Variance of the test of Baek and Brock modified by Hiemstra and Jones

In this supplementary material, we present the variance of the statistics of Baek and Brock given in eq. 32. We follow the approach of the appendix of Hiemstra and Jones article (Hiemstra & Jones, 1994) and we use notations of section 2.4. Firstly, let us define quantities

$$C1 = P\left(\left\|\mathrm{X}_2^f(t) - \mathrm{X}_2^f(s)\right\| < r, \|\mathrm{X}_2^p(t) - \mathrm{X}_2^p(s)\| < r, \|\mathrm{X}_1^p(t) - \mathrm{X}_1^p(s)\| < r\right) \tag{SM.1}$$

$$C2 = P\left(\|\mathrm{X}_2^p(t) - \mathrm{X}_2^p(s)\| < r, \|\mathrm{X}_1^p(t) - \mathrm{X}_1^p(s)\| < r\right) \tag{SM.2}$$

$$C3 = P\left(\left\|\mathrm{X}_2^f(t) - \mathrm{X}_2^f(s)\right\| < r, \|\mathrm{X}_2^p(t) - \mathrm{X}_2^p(s)\| < r\right) \tag{SM.3}$$

$$C4 = P\left(\|\mathrm{X}_2^p(t) - \mathrm{X}_2^p(s)\| < r\right) \tag{SM.4}$$

such that the expression tested by Hiemstra and Jones (eq. 32) writes

$$\frac{C1}{C2} - \frac{C3}{C4} \tag{SM.5}$$

Then, let us define

$H1(\mathrm{x}_2^f(t), \mathrm{x}_2^p(t), \mathrm{x}_1^p(t), r) \equiv P\left(\left\|\mathrm{x}_2^f(t) - \mathrm{X}_2^f(s)\right\| < r, \|\mathrm{x}_2^p(t) - \mathrm{X}_2^p(s)\| < r, \|\mathrm{x}_1^p(t) - \mathrm{X}_1^p(s)\| < r\right)$ (noted $H1$ in the following)

$H2(\mathrm{x}_2^p(t), \mathrm{x}_1^p(t), r) \equiv P\left(\|\mathrm{x}_2^p(t) - \mathrm{X}_2^p(s)\| < r, \|\mathrm{x}_1^p(t) - \mathrm{X}_1^p(s)\| < r\right)$ (noted $H2$)

$H3(\mathrm{x}_2^f(t), \mathrm{x}_2^p(t), r) \equiv P\left(\left\|\mathrm{x}_2^f(t) - \mathrm{X}_2^f(s)\right\| < r, \|\mathrm{x}_2^p(t) - \mathrm{X}_2^p(s)\| < r\right)$ (noted $H3$)

$H4(\mathrm{x}_2^p(t), r) \equiv P\left(\|\mathrm{x}_2^p(t) - \mathrm{X}_2^p(s)\| < r\right)$ (noted $H4$)

Under the assumptions that the underlying series are strictly stationary, weakly dependent, and satisfy the mixing conditions of Denker and Keller (Denker & Keller, 1983), an expression for the variance of the Baek and Brock test is given by

$$\sigma^2(\tau, L_1, L_2, r) = d.\Sigma.d^T \tag{SM.6}$$

where ^{T} is transposition, and with

$$d = \left[\frac{1}{C2}, -\frac{C1}{C2^2}, -\frac{1}{C4}, \frac{C3}{C4^2}\right] \tag{SM.7}$$

$$\Sigma = [\Sigma_{i,j}]_{i,j\in\{1,..,4\}} = \left[4\sum_{k\geq 1} w_k E\left[A_{i,t}A_{j,t+k-1}\right]\right]_{i,j\in\{1,..,4\}} \tag{SM.8}$$

where $w_k = 1 + 1_{\{k>1\}}$, E denotes expected value and

$$A_{1,t} = H1 - C1 \tag{SM.9}$$

$$A_{2,t} = H2 - C2 \tag{SM.10}$$

$$A_{3,t} = H3 - C3 \tag{SM.11}$$

$$A_{4,t} = H4 - C4. \tag{SM.12}$$

A consistent estimator for d is given by

$$\widehat{d}(n) = \left[\frac{1}{\widehat{C2}}, -\frac{\widehat{C1}}{\widehat{C2}^2}, -\frac{1}{\widehat{C4}}, \frac{\widehat{C3}}{\widehat{C4}^2}\right] \tag{SM.13}$$

with

$$\widehat{C1} = C^2(\mathrm{X}_2^f, \mathrm{X}_2^p, \mathrm{X}_1^p) \tag{SM.14}$$

$$\widehat{C2} = C^2(\mathrm{X}_2^p, \mathrm{X}_1^p) \tag{SM.15}$$

$$\widehat{C3} = C^2(\mathrm{X}_2^f, \mathrm{X}_2^p) \tag{SM.16}$$

$$\widehat{C4} = C^2(\mathrm{X}_2^p). \tag{SM.17}$$

Using the results of Denker and Keller (Denker & Keller, 1983), and Newey and West (Newey & West, 1987), a consistent estimator of $\Sigma_{i,j}$ is given by

$$\widehat{\Sigma}_{i,j}(n) = 4\sum_{k=1}^{K(n)} w_k(n)\left[\frac{1}{2(n-k+1)}\sum_t \widehat{A}_{i,t}(n).\widehat{A}_{j,t-k+1}(n) + \widehat{A}_{i,t-k+1}(n).\widehat{A}_{j,t}(n)\right] \quad \text{(SM.18)}$$

with

$$t = \max(L_1, L_2) + k, ..., T - \tau + 1 \quad \text{(SM.19)}$$

$$n = T + 1 - \tau - \max(L_1, L_2) \quad \text{(SM.20)}$$

$$K(n) = \left[n^{1/4}\right] \quad \text{(SM.21)}$$

$$w_k(n) = \begin{cases} 1, & \text{if } k = 1 \\ 2\left(1 - \frac{k-1}{K(n)}\right) & \text{otherwise} \end{cases} \quad \text{(SM.22)}$$

where [.] denotes the integer part and

$$\widehat{A}_{1,t}(n) = \frac{1}{n-1}\left(\sum_{s\neq t} 1_{\{\|x_2^f(t)-x_2^f(s)\|<r\}}.1_{\{\|x_2^p(t)-x_2^p(s)\|<r,\}}.1_{\{\|x_1^p(t)-x_1^p(s)\|<r\}}\right) - \widehat{C1} \quad \text{(SM.23)}$$

$$\widehat{A}_{2,t}(n) = \frac{1}{n-1}\left(\sum_{s\neq t} 1_{\{\|x_2^p(t)-x_2^p(s)\|<r,\}}.1_{\{\|x_1^p(t)-x_1^p(s)\|<r\}}\right) - \widehat{C2} \quad \text{(SM.24)}$$

$$\widehat{A}_{3,t}(n) = \frac{1}{n-1}\left(\sum_{s\neq t} 1_{\{\|x_2^f(t)-x_2^f(s)\|<r\}}.1_{\{\|x_2^p(t)-x_2^p(s)\|<r,\}}\right) - \widehat{C3} \quad \text{(SM.25)}$$

$$\widehat{A}_{4,t}(n) = \frac{1}{n-1}\left(\sum_{s\neq t} 1_{\{\|x_2^p(t)-x_2^p(s)\|<r,\}}\right) - \widehat{C4} \quad \text{(SM.26)}$$

for $t, s = \max(L_1, L_2) + 1, ..., T - \tau + 1$. Finally, a consistent estimator for $\sigma^2(\tau, L_1, L_2, r)$ can be expressed as

$$\widehat{\sigma}^2(\tau, L_1, L_2, r) = \widehat{d}(n).\widehat{\Sigma}(n).\widehat{d}(n)^T. \quad \text{(SM.27)}$$

For multidimensional estimator of eq. 33, replace X_2^f by X_n^f and X_1^p, X_2^p by $X_1^p, \dots, X_Q^p$ in each estimator.

References

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