	1	0.20	0.19	0.73	0.94	0.96	0.96	0.02	0.12	1.03	1.06	1.14	1.14	0.04	0.13	2.90	2.07	1.85	1.87
	2	0.36	0.33	0.65	0.98	0.96	0.96	0.04	0.20	0.78	0.97	1.00	1.02	0.04	0.15	1.58	1.35	1.42	1.43
	5	0.70	0.65	0.52	1.00	0.95	0.95	0.07	0.36	0.61	0.98	0.98	0.98	0.06	0.22	0.95	1.10	1.19	1.19
	10	1.08	1.01	0.40	1.00	0.93	0.93	0.11	0.56	0.47	0.98	0.95	0.95	0.08	0.31	0.65	0.98	1.06	1.06
	20	1.46	1.37	0.27	1.00	0.90	0.90	0.16	0.78	0.32	0.97	0.91	0.91	0.11	0.44	0.44	0.93	0.97	0.97
	30	1.59	1.49	0.20	1.00	0.88	0.88	0.18	0.93	0.25	0.98	0.88	0.88	0.14	0.50	0.35	0.91	0.94	0.94
0.9	0	0.01	0.01	∞	1.81	1.89	1.92	0.01	0.04	∞	8.13	3.55	4.08	0.02	0.08	Inf	12.28	2.91	4.09
	0.01	0.01	0.01	9.99	1.60	1.73	1.75	0.01	0.05	78.72	2.95	2.63	2.81	0.03	0.09	384.66	18.01	2.71	4.31
	0.1	0.02	0.02	1.60	1.06	1.21	1.22	0.01	0.04	7.12	3.13	2.00	2.56	0.03	0.10	42.75	11.10	3.40	3.98
	0.5	0.06	0.06	0.90	0.91	1.00	1.00	0.01	0.07	2.38	1.53	1.57	1.58	0.03	0.10	8.76	4.19	2.49	2.62
	1	0.11	0.10	0.78	0.93	0.98	0.98	0.02	0.08	1.38	1.18	1.28	1.28	0.03	0.11	5.00	2.64	2.15	2.17
	2	0.19	0.18	0.70	0.97	0.97	0.97	0.02	0.12	0.94	1.01	1.09	1.09	0.03	0.12	2.50	1.64	1.65	1.66
	5	0.41	0.38	0.60	1.00	0.96	0.96	0.05	0.22	0.74	1.00	1.03	1.03	0.04	0.16	1.39	1.25	1.34	1.35
	10	0.69	0.65	0.51	1.00	0.96	0.96	0.07	0.36	0.60	0.99	0.99	0.99	0.06	0.22	0.90	1.07	1.16	1.16
	20	1.08	1.01	0.39	1.00	0.94	0.94	0.11	0.54	0.44	0.98	0.95	0.95	0.08	0.31	0.62	0.97	1.05	1.05
	30	1.27	1.19	0.31	1.00	0.92	0.92	0.14	0.70	0.37	0.99	0.93	0.93	0.10	0.36	0.51	0.96	1.01	1.01
0.99	0	0.01	0.01	∞	1.72	1.85	1.87	0.01	0.04	∞	8.53	3.59	4.24	0.02	0.08	∞	13.24	2.95	4.07
	0.01	0.01	0.01	79.43	1.67	1.82	1.84	0.01	0.05	740.15	2.98	2.69	2.86	0.02	0.09	3678.00	20.67	2.65	4.28
	0.1	0.01	0.01	8.11	1.36	1.59	1.60	0.01	0.04	60.92	4.69	2.37	3.16	0.03	0.09	412.51	16.55	3.55	4.24
	0.5	0.02	0.02	2.35	1.05	1.25	1.26	0.01	0.05	17.03	2.37	2.13	2.15	0.03	0.09	79.76	6.71	2.92	3.09
	1	0.02	0.02	1.52	0.97	1.13	1.13	0.01	0.04	7.48	1.84	1.85	1.86	0.03	0.10	42.67	3.87	2.67	2.69
	2	0.03	0.03	1.04	0.98	1.06	1.06	0.01	0.05	3.53	1.29	1.50	1.50	0.02	0.09	18.94	2.71	2.17	2.19
	5	0.06	0.05	0.82	1.00	1.01	1.01	0.01	0.06	2.12	1.12	1.33	1.33	0.03	0.10	8.49	1.78	1.82	1.82
	10	0.10	0.09	0.74	1.00	1.00	1.00	0.02	0.08	1.39	1.09	1.21	1.21	0.03	0.10	4.22	1.42	1.60	1.60
	20	0.19	0.17	0.67	1.00	0.99	0.99	0.02	0.11	0.92	1.02	1.09	1.09	0.03	0.11	2.23	1.14	1.38	1.38
	30	0.26	0.24	0.63	1.00	0.99	0.99	0.03	0.15	0.81	1.00	1.06	1.06	0.03	0.12	1.75	1.09	1.31	1.31

Table S3: The relative efficiencies for the proposed estimators in case of $n = 20$, $p = 10$, and $\alpha = 0.1$

		No Outliers						10% Outliers						20% Outliers					
ρ^2	Δ^2	OLS	M	RRME	PTRME	SRME	RPRME	OLS	M	RRME	PTRME	SRME	PRRME	OLS	M	RRME	PTRME	SRME	PRRME
0	0	1.10	1.00	∞	1.70	2.62	2.74	0.39	1.00	∞	5.31	3.59	7.17	0.53	1.00	∞	13.89	3.59	12.87
	0.01	1.10	1.00	283.32	1.67	2.60	2.72	0.38	1.00	1056.80	5.50	4.03	6.95	0.55	1.00	2978.21	14.74	3.87	13.22
	0.1	1.09	1.00	28.13	1.48	2.32	2.41	0.39	1.00	110.77	4.96	3.51	6.47	0.55	1.00	294.93	13.12	4.43	11.54
	0.5	1.10	1.00	5.59	1.11	1.81	1.83	0.39	1.00	22.02	2.94	3.61	4.28	0.53	1.00	55.94	7.07	4.61	7.29
	1	1.09	1.00	2.84	1.01	1.56	1.57	0.38	1.00	10.93	2.22	3.22	3.40	0.54	1.00	30.02	4.50	4.63	5.37
	2	1.09	1.00	1.41	0.99	1.39	1.39	0.36	1.00	5.17	1.65	2.48	2.51	0.53	1.00	14.22	2.78	3.64	3.77
	5	1.10	1.00	0.56	1.00	1.25	1.25	0.39	1.00	2.18	1.26	1.98	1.98	0.55	1.00	6.10	1.81	2.83	2.83
	10	1.09	1.00	0.28	1.00	1.18	1.18	0.39	1.00	1.11	1.11	1.76	1.76	0.54	1.00	2.92	1.40	2.34	2.34
	20	1.09	1.00	0.14	1.00	1.10	1.10	0.39	1.00	0.55	1.04	1.58	1.58	0.54	1.00	1.47	1.18	2.06	2.06
	30	1.10	1.00	0.10	1.00	1.05	1.05	0.37	1.00	0.35	1.02	1.46	1.46	0.54	1.00	0.98	1.08	1.91	1.91
0.5	0	0.03	0.03	∞	1.31	2.10	2.11	0.04	0.11	∞	6.55	4.19	7.98	0.07	0.14	∞	10.96	4.39	10.44
	0.01	0.03	0.03	13.31	1.22	1.91	1.92	0.04	0.10	177.06	6.44	4.36	7.26	0.08	0.15	753.07	8.43	3.94	9.09
	0.1	0.04	0.04	1.98	1.05	1.38	1.38	0.04	0.11	20.34	4.34	3.84	5.37	0.07	0.14	70.52	9.73	5.64	8.94
	0.5	0.11	0.10	0.96	0.97	1.05	1.05	0.05	0.12	4.68	2.31	2.64	2.95	0.07	0.14	13.16	4.52	4.30	4.94
	1	0.18	0.16	0.82	0.98	1.00	1.00	0.05	0.13	2.52	1.74	2.02	2.06	0.08	0.15	7.60	3.05	3.44	3.61
	2	0.31	0.28	0.71	1.00	0.97	0.97	0.06	0.16	1.44	1.31	1.49	1.50	0.08	0.16	3.94	1.94	2.45	2.47
	5	0.63	0.57	0.57	1.00	0.95	0.95	0.09	0.25	0.94	1.08	1.21	1.21	0.11	0.20	2.05	1.43	1.81	1.82
	10	0.98	0.89	0.45	1.00	0.93	0.93	0.14	0.36	0.69	1.04	1.09	1.09	0.13	0.25	1.27	1.15	1.47	1.47
	20	1.40	1.28	0.32	1.00	0.89	0.89	0.19	0.50	0.48	0.99	0.98	0.98	0.16	0.32	0.80	1.02	1.22	1.22
	30	1.61	1.45	0.25	1.00	0.86	0.86	0.22	0.62	0.37	0.97	0.92	0.92	0.20	0.38	0.63	0.94	1.12	1.12
0.8	0	0.03	0.03	∞	1.30	2.04	2.04	0.04	0.10	∞	6.46	4.22	8.13	0.07	0.14	∞	11.28	4.49	10.49
	0.01	0.03	0.02	30.64	1.26	2.01	2.01	0.04	0.10	437.64	6.68	4.39	7.53	0.08	0.14	1831.16	9.34	4.01	9.41
	0.1	0.03	0.03	3.77	1.09	1.60	1.60	0.04	0.10	49.14	4.38	4.28	5.90	0.07	0.14	172.86	9.53	6.01	9.28
	0.5	0.06	0.05	1.31	0.98	1.16	1.16	0.04	0.11	10.68	2.75	3.24	3.64	0.07	0.13	31.43	5.23	4.99	5.66
	1	0.09	0.08	1.02	0.99	1.07	1.07	0.04	0.11	5.26	2.03	2.61	2.66	0.07	0.14	18.02	3.42	3.88	4.12
	2	0.15	0.13	0.84	1.00	1.01	1.01	0.04	0.12	2.61	1.57	1.90	1.91	0.07	0.14	8.81	2.40	3.04	3.08
	5	0.31	0.28	0.69	1.00	0.98	0.98	0.06	0.17	1.54	1.19	1.47	1.47	0.09	0.17	4.24	1.69	2.28	2.29
	10	0.52	0.47	0.59	1.00	0.96	0.96	0.08	0.23	1.08	1.12	1.29	1.29	0.10	0.19	2.43	1.41	1.87	1.87
	20	0.84	0.76	0.48	1.00	0.94	0.94	0.12	0.31	0.74	1.03	1.12	1.12	0.12	0.22	1.39	1.22	1.52	1.53
	30	1.07	0.96	0.41	1.00	0.92	0.92	0.14	0.40	0.59	1.02	1.04	1.04	0.14	0.26	1.08	1.10	1.38	1.38
0.9	0	0.03	0.02	∞	1.30	2.08	2.08	0.04	0.10	∞	6.67	4.27	8.33	0.07	0.14	∞	11.09	4.52	10.66
	0.01	0.03	0.02	59.22	1.26	2.03	2.03	0.04	0.10	875.61	6.86	4.42	7.60	0.08	0.14	3625.13	9.85	4.04	9.53
	0.1	0.03	0.03	6.84	1.12	1.74	1.74	0.04	0.10	97.33	4.55	4.45	6.10	0.07	0.14	340.63	9.17	6.10	9.44
	0.5	0.04	0.04	1.87	1.00	1.27	1.27	0.04	0.11	20.64	2.98	3.55	3.99	0.07	0.13	62.26	5.35	5.29	5.94
	1	0.06	0.05	1.33	0.99	1.14	1.14	0.04	0.10	9.83	2.30	3.01	3.06	0.07	0.14	35.29	3.63	4.08	4.38
	2	0.09	0.08	1.00	1.00	1.06	1.06	0.04	0.10	4.52	1.72	2.22	2.24	0.07	0.14	16.85	2.51	3.34	3.38
	5	0.17	0.16	0.78	1.00	1.01	1.01	0.05	0.13	2.48	1.28	1.71	1.71	0.08	0.15	7.80	1.86	2.58	2.59
	10	0.30	0.27	0.69	1.00	0.98	0.98	0.06	0.17	1.61	1.19	1.48	1.48	0.09	0.17	4.28	1.56	2.17	2.18
	20	0.52	0.47	0.59	1.00	0.97	0.97	0.08	0.22	1.04	1.09	1.26	1.26	0.09	0.18	2.29	1.37	1.80	1.81
	30	0.69	0.62	0.53	1.00	0.95	0.95	0.10	0.27	0.81	1.06	1.15	1.15	0.11	0.21	1.71	1.23	1.63	1.63
0.99	0	0.03	0.02	∞	1.27	2.07	2.08	0.04	0.10	∞	6.20	4.24	8.19	0.07	0.14	∞	10.01	4.56	10.99
	0.01	0.03	0.02	580.49	1.28	2.06	2.07	0.04	0.10	8795.16	6.95	4.43	7.63	0.07	0.14	35987.03	10.38	4.03	9.50
	0.1	0.03	0.02	60.89	1.15	1.91	1.92	0.04	0.10	967.69	4.75	4.62	6.31	0.07	0.14	3396.01	8.51	6.03	9.32
	0.5	0.03	0.02	12.14	1.03	1.55	1.55	0.04	0.10	202.54	3.34	3.95	4.40	0.07	0.13	616.45	5.37	5.54	6.16
	1	0.03	0.03	6.94	1.01	1.39	1.39	0.04	0.10	92.39	2.69	3.56	3.60	0.07	0.14	346.43	3.79	4.32	4.64
	2	0.03	0.03	3.67	1.00	1.27	1.27	0.03	0.09	39.97	1.91	2.76	2.78	0.07	0.13	162.25	2.74	3.69	3.74
	5	0.04	0.04	1.86	1.00	1.15	1.15	0.04	0.10	18.87	1.47	2.29	2.29	0.07	0.14	71.94	2.04	2.99	3.00
	10	0.06	0.05	1.28	1.00	1.09	1.09	0.04	0.11	10.39	1.40	2.10	2.10	0.08	0.15	37.15	1.68	2.67	2.67
	20	0.09	0.08	1.01	1.00	1.05	1.05	0.04	0.11	5.23	1.24	1.83	1.83	0.07	0.14	17.61	1.55	2.41	2.41
	30	0.12	0.10	0.89	1.00	1.03	1.03	0.04	0.11	3.37	1.14	1.66	1.66	0.08	0.14	11.95	1.43	2.33	2.33

Table S4: The relative efficiencies for the proposed estimators in case of $n = 30$, $p = 10$, and $\alpha = 0.05$

		No Outliers						10% Outliers						20% Outliers					
ρ^2	Δ^2	OLS	M	RRME	PTRME	SRME	RPRME	OLS	M	RRME	PTRME	SRME	PRRME	OLS	M	RRME	PTRME	SRME	PRRME
0	0	1.08	1.00	∞	2.55	3.39	3.85	0.21	1.00	∞	5.26	4.49	6.28	0.30	1.00	∞	17.66	4.50	12.69
	0.01	1.08	1.00	131.07	2.44	3.26	3.63	0.21	1.00	277.40	5.19	4.35	5.99	0.30	1.00	742.99	15.73	4.44	11.33
	0.1	1.08	1.00	13.19	1.64	2.65	2.77	0.21	1.00	28.38	3.28	3.85	4.56	0.30	1.00	76.41	8.98	4.99	8.76
	0.5	1.08	1.00	2.59	1.01	1.76	1.76	0.22	1.00	5.80	1.55	2.60	2.67	0.30	1.00	15.44	3.65	4.17	4.66
	1	1.07	1.00	1.31	0.97	1.50	1.50	0.21	1.00	2.89	1.23	2.10	2.11	0.30	1.00	7.56	2.21	3.06	3.46
	2	1.07	1.00	0.65	1.00	1.34	1.34	0.21	1.00	1.40	1.05	1.72	1.72	0.30	1.00	3.85	1.56	2.60	2.61
	5	1.07	1.00	0.26	1.00	1.19	1.19	0.21	1.00	0.56	1.01	1.45	1.45	0.29	1.00	1.47	1.12	1.96	1.97
	10	1.08	1.00	0.13	1.00	1.11	1.11	0.21	1.00	0.29	1.00	1.29	1.29	0.30	1.00	0.75	1.02	1.69	1.69
	20	1.08	1.00	0.07	1.00	0.99	0.99	0.21	1.00	0.14	0.99	1.13	1.13	0.30	1.00	0.38	0.97	1.45	1.45
	30	1.07	1.00	0.04	1.00	0.92	0.92	0.22	1.00	0.10	0.99	1.03	1.03	0.30	1.00	0.26	0.95	1.31	1.31
0.5	0	0.01	0.01	∞	1.53	2.60	2.63	0.00	0.02	∞	5.05	5.39	6.14	0.02	0.05	∞	20.21	5.41	12.76
	0.01	0.01	0.01	2.42	1.22	1.59	1.60	0.01	0.02	12.85	3.91	3.43	4.07	0.02	0.05	70.19	13.19	5.38	10.06
	0.1	0.04	0.04	0.96	0.95	1.01	1.02	0.01	0.04	1.99	1.61	1.56	1.79	0.02	0.05	7.58	5.08	3.88	4.73
	0.5	0.18	0.17	0.80	0.97	0.95	0.95	0.02	0.09	1.00	1.06	1.11	1.12	0.02	0.08	2.25	1.89	2.00	2.03
	1	0.33	0.31	0.75	0.99	0.95	0.95	0.03	0.15	0.85	1.01	1.02	1.02	0.03	0.10	1.46	1.38	1.49	1.51
	2	0.60	0.55	0.68	1.00	0.95	0.95	0.06	0.28	0.73	0.99	0.97	0.97	0.04	0.14	1.05	1.10	1.21	1.21
	5	1.21	1.13	0.55	1.00	0.94	0.94	0.12	0.55	0.59	1.00	0.94	0.94	0.08	0.26	0.73	1.00	1.02	1.02
	10	1.88	1.74	0.43	1.00	0.93	0.93	0.19	0.88	0.48	1.00	0.92	0.92	0.12	0.40	0.58	0.99	0.96	0.96
	20	2.67	2.48	0.30	1.00	0.90	0.90	0.28	1.30	0.35	1.00	0.88	0.88	0.19	0.62	0.44	0.99	0.91	0.91
	30	3.05	2.84	0.23	1.00	0.87	0.87	0.34	1.55	0.28	1.00	0.85	0.85	0.23	0.75	0.37	0.99	0.88	0.88
0.8	0	0.01	0.01	∞	1.56	2.61	2.64	0.00	0.02	∞	5.48	5.43	6.22	0.01	0.05	∞	21.36	5.33	12.67
	0.01	0.01	0.01	4.51	1.36	1.94	1.95	0.00	0.02	28.69	4.51	4.04	4.70	0.01	0.05	164.56	12.89	5.53	10.76
	0.1	0.02	0.02	1.16	0.99	1.12	1.12	0.01	0.03	3.69	2.26	2.04	2.60	0.01	0.05	16.88	7.57	4.98	6.58
	0.5	0.08	0.07	0.85	0.97	0.98	0.98	0.01	0.05	1.35	1.22	1.35	1.36	0.02	0.06	4.35	2.70	2.86	2.95
	1	0.14	0.13	0.80	0.99	0.97	0.97	0.02	0.08	1.04	1.08	1.13	1.14	0.02	0.07	2.53	1.87	2.02	2.09
	2	0.27	0.25	0.75	1.00	0.97	0.97	0.03	0.13	0.85	1.01	1.03	1.03	0.03	0.09	1.60	1.26	1.51	1.51
	5	0.59	0.55	0.67	1.00	0.97	0.97	0.06	0.27	0.72	1.00	0.98	0.98	0.04	0.14	0.99	1.05	1.16	1.16
	10	1.03	0.96	0.59	1.00	0.96	0.96	0.10	0.46	0.63	1.00	0.96	0.96	0.07	0.22	0.78	1.01	1.06	1.06
	20	1.67	1.55	0.47	1.00	0.94	0.94	0.16	0.77	0.51	1.00	0.94	0.94	0.11	0.35	0.62	1.00	0.99	0.99
	30	2.09	1.95	0.39	1.00	0.93	0.93	0.21	0.99	0.44	1.00	0.92	0.92	0.14	0.45	0.54	1.00	0.96	0.96
0.9	0	0.01	0.01	∞	1.54	2.59	2.62	0.00	0.02	∞	5.44	5.39	6.23	0.01	0.05	∞	21.66	5.41	12.69
	0.01	0.01	0.01	7.79	1.42	2.16	2.18	0.00	0.02	54.58	4.77	4.27	5.03	0.01	0.05	322.25	13.30	5.60	11.03
	0.1	0.01	0.01	1.48	1.04	1.26	1.27	0.01	0.02	6.55	2.87	2.45	3.38	0.01	0.05	32.63	9.65	5.55	7.73
	0.5	0.04	0.04	0.92	0.97	1.01	1.01	0.01	0.04	1.92	1.42	1.66	1.69	0.02	0.05	7.87	3.41	3.64	3.80
	1	0.07	0.07	0.84	0.99	0.99	0.99	0.01	0.05	1.32	1.17	1.28	1.31	0.02	0.06	4.31	2.38	2.57	2.70
	2	0.14	0.13	0.79	1.00	0.98	0.98	0.02	0.08	1.00	1.02	1.11	1.11	0.02	0.07	2.48	1.44	1.85	1.85
	5	0.32	0.30	0.73	1.00	0.98	0.98	0.03	0.15	0.81	1.00	1.02	1.02	0.03	0.09	1.33	1.10	1.32	1.32
	10	0.59	0.55	0.67	1.00	0.97	0.97	0.06	0.27	0.72	1.00	0.99	0.99	0.04	0.14	0.99	1.02	1.16	1.16
	20	1.04	0.96	0.59	1.00	0.96	0.96	0.10	0.47	0.62	1.00	0.97	0.97	0.07	0.22	0.78	1.01	1.06	1.06
	30	1.38	1.28	0.52	1.00	0.95	0.95	0.14	0.63	0.56	1.00	0.95	0.95	0.09	0.29	0.69	1.00	1.02	1.02
0.99	0	0.01	0.01	∞	1.51	2.55	2.57	0.00	0.02	∞	5.15	5.40	6.24	0.01	0.05	∞	23.33	5.46	12.70
	0.01	0.01	0.01	65.34	1.49	2.48	2.50	0.00	0.02	515.64	5.03	4.57	5.47	0.01	0.05	3158.59	13.76	5.73	11.26
	0.1	0.01	0.01	7.19	1.28	1.98	1.99	0.00	0.02	57.66	4.12	3.28	5.08	0.01	0.05	314.98	13.28	6.17	9.33
	0.5	0.01	0.01	2.05	1.02	1.33	1.33	0.00	0.02	12.06	2.25	3.14	3.28	0.02	0.05	71.01	5.03	5.21	5.64
	1	0.01	0.01	1.43	1.00	1.16	1.16	0.01	0.02	6.35	1.60	2.13	2.28	0.02	0.05	36.36	3.72	3.98	4.32
	2	0.02	0.02	1.11	1.00	1.07	1.07	0.01	0.03	3.30	1.15	1.68	1.68	0.02	0.05	18.16	2.09	2.95	2.95
	5	0.04	0.04	0.91	1.00	1.02	1.02	0.01	0.03	1.79	1.04	1.34	1.34	0.01	0.05	6.89	1.28	2.13	2.13
	10	0.07	0.07	0.84	1.00	1.00	1.00	0.01	0.05	1.32	1.02	1.21	1.21	0.02	0.06	3.98	1.15	1.81	1.81
	20	0.14	0.13	0.79	1.00	0.99	0.99	0.02	0.08	1.00	1.00	1.09	1.09	0.02	0.07	2.35	1.09	1.56	1.56
	30	0.20	0.19	0.77	1.00	0.99	0.99	0.02	0.10	0.90	1.00	1.05	1.05	0.02	0.08	1.90	1.03	1.42	1.42

Table S5: The relative efficiencies for the proposed estimators in case of $n = 50$, $p = 10$, and $\alpha = 0.05$

		No Outliers						10% Outliers						20% Outliers					
ρ^2	Δ^2	OLS	M	RRME	PTRME	SRME	RPRME	OLS	M	RRME	PTRME	SRME	PRRME	OLS	M	RRME	PTRME	SRME	PRRME
0	0	1.07	1.00	∞	4.10	4.16	5.18	0.16	1.00	∞	6.09	4.59	6.51	0.17	1.00	Inf	12.96	5.65	9.37
	0.01	1.07	1.00	55.39	3.21	3.88	4.48	0.16	1.00	92.13	4.88	4.51	5.79	0.18	1.00	193.39	10.15	5.30	8.36
	0.1	1.06	1.00	5.48	1.35	2.38	2.42	0.15	1.00	9.02	1.96	3.02	3.21	0.17	1.00	19.24	3.61	4.06	4.70
	0.5	1.06	1.00	1.12	0.99	1.45	1.45	0.16	1.00	1.81	0.99	1.72	1.72	0.17	1.00	3.72	1.27	2.29	2.30
	1	1.07	1.00	0.56	1.00	1.29	1.29	0.16	1.00	0.91	0.99	1.45	1.45	0.17	1.00	1.86	1.03	1.80	1.80
	2	1.06	1.00	0.28	1.00	1.19	1.19	0.16	1.00	0.46	1.00	1.30	1.30	0.17	1.00	0.96	1.00	1.55	1.55
	5	1.07	1.00	0.11	1.00	1.10	1.10	0.16	1.00	0.19	1.00	1.16	1.16	0.17	1.00	0.38	0.99	1.33	1.33
	10	1.06	1.00	0.06	1.00	1.03	1.03	0.16	1.00	0.09	1.00	1.07	1.07	0.17	1.00	0.19	1.00	1.19	1.19
	20	1.06	1.00	0.03	1.00	0.92	0.92	0.16	1.00	0.05	1.00	0.94	0.94	0.17	1.00	0.09	1.00	1.04	1.04
	30	1.06	1.00	0.02	1.00	0.84	0.84	0.16	1.00	0.03	1.00	0.86	0.86	0.17	1.00	0.06	1.00	0.96	0.96
0.5	0	0.00	0.00	∞	2.12	3.30	3.35	0.00	0.00	∞	3.23	4.17	4.47	0.00	0.01	∞	8.60	6.57	8.04
	0.01	0.01	0.01	1.06	1.00	1.05	1.05	0.00	0.01	1.48	1.27	1.35	1.37	0.00	0.01	4.94	3.23	3.10	3.39
	0.1	0.09	0.09	0.88	0.95	0.95	0.95	0.01	0.05	0.90	0.93	0.96	0.96	0.01	0.03	1.21	1.15	1.22	1.23
	0.5	0.41	0.38	0.80	0.99	0.95	0.95	0.04	0.24	0.80	0.97	0.94	0.94	0.02	0.12	0.84	0.95	0.97	0.97
	1	0.75	0.71	0.74	1.00	0.95	0.95	0.07	0.43	0.74	0.99	0.94	0.94	0.04	0.21	0.76	0.96	0.94	0.94
	2	1.32	1.24	0.64	1.00	0.94	0.94	0.12	0.77	0.66	1.00	0.93	0.93	0.07	0.37	0.68	0.98	0.92	0.92
	5	2.39	2.24	0.47	1.00	0.92	0.92	0.23	1.41	0.49	1.00	0.90	0.90	0.13	0.75	0.53	0.99	0.89	0.89
	10	3.20	3.00	0.32	1.00	0.89	0.89	0.32	1.99	0.35	1.00	0.87	0.87	0.19	1.12	0.39	1.00	0.85	0.85
	20	3.84	3.60	0.19	1.00	0.83	0.83	0.41	2.56	0.22	1.00	0.81	0.81	0.26	1.50	0.26	1.00	0.80	0.80
	30	4.03	3.78	0.13	1.00	0.79	0.79	0.43	2.71	0.16	1.00	0.77	0.77	0.29	1.65	0.20	1.00	0.76	0.76
0.8	0	0.00	0.00	∞	2.23	3.40	3.47	0.00	0.00	∞	3.34	4.27	4.57	0.00	0.01	∞	8.63	6.58	8.02
	0.01	0.01	0.01	1.30	1.10	1.21	1.22	0.00	0.01	2.38	1.66	1.82	1.85	0.00	0.01	10.53	4.57	4.23	4.89
	0.1	0.04	0.03	0.89	0.93	0.96	0.96	0.00	0.02	0.96	0.95	1.01	1.01	0.00	0.02	1.75	1.48	1.58	1.62
	0.5	0.17	0.16	0.82	0.99	0.96	0.96	0.02	0.10	0.83	0.96	0.95	0.95	0.01	0.05	0.94	0.98	1.04	1.04
	1	0.33	0.31	0.79	1.00	0.96	0.96	0.03	0.19	0.79	0.99	0.95	0.95	0.02	0.10	0.84	0.97	0.98	0.98
	2	0.62	0.58	0.74	1.00	0.96	0.96	0.06	0.35	0.74	1.00	0.95	0.95	0.03	0.17	0.77	0.98	0.96	0.96
	5	1.31	1.23	0.63	1.00	0.95	0.95	0.12	0.74	0.64	1.00	0.94	0.94	0.07	0.38	0.66	0.99	0.93	0.93
	10	2.07	1.94	0.50	1.00	0.94	0.94	0.20	1.22	0.53	1.00	0.92	0.92	0.11	0.64	0.56	1.00	0.91	0.91
	20	2.95	2.77	0.36	1.00	0.91	0.91	0.30	1.86	0.39	1.00	0.89	0.89	0.17	1.00	0.43	1.00	0.88	0.88
	30	3.44	3.23	0.28	1.00	0.88	0.88	0.35	2.17	0.31	1.00	0.86	0.86	0.21	1.20	0.36	1.00	0.85	0.85
0.9	0	0.00	0.00	∞	2.26	3.41	3.50	0.00	0.00	∞	3.32	4.26	4.56	0.00	0.01	∞	8.65	6.60	8.04
	0.01	0.00	0.00	1.68	1.21	1.40	1.42	0.00	0.00	3.71	2.01	2.27	2.33	0.00	0.01	19.48	5.43	4.96	5.93
	0.1	0.02	0.02	0.92	0.93	0.98	0.98	0.00	0.01	1.07	1.00	1.09	1.10	0.00	0.01	2.62	1.88	2.04	2.12
	0.5	0.09	0.08	0.83	0.98	0.96	0.96	0.01	0.05	0.85	0.95	0.97	0.97	0.01	0.03	1.10	1.06	1.16	1.16
	1	0.17	0.16	0.81	1.00	0.97	0.97	0.02	0.10	0.81	0.99	0.96	0.96	0.01	0.05	0.93	0.99	1.05	1.05
	2	0.33	0.31	0.78	1.00	0.97	0.97	0.03	0.18	0.78	1.00	0.96	0.96	0.02	0.09	0.83	0.99	1.00	1.00
	5	0.75	0.70	0.72	1.00	0.97	0.97	0.07	0.42	0.71	1.00	0.96	0.96	0.04	0.21	0.73	0.99	0.96	0.96
	10	1.29	1.21	0.63	1.00	0.96	0.96	0.12	0.74	0.64	1.00	0.95	0.95	0.06	0.38	0.66	1.00	0.94	0.94
	20	2.08	1.95	0.51	1.00	0.94	0.94	0.20	1.26	0.53	1.00	0.93	0.93	0.11	0.65	0.56	1.00	0.92	0.92
	30	2.62	2.47	0.42	1.00	0.93	0.93	0.26	1.59	0.45	1.00	0.91	0.91	0.14	0.82	0.49	1.00	0.91	0.91
0.99	0	0.00	0.00	∞	2.07	3.23	3.30	0.00	0.00	∞	3.10	4.15	4.37	0.00	0.01	∞	8.57	6.70	8.07
	0.01	0.00	0.00	7.11	1.67	2.38	2.42	0.00	0.00	25.27	2.83	3.59	3.73	0.00	0.01	176.85	6.99	6.13	7.66
	0.1	0.00	0.00	1.45	1.05	1.26	1.27	0.00	0.00	3.00	1.57	1.96	1.98	0.00	0.01	17.95	4.49	4.16	4.56
	0.5	0.01	0.01	0.95	0.98	1.02	1.02	0.00	0.01	1.26	1.02	1.19	1.19	0.00	0.01	3.78	1.69	2.23	2.23
	1	0.02	0.02	0.88	1.00	1.00	1.00	0.00	0.01	1.02	1.00	1.07	1.07	0.00	0.01	2.34	1.24	1.66	1.66
	2	0.04	0.03	0.85	1.00	0.99	0.99	0.00	0.02	0.92	1.00	1.02	1.02	0.00	0.02	1.61	1.08	1.36	1.36
	5	0.09	0.08	0.82	1.00	0.99	0.99	0.01	0.05	0.84	1.00	1.00	1.00	0.01	0.03	1.04	1.00	1.10	1.10
	10	0.17	0.16	0.80	1.00	0.99	0.99	0.02	0.09	0.81	1.00	0.99	0.99	0.01	0.05	0.90	1.00	1.04	1.04
	20	0.32	0.30	0.78	1.00	0.99	0.99	0.03	0.18	0.78	1.00	0.98	0.98	0.02	0.09	0.81	1.00	1.00	1.00
	30	0.47	0.45	0.75	1.00	0.98	0.98	0.04	0.27	0.75	1.00	0.98	0.98	0.02	0.13	0.78	1.00	1.00	1.00

Table S6: The relative efficiencies for the proposed estimators in case of $n = 50$, $p = 20$, and $\alpha = 0.05$

		No Outliers						10% Outliers						20% Outliers					
ρ^2	Δ^2	OLS	M	RRME	PTRME	SRME	RPRME	OLS	M	RRME	PTRME	SRME	PRRME	OLS	M	RRME	PTRME	SRME	PRRME
0	0	1.09	1.00	∞	1.76	4.42	4.53	0.21	1.00	∞	3.69	7.09	8.57	0.31	1.00	∞	14.06	8.04	22.77
	0.01	1.09	1.00	144.03	1.68	4.18	4.29	0.22	1.00	312.85	3.47	6.63	8.15	0.32	1.00	887.73	12.62	8.06	21.79
	0.1	1.09	1.00	14.41	1.30	3.14	3.16	0.21	1.00	31.22	2.50	5.45	5.89	0.32	1.00	89.77	8.24	8.53	14.74
	0.5	1.09	1.00	2.90	1.01	1.94	1.94	0.21	1.00	6.14	1.27	3.01	3.02	0.31	1.00	16.96	2.90	5.99	6.28
	1	1.09	1.00	1.45	1.00	1.63	1.63	0.21	1.00	3.11	1.13	2.40	2.41	0.31	1.00	8.50	1.89	4.34	4.38
	2	1.09	1.00	0.73	1.00	1.43	1.43	0.21	1.00	1.56	1.02	1.91	1.91	0.32	1.00	4.42	1.38	3.26	3.28
	5	1.09	1.00	0.29	1.00	1.24	1.24	0.21	1.00	0.63	1.00	1.55	1.55	0.31	1.00	1.71	1.09	2.32	2.32
	10	1.08	1.00	0.14	1.00	1.12	1.12	0.22	1.00	0.31	1.00	1.33	1.33	0.31	1.00	0.88	1.02	1.88	1.88
	20	1.09	1.00	0.07	1.00	0.96	0.96	0.22	1.00	0.16	1.00	1.09	1.09	0.31	1.00	0.42	0.99	1.47	1.47
	30	1.09	1.00	0.05	1.00	0.85	0.85	0.21	1.00	0.10	1.00	0.95	0.95	0.31	1.00	0.30	0.98	1.29	1.29
0.5	0	0.00	0.00	∞	1.32	3.36	3.37	0.00	0.01	∞	3.38	7.29	8.40	0.01	0.02	∞	18.67	9.13	25.29
	0.01	0.01	0.01	1.60	1.08	1.38	1.38	0.00	0.01	6.30	2.65	3.55	4.14	0.01	0.02	41.02	11.34	8.11	16.90
	0.1	0.04	0.03	0.93	0.97	0.98	0.98	0.00	0.02	1.39	1.29	1.38	1.40	0.01	0.03	4.81	3.65	3.69	4.20
	0.5	0.17	0.16	0.85	0.99	0.95	0.95	0.02	0.08	0.90	0.99	0.99	0.99	0.01	0.05	1.47	1.42	1.51	1.53
	1	0.32	0.30	0.81	1.00	0.95	0.95	0.03	0.14	0.84	1.00	0.96	0.96	0.02	0.07	1.10	1.14	1.20	1.20
	2	0.60	0.54	0.75	1.00	0.94	0.94	0.06	0.26	0.77	0.99	0.94	0.94	0.03	0.11	0.90	1.03	1.05	1.05
	5	1.25	1.15	0.62	1.00	0.92	0.92	0.12	0.53	0.65	1.00	0.90	0.90	0.07	0.22	0.72	0.98	0.93	0.93
	10	1.89	1.75	0.48	1.00	0.89	0.89	0.19	0.87	0.52	1.00	0.87	0.87	0.11	0.36	0.60	0.98	0.87	0.87
	20	2.58	2.37	0.32	1.00	0.83	0.83	0.28	1.27	0.38	1.00	0.80	0.80	0.18	0.56	0.46	0.99	0.80	0.80
	30	2.89	2.66	0.24	1.00	0.78	0.78	0.32	1.48	0.30	1.00	0.76	0.76	0.21	0.67	0.38	0.99	0.76	0.76
0.8	0	0.00	0.00	∞	1.32	3.35	3.37	0.00	0.01	∞	3.35	7.23	8.35	0.01	0.02	∞	19.71	9.30	25.82
	0.01	0.00	0.00	2.57	1.14	1.76	1.76	0.00	0.01	13.84	3.18	4.71	5.94	0.01	0.02	98.61	13.58	8.96	21.62
	0.1	0.02	0.01	1.02	0.98	1.04	1.04	0.00	0.01	2.17	1.66	1.95	2.01	0.01	0.02	10.53	5.68	5.78	7.29
	0.5	0.07	0.06	0.87	0.99	0.97	0.97	0.01	0.04	1.03	1.04	1.10	1.10	0.01	0.03	2.49	2.02	2.29	2.35
	1	0.13	0.12	0.84	1.00	0.96	0.96	0.01	0.06	0.93	1.03	1.04	1.04	0.01	0.04	1.61	1.42	1.61	1.62
	2	0.26	0.24	0.81	1.00	0.96	0.96	0.02	0.11	0.84	1.00	0.98	0.98	0.02	0.06	1.19	1.14	1.29	1.29
	5	0.60	0.55	0.74	1.00	0.95	0.95	0.05	0.25	0.75	1.00	0.95	0.95	0.03	0.11	0.88	1.01	1.04	1.04
	10	1.03	0.95	0.64	1.00	0.94	0.94	0.10	0.45	0.67	1.00	0.93	0.93	0.06	0.18	0.75	1.00	0.97	0.97
	20	1.65	1.52	0.52	1.00	0.91	0.91	0.16	0.75	0.56	1.00	0.89	0.89	0.10	0.31	0.62	1.00	0.90	0.90
	30	2.06	1.89	0.43	1.00	0.89	0.89	0.21	0.96	0.48	1.00	0.87	0.87	0.12	0.39	0.56	1.00	0.88	0.88
0.9	0	0.00	0.00	∞	1.31	3.33	3.34	0.00	0.01	∞	3.28	7.21	8.30	0.01	0.02	Inf	19.85	9.34	25.98
	0.01	0.00	0.00	4.12	1.19	2.12	2.12	0.00	0.01	26.39	3.46	5.41	7.19	0.01	0.02	194.27	14.35	9.33	23.89
	0.1	0.01	0.01	1.16	1.00	1.13	1.13	0.00	0.01	3.47	2.05	2.66	2.81	0.01	0.02	20.07	7.51	7.46	10.29
	0.5	0.04	0.03	0.91	0.99	0.99	0.99	0.00	0.02	1.25	1.12	1.26	1.27	0.01	0.03	4.20	2.66	3.26	3.40
	1	0.07	0.06	0.87	1.00	0.98	0.98	0.01	0.04	1.07	1.07	1.14	1.14	0.01	0.03	2.46	1.72	2.16	2.18
	2	0.13	0.12	0.84	1.00	0.97	0.97	0.01	0.06	0.92	1.00	1.03	1.03	0.01	0.04	1.65	1.27	1.60	1.60
	5	0.32	0.29	0.79	1.00	0.97	0.97	0.03	0.14	0.81	1.00	0.98	0.98	0.02	0.07	1.07	1.04	1.18	1.18
	10	0.58	0.54	0.73	1.00	0.96	0.96	0.06	0.25	0.75	1.00	0.96	0.96	0.03	0.11	0.89	1.01	1.06	1.06
	20	1.03	0.94	0.64	1.00	0.95	0.95	0.10	0.45	0.67	1.00	0.94	0.94	0.06	0.18	0.74	1.00	0.97	0.97
	30	1.37	1.26	0.57	1.00	0.93	0.93	0.13	0.61	0.61	1.00	0.92	0.92	0.08	0.24	0.68	1.00	0.95	0.95
0.99	0	0.00	0.00	∞	1.30	3.28	3.29	0.00	0.01	∞	3.29	7.20	8.30	0.01	0.02	∞	19.42	9.31	25.99
	0.01	0.00	0.00	31.75	1.25	2.97	2.98	0.00	0.01	251.89	3.65	6.34	8.99	0.01	0.02	1914.34	15.50	9.75	26.63
	0.1	0.00	0.00	3.79	1.11	1.93	1.93	0.00	0.01	26.86	3.12	5.63	6.60	0.01	0.02	191.97	10.15	10.39	17.22
	0.5	0.01	0.01	1.44	1.00	1.22	1.22	0.00	0.01	5.18	1.48	2.65	2.66	0.01	0.02	34.95	4.43	7.16	7.84
	1	0.01	0.01	1.16	1.00	1.09	1.09	0.00	0.01	3.39	1.29	2.07	2.07	0.01	0.02	17.57	2.35	4.74	4.84
	2	0.02	0.01	0.99	1.00	1.03	1.03	0.00	0.01	2.00	1.03	1.48	1.48	0.01	0.02	9.69	1.56	3.32	3.33
	5	0.04	0.03	0.90	1.00	1.00	1.00	0.00	0.02	1.30	1.00	1.20	1.20	0.01	0.03	4.12	1.11	2.13	2.13
	10	0.07	0.06	0.86	1.00	0.99	0.99	0.01	0.04	1.06	1.00	1.10	1.10	0.01	0.03	2.57	1.02	1.72	1.72
	20	0.13	0.12	0.83	1.00	0.99	0.99	0.01	0.06	0.91	1.00	1.03	1.03	0.01	0.04	1.62	1.00	1.39	1.39
	30	0.20	0.18	0.82	1.00	0.98	0.98	0.02	0.09	0.86	1.00	1.01	1.01	0.02	0.05	1.39	1.00	1.31	1.31

Table S7: The relative efficiencies for the proposed estimators in case of $n = 50$, $p = 30$,and $\alpha = 0.05$.

		No Outliers						10% Outliers						20% Outliers					
ρ^2	Δ^2	OLS	M	RRME	PTRME	SRME	RPRME	OLS	M	RRME	PTRME	SRME	PRRME	OLS	M	RRME	PTRME	SRME	PRRME
0	0	1.13	1.00	∞	1.08	2.72	2.73	0.37	1.00	∞	2.31	6.22	7.31	0.58	1.00	∞	5.86	8.53	18.09
	0.01	1.14	1.00	416.44	1.07	2.67	2.67	0.37	1.00	1498.09	2.22	5.94	6.95	0.57	1.00	4361.56	5.65	8.95	17.13
	0.1	1.14	1.00	41.99	1.04	2.45	2.45	0.37	1.00	149.03	2.02	5.53	6.27	0.59	1.00	454.54	5.13	8.88	15.32
	0.5	1.14	1.00	8.24	1.00	1.94	1.94	0.37	1.00	29.91	1.52	4.08	4.18	0.57	1.00	87.47	3.00	7.98	8.99
	1	1.13	1.00	4.12	1.00	1.69	1.69	0.37	1.00	15.13	1.30	3.31	3.32	0.57	1.00	43.44	2.30	6.50	6.67
	2	1.14	1.00	2.08	1.00	1.50	1.50	0.38	1.00	7.74	1.16	2.65	2.65	0.58	1.00	22.35	1.74	4.90	4.91
	5	1.15	1.00	0.83	1.00	1.34	1.34	0.38	1.00	3.13	1.05	2.11	2.11	0.57	1.00	8.62	1.28	3.41	3.41
	10	1.14	1.00	0.42	1.00	1.25	1.25	0.37	1.00	1.48	1.01	1.82	1.82	0.58	1.00	4.44	1.11	2.82	2.82
	20	1.13	1.00	0.21	1.00	1.18	1.18	0.38	1.00	0.77	1.00	1.63	1.63	0.58	1.00	2.18	1.04	2.35	2.35
	30	1.14	1.00	0.14	1.00	1.12	1.12	0.37	1.00	0.50	1.00	1.52	1.52	0.59	1.00	1.52	1.02	2.16	2.16
0.5	0	0.01	0.01	∞	1.03	2.38	2.38	0.01	0.04	∞	2.20	6.45	7.23	0.03	0.06	Inf	4.02	8.23	13.28
	0.01	0.01	0.01	8.89	1.03	2.08	2.08	0.01	0.03	103.28	2.39	6.40	7.13	0.03	0.06	495.54	3.90	8.46	12.29
	0.1	0.02	0.02	1.59	1.00	1.30	1.30	0.01	0.04	11.57	1.73	4.06	4.42	0.03	0.06	52.86	3.34	7.24	9.60
	0.5	0.07	0.06	0.95	1.00	1.03	1.03	0.02	0.05	2.83	1.32	2.17	2.19	0.03	0.06	10.40	2.22	4.66	4.90
	1	0.12	0.10	0.85	1.00	0.99	0.99	0.02	0.06	1.81	1.14	1.63	1.63	0.04	0.06	5.56	1.68	3.31	3.35
	2	0.22	0.19	0.78	1.00	0.97	0.97	0.03	0.08	1.28	1.07	1.32	1.32	0.04	0.07	3.27	1.35	2.40	2.40
	5	0.48	0.42	0.69	1.00	0.95	0.95	0.05	0.14	0.89	1.01	1.08	1.08	0.05	0.09	1.58	1.10	1.56	1.56
	10	0.82	0.72	0.59	1.00	0.94	0.94	0.09	0.24	0.70	1.00	0.98	0.98	0.07	0.13	1.13	1.02	1.28	1.28
	20	1.29	1.15	0.47	1.00	0.92	0.92	0.14	0.37	0.57	1.00	0.93	0.93	0.11	0.18	0.80	1.00	1.07	1.07
	30	1.61	1.42	0.39	1.00	0.90	0.90	0.18	0.48	0.49	1.00	0.90	0.90	0.13	0.23	0.69	1.00	1.01	1.01
0.8	0	0.01	0.01	∞	1.02	2.36	2.36	0.01	0.04	∞	2.19	6.37	7.15	0.03	0.06	∞	4.13	8.35	13.46
	0.01	0.01	0.01	20.26	1.03	2.25	2.25	0.01	0.03	257.35	2.31	6.59	7.34	0.03	0.06	1214.13	4.19	8.77	12.89
	0.1	0.02	0.01	2.69	1.00	1.54	1.54	0.01	0.04	27.01	1.79	4.77	5.25	0.03	0.06	129.16	3.33	7.72	10.29
	0.5	0.03	0.03	1.16	1.00	1.12	1.12	0.02	0.04	5.92	1.42	2.92	2.95	0.03	0.06	24.93	2.24	5.65	5.92
	1	0.05	0.05	0.98	1.00	1.04	1.04	0.02	0.05	3.47	1.18	2.16	2.16	0.03	0.06	12.83	1.74	4.22	4.27
	2	0.10	0.09	0.87	1.00	1.00	1.00	0.02	0.06	2.14	1.11	1.67	1.67	0.04	0.06	7.10	1.43	3.14	3.15
	5	0.22	0.19	0.78	1.00	0.98	0.98	0.03	0.08	1.26	1.03	1.27	1.27	0.04	0.07	2.99	1.16	2.08	2.08
	10	0.40	0.35	0.71	1.00	0.97	0.97	0.05	0.13	0.92	1.01	1.09	1.09	0.05	0.09	1.94	1.06	1.67	1.67
	20	0.69	0.61	0.62	1.00	0.95	0.95	0.08	0.20	0.75	1.00	1.02	1.02	0.07	0.11	1.24	1.02	1.32	1.32
	30	0.93	0.82	0.56	1.00	0.94	0.94	0.10	0.27	0.67	1.00	0.98	0.98	0.08	0.14	1.02	1.01	1.21	1.21
0.9	0	0.01	0.01	∞	1.02	2.36	2.36	0.01	0.04	∞	2.17	6.31	7.08	0.03	0.06	∞	4.19	8.34	13.49
	0.01	0.01	0.01	39.73	1.03	2.32	2.32	0.01	0.03	509.48	2.24	6.67	7.43	0.03	0.06	2412.95	4.17	8.82	13.00
	0.1	0.01	0.01	4.55	1.01	1.75	1.75	0.01	0.04	53.23	1.82	5.06	5.58	0.03	0.06	256.42	3.36	7.93	10.58
	0.5	0.02	0.02	1.56	1.00	1.24	1.24	0.01	0.04	11.09	1.44	3.41	3.44	0.03	0.06	49.02	2.26	6.12	6.41
	1	0.03	0.03	1.17	1.00	1.10	1.10	0.02	0.04	6.22	1.20	2.56	2.56	0.03	0.06	24.71	1.79	4.72	4.77
	2	0.05	0.05	0.98	1.00	1.04	1.04	0.02	0.05	3.56	1.12	2.00	2.00	0.04	0.06	13.43	1.46	3.61	3.62
	5	0.12	0.10	0.84	1.00	1.00	1.00	0.02	0.06	1.83	1.04	1.48	1.48	0.04	0.06	5.36	1.21	2.51	2.51
	10	0.22	0.19	0.78	1.00	0.98	0.98	0.03	0.08	1.20	1.01	1.22	1.22	0.04	0.07	3.24	1.08	2.05	2.05
	20	0.40	0.35	0.71	1.00	0.97	0.97	0.05	0.12	0.93	1.00	1.10	1.10	0.05	0.09	1.87	1.04	1.59	1.59
	30	0.55	0.48	0.66	1.00	0.96	0.96	0.06	0.17	0.83	1.00	1.05	1.05	0.06	0.10	1.48	1.03	1.46	1.46
0.99	0	0.01	0.01	∞	1.02	2.35	2.35	0.01	0.04	∞	2.25	6.38	7.21	0.03	0.06	∞	4.15	8.32	13.32
	0.01	0.01	0.01	387.31	1.05	2.39	2.39	0.01	0.03	5095.79	2.15	6.60	7.36	0.03	0.06	23968.83	4.14	8.87	13.12
	0.1	0.01	0.01	38.76	1.01	2.05	2.05	0.01	0.04	522.13	1.91	5.40	5.95	0.03	0.06	2554.61	3.39	8.14	10.85
	0.5	0.01	0.01	7.92	1.00	1.63	1.63	0.01	0.04	104.78	1.48	4.09	4.12	0.03	0.06	484.42	2.25	6.59	6.88
	1	0.01	0.01	4.61	1.00	1.44	1.44	0.01	0.04	56.35	1.23	3.21	3.21	0.03	0.06	240.69	1.80	5.23	5.28
	2	0.02	0.01	2.70	1.00	1.28	1.28	0.01	0.04	28.92	1.15	2.69	2.69	0.03	0.06	127.86	1.49	4.19	4.19
	5	0.02	0.02	1.53	1.00	1.14	1.14	0.01	0.04	11.91	1.07	2.19	2.19	0.03	0.06	47.07	1.25	3.32	3.32
	10	0.03	0.03	1.17	1.00	1.07	1.07	0.01	0.04	5.78	1.03	1.84	1.84	0.04	0.06	26.38	1.12	2.91	2.91
	20	0.05	0.05	0.97	1.00	1.03	1.03	0.02	0.04	3.35	1.00	1.63	1.63	0.03	0.06	12.88	1.07	2.48	2.48
	30	0.08	0.07	0.91	1.00	1.01	1.01	0.02	0.05	2.58	1.01	1.50	1.50	0.04	0.06	9.21	1.05	2.40	2.40

Table S8: Summary statistics for the V02 data set

Variables	Min	Q1	Median	Mean	Q3	Max
AGE	34.00	45.00	52.00	51.76	58.00	73.00
MHR	94.0	140.0	162.0	158.3	178.0	200.0
DUR	36.0	289.0	478.0	444.8	599.0	798.0
MXS	105.0	150.0	175.0	175.6	196.0	290.0
MXD	30.0	70.0	80.00	78.6	90.0	152.0
FAI	−22.00	−1.00	12.00	17.99	37.00	71.00
V02	4.70	20.10	30.60	29.17	37.30	59.70

Table S9: VIF values for the V02 data set

Predictors	AGE	MHR	DUR	MXS	MXD	FAI
VIF	2.0709	2.4013	18.8488	1.2082	1.2702	16.9611

The Figure S9 shows the observations 31, 33, 75 may be outliers. The results of apply outlierTest function in the **car** package in R confirm that observations 33 is an outlier.

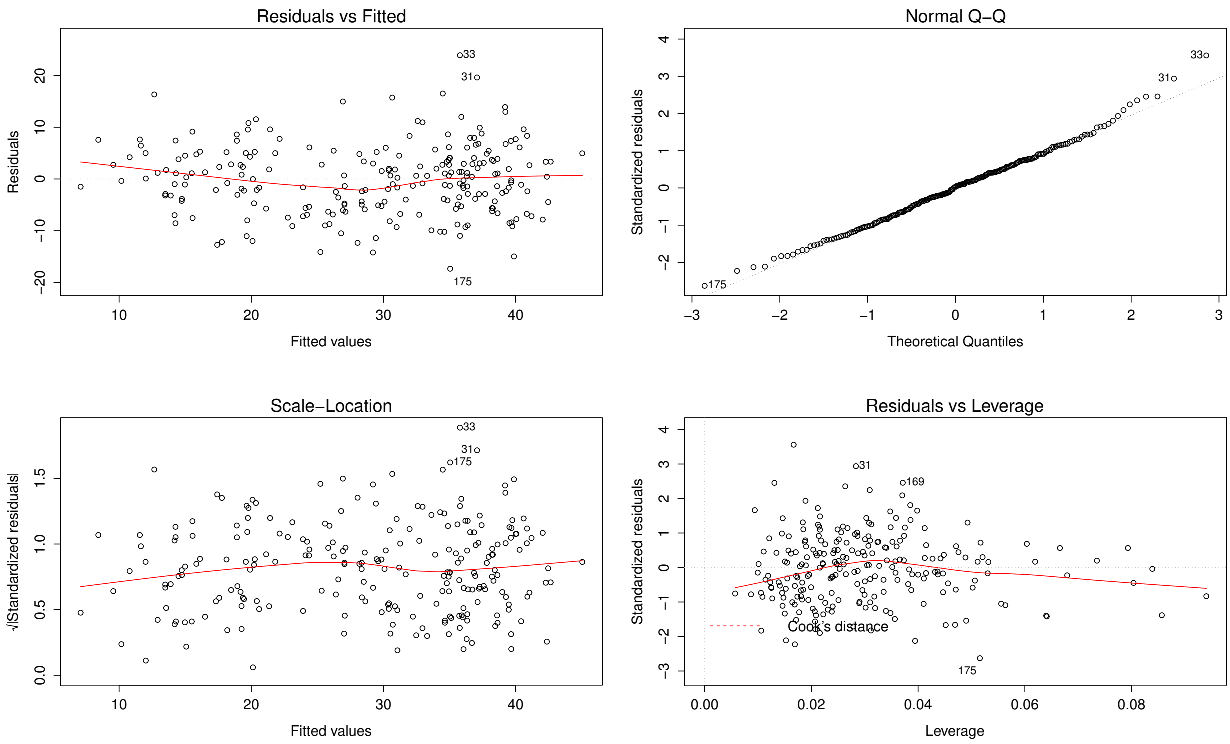

Figure S9: The plots for identifying outliers in V02 dataset

Thus, this dataset suffers the problems of multicollinearity and outliers simultaneously.

S5.2 Bodyfat Dataset

The summary statistics of all variables is shown in Table S10. The visual correlation matrix for the covariates is shown in Figure S10.

Table S10: Summary statistics for the bodyfat data

Variables	Min	Q1	Median	Mean	Q3	Max
DENSITY	0.995	1.041	1.055	1.056	1.070	1.109
AGE	22.00	35.75	43.00	44.88	54.00	81.00
WEIGHT	118.5	159.0	176.5	178.9	197.0	363.1
HEIGHT	29.50	68.25	70.00	70.15	72.25	77.75
NECK	31.10	36.40	38.00	37.99	39.42	51.20
CHEST	79.30	94.35	99.65	100.82	105.38	136.20
ABDOMEN	69.40	84.58	90.95	92.56	99.33	148.10
HIP	85.0	95.5	99.3	99.9	103.5	147.7
THIGH	47.20	56.00	59.00	59.41	62.35	87.30
KNEE	33.00	36.98	38.50	38.59	39.92	49.10
ANKLE	19.1	22.0	22.8	23.1	24.0	33.9
BICEPS	24.80	30.20	32.05	32.27	34.33	45.00
FOREARM	21.00	27.30	28.70	28.66	30.00	34.90
WRIST	15.80	17.60	18.30	18.23	18.80	21.40
SIRI	0.00	12.47	19.20	19.15	25.30	47.50

From Figure S10, we may guess the existence of multicollinearity. VIF values are shown in Table S11.

Table S11: VIF values for the body fat data

Predictors	density	age	weight	height	neck	chest	abdomen
VIF	3.8182	2.2747	34.0317	1.6778	4.3965	9.4722	18.1199
Predictors	hip	thigh	knee	ankle	biceps	forearm	wrist
VIF	14.9609	7.8877	4.6123	1.9200	3.6516	2.2370	3.5215

The values of Table S11 and $\kappa = 189095237$ shows that there is a strong multicollinearity between independent variables.

The Figure S11 shows the observations 48, 76, 86 and 96 may be outliers. The results of apply outlierTest function in the **car** package in R confirm that observations 48 and 96 are outliers.

Thus, this dataset suffers the problems of multicollinearity and outliers simultaneously.

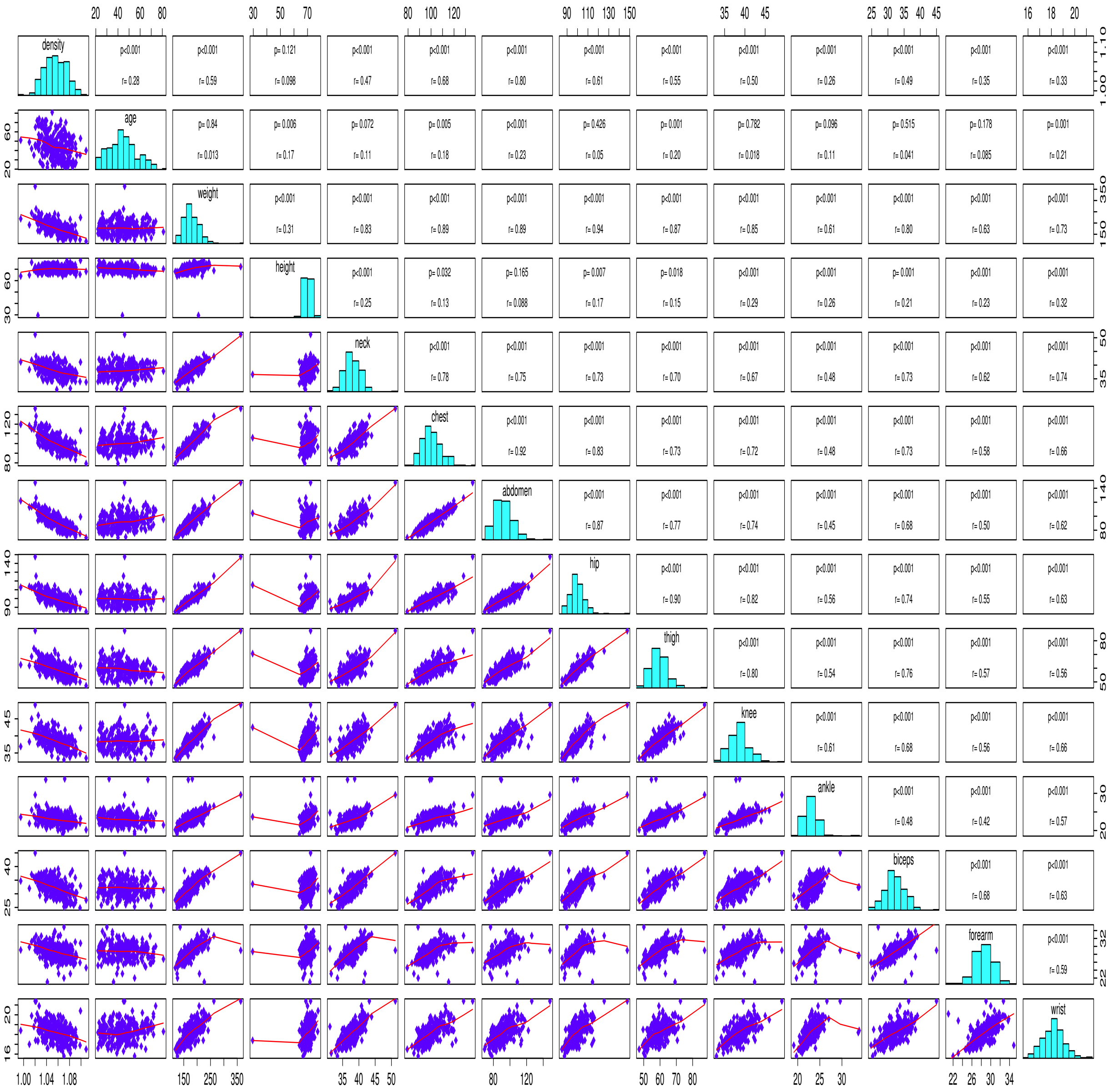

Figure S10: The scatter plot for independent variables of bodyfat datasets