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Electronic Supplementary Material to the paper *Resample-smoothing of Voronoi intensity estimators*

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1 Proofs of the results in the paper

1.1 Proof of Lemma 1

Conditionally on $\mathbf{x}$, all $\widehat{\rho}_i^V(u)$ are non-negative i.i.d. random variables. Hence, by the (conditional) law of large numbers, $\widehat{\rho}_{p,m}^V(u)$ converges a.s. to the conditional expectation of $\widehat{\rho}_i^V(u)$ given $\mathbf{x}$. By using equation (6) in the main text, we obtain that this conditional expectation is a.s. finite.

1.2 Proof of Theorem 1

A p-thinning of X is again stationary with intensity $p\rho$. By Daley and Vere-Jones (2008, Expression (11.3.2)),

$$G_{\theta_v X_p}(\cdot) = G_{(\theta_v X)_p}(\cdot) = G_{\theta_v X}(p\cdot{+}1-p) = G_X(p\cdot{+}1-p) = G_{X_p}(\cdot), \quad v \in S,$$

where $G_X(\cdot)$ is the generating functional of X. Using Last (2010, Corollary 8.7) we immediately obtain that

$$\mathbb{E}[\widehat{\rho}_{p,m}^V(u)] = \frac{\mathbb{E}[\widehat{\rho}^V(u; X_p, S)]}{p} = \mathbb{E}[1/|\{\text{cell of } X_p \text{ containing } u\}|]/p = \frac{p\rho}{p} = \rho.$$

1.3 Proof of Theorem 2

We denote by $x_u(X) \in X$ the centre of the Voronoi cell $C_u(X)$, the cell containing $u \in \mathbb{R}^d$. Let $\varepsilon > 0$, $\mu = \mu_u$ and $\rho_- = \min_{v \in B(u,\varepsilon)} \rho(v)$, such that $\rho(v)/2 \leq \rho(v) - \mu\varepsilon \leq \rho_- \leq \rho(v)$ on $B(u, \varepsilon)$. Let X_- be obtained by independently removing/adding points at rate $\rho_- - \rho(v)$, $v \in \mathbb{R}^d$. Note that X_- is a homogeneous Poisson process with intensity ρ_- and $X_- \subseteq X$ on $B(u, \varepsilon)$ a.s..

We call Voronoi neighbours in some configuration $\mathbf{x}$ the centres of cells of $\mathbf{x}$ which are neighbours of $C_u(\mathbf{x})$. Denote by $R(\mathbf{x})$ the maximal Euclidean distance between $x_u(\mathbf{x})$ and its Voronoi neighbours. Remark that if $R(\mathbf{x}) \leq \varepsilon$, then $C_u(\mathbf{x}) \subseteq B(u, \varepsilon)$. One can find a finite number of balls such that if any such ball contains a point of $\mathbf{x}$, then $R(\mathbf{x}) \leq 1$. Hence, using the void probabilities of X, we have at the scale ε for X that

$$\mathbb{P}(R(X) \geq \varepsilon) \leq C_d \, \mathrm{e}^{-c_d \rho_- \varepsilon^d}$$

for some $C_d, c_d > 0$.

Now, let Ω be the event that X and X_- coincide on $B(u, \varepsilon)$ and $R(X) \leq \varepsilon$. Conditionally on Ω, $C_u(X) = C_u(X_-) \subseteq B(u, \varepsilon)$. We obtain

$$\mathbf{1}_{\{\Omega^c\}} \leq \mathbf{1}_{\{R(X)>\varepsilon\}} + \sum_{x \in X_- \cap B(u,\varepsilon)} \mathbf{1}_{\{x \text{ eliminated at thinning}\}},$$

$$\mathbb{P}(\Omega^c) \leq \mathbb{P}(R(X) > \varepsilon) + \int_{B(u,\varepsilon)} \mu\varepsilon \mathrm{d}x \leq C_d \, \mathrm{e}^{-c_d \rho_- \varepsilon^d} + c\varepsilon^d \mu\varepsilon.$$

Let further $\kappa' = (1-\kappa^{-1})^{-1} \le d+1$. By Hölder's inequality and Theorem 1 we have that

$$\begin{aligned}
\left|\mathbb{E}\left[\widehat{\rho}^V(u)\right] - \rho(u)\right| \le & \left|\mathbb{E}\left[\mathbf{1}_{\{\Omega\}}\frac{1}{|C_u(X)|}\right] - \rho(u)\right| + \mathbb{E}\left[\mathbf{1}_{\{\Omega^c\}}\frac{1}{|C_u(X)|}\right] \\
\le & \left|\mathbb{E}\left[\mathbf{1}_{\{\Omega\}}\frac{1}{|C_u(X_-)|} - \rho(u)\right]\right| + (\mathbb{E}|C_u(X)|^{-\kappa})^{1/\kappa}\mathbb{P}(\Omega^c)^{1/\kappa'} \\
\le & \underbrace{\mathbb{E}\left[\frac{1}{|C_u(X_-)|} - \rho_-\right]}_{=0} + \mathbf{1}_{\{\Omega^c\}}\frac{1}{|C_u(X_-)|} + |\rho(u)-\rho_-| \\
& + m(c_d\mu\varepsilon^d\varepsilon + C_d\,\mathrm{e}^{-c_d\rho_-\varepsilon^d})^{1/\kappa'} \\
\le & \mu\varepsilon + 2m(c_d\mu\varepsilon^d\varepsilon + C_d\,\mathrm{e}^{-c_d\rho_-\varepsilon^d})^{1/\kappa'}.
\end{aligned}$$

Setting $\varepsilon = \rho_-^{-1/d}\log(\rho_-)^{2/d}$ and recalling that $\rho(u)/2 \le \rho_-$, using that $\kappa' \le d+1$, proves the result for the original Voronoi intensity estimator.

As a p-thinning X_p, $p \in (0,1]$, of X is a Poisson process with intensity $p\rho(\cdot)$, we finally note that

$$\begin{aligned}
p|\mathbb{E}[\widehat{\rho}^V_{p,m}(u)] - \rho(u)| &= \left|\mathbb{E}\left[\widehat{\rho}^V(u;X_p,\mathbb{R}^d)\right] - p\rho(u)\right| \\
&\le \mu p^{-1}\varepsilon + 2m(c_d\mu p^{-1}\varepsilon^d\varepsilon + C_d\,\mathrm{e}^{-c_d p\rho(u)\varepsilon^d})^{1/\kappa'},
\end{aligned}$$

since $\mathbb{E}|C_u(X_p)|^{-\kappa} \le \mathbb{E}|C_u(X)|^{-\kappa}$.

1.4 Proof of Theorem 3

Note first that

$$\begin{aligned}
\mathrm{Var}(\widehat{\rho}^V_{p,m}(u)) &= \frac{1}{(mp)^2}\sum_{i=1}^m \mathrm{Var}(\widehat{\rho}^V_1(u)) + \frac{1}{(mp)^2}\sum_{i\neq j}\mathrm{Cov}(\widehat{\rho}^V_i(u),\widehat{\rho}^V_j(u)) \qquad (1) \\
&= \frac{1}{m}\mathrm{Var}(\widehat{\rho}^V_1(u)/p) + \frac{m-1}{m}\mathrm{Cov}(\widehat{\rho}^V_1(u)/p,\widehat{\rho}^V_2(u)/p) \\
&= \mathrm{Var}(\widehat{\rho}^V_{p,1}(u))\frac{1+(m-1)\mathrm{Corr}(\widehat{\rho}^V_1(u),\widehat{\rho}^V_2(u))}{m},
\end{aligned}$$

where Cov and Corr denote covariance and correlation, respectively. Since the variance is non-negative, by (1) we must have that $\mathrm{Corr}(\widehat{\rho}^V_1(u),\widehat{\rho}^V_2(u)) \ge -1/(m-1)$ for every single $m \ge 1$. Hence, the correlation must be non-negative, whereby $\mathrm{Var}(\widehat{\rho}^V_{p,1}(u))/m \le \mathrm{Var}(\widehat{\rho}^V_{p,m}(u)) \le \mathrm{Var}(\widehat{\rho}^V_{p,1}(u))$; this is obtained by setting $\mathrm{Corr}(\widehat{\rho}^V_1(u),\widehat{\rho}^V_2(u)) = 0, 1$ in expression (1). Also, letting $m\to\infty$ in (1), the limit of (1) is given by $\mathrm{Cov}(\widehat{\rho}^V_1(u),\widehat{\rho}^V_2(u))/p^2$ since $\mathrm{Var}(\widehat{\rho}^V_1(u)) < \infty$.

Regarding the variance tending to 0, it is sufficient to show it for $m=1$ since $\mathrm{Var}(\widehat{\rho}^V_{p,m}(u)) \le \mathrm{Var}(\widehat{\rho}^V_{p,1}(u))$.

Let $(X_p)_{p\in(0,1]}$ be a coupling such that X_p is non-increasing in terms of inclusion: assign independent $U(0,1)$-distributed labels to the points of X and generate X_p by keeping all points with labels smaller than p. For a bounded W there is a.s. some $p_0 \in (0,1)$ such that $X_p = \emptyset$ for all $p \in (0, p_0)$. Hence, $\widehat{\rho}^V_{p,1}(u) = \widehat{\rho}^V(u; X_p, W)/p = 0/p = 0$ (by definition) for such p, which means that the limit $\widehat{\rho}^V_{p,1}(u) \downarrow 0$ is deterministic. Since there are $p \in (0,1]$ such that $\mathbb{E}[\widehat{\rho}^V_{p,1}(u)^2] < \infty$, by the dominated convergence theorem it follows that $\mathrm{Var}(\widehat{\rho}^V_{p,1}(u)) \to 0$ as $p \to 0$.

Consider a sequence of windows $(W_p)_{p\in(0,1]}$ which increases (in terms of inclusion) as p decreases and satisfies $\mathbb{E}[N(X_p \cap W_p)] = p\int_{W_p} \rho(u)\mathrm{d}u \to 0$ as $p \to 0$. This implies that $\mathbb{P}(X_p \cap W_p \neq \emptyset) \to 0$ and writing $\widehat{\rho}^V_{p,1}(u) = \widehat{\rho}^V_{p,1}(u; X_p, W_p)$, we obtain that $\mathbb{P}(\widehat{\rho}^V_{p,1}(u) \neq 0) = \mathbb{P}(X_p \cap W_p \neq \emptyset) \to 0$. Since $\widehat{\rho}^V_{p,1}(u) \leq \widehat{\rho}^V_{1,1}(u) = \widehat{\rho}^V(u)$ a.s., which has finite variance by assumption, and since $\widehat{\rho}^V_{p,1}(u)$ is a.s. decreasing as $p \to 0$ (even with changing study region), the monotone convergence theorem yields that $\widehat{\rho}^V_{p,1}(u) \to^{L^2} 0$ as $p \to 0$, whereby $\mathrm{Var}(\widehat{\rho}^V_{p,1}(u)) \to 0$ as $p \to 0$.

1.5 Proof of Lemma 2

Recall that X_p is a homogeneous Poisson process with intensity $p\rho$. For a typical point of X_p, let Δ_- and Δ_+ be the distances to the point's nearest neighbours to the left and to the right, respectively; they are independent and exponentially distributed with mean $p\rho$. Since $\Delta_-/2$ and $\Delta_+/2$ are independent and exponentially distributed with mean $2p\rho$, the typical cell size, $\Delta_-/2 + \Delta_+/2$, follows an Erlang/Gamma distribution with shape parameter 2 and rate $2p\rho$, whereby the density of $P_{|\mathcal{V}_o(X_p)|}(\cdot)$ is given by $f_{|\mathcal{V}_o(X_p)|}(t) = (2p\rho)^2 t\,\mathrm{e}^{-2p\rho t}$. Through equation (6) in the paper, we now obtain

$$\begin{aligned}\mathbb{E}[\widehat{\rho}^V_{p,1}(u)^2] &= \frac{\rho}{p}\mathbb{E}[1/|\mathcal{V}_o(X_p)|] = \frac{\rho}{p}\int_0^\infty \frac{1}{t}(2p\rho)^2 t\,\mathrm{e}^{-2p\rho t}\,\mathrm{d}t = 4p\rho^3\int_0^\infty \mathrm{e}^{-2p\rho t}\,\mathrm{d}t \\ &= \frac{4p\rho^3}{2p\rho} = 2\rho^2,\end{aligned}$$

i.e., $\mathrm{Var}(\widehat{\rho}^V_{p,m}(u)) \leq \mathrm{Var}(\widehat{\rho}^V_{p,1}(u)) = 2\rho^2 - \rho^2 = \rho^2$ by Theorem 3.

2 Estimation error plots

This section provides plots of the estimated bias and variance for $\widehat{\rho}_{p,m}^{V}(u)$, for each of the models described in Section 4 of the paper, when $m = 400$ and $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$. The estimates are generated by means of 500 realisations of each model. Also, box plots of average, average absolute, minimum, and maximum point-wise errors for each model are presented. We additionally provide the above for kernel intensity estimates, with bandwidths selected by means of Poisson likelihood cross-validation (Baddeley et al., 2015; Loader, 1999), hereinafter `ppl`, and the method of Cronie and Van Lieshout (2018), hereinafter `CvL`.

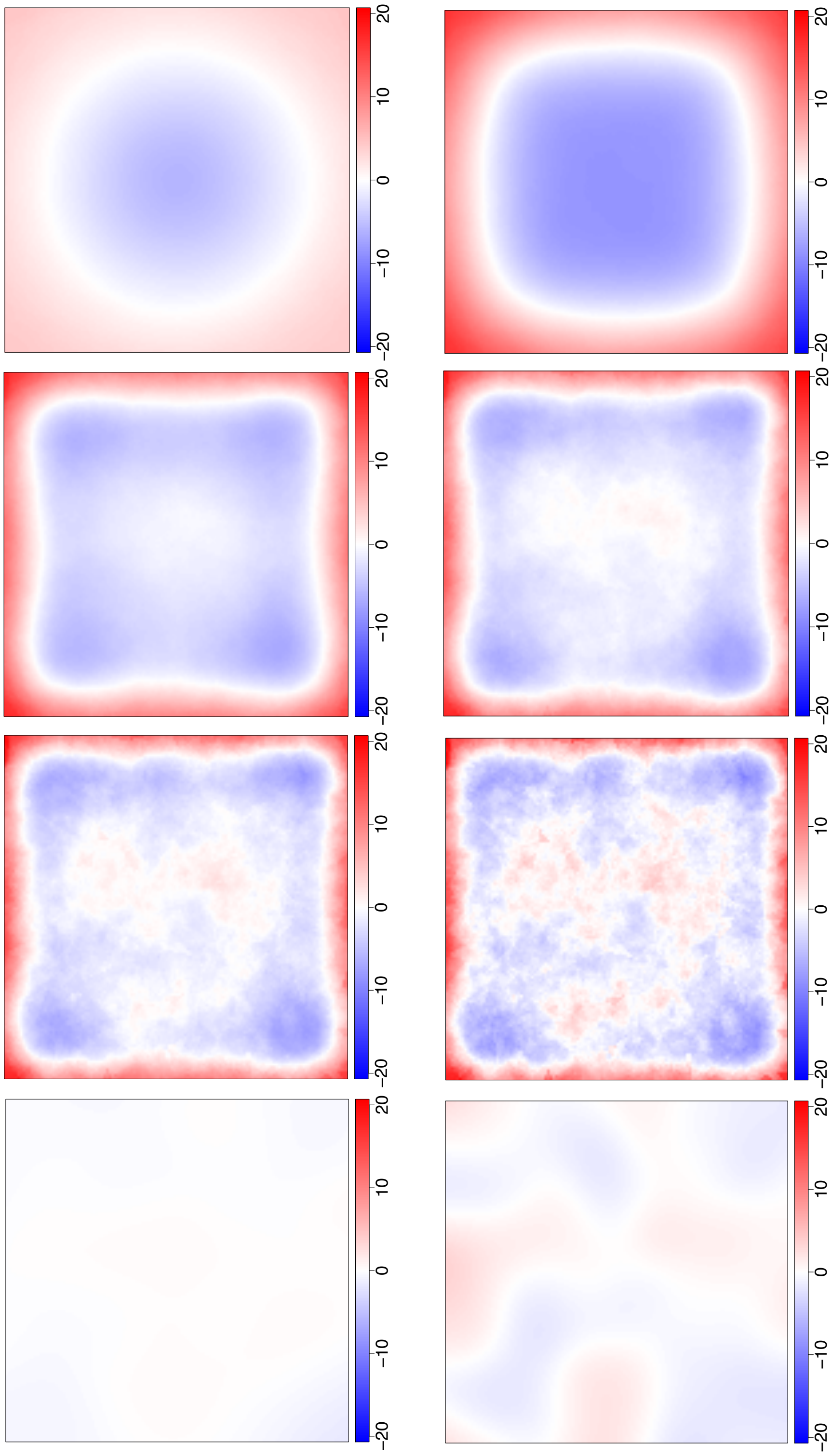

Fig. 1 Estimated bias for $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of a homogeneous Poisson process $X \subset W = [0,1]^2$ with intensity $\rho = 60$. From top-left to bottom-right: $\widehat{\rho}^{V}_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row.

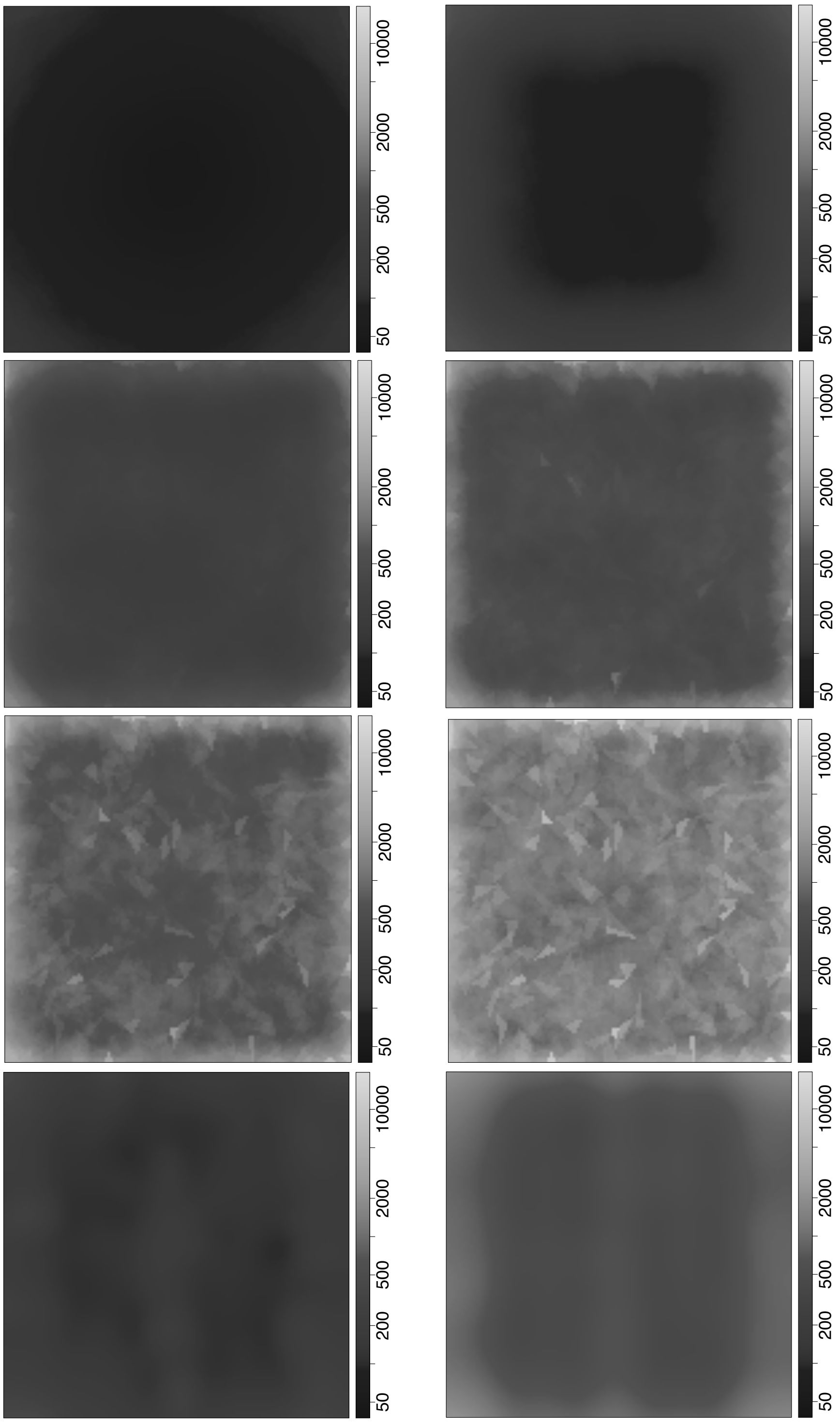

Fig. 2 Estimated variance for $\widehat{\rho}^V_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of a homogeneous Poisson process $X \subset W = [0,1]^2$ with intensity $\rho = 60$. From top-left to bottom-right: $\widehat{\rho}^V_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row. Logarithmic colour map.

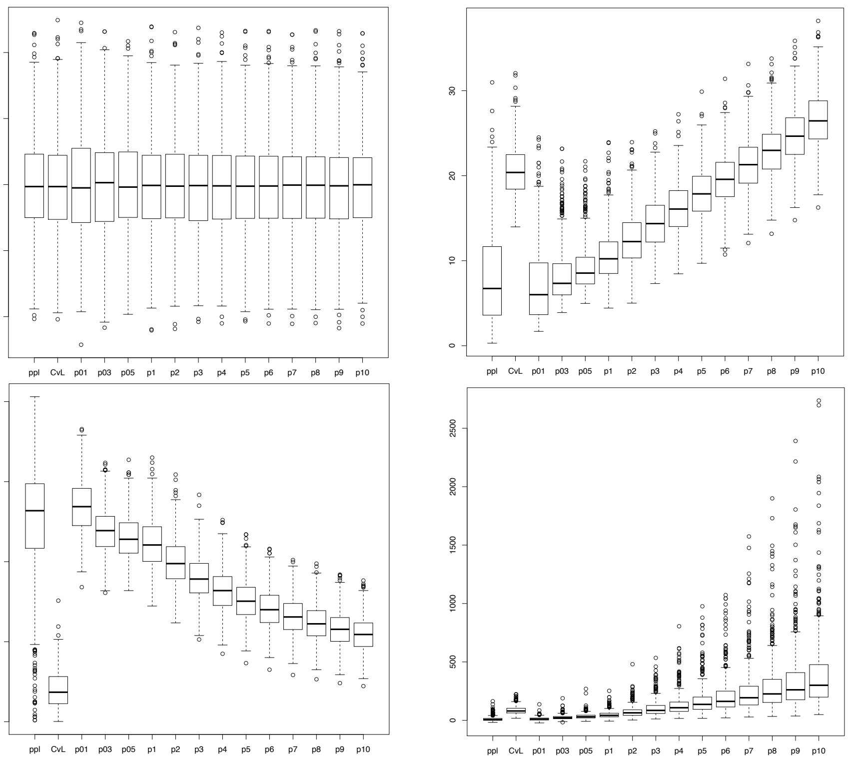

Fig. 3 Box plots for point-wise errors of $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimates, based on 500 realisations of a homogeneous Poisson process $X \subset W = [0,1]^2$ with intensity $\rho = 60$. From top-left to bottom-right: average; average absolute; minimum and maximum. x-axis labels from left to right: ppl, CvL, $p = 0.01, 0.03, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

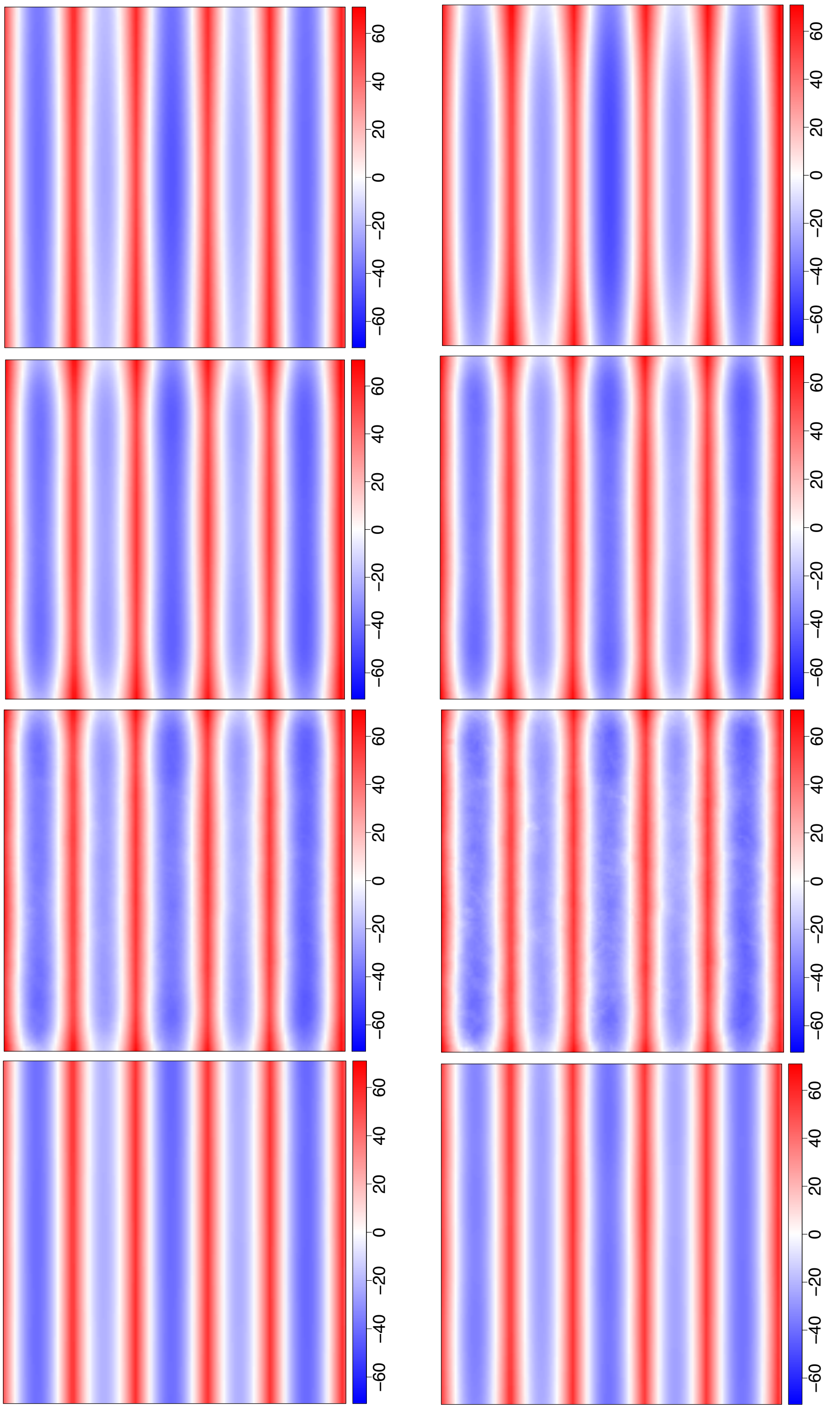

Fig. 4 Estimated bias for $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of an inhomogeneous Poisson process $X \subset W = [0,1]^2$ with intensity $\rho(x,y) = |10 + 90\sin(16x)|$. From top-left to bottom-right: $\widehat{\rho}^{V}_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row.

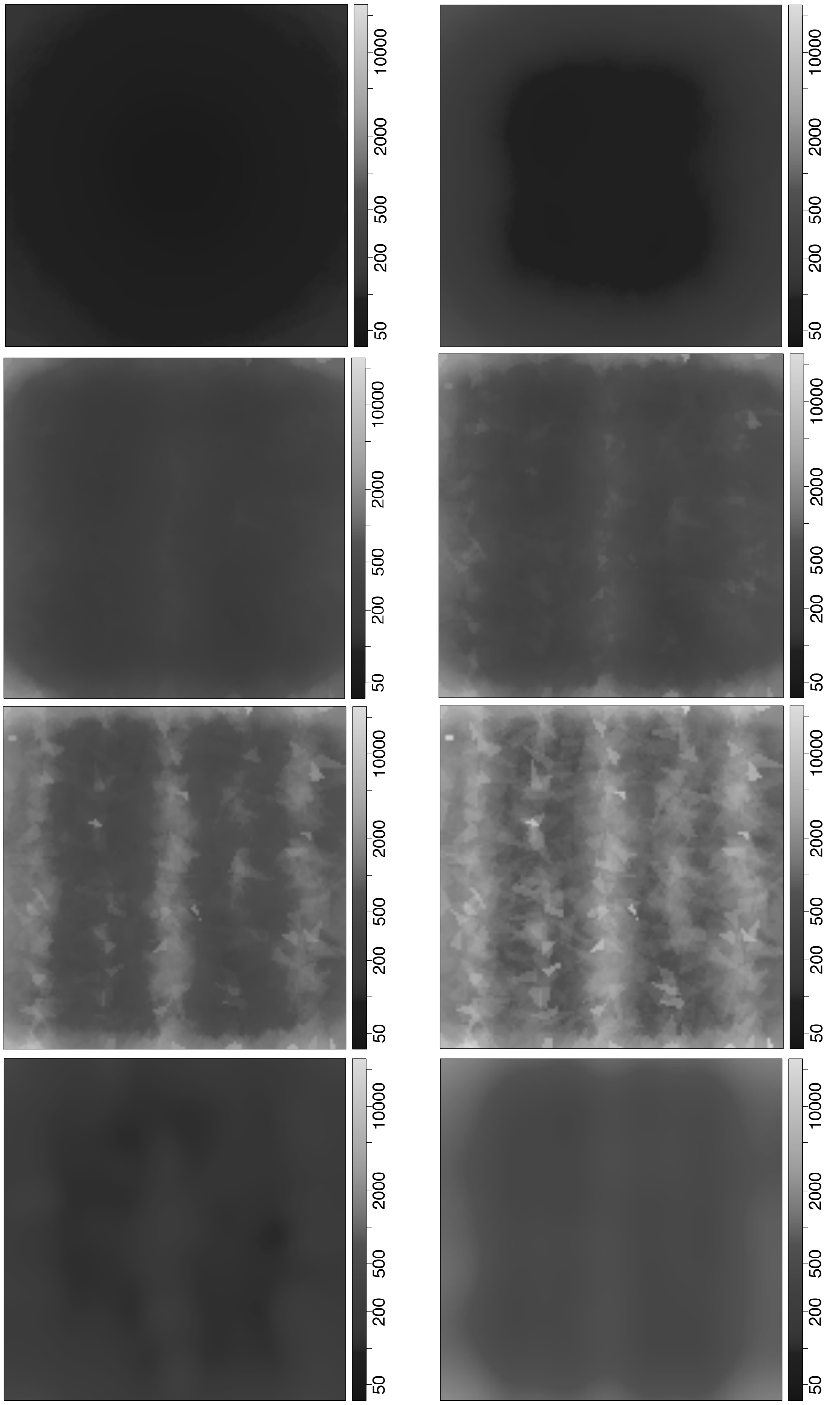

Fig. 5 Estimated bias for $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 200$, and kernel estimators, based on 500 realisations of an inhomogeneous Poisson process $X \subset W = [0,1]^2$ with intensity $\rho(x,y) = |10 + 90\sin(16x)|$. From top-left to bottom-right: $\widehat{\rho}^{V}_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row. Logarithmic colour map.

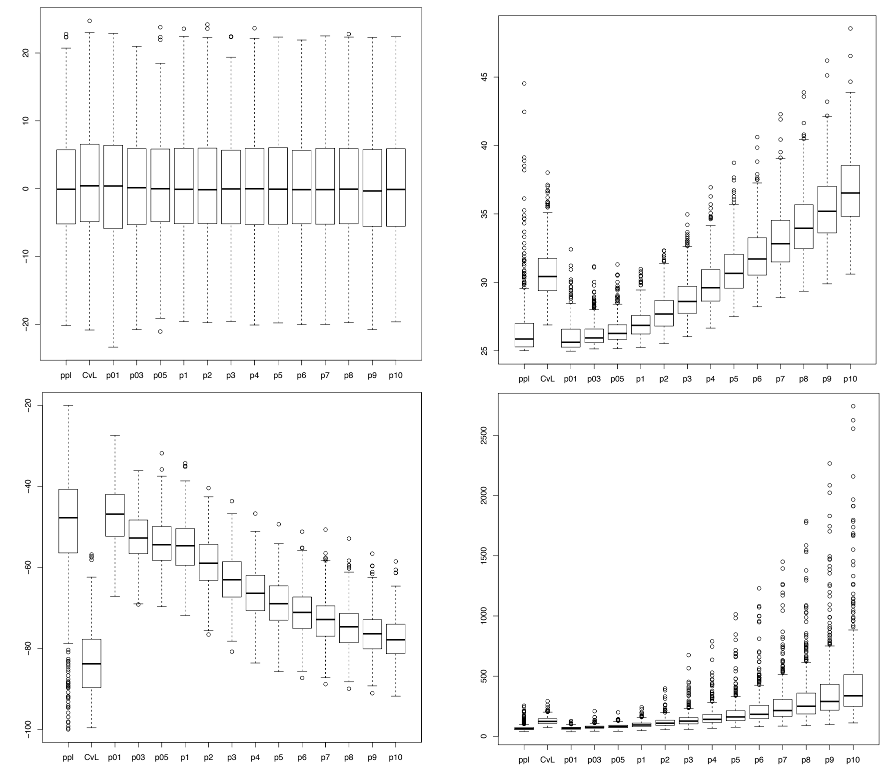

Fig. 6 Box plots for point-wise errors of $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of an inhomogeneous Poisson process $X \subset W = [0,1]^2$ with intensity $\rho(x,y) = |10 + 90\sin(16x)|$. From top-left to bottom-right: average; average absolute; minimum and maximum. x-axis labels from left to right: ppl, CvL, $p = 0.01, 0.03, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

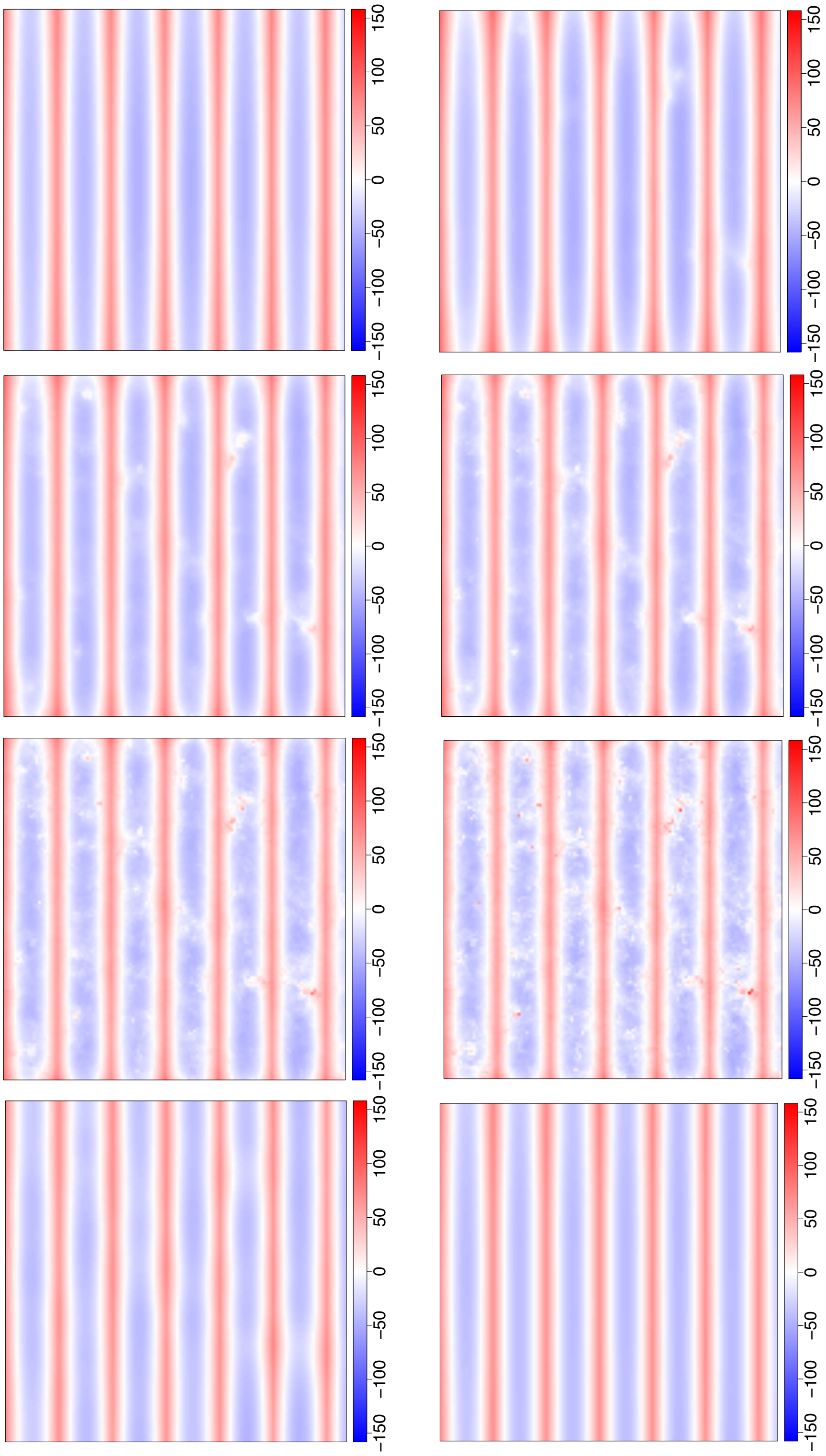

Fig. 7 Estimated bias for $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of a log-Gaussian Cox process $X \subset W = [0,1]^2$ where the driving Gaussian random field has mean function $(x, y) \mapsto \log(40|\sin(20x)|)$ and covariance function $((x_1, y_1), (x_2, y_2)) \mapsto 2\exp\{-\|(x_1, y_1) - (x_2, y_2)\|/0.1\}$. From top-left to bottom-right: $\widehat{\rho}^{V}_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row.

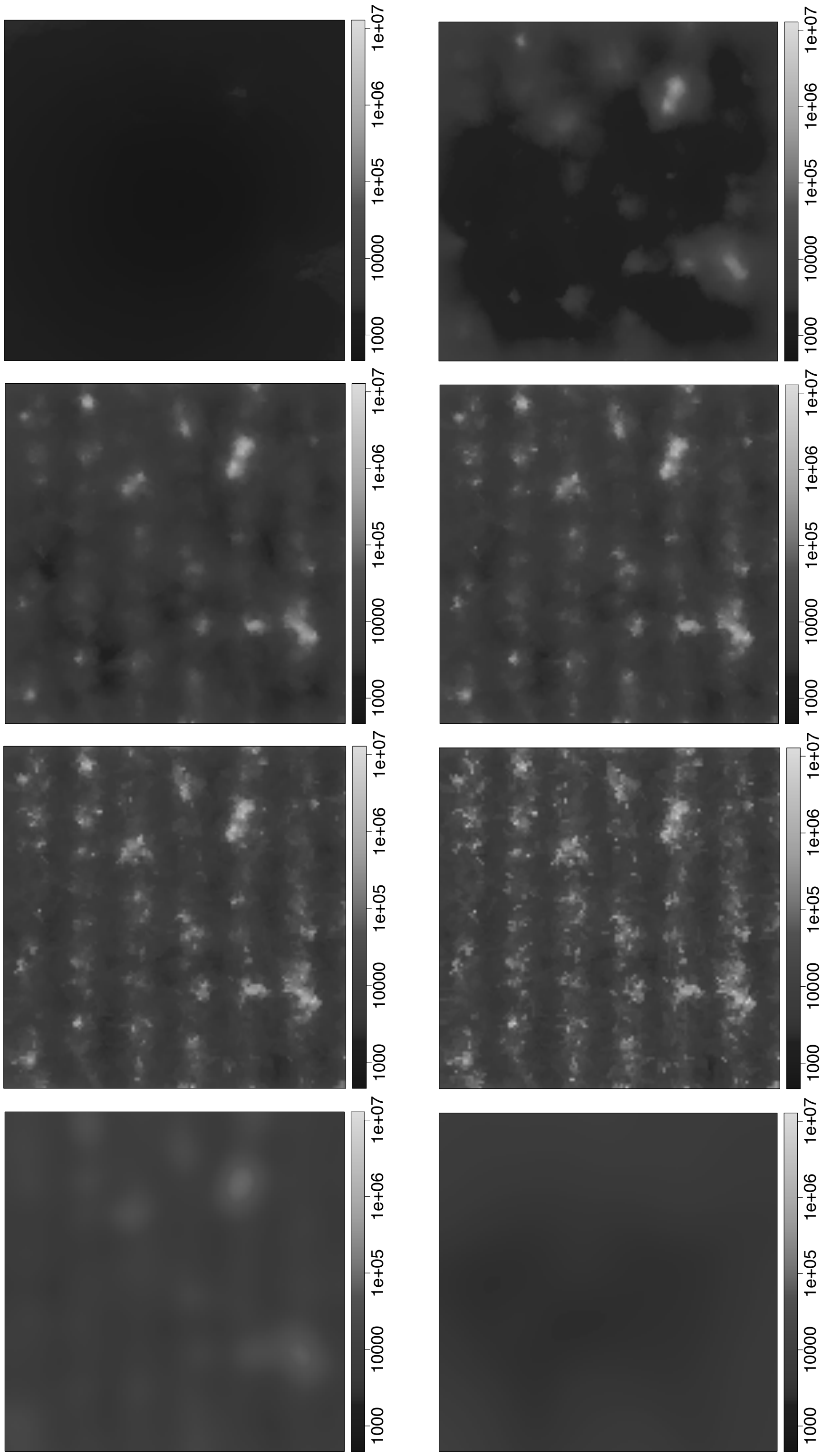

Fig. 8 Estimated variance for $\widehat{\rho}^V_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of a log-Gaussian Cox process $X \subset W = [0,1]^2$ where the driving Gaussian random field has mean function $(x,y) \mapsto \log(40|\sin(20x)|)$ and covariance function $((x_1,y_1),(x_2,y_2)) \mapsto 2\exp\{-\|(x_1,y_1)-(x_2,y_2)\|/0.1\}$. From top-left to bottom-right: $\widehat{\rho}^V_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row. Logarithmic colour map.

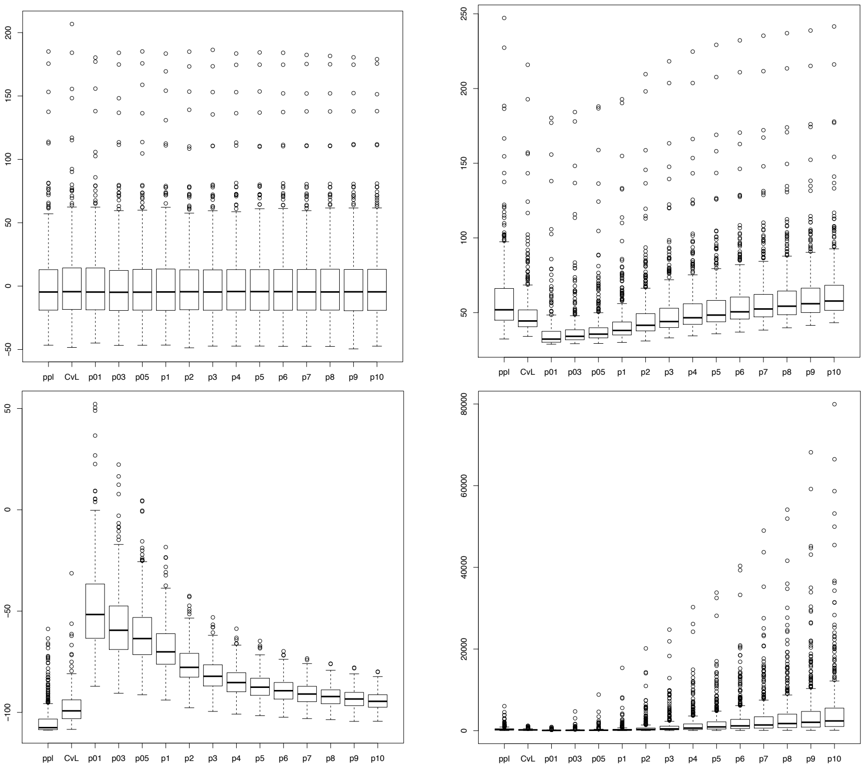

Fig. 9 Box plots for point-wise errors of $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of a log-Gaussian Cox process $X \subset W = [0,1]^2$ where the driving Gaussian random field has mean function $(x, y) \mapsto \log(40|\sin(20x)|)$ and covariance function $((x_1, y_1), (x_2, y_2)) \mapsto 2\exp\{-\|(x_1, y_1) - (x_2, y_2)\|/0.1\}$. From top-left to bottom-right: average; average absolute; minimum and maximum. x-axis labels from left to right: ppl, CvL, $p = 0.01, 0.03, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

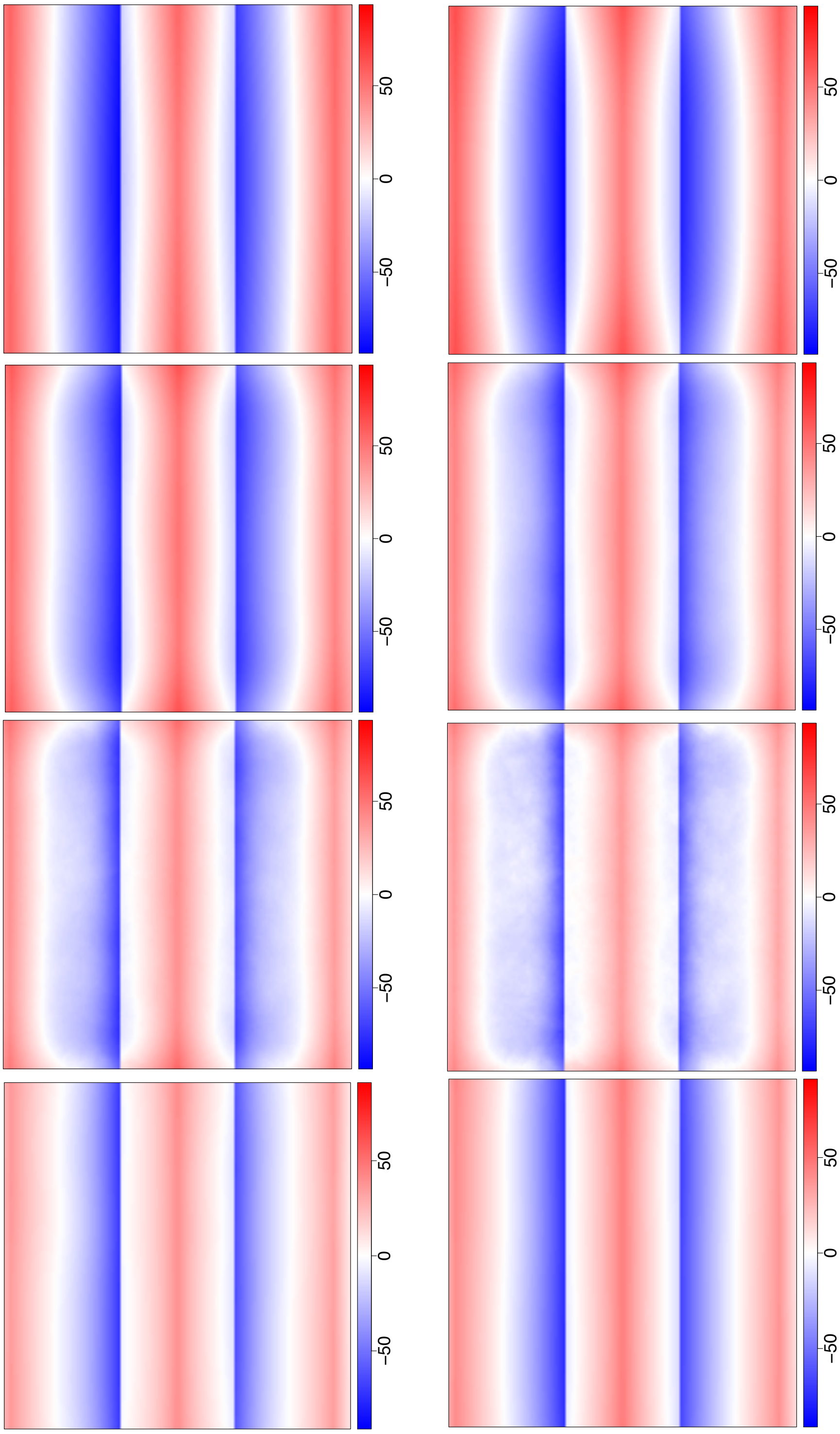

Fig. 10 Estimated bias for $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of an independently thinned simple sequential inhibition process in $W = [0,1]^2$ with intensity $\rho(x,y) = 450p(x,y)$, $p(x,y) = \mathbf{1}\{x < 1/3\}|x - 0.02| + \mathbf{1}\{1/3 \leq x < 2/3\}|x - 0.5| + \mathbf{1}\{x \geq 2/3\}|x - 0.95|$, $x, y \in W$. From top-left to bottom-right: $\widehat{\rho}^{V}_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row.

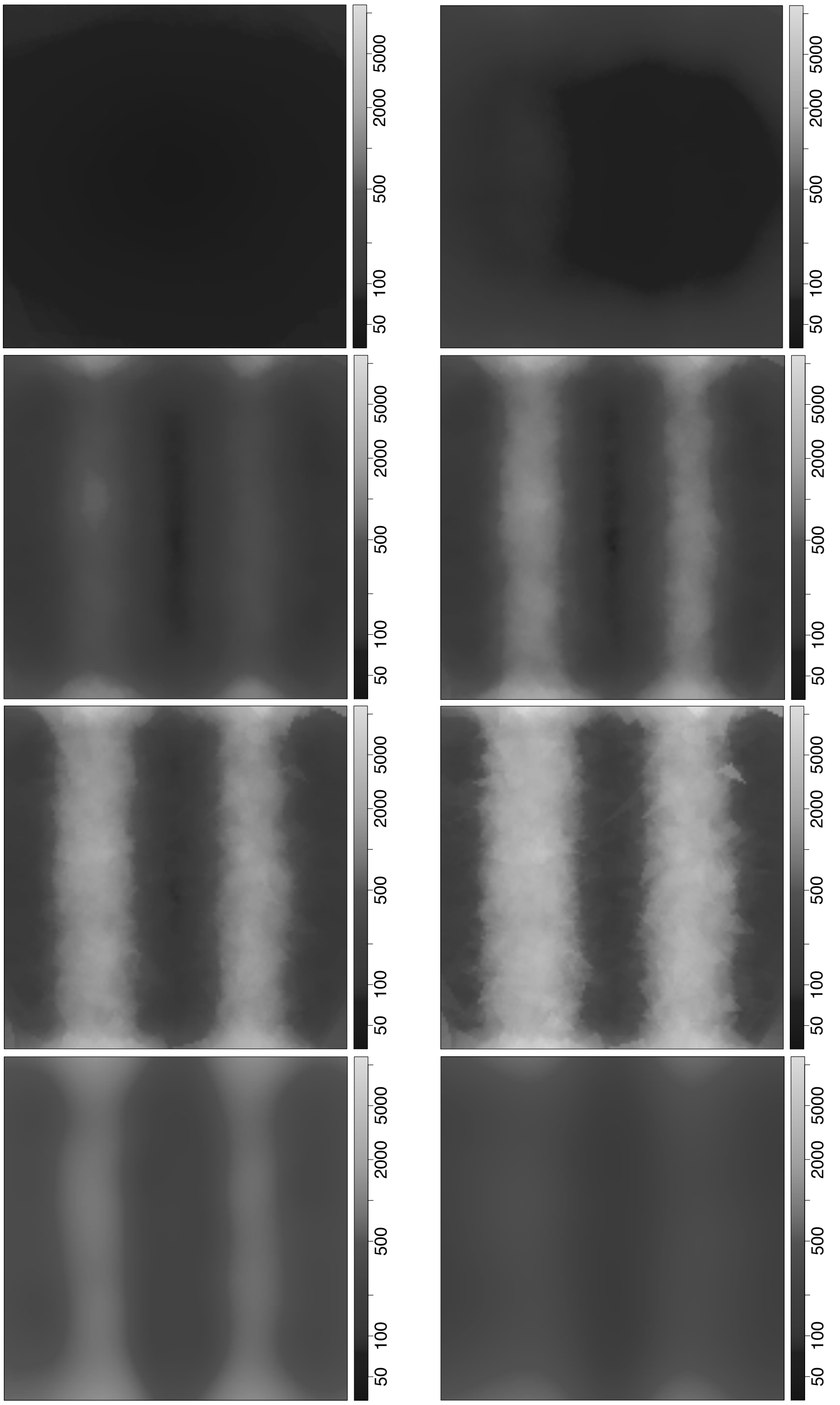

Fig. 11 Estimated variance for $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of an independently thinned simple sequential inhibition process in $W = [0,1]^2$ with intensity $\rho(x,y) = 450p(x,y)$, $p(x,y) = \mathbf{1}\{x < 1/3\}|x - 0.02| + \mathbf{1}\{1/3 \leq x < 2/3\}|x - 0.5| + \mathbf{1}\{x \geq 2/3\}|x - 0.95|$, $x, y \in W$. From top-left to bottom-right: $\widehat{\rho}^{V}_{p,m}(u)$ with $p = 0.01, 0.1, 0.3, 0.5, 0.7, 1$; kernel estimators with bandwidths selected using ppl (left) CvL (right) are on the last row. Logarithmic colour map.

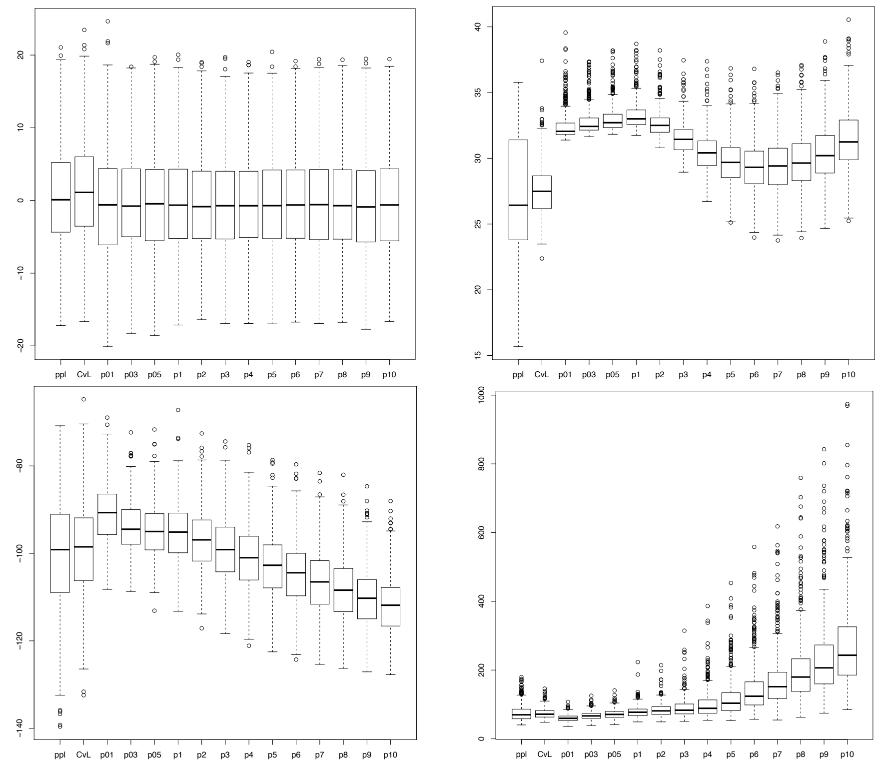

Fig. 12 Box plots for point-wise errors of $\widehat{\rho}^{V}_{p,m}(u)$, $u \in W = [0,1]^2$, $m = 400$, and kernel estimators, based on 500 realisations of an independently thinned simple sequential inhibition process in $W = [0,1]^2$ with intensity $\rho(x,y) = 450p(x,y)$, $p(x,y) = \mathbf{1}\{x < 1/3\}|x - 0.02| + \mathbf{1}\{1/3 \leq x < 2/3\}|x - 0.5| + \mathbf{1}\{x \geq 2/3\}|x - 0.95|$, $x, y \in W$. From top-left to bottom-right: average; average absolute; minimum and maximum. x-axis labels from left to right: ppl, CvL, $p = 0.01, 0.03, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

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