Supplementary Material: A Dual Gradient-Projection Method for Large-Scale Strictly Convex Quadratic Problems *

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Abstract

This is the supplementary material for the manuscript titled "A Dual Gradient-Projection Method for Large-Scale Strictly Convex Quadratic Problems". In Section 1 we present a method for computing a direction of linear infinite descent that is an alternative to the strategy described in the main manuscript. In Section 2 we provide the detailed output from our solver DQP on the entire test set considered in the main manuscript. Finally, in Section 3 we provide the detailed output for the number of required iterations and the required time to reach the various tolerance levels, as described in Section 5.2 of the main manuscript.

1 Directions of linear infinite descent using A^F

We show how one might find a suitable Δy^F_∞ satisfying (32) from the manuscript when c^F is not in the range space of A^F (i.e., when (28) from the manuscript is inconsistent). Recall that A^F has m_k rows, and suppose that

$$A^F = P \begin{pmatrix} L \\ M \end{pmatrix} (U \;\; N), \tag{1}$$

where P is a permutation matrix, L and U are, respectively, r_k by r_k unit lower and upper triangular matrices, and the rank $r_k \le \min(n, m_k)$; such a "rectangular" LU factorization is implemented in many modern sparse-matrix packages (e.g., `LUSOL` [3], `MA48` [2] and `MUMPS` [1]). Let

$$\begin{pmatrix} c^F_1 \\ c^F_2 \end{pmatrix} = P^T c^F, \tag{2}$$

where c^F_1 has r_k components. The following holds.

Lemma 1. *Suppose that c^F does not lie in the range of A^F. Then*

$$d^F := c^F_2 - M L^{-1} c^F_1 \neq 0. \tag{3}$$

Proof. For a contrapositive proof, suppose that $c^F_2 = M L^{-1} c^F_1$. Writing $w = L^{-1} c^F_1$, we have

$$\begin{pmatrix} L \\ M \end{pmatrix} w = \begin{pmatrix} c^F_1 \\ c^F_2 \end{pmatrix}$$

or equivalently $A^F v = c^F$, where $v = \begin{pmatrix} U^{-1} w \\ 0 \end{pmatrix}$. This completes the proof since this shows that c^F lies in the range of A^F. □

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This result allows us to determine whether there is a direction of linear infinite descent by checking whether d^F is zero or not; in practice small rounding errors need to be taken into account.

Next, let d^F be nonzero, D be any diagonal matrix with $D_{ii} \geq 0$, and $Dd^F \neq 0$, and then define

$$\Delta y^F_{D2} := Dd^F \tag{4}$$

and

$$\Delta y^F_{D1} := -L^{-T} M^T \Delta y^F_{D2}. \tag{5}$$

Then we have the following crucial result.

Lemma 2. *Suppose that c^F does not lie in the range of A^F, and that Δy^F_{D2} and Δy^F_{D1} are defined by (4) and (5). Then, the vector*

$$\Delta y^F_\infty = P \begin{pmatrix} \Delta y^F_{D1} \\ \Delta y^F_{D2} \end{pmatrix} \tag{6}$$

satisfies (32) from the manuscript.

Proof. It follows trivially from (1), (6), and (5) that

$$(A^F)^T \Delta y^F_\infty = \begin{pmatrix} U^T \\ N^T \end{pmatrix} (L^T \;\; M^T) P^T \Delta y^F_\infty = \begin{pmatrix} U^T \\ N^T \end{pmatrix} (L^T \Delta y^F_{D1} + M^T \Delta y^F_{D2}) = 0.$$

Moreover (5) implies that

$$\langle \Delta y^F_{D1}, c^F_1 \rangle = -\langle L^{-T} M^T \Delta y^F_{D2}, c^F_1 \rangle = -\langle \Delta y^F_{D2}, M L^{-1} c^F_1 \rangle,$$

which together with (6), (2), and the definition of D shows that

$$\langle \Delta y^F_\infty, c^F \rangle = \langle \Delta y^F_{D1}, c^F_1 \rangle + \langle \Delta y^F_{D2}, c^F_2 \rangle = \langle \Delta y^F_{D2}, c^F_2 - M L^{-1} c^F_1 \rangle = \langle Dd^F, d^F \rangle > 0,$$

where we used Lemma 1 to derive the last inequality. This completes the proof. □

Of course the action of the inverses in (3) and (5) would be implemented in practice as forward and back solves with the triangular L and its transpose. The most obvious choices for D are the identity matrix, in which case $\Delta y^F_{D2} = d^F$, or the rank-one matrix $e_j e_j^T$ for some j satisfying $d^F_j \neq 0$, in which case $y^F_{D2} = d^F_j e_j$. We summarize our procedure in terms of general data A and c in Algorithm 1.

In practice, the LU factorization of A^F is preceded by block triangularization. Specifically, row and column permutations P and Q are applied so that

$$A^F = P \begin{pmatrix} A^F_{11} & 0 & \cdots & 0 \\ A^F_{21} & A^F_{22} & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^F_{\ell 1} & A^F_{\ell 2} & \cdots & A^F_{\ell\ell} \end{pmatrix} Q, \tag{7}$$

where each diagonal block, except perhaps the last, is square and order n_i, say. Vitally, rather than computing the LU factors of A^F, only the diagonal blocks are factorized. That is, rather than finding (1), the factorizations

$$A^F_{ii} = \begin{pmatrix} L_{ii} \\ M_{ii} \end{pmatrix} (U_{ii} \;\; N_{ii}), \quad \text{for} \;\; i = 1, \ldots, \ell, \tag{8}$$

involving triangular matrices L_{ii} and U_{ii} of order $r_i \leq n_i$ are obtained (any further row permutations have been absorbed into P). To discover whether there are directions of infinite descent and when they exist to find one, we proceed by considering the blocks.

Algorithm 1 Computing a feasible direction or a direction of linear infinite descent satisfying the conditions in (32) from the manuscript.

procedure $[\Delta x, \Delta y, flag] = \text{DOLID}(A,c)$

input: $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^m$.

1. (compute the LU factorization of A)

Compute nonsingular matrices L and U in $\mathbb{R}^{r\times r}$ and a permutation matrix P so that

$$A = P \begin{pmatrix} L \\ M \end{pmatrix} (U \;\; N),$$

2. (compute a feasible point for $A(x+\Delta x) = c$ or a direction of linear infinite descent)

Compute

$$d = c_2 - Mw, \;\; \text{where} \;\; \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = P^T c, \;\; Lw = c_1, \;\; \text{and} \;\; c_1 \in \mathbb{R}^r.$$

if $d = 0$ **then**

Set $flag$ to `false` and $\Delta y = 0$, and compute the feasible point

$$x + \Delta x = \begin{pmatrix} U^{-1}w \\ 0 \end{pmatrix}.$$

else

Compute diagonal matrix D such that $D_{ii} \geq 0$ and $Dd \neq 0$.

Set $flag$ to `true` and $\Delta x = 0$, and compute the direction of linear infinite descent

$$\Delta y = P \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}, \;\; \text{where} \;\; \Delta y_2 = Dd \;\; \text{and} \;\; \Delta y_1 = -L^{-T} M^T \Delta y_2.$$

end if

end procedure

To illustrate the general procedure, suppose that after suitable row and column permutations

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \;\; \text{and} \;\; c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

We first use Algorithm 1 to compute $[\Delta x_1, \Delta y_1, flag] = \text{DOLID}(A_{11}, c_1)$, and if $flag$ is true, then

$$\Delta y := \begin{pmatrix} \Delta y_1 \\ 0 \end{pmatrix}$$

is a direction of linear infinite descent. If $flag$ has the value false, then the vector c_1 lies in the range of A_{11}. So now suppose there is a direction of linear infinite descent

$$\Delta y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

with $\Delta y_2 \neq 0$. In particular, this means that $A^T \Delta y = 0$, or equivalently that

$$A_{22}^T \Delta y_2 = 0 \;\; \text{and} \;\; A_{11}^T \Delta y_1 = -A_{21}^T \Delta y_2.$$

In this case $\Delta y_1 = -A_{11}^{+T} A_{21}^T \Delta y_2$, where A_{11}^+ is the generalized inverse of A_{11}, and

$$\langle c, \Delta y \rangle = \langle c_2 - A_{21} A_{11}^+ c_1, \Delta y_2 \rangle.$$

Thus we seek Δy_2 for which

$$A_{22}^T \Delta y_2 = 0 \;\text{ and }\; \langle c_2^{(2)}, \Delta y_2 \rangle \neq 0, \;\text{ where }\; c_2^{(2)} := c_2 - A_{21} A_{11}^+ c_1.$$

We may check this by calling DOLID$(A_{22}, c_2^{(2)})$. Clearly this idea may be applied recursively by partitioning A_{21}, A_{22}, and c_2. We summarize the procedure in Algorithm 2.

Algorithm 2 Computing a feasible direction or a direction of linear infinite descent satisfying the conditions in (32) from the manuscript.

input: Matrix A^F factorized as in (7) and c^F.
Let c_j denote the jth block of components of $P^T c^F$ for $j = 1, 2, \dots \ell$ and some ℓ.
for $i = 1, \dots, \ell$ **do**
  Set $c_i \leftarrow c_i - \sum_{j=1}^{i-1} A_{ij} \Delta x_j$ and compute $[\Delta x_i, \Delta y_i, flag] =$**dolid**$(A_{ii}, c_i)$.
  if $flag$ has the value true **then**
    Back solve by performing $A_{jj}^T \Delta y_j = -\sum_{k=j+1}^{i} A_{jk}^T \Delta y_k$ for $j = i-1, \dots, 1$.
    return the direction of infinite descent

$$\Delta y^F = P \begin{pmatrix} \Delta y_1 \\ \vdots \\ \Delta y_i \\ 0 \end{pmatrix}.$$

  end if
  if $i = \ell$ **then**
    return there does not exist a direction of infinite descent.
  end if
end for

Note that Algorithm 2 only provides the direction of linear infinite descent or confirms that there is no such direction. In the latter case, one has then to solve (30) from the manuscript to recover a solution (27) from the manuscript. Clearly it is inefficient to check for a direction of linear infinite descent when it is unlikely that there will be one, and equally it is wasteful to try to compute a finite minimizer when this is unlikely to happen. Pragmatically, therefore, we would apply the method of §2.3.1 from the manuscript when $m_k \leq n$ and only resort to that from §2.3.2 from the manuscript when (30) from the manuscript is reported to be inconsistent. When $m_k > n$, the method in §2.3.2 of the manuscipt takes precedence, and we only use (30) from the manuscript when we have failed to find a direction of linear infinite descent.

2 Detailed Summary of Numerical Results

Tables 1 (iterative subproblem solver) and 2 (direct subproblem solver) summarize the results of DQP for the test problems. We provide values for the objective value (Obj. Value), primal violation (Primal Viol.), dual violation (Dual Viol.), and complementarity violation (Comp. Viol.) at the final iterate. The number of iterations computed (Iters.) and time required (Time) are also supplied. The final column (Status) indicates the outcome of the solution process; see Table 3 for the meaning of each status value.

Table 1: Results for package DQP when an iterative subproblem solver is used.

Name	n	m	Obj. Value	Primal Viol.	Dual Viol.	Comp. Viol.	Iters.	Time	Status
AUG2DC	20200	10000	1.81836806E+06	5.5E-08	5.7E-14	5.3E-06	4	0.20	0
AUG2DCQP	20200	10000	6.49813468E+06	4.7E-07	2.3E-13	1.3E-03	35	1.00	0
AUG3DC	27543	8000	2.76540711E+04	1.2E-07	1.8E-15	1.8E-06	1	0.03	0
AUG3DCQP	27543	8000	6.15603837E+04	1.6E-08	3.6E-15	5.9E-07	16	0.56	0
BTS4	36570	36310	6.90514312E+07	6.8E-08	2.3E-13	1.4E-04	10	1.29	0
CBS	11163	244	5.36161663E+06	4.7E-07	1.1E-13	1.7E-04	7	0.13	0
CONT-050	2597	2401	-4.56385182E+00	9.4E-07	4.1E-17	7.8E-08	1694	13.01	0
CONT1-100	10197	9801	-4.64440164E+00	9.9E-07	1.1E-16	1.1E-07	28743	894.66	0
CONT1-200	40397	39601	-4.73584419E+00	2.9E-01	2.8E-18	2.0E-03	-13999	-1800.10	-19
DALE	16514	405	1.82596990E+04	3.0E-08	1.8E-15	5.2E-08	6	0.20	0
DEGENQP	50	125025	1.13968254E-09	1.1E-08	5.9E-14	5.5E-09	102	7.46	0
DUAL1	85	1	3.50129677E-02	2.2E-11	3.3E-15	3.2E-10	10	0.51	0
DUAL2	96	1	3.37336714E-02	8.9E-16	2.0E-15	3.2E-17	8	0.01	0
DUAL3	111	1	1.35755832E-01	3.7E-12	6.0E-15	2.7E-10	8	0.01	0
DUAL4	75	1	7.46090652E-01	4.9E-11	3.8E-15	1.6E-09	4	0.00	0
FIVE20B	34552	52983	1.37339055E+08	9.7E-07	2.3E-13	1.5E-03	21	3.36	0
FIVE20C	34501	58825	1.62654515E+08	1.6E-07	2.3E-13	1.9E-03	46	8.03	0
HIE1327D	1183	1443	2.48905243E+06	7.3E-08	1.1E-13	9.6E-06	21	0.09	0
HIE1372D	637	525	8.06298962E+05	8.4E-08	5.7E-14	3.8E-05	7	0.02	0
HIER13	2020	3313	7.50704295E+06	8.1E-08	1.1E-13	6.4E-04	6	0.06	0
HIER133A	2197	3549	7.28073418E+06	3.9E-07	2.3E-13	1.5E-03	18	0.17	0
HIER133B	2197	3549	7.28073418E+06	3.9E-07	2.3E-13	1.5E-03	18	0.17	0
HIER133C	2197	3549	7.28073418E+06	3.9E-07	2.3E-13	1.5E-03	18	0.17	0
HIER133D	2197	3549	2.99559854E+07	4.6E-07	4.5E-13	1.7E-03	20	0.18	0
HIER133E	2197	3549	2.99559854E+07	7.5E-08	2.3E-13	3.9E-04	23	0.21	0
HIER16	3564	5484	1.04347325E+07	8.2E-09	1.1E-13	5.7E-05	8	0.12	0
HIER163A	4096	5376	8.69609949E+06	3.5E-09	1.1E-13	6.1E-06	24	0.38	0
HIER163B	4096	5376	8.69609949E+06	3.5E-09	1.1E-13	6.1E-06	24	0.38	0
HIER163C	4096	5376	8.69609949E+06	3.5E-09	1.1E-13	6.1E-06	24	0.38	0
HIER163D	4096	5376	3.47843980E+07	3.1E-07	1.1E-13	1.7E-04	26	0.41	0
HIER163E	4096	5376	3.47843980E+07	3.1E-07	1.1E-13	1.7E-04	26	0.41	0
HUES-MOD	5000	2	3.48244898E+07	3.0E-12	4.4E-16	2.5E-07	4	0.01	0
HUESTIS	1000	2	3.48245418E+10	1.9E-12	0.0E+00	1.8E-04	4	0.00	0
JJTABEL3	3025	1650	1.30980707E+14	3.7E-07	9.3E-10	1.2E+00	6544	50.55	0
KSIP	20	1001	5.75815430E-01	5.3E-07	2.2E-16	1.1E-05	-100001	-21.28	-18
LASER	1002	1000	1.14592952E+02	3.5E+02	9.5E-07	4.4E+02	-100001	-1722.22	-18
LISWET1	2002	2000	5.07215863E+00	3.6E-05	2.8E-14	1.1E-02	-100001	-410.40	-18
LISWET2	2002	2000	4.99798638E+00	9.0E-07	5.6E-16	1.7E-06	5920	24.17	0
LISWET3	2002	2000	4.99776948E+00	9.3E-07	2.2E-16	1.5E-06	1649	6.53	0
LISWET4	2002	2000	4.99779938E+00	9.7E-07	2.2E-16	2.5E-06	5786	22.83	0
LISWET5	2002	2000	4.99782483E+00	9.8E-07	6.2E-16	2.0E-06	3208	12.77	0
LISWET6	2002	2000	4.99788651E+00	9.6E-07	4.4E-16	1.3E-06	2360	9.40	0
LISWET7	2002	2000	5.00215132E+00	1.4E-04	4.1E-14	5.4E-02	-100001	-411.09	-18
LISWET8	2002	2000	5.19206925E+00	1.4E-02	2.4E-13	1.4E-01	-100001	-397.44	-18
LISWET9	2002	2000	2.21560671E+01	7.5E-03	8.2E-13	1.7E+00	-100001	-398.40	-18
LISWET10	2002	2000	5.02426900E+00	1.6E-03	2.8E-14	6.0E-03	-100001	-400.56	-18
LISWET12	2002	2000	1.21770710E+01	8.0E-03	6.8E-13	3.4E+00	-100001	-401.34	-18
MOSARQP1	2500	700	-3.82140982E+03	8.7E-07	7.1E-15	9.6E-07	62	0.98	0
MOSARQP2	2500	700	-5.05259194E+03	2.7E-07	7.1E-15	2.3E-06	20	0.37	0
NINE12	10399	11362	3.27678548E+07	5.2E-07	1.1E-13	2.0E-04	12	0.47	0
NINE5D	10733	17295	3.79459871E+07	1.1E-07	1.1E-13	3.6E-05	13	0.62	0
NINENEW	6546	7340	2.73604976E+07	4.7E-07	2.3E-13	6.6E-04	12	0.30	0
OSORIO	10201	202	1.13368430E+01	6.5E-09	2.5E-16	2.7E-08	7	0.07	0
POWELL20	10000	10000	5.20895823E+10	1.0E-06	0.0E+00	5.0E+00	24611	419.11	0
QPBAND	50000	25000	-3.21962814E+06	0.0E+00	3.3E-01	2.9E+02	-398	-1800.56	-19
STCQP1	8193	4095	3.67100485E+05	1.2E-08	1.8E-12	2.9E-06	4	0.79	0
STCQP2	8193	4095	3.71892743E+04	2.4E-07	4.8E-13	1.9E-05	32	2.88	0
TABLE1	1584	510	2.78623226E+12	1.1E-08	2.3E-10	4.2E-03	1122	4.14	0
TABLE3	4992	2464	2.93000638E+09	6.8E-09	1.8E-12	1.1E-04	29	0.43	0
TABLE4	4992	2464	2.93000638E+09	6.8E-09	1.8E-12	1.1E-04	29	0.43	0
TABLE5	4992	2464	2.93000638E+09	6.8E-09	1.8E-12	1.1E-04	29	0.43	0
TABLE6	1584	510	2.78623226E+12	4.7E-08	1.2E-10	3.6E-02	1098	4.06	0
TABLE7	624	230	7.79587339E+08	8.1E-08	3.6E-12	7.8E-05	13	0.02	0
TABLE8	1271	72	1.91933038E+02	8.9E-09	1.7E-15	9.2E-10	3	0.01	0
TARGUS	162	63	1.68744463E+07	1.3E-07	4.5E-13	2.7E-04	185	0.11	0
TOYSARAH	2890	1649	5.89122424E+19	4.0E+02	1.2E-07	2.1E+11	-100001	-894.75	-18
TWO5IN6	5681	9629	1.91002932E+07	3.5E-07	2.3E-13	2.6E-04	10	0.27	0
YAO	2002	2000	2.48349703E+01	5.7E-04	3.1E-13	8.0E-01	-100001	-411.01	-18

Table 2: Results for package DQP when a direct subproblem solver is used.

Name	Obj. Value	Primal Viol.	Dual Viol.	Comp. Viol.	Iters.	Time	Status
AUG2DC	1.81836807E+06	5.7E-13	5.7E-14	3.6E-10	1	0.07	0
AUG2DCQP	6.49813474E+06	1.4E-12	2.3E-13	3.8E-09	6815	217.79	0
AUG3DC	2.76540711E+04	1.5E-13	1.8E-15	6.1E-13	1	0.08	0
AUG3DCQP	6.15603837E+04	3.6E-14	3.6E-15	1.9E-12	135	8.76	0
BTS4	6.90514292E+07	5.8E-05	2.3E-13	2.3E-01	-4398	-1800.10	-19
CBS	5.36161663E+06	1.4E-12	1.7E-13	3.8E-11	6	0.26	0
CONT-050	0.00000000E+00	6.8E+00	3.2E+62	4.6E+62	-2429	-45.54	-5
CONT1-100	0.00000000E+00	9.7E+00	2.7E+61	1.7E+62	-3716	-339.94	-5
CONT1-200	0.00000000E+00	8.7E+00	1.0E+62	1.8E+62	-3622	-1443.41	-5
DALE	1.82596990E+04	1.1E-10	1.8E-15	2.9E-10	6	0.16	0
DEGENQP	2.35828854E+01	7.6E+00	4.5E+03	1.2E+01	-17642	-1800.10	-19
DUAL1	3.50129678E-02	2.1E-13	4.8E-15	7.6E-15	12	0.01	0
DUAL2	3.37336714E-02	6.5E-14	2.1E-15	2.3E-15	8	0.01	0
DUAL3	1.35755833E-01	5.6E-14	4.0E-15	8.1E-15	8	0.02	0
DUAL4	7.46090649E-01	2.2E-15	4.6E-15	1.9E-15	4	0.00	0
HIE1327D	2.48905242E+06	8.5E-07	1.1E-13	3.1E-03	3062	255.06	0
HIE1372D	8.06298962E+05	5.4E-11	2.8E-14	7.3E-08	6	0.06	0
HIER13	7.50699445E+06	9.2E-04	2.3E-13	1.5E+01	-4251	-1800.17	-19
HIER133A	7.28065677E+06	1.2E-03	1.1E-13	9.3E+00	-3409	-1800.35	-19
HIER133B	7.28066894E+06	2.3E-03	2.3E-13	1.0E+01	-3433	-1800.45	-19
HIER133C	7.28068850E+06	8.2E-04	1.1E-13	6.3E+00	-3410	-1800.31	-19
HIER133D	2.99558560E+07	1.7E-03	2.3E-13	2.7E+01	-3694	-1800.31	-19
HIER133E	2.99555414E+07	7.9E-03	2.3E-13	7.6E+01	-3443	-1800.12	-19
HIER16	1.04280522E+07	2.4E-02	1.1E-13	9.4E+02	-903	-1801.11	-19
HIER163A	8.68585956E+06	4.1E-02	5.7E-14	8.7E+02	-796	-1801.97	-19
HIER163B	8.68605451E+06	5.3E-02	1.1E-13	8.4E+02	-787	-1800.68	-19
HIER163C	8.68488527E+06	6.0E-02	5.7E-14	9.5E+02	-789	-1800.34	-19
HIER163D	3.47533066E+07	6.1E-02	2.3E-13	2.6E+03	-794	-1801.22	-19
HIER163E	3.47729480E+07	3.4E-02	2.3E-13	1.0E+03	-795	-1801.47	-19
HUES-MOD	3.48244898E+07	2.7E-12	4.4E-16	2.6E-07	4	0.07	0
HUESTIS	3.48245418E+10	1.1E-12	0.0E+00	1.1E-04	4	0.02	0
JJTABEL3	1.30980708E+14	4.0E-01	4.7E-10	1.1E+07	-100001	-839.10	-18
KSIP	5.75797349E-01	5.6E-07	2.8E-17	1.1E-16	81304	23.90	0
LASER	0.00000000E+00	2.1E+02	2.5E+62	1.5E+64	-64	-0.61	-5
LISWET1	7.22189448E+00	1.3E-11	1.3E-12	1.5E-07	1169	2.23	0
LISWET2	4.99808204E+00	5.0E-07	2.8E-15	4.6E-08	482	1.03	0
LISWET3	4.99777877E+00	1.6E-15	2.1E-16	1.8E-15	52	0.19	0
LISWET4	4.99781511E+00	2.2E-15	2.6E-16	6.8E-15	87	0.24	0
LISWET5	4.99783185E+00	2.7E-15	6.1E-16	5.9E-15	68	0.22	0
LISWET6	4.99789739E+00	2.2E-15	2.2E-16	4.8E-15	69	0.22	0
LISWET7	9.98951642E+01	8.7E-11	7.3E-12	7.6E-06	1659	3.62	0
LISWET8	1.43130575E+02	8.7E-11	7.3E-12	5.9E-06	897	2.63	0
LISWET9	3.92920161E+02	8.7E-11	7.8E-12	6.6E-06	905	4.00	0
LISWET10	9.89648678E+00	1.1E-11	1.0E-12	1.2E-07	622	1.46	0
LISWET12	3.47518978E+02	8.7E-11	7.3E-12	7.0E-06	936	3.63	0
MOSARQP1	-3.82140980E+03	5.4E-15	7.1E-15	3.0E-15	84	0.55	0
MOSARQP2	-5.05259194E+03	2.2E-14	7.1E-15	7.4E-15	34	0.21	0
NINE12	3.27678548E+07	2.9E-08	2.3E-13	1.1E-03	176	201.30	0
NINE5D	9.75187214E+06	3.7E+02	1.1E-13	1.5E+05	-168	-1807.39	-19
NINENEW	2.73588127E+07	2.2E-02	1.1E-13	6.0E+02	-2776	-1800.03	-19
OSORIO	1.13368430E+01	1.8E-14	4.4E-16	1.8E-14	7	0.33	0
POWELL20	5.20895828E+10	9.3E-10	0.0E+00	5.8E-03	2501	38.16	0
QPBAND	0.00000000E+00	0.0E+00	2.3E+62	4.9E+65	-4210	-1491.66	-5
STCQP1	3.67100485E+05	3.2E-13	2.3E-12	7.7E-10	4	2.90	0
STCQP2	3.71892743E+04	1.8E-14	6.5E-13	1.3E-12	17	2.39	0
TABLE1	2.78623226E+12	4.3E-08	2.9E-11	3.3E-02	279	1.25	0
TABLE3	2.93000638E+09	6.7E-07	3.6E-12	2.7E-02	21840	505.33	0
TABLE4	2.93000638E+09	8.1E-07	3.6E-12	2.4E-02	22498	533.54	0
TABLE5	2.93000638E+09	3.9E-07	1.8E-12	1.7E-02	24521	573.97	0
TABLE6	2.78623225E+12	1.4E-04	1.2E-10	1.5E+07	-100001	-438.35	-18
TABLE7	7.79587339E+08	2.2E-11	9.1E-13	4.6E-07	22	0.04	0
TABLE8	1.91933038E+02	1.8E-14	1.8E-15	1.8E-13	3	0.01	0
TARGUS	1.68744463E+07	1.7E-12	2.3E-13	8.3E-09	201	0.07	0
TOYSARAH	5.88881391E+19	5.9E+05	1.2E-07	2.9E+15	-73844	-1800.00	-19
TWO5IN6	9.47197659E+06	1.3E+02	1.1E-13	1.3E+05	-366	-1801.26	-19
YAO	1.97704616E+02	2.3E-10	1.6E-11	3.2E-05	3	0.01	0

Status	Meaning
0	Optimality conditions met.
-5	Problem was determined to be locally infeasible.
-18	Maximum iteration limit was reached.
-19	Maximum time limit was reached.

Table 3: Meaning of the status column in Tables 1 and 2.

3 Required Iterations and Time to Reach Various Accuracies

Tables 4 (iterative subproblem solver) and 5 (direct subproblem solver) give the total number of iterations (Iters.) and total time (Time) needed to meet termination tolerances 10^{-1} through 10^{-6}.

Table 4: Checkpoint information for package DQP when an iterative subproblem solver is used.

Name	Iters.						Time					
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
AUG2DC	2	2	2	3	3	4	0.05	0.05	0.05	0.08	0.08	0.10
AUG2DCQP	23	25	27	31	33	35	0.69	0.74	0.79	0.90	0.95	1.00
AUG3DC	1	1	1	1	1	1	0.02	0.02	0.02	0.02	0.02	0.02
AUG3DCQP	16	16	16	16	16	16	0.54	0.54	0.54	0.54	0.54	0.54
BTS4	8	8	9	9	10	10	1.03	1.03	1.16	1.16	1.28	1.28
CBS	6	6	6	7	7	7	0.11	0.11	0.11	0.13	0.13	0.13
CONT-050	258	384	1126	1274	1462	1694	2.07	3.03	8.68	9.80	11.23	13.00
CONT1-100	34	8565	10864	24240	25950	28743	1.13	267.10	338.58	754.25	807.37	894.13
CONT1-200	2583	2703	-1	-1	-1	-1	355.00	370.17	-1.00	-1.00	-1.00	-1.00
DALE	5	5	5	5	6	6	0.16	0.16	0.16	0.16	0.19	0.19
DEGENQP	96	96	98	102	102	102	7.28	7.28	7.33	7.44	7.44	7.44
DUAL1	8	9	10	10	10	10	0.01	0.01	0.01	0.01	0.01	0.01
DUAL2	6	8	8	8	8	8	0.01	0.01	0.01	0.01	0.01	0.01
DUAL3	4	7	7	8	8	8	0.00	0.01	0.01	0.01	0.01	0.01
DUAL4	1	3	3	3	4	4	0.00	0.00	0.00	0.00	0.00	0.00
FIVE20B	12	14	15	17	19	21	1.93	2.24	2.40	2.72	3.03	3.35
FIVE20C	36	38	40	42	44	46	6.28	6.62	6.97	7.32	7.66	8.01
HIE1327D	11	14	17	18	20	21	0.05	0.06	0.07	0.08	0.08	0.09
HIE1372D	5	5	7	7	7	7	0.01	0.01	0.02	0.02	0.02	0.02
HIER13	4	4	4	5	6	6	0.04	0.04	0.04	0.05	0.06	0.06
HIER133A	10	11	11	18	18	18	0.09	0.10	0.10	0.17	0.17	0.17
HIER133B	10	11	11	18	18	18	0.09	0.10	0.10	0.17	0.17	0.17
HIER133C	10	11	11	18	18	18	0.10	0.11	0.11	0.17	0.17	0.17
HIER133D	13	15	15	15	18	20	0.12	0.14	0.14	0.14	0.17	0.18
HIER133E	14	17	19	20	21	23	0.13	0.16	0.18	0.19	0.19	0.21
HIER16	6	6	6	7	8	8	0.09	0.09	0.09	0.11	0.12	0.12
HIER163A	14	15	18	21	22	24	0.22	0.24	0.29	0.33	0.35	0.38
HIER163B	14	15	18	21	22	24	0.22	0.24	0.29	0.33	0.35	0.38
HIER163C	14	15	18	21	22	24	0.22	0.24	0.29	0.33	0.35	0.38
HIER163D	17	19	22	22	24	26	0.27	0.30	0.35	0.35	0.38	0.41
HIER163E	17	19	22	22	24	26	0.27	0.30	0.35	0.35	0.38	0.41
HUES-MOD	4	4	4	4	4	4	0.01	0.01	0.01	0.01	0.01	0.01
HUESTIS	4	4	4	4	4	4	0.00	0.00	0.00	0.00	0.00	0.00
JJTABEL3	6541	6541	6541	6543	6544	6544	50.49	50.49	50.49	50.51	50.52	50.52
KSIP	5	103	1519	8884	-1	-1	0.02	0.28	1.50	3.68	-1.00	-1.00
LASER	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
LISWET1	0	6	21	295	-1	-1	0.00	0.02	0.08	1.21	-1.00	-1.00
LISWET2	0	4	22	54	441	5920	0.00	0.01	0.09	0.22	1.79	24.15
LISWET3	0	4	20	77	436	1649	0.00	0.01	0.08	0.30	1.73	6.53
LISWET4	0	4	29	76	625	5786	0.00	0.01	0.11	0.30	2.46	22.81
LISWET5	0	4	17	117	522	3208	0.00	0.01	0.07	0.46	2.07	12.76
LISWET6	0	4	26	148	590	2360	0.00	0.01	0.10	0.59	2.35	9.40
LISWET7	0	5	16	783	-1	-1	0.00	0.02	0.06	3.21	-1.00	-1.00
LISWET8	0	4	316	-1	-1	-1	0.00	0.01	1.25	-1.00	-1.00	-1.00
LISWET9	0	37	18665	-1	-1	-1	0.00	0.14	73.61	-1.00	-1.00	-1.00
LISWET10	0	5	60	1767	-1	-1	0.00	0.02	0.24	7.05	-1.00	-1.00
LISWET12	0	40	3395	-1	-1	-1	0.00	0.15	13.34	-1.00	-1.00	-1.00
MOSARQP1	46	51	51	53	57	62	0.74	0.82	0.82	0.84	0.90	0.97
MOSARQP2	20	20	20	20	20	20	0.37	0.37	0.37	0.37	0.37	0.37
NINE12	8	8	9	10	11	12	0.32	0.32	0.35	0.39	0.43	0.47
NINE5D	8	11	11	12	12	13	0.38	0.52	0.52	0.57	0.57	0.62
NINENEW	7	8	9	10	12	12	0.17	0.20	0.22	0.24	0.29	0.29
OSORIO	2	3	3	4	6	7	0.04	0.05	0.05	0.05	0.06	0.06
POWELL20	9054	13469	16136	18570	21666	24611	143.16	221.31	268.59	311.74	366.65	418.85
QPBAND	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
STCQP1	4	4	4	4	4	4	2.90	2.90	2.90	2.90	2.90	2.90
STCQP2	25	25	27	28	30	32	15.72	15.72	16.90	17.50	18.68	19.87
TABLE1	33	1119	1119	1121	1121	1122	0.12	4.12	4.12	4.13	4.13	4.13
TABLE3	19	21	22	25	27	29	0.28	0.31	0.33	0.37	0.40	0.43
TABLE4	19	21	22	25	27	29	0.28	0.31	0.32	0.37	0.40	0.43
TABLE5	19	21	22	25	27	29	0.28	0.31	0.32	0.37	0.40	0.43
TABLE6	23	1096	1097	1097	1097	1098	0.09	4.05	4.05	4.05	4.05	4.06
TABLE7	9	10	11	12	12	13	0.02	0.02	0.02	0.02	0.02	0.02
TABLE8	3	3	3	3	3	3	0.01	0.01	0.01	0.01	0.01	0.01
TARGUS	185	185	185	185	185	185	0.11	0.11	0.11	0.11	0.11	0.11
TOYSARAH	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
TWO5IN6	6	8	8	9	10	10	0.16	0.22	0.22	0.24	0.27	0.27
YAO	47	71	3778	-1	-1	-1	0.19	0.29	15.45	-1.00	-1.00	-1.00

Table 5: Checkpoint information for package DQP when a direct subproblem solver is used.

	Iters.						Time					
Name	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
AUG2DC	1	1	1	1	1	1	0.27	0.27	0.27	0.27	0.27	0.27
AUG2DCQP	6815	6815	6815	6815	6815	6815	890.77	890.77	890.77	890.77	890.77	890.77
AUG3DC	1	1	1	1	1	1	0.31	0.31	0.31	0.31	0.31	0.31
AUG3DCQP	134	134	134	135	135	135	33.76	33.76	33.76	35.20	35.20	35.20
BTS4	486	1314	2484	4030	-1	-1	863.80	2321.94	4362.33	7062.49	-1.00	-1.00
CBS	6	6	6	6	6	6	1.35	1.35	1.35	1.35	1.35	1.35
CONT-050	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
CONT1-100	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
CONT1-200	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
DALE	5	5	6	6	6	6	0.53	0.53	0.64	0.64	0.64	0.64
DEGENQP	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
DUAL1	8	10	10	11	12	12	0.01	0.01	0.01	0.01	0.01	0.01
DUAL2	6	7	8	8	8	8	0.01	0.01	0.01	0.01	0.01	0.01
DUAL3	4	6	7	8	8	8	0.01	0.01	0.01	0.01	0.01	0.01
DUAL4	1	3	3	3	4	4	0.00	0.00	0.00	0.00	0.00	0.00
HIE1327D	87	189	603	1222	2500	3062	20.99	44.45	138.72	280.88	577.09	704.46
HIE1372D	6	6	6	6	6	6	0.15	0.15	0.15	0.15	0.15	0.15
HIER13	1006	2481	-1	-1	-1	-1	1132.75	2767.33	-1.00	-1.00	-1.00	-1.00
HIER133A	965	1829	-1	-1	-1	-1	1340.32	2544.38	-1.00	-1.00	-1.00	-1.00
HIER133B	793	1926	-1	-1	-1	-1	1096.55	2643.27	-1.00	-1.00	-1.00	-1.00
HIER133C	1302	2060	-1	-1	-1	-1	1827.50	2873.03	-1.00	-1.00	-1.00	-1.00
HIER133D	1997	2743	-1	-1	-1	-1	2599.48	3525.58	-1.00	-1.00	-1.00	-1.00
HIER133E	1645	2716	-1	-1	-1	-1	2267.25	3750.37	-1.00	-1.00	-1.00	-1.00
HIER16	846	-1	-1	-1	-1	-1	5554.50	-1.00	-1.00	-1.00	-1.00	-1.00
HIER163A	721	-1	-1	-1	-1	-1	5508.16	-1.00	-1.00	-1.00	-1.00	-1.00
HIER163B	731	-1	-1	-1	-1	-1	5602.53	-1.00	-1.00	-1.00	-1.00	-1.00
HIER163C	763	-1	-1	-1	-1	-1	5823.36	-1.00	-1.00	-1.00	-1.00	-1.00
HIER163D	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
HIER163E	381	-1	-1	-1	-1	-1	2889.72	-1.00	-1.00	-1.00	-1.00	-1.00
HUES-MOD	4	4	4	4	4	4	0.07	0.07	0.07	0.07	0.07	0.07
HUESTIS	4	4	4	4	4	4	0.02	0.02	0.02	0.02	0.02	0.02
JJTABEL3	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
KSIP	290	367	3005	21811	81299	81304	6.38	6.77	8.55	13.87	27.70	27.70
LASER	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
LISWET1	0	5	1169	1169	1169	1169	0.00	0.04	2.21	2.21	2.21	2.21
LISWET2	0	4	23	127	320	482	0.00	0.04	0.16	0.42	0.80	1.03
LISWET3	0	4	15	48	52	52	0.00	0.04	0.13	0.18	0.19	0.19
LISWET4	0	5	15	76	87	87	0.00	0.05	0.12	0.22	0.24	0.24
LISWET5	0	4	15	55	66	68	0.00	0.05	0.13	0.20	0.22	0.22
LISWET6	0	4	14	53	69	69	0.00	0.04	0.12	0.19	0.22	0.22
LISWET7	0	4	1659	1659	1659	1659	0.00	0.04	3.61	3.61	3.61	3.61
LISWET8	0	4	897	897	897	897	0.00	0.04	2.63	2.63	2.63	2.63
LISWET9	0	901	904	905	905	905	0.00	3.99	3.99	3.99	3.99	3.99
LISWET10	0	5	51	622	622	622	0.00	0.04	0.22	1.46	1.46	1.46
LISWET12	0	936	936	936	936	936	0.00	3.62	3.62	3.62	3.62	3.62
MOSARQP1	79	83	84	84	84	84	0.52	0.55	0.55	0.55	0.55	0.55
MOSARQP2	33	34	34	34	34	34	0.21	0.21	0.21	0.21	0.21	0.21
NINE12	175	176	176	176	176	176	479.41	481.79	481.79	481.79	481.79	481.79
NINE5D	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
NINENEW	646	2706	-1	-1	-1	-1	1037.07	4417.92	-1.00	-1.00	-1.00	-1.00
OSORIO	3	3	4	5	7	7	0.46	0.46	0.71	1.40	1.90	1.90
POWELL20	2501	2501	2501	2501	2501	2501	186.30	186.30	186.30	186.30	186.30	186.30
QPBAND	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
STCQP1	4	4	4	4	4	4	6.09	6.09	6.09	6.09	6.09	6.09
STCQP2	16	17	17	17	17	17	6.41	6.65	6.65	6.65	6.65	6.65
TABLE1	43	46	47	63	279	279	0.80	0.86	0.88	1.17	5.11	5.11
TABLE3	4606	8008	11582	14922	18186	21840	456.39	791.98	1153.32	1497.13	1829.26	2198.03
TABLE4	4208	8320	11752	15204	18682	22498	439.86	872.43	1219.74	1573.94	1927.13	2322.22
TABLE5	4366	8589	12901	16749	20515	24521	444.67	874.73	1318.60	1717.90	2104.66	2511.45
TABLE6	60	63	64	68	161	-1	1.05	1.10	1.12	1.19	2.76	-1.00
TABLE7	22	22	22	22	22	22	0.04	0.04	0.04	0.04	0.04	0.04
TABLE8	3	3	3	3	3	3	0.01	0.01	0.01	0.01	0.01	0.01
TARGUS	201	201	201	201	201	201	0.07	0.07	0.07	0.07	0.07	0.07
TOYSARAH	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
TWO5IN6	-1	-1	-1	-1	-1	-1	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
YAO	3	3	3	3	3	3	0.01	0.01	0.01	0.01	0.01	0.01

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