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Depth Image Upsampling based on Guided Filter with Low Gradient Minimization

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1 Overview

In this supplementary material, we consider the solution of following equation:

$$\sum_i \min_{x_i}\{(x_i - p_i)^2 + \alpha H^t(p_i)\} \tag{1}$$

where

$$H^t(p) = \begin{cases} 0 & \text{if p = 0} \\ t & \text{if } 0 <\mid p \mid\leq 1 \\ 1 & \text{if } \mid p \mid > 1 \end{cases} \tag{2}$$

Each single term w.r.t. pixel p_i in Eq. (2) is

$$E(p) = \min_p (x-p)^2 + \alpha H^t(p) \tag{3}$$

Theorem 1 *When $|x| \geq 1$, Eq.(3) reaches its minimum E_p^* under the condition*

$$p = \begin{cases} 0 & if \mid x \mid \leq \min(\frac{1+\alpha t}{2}, \sqrt{\alpha}) \\ sgn(x) & if \frac{1+\alpha t}{2} <\mid x \mid \leq 1+\sqrt{\alpha(1-t)} \\ x & if \mid x \mid > \max(1+\sqrt{\alpha(1-t)}, \sqrt{\alpha}) \end{cases} \tag{4}$$

In our submission, we have give the proofs of Theorem 1.1.

In order to make the proof more intuitive, in Fig.2, we show the relationship of three functions $1+\sqrt{\alpha(1-t)}$, $\sqrt{\alpha}$ and $\frac{1+\alpha t}{2}$.

Theorem 2 *When $|x| \geq 1$, and $\alpha > \frac{2-t+2\sqrt{1-t}}{t^2}$, Eq.(3) reaches its minimum E_p^* under the condition*

$$p = \begin{cases} 0 & if \mid x \mid \leq \sqrt{\alpha} \\ x & if \mid x \mid > \sqrt{\alpha} \end{cases} \tag{5}$$

When $|x| \geq 1$, and $\frac{2-t+2\sqrt{1-t}}{t^2} \geq \alpha \geq \frac{2-t-2\sqrt{1-t}}{t^2}$, Eq.(3) reaches its minimum E_p^* under the condition

$$p = \begin{cases} 0 & if \mid x \mid \leq \frac{1+\alpha t}{2} \\ sgn(x) & if \frac{1+\alpha t}{2} <\mid x \mid \leq 1+\sqrt{\alpha(1-t)} \\ x & if \mid x \mid > 1+\sqrt{\alpha(1-t)} \end{cases} \tag{6}$$

When $|x| \geq 1$, and $\frac{2-t-2\sqrt{1-t}}{t^2} > \alpha > 0$, Eq.(3) reaches its minimum E_p^* under the condition

$$p = \begin{cases} sgn(x) & if\ 1 \leq\mid x \mid \leq 1+\sqrt{\alpha(1-t)} \\ x & if \mid x \mid > 1+\sqrt{\alpha(1-t)} \end{cases} \tag{7}$$

Proof According to Lemma 1.2, we can see that the Eq.(6) is true obviously. We only need to proof the Eq.(5) and (7).

When $\frac{2-t-2\sqrt{1-t}}{t^2} > \alpha > 0$, according to Lemma 1.2 and Theorem 1.1, Eq.(3) reaches its minimum E_p^* under the condition

$$p = \begin{cases} 0 & \text{if} \mid x \mid \leq \sqrt{\alpha} \\ sgn(x) & \text{if } \frac{1+\alpha t}{2} <\mid x \mid \leq 1+\sqrt{\alpha(1-t)} \\ x & \text{if} \mid x \mid > \sqrt{\alpha} \end{cases} \tag{8}$$

but according to Lemma 1.2, in this case, $\frac{1+\alpha t}{2} \leq 1 + \sqrt{\alpha(1-t)}$ is false (see Eq.(12). Then, Eq.(8) can be reducible to Eq.(5).

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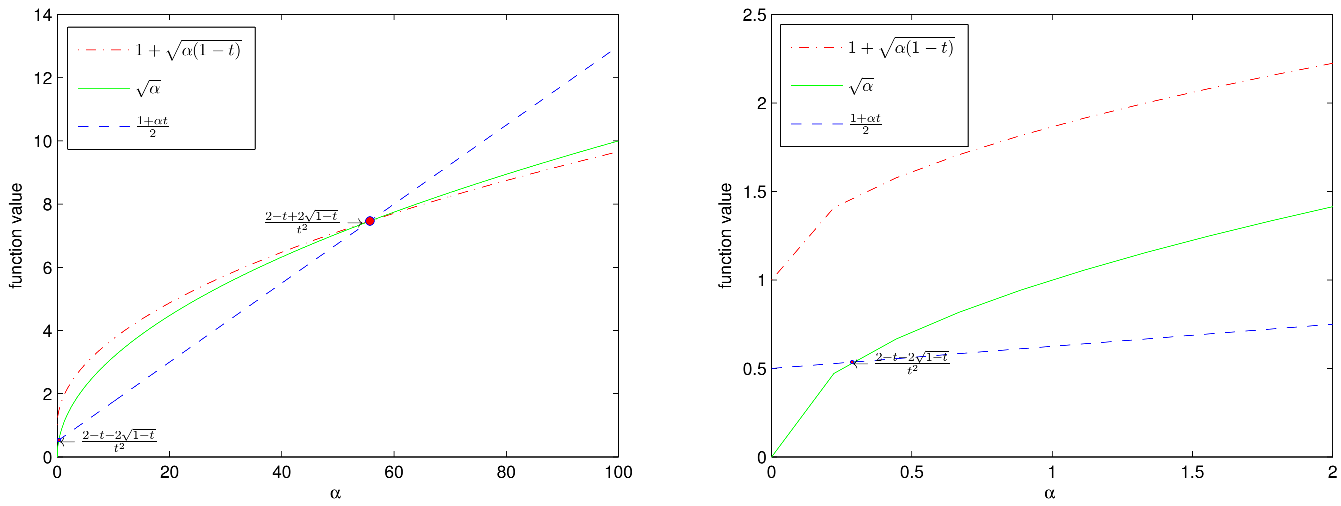

Fig. 1 The relation of the three curves when $t = 0.25$. Left subplot: $\alpha \in [0, 100]$. Right subplot: $\alpha \in [0, 2]$.

According to Lemma 1.2, when $\frac{2-t-2\sqrt{1-t}}{t^2} > \alpha > 0$, Eq.(3) reaches its minimum E_p^* under the condition

$$p = \begin{cases} 0 & \text{if } |x| \leq \sqrt{\alpha} \\ sgn(x) & \text{if } \frac{1+\alpha t}{2} < |x| \leq 1+\sqrt{\alpha(1-t)} \\ x & \text{if } |x| > 1+\sqrt{\alpha(1-t)} \end{cases} \tag{9}$$

but one can see that $\sqrt{\alpha} < 1$, $\frac{1+\alpha t}{2} < 1$ in this case, and $|x| \geq 1$, so $|x| \leq \sqrt{\alpha}$ is false. Then, Eq.(9) can be written as Eq.(7). ■

Lemma 1 *If* $0 < t < 1$, *then* $0 < \frac{2-t-2\sqrt{1-t}}{t^2} < 1$

Proof

$$\begin{aligned} -3t^2 - t^3 &< 0 \\ (t+2)^2(1-t) &< 4 \\ (1-t)(2+t) &< 2\sqrt{(1-t)} \\ 2-t-2\sqrt{(1-t)} &< t^2 \\ \frac{2-t-2\sqrt{1-t}}{t^2} &< 1 \end{aligned} \tag{10}$$

$$\begin{aligned} t^2 &> 0 \\ (2-t)^2 &> 4(1-t) \\ 2-t-2\sqrt{(1-t)} &> 0 \\ \frac{2-t-2\sqrt{1-t}}{t^2} &> 0 \end{aligned} \tag{11}$$

■

Lemma 2 *If* $\alpha > \frac{2-t+2\sqrt{1-t}}{t^2}$, *we have*

$$\frac{1+\alpha t}{2} > \sqrt{\alpha} > 1+\sqrt{\alpha(1-t)} \tag{12}$$

If $\frac{2-t+2\sqrt{1-t}}{t^2} \geq \alpha \geq \frac{2-t-2\sqrt{1-t}}{t^2}$, *we have*

$$1+\sqrt{\alpha(1-t)} \geq \sqrt{\alpha} \geq \frac{1+\alpha t}{2} \tag{13}$$

If $\frac{2-t-2\sqrt{1-t}}{t^2} > \alpha > 0$, *we have*

$$1+\sqrt{\alpha(1-t)} > \frac{1+\alpha t}{2} > \sqrt{\alpha} \tag{14}$$

Proof First, we give a fact,

$$t^2\alpha^2 + (2t-4)\alpha + 1 = 0 \tag{15}$$

because $t < 1$ and the discriminant value of the equation is $(2t-4)^2 - 4t^2 = 16(1-t) > 0$. Then,this equation has two real solutions, they are

$$\alpha = \frac{2-t \pm 2\sqrt{1-t}}{t^2} \tag{16}$$

where $\frac{2-t+2\sqrt{1-t}}{t^2} > 1$ and according to Lemma 1.1, we have $\frac{2-t-2\sqrt{1-t}}{t^2} < 1$

1) When $\alpha > \frac{2-t+2\sqrt{1-t}}{t^2}$, we can see that $\alpha > 1$

$$\begin{aligned} t^2\alpha^2 + (2t-4)\alpha + 1 &> 0 \\ (t\alpha+1)^2 &> 4\alpha \\ t\alpha + 1 &> 2\sqrt{\alpha} \\ \alpha + 1 - 2\sqrt{\alpha} &> \alpha(1-t) \\ \sqrt{\alpha} &> 1+\sqrt{\alpha(1-t)} \end{aligned} \tag{17}$$

and

$$\begin{aligned} t^2\alpha^2 + (2t-4)\alpha + 1 &> 0 \\ (t\alpha+1)^2 &> 4\alpha \\ \frac{1+\alpha t}{2} &> \sqrt{\alpha} \end{aligned} \tag{18}$$

2)When $\frac{2-t+2\sqrt{1-t}}{t^2} \geq \alpha \geq \frac{2-t-2\sqrt{1-t}}{t^2}$

$$\begin{aligned} t^2\alpha^2 + (2t-4)\alpha + 1 &\leq 0 \\ t\alpha + 1 &\leq 2\sqrt{\alpha} \\ \frac{1+\alpha t}{2} &\leq \sqrt{\alpha} \end{aligned} \tag{19}$$

and when $\frac{2-t+2\sqrt{1-t}}{t^2} \geq \alpha \geq 1$, we have

$$\begin{aligned} t^2\alpha^2 + (2t-4)\alpha + 1 &\leq 0 \\ (\sqrt{\alpha}-1)^2 &\leq \alpha(1-t) \\ \sqrt{\alpha} &\leq 1 + \sqrt{\alpha(1-t)} \end{aligned} \tag{20}$$

When $\frac{2-t-2\sqrt{1-t}}{t^2} \leq \alpha < 1$, we have

$$\sqrt{\alpha} < 1 < 1 + \sqrt{\alpha(1-t)} \tag{21}$$

So, in summary, when $\frac{2-t+2\sqrt{1-t}}{t^2} \geq \alpha \geq \frac{2-t-2\sqrt{1-t}}{t^2}$, we have Eq.(21).

3) When $\frac{2-t-2\sqrt{1-t}}{t^2} > \alpha > 0$,

$$t^2\alpha^2 + (2t-4)\alpha + 1 > 0 \tag{22}$$

similar to the Eq.(18), we can obtain $\frac{1+\alpha t}{2} > \sqrt{\alpha}$.

And according to Lemma 1.1, $\alpha < \frac{2-t-2\sqrt{1-t}}{t^2} < 1$,

$$\frac{1+\alpha t}{2} < 1 < 1 + \sqrt{\alpha(1-t)} \tag{23}$$

■