Appendix 1

Theorem 1: On the complex plane, let $\omega=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $D_{\rm o}=\{0,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6\}$, and $D_{\rm e}=\{0,\omega+\omega^2,\omega^2+\omega^3,\omega^3+\omega^4,\omega^4+\omega^5,\omega^5+\omega^6,\omega^6+\omega\}$.

The set of cell centers of A3H grid system is given by

$$\mathbb{L}_n=\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\frac{1}{3^i}D_{\rm o}+\sum_{j=2}^{\left\lceil\frac{n+1}{2}\right\rceil}\frac{1}{3^j}D_{\rm e}\quad(n\geqslant 1),$$

where $\sum$ and + denote accumulation among sets, $\lfloor\ \rfloor$ is the operator for rounding down, and $\lceil\ \rceil$ is the operator for rounding up.

Proof: Let r_{n-1} denote the radius of the circumcircle of the n−1th level and $\mathbb{L}_n$ be the set of cell centers at the nth level. According to the A3H grid system generation rule, $\mathbb{L}_n$ can be expressed as

$$\mathbb{L}_n=\begin{cases}\mathbb{L}_{n-1}+r_{n-1}D_{\rm o} & n=2k-1\\ \mathbb{L}_{n-1}+\frac{1}{\sqrt{3}}r_{n-1}D_{\rm e} & n=2k\end{cases}\quad(k=1,2,3,\cdots).\tag{a1}$$

Two cases are discussed as follows.

1° If n=2k, in terms of eq. (a1), then cell center sets have recursive relations between odd levels and even levels as follows:

$$\begin{aligned}\mathbb{L}_n&=\mathbb{L}_{2k}\\&=\mathbb{L}_{2k-1}+\frac{1}{\sqrt{3}}r_{2k-1}D_{\rm e}\\&=\left(\mathbb{L}_{2k-2}+r_{2k-2}D_{\rm o}\right)+\frac{1}{\sqrt{3}}r_{2k-1}D_{\rm e}\\&=\left(\mathbb{L}_{2k-3}+\frac{1}{\sqrt{3}}r_{2k-3}D_{\rm e}\right)+r_{2k-2}D_o+\frac{1}{\sqrt{3}}r_{2k-1}D_{\rm e}\\&=\left(\mathbb{L}_{2k-4}+r_{2k-4}D_{\rm o}\right)+\frac{1}{\sqrt{3}}r_{2k-3}D_{\rm e}+r_{2k-2}D_o+\frac{1}{\sqrt{3}}r_{2k-1}D_{\rm e}\\&\vdots\\&=\mathbb{L}_1+\left(r_2D_{\rm o}+r_4D_{\rm o}+\cdots+r_{2k-2}D_{\rm o}\right)+\frac{1}{\sqrt{3}}\left(r_1D_{\rm e}+r_3D_{\rm e}+\cdots+r_{2k-1}D_{\rm e}\right)\\&=\mathbb{L}_1+\sum_{i=1}^{k-1}r_{2i}D_{\rm o}+\frac{1}{\sqrt{3}}\sum_{j=1}^{k}r_{2j-1}D_{\rm e},\end{aligned}$$

where $\sum$ represents accumulation among sets and + is the addition operator.

In addition, since $\frac{r_{i-1}}{r_i}=\sqrt{3}$,

$$r_i=\frac{1}{\sqrt{3}}r_{i-1}=\left(\frac{1}{\sqrt{3}}\right)^2r_{i-2}=\left(\frac{1}{\sqrt{3}}\right)^3r_{i-3}=\cdots=\left(\frac{1}{\sqrt{3}}\right)^{i-2}r_2=\left(\frac{1}{\sqrt{3}}\right)^{i-1}r_1,$$

so

$$\mathbb{L}_n=\mathbb{L}_1+\sum_{i=1}^{k-1} r_{2i}D_{\mathrm{o}}+\frac{1}{\sqrt{3}}\sum_{j=1}^{k} r_{2j-1}D_{\mathrm{e}}=\mathbb{L}_1+\sqrt{3}r_1\sum_{i=1}^{k-1}\frac{1}{3^i}D_{\mathrm{o}}+\sqrt{3}r_1\sum_{j=1}^{k}\frac{1}{3^j}D_{\mathrm{e}}.$$

Let $\mathbb{L}_1=\frac{1}{3}D_{\mathrm{o}}$; thus, $r_1=\frac{1}{3\sqrt{3}}$, and now

$$\mathbb{L}_n=\frac{1}{3}D_{\mathrm{o}}+\frac{1}{3}\sum_{i=1}^{k-1}\frac{1}{3^i}D_{\mathrm{o}}+\frac{1}{3}\sum_{j=1}^{k}\frac{1}{3^j}D_{\mathrm{e}}=\frac{1}{3}D_{\mathrm{o}}+\sum_{i=1}^{k-1}\frac{1}{3^{i+1}}D_{\mathrm{o}}+\sum_{j=1}^{k}\frac{1}{3^{j+1}}D_{\mathrm{e}}=\sum_{i=1}^{k}\frac{1}{3^i}D_{\mathrm{o}}+\sum_{j=2}^{k+1}\frac{1}{3^j}D_{\mathrm{e}}. \tag{a2}$$

Owing to n=2k, it is easy to see that $\left\lfloor\frac{n+1}{2}\right\rfloor=\left\lfloor k+\frac{1}{2}\right\rfloor=k$ and $\left\lceil\frac{n+1}{2}\right\rceil=\left\lceil k+\frac{1}{2}\right\rceil=k+1$, hence eq. (a2) is equivalent to

$$\mathbb{L}_n=\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\frac{1}{3^i}D_{\mathrm{o}}+\sum_{j=2}^{\left\lceil\frac{n+1}{2}\right\rceil}\frac{1}{3^j}D_{\mathrm{e}}\ (n=2k,k=1,2,3,\cdots).$$

2° If n=2k−1, it can likewise be obtained that

$$\mathbb{L}_n=\mathbb{L}_{2k-1}=\sum_{i=1}^{k}\frac{1}{3^i}D_{\mathrm{o}}+\sum_{j=2}^{k}\frac{1}{3^j}D_{\mathrm{e}}. \tag{a3}$$

Since n=2k−1, then $\left\lfloor\frac{n+1}{2}\right\rfloor=\lfloor k\rfloor=k$ and $\left\lceil\frac{n+1}{2}\right\rceil=\lceil k\rceil=k$; thus, eq. (a3) is equivalent to

$$\mathbb{L}_n=\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\frac{1}{3^i}D_{\mathrm{o}}+\sum_{j=2}^{\left\lceil\frac{n+1}{2}\right\rceil}\frac{1}{3^j}D_{\mathrm{e}}\ (n=2k-1,k=1,2,3,\cdots).$$

Therefore, Theorem 1 holds whether n is even or odd.

Theorem 2: Let $\mathbb{P}_0=\{0\}$ and $\mathbb{P}_1=\frac{1}{3}D_{\mathrm{o}}^*\cup\{0\}$. A portion of cell center set $\mathbb{L}_n$ can be expressed as

$$\mathbb{P}_n=\begin{cases}\mathbb{P}_{n-1}\cup\left(\mathbb{P}_{n-2}+\dfrac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^*\right) & n=2k-1\\ \mathbb{P}_{n-1}\cup\left(\mathbb{P}_{n-2}+\dfrac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^*\right) & n=2k\end{cases}\quad(k=1,2,3,\cdots).$$

In addition, there exists unique and definite

$$\mathrm{e}_i\in\begin{cases}\dfrac{1}{3^{\frac{i+1}{2}}}D_{\mathrm{o}}^*\cup\{0\} & i=2k-1\\ \dfrac{1}{3^{\frac{i+2}{2}}}D_{\mathrm{e}}^*\cup\{0\} & i=2k\end{cases}\quad(k=1,2,3,\cdots)$$

to let $\sum_{i=1}^{n}\mathrm{e}_i$ denote $\beta\in\mathbb{P}_n$, where $\mathrm{e}_i=0$ or $\mathrm{e}_{i+1}=0$ $(1\leqslant i\leqslant n-1)$.

Proof: According to eq. (2), odd levels of A3H grid system establish a radix system with base 3 and digit set D_{o}, and even levels of A3H grid system establish a radix system with base 3 and digit set D_{e}. The aperture of the two radix systems is 9 for both, and their directions are both constant. Let $\mathbb{L}_{n-2}(n\geqslant 2)$ be β_{n-2}. Under the original condition of the grid cells in Theorem 1, new generated cell centers of β_{n-2} at $\mathbb{L}_n$ are computed as follows:

$$\beta_{n,k}=\begin{cases}\beta_{n-2}+\dfrac{1}{3^{\frac{n+1}{2}}}\omega^k=\beta_{n-2}+\dfrac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^* & n=1,3,5,\cdots\\ \beta_{n-2}+\dfrac{1}{3^{\frac{n+2}{2}}}(\omega^k+\omega^{k+1})=\beta_{n-2}+\dfrac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^* & n=0,2,4,\cdots\end{cases}\quad(k=1,2,\cdots,6). \tag{a4}$$

Eq. (a4) is obtained by recursion between interval levels (i.e., the top line or bottom line of eq. (2)). Compared with eq. (1)

using neighboring levels, it has two main advantages. First, only cell centers at $\mathbb{L}_{n-2}$ can generate new cell centers at the nth level; hence, sets of cell centers at every level are subsets of $\mathbb{L}_n$, and the numbers of cell centers are greatly decreased. Second, grid cells corresponding to $\beta_{n,k}$ are always inside the grid cells corresponding to β_{n-2} and never exceed the cell edge of the first level, which facilitates extension onto the sphere.

Define the set of cell centers at the nth ($n\geqslant 0$) level generated by the method above to be $\mathbb{P}_n$. Taking the nested relation of sets of cell centers into account, $\mathbb{P}_n$ comprises new cell centers generated by $\mathbb{P}_{n-1}$ and $\mathbb{P}_{n-2}$ at $\mathbb{P}_n$. By known conditions $\mathbb{P}_0=\mathbb{L}_0=\{0\}$ and $\mathbb{P}_1=\mathbb{L}_1=\frac{1}{3}D_{\mathrm{o}}^{*}\cup\{0\}$, according to eq. (a4), $\mathbb{P}_n$ is given by

$$\mathbb{P}_n=\begin{cases}\mathbb{P}_{n-1}\cup\left(\mathbb{P}_{n-2}+\dfrac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^{*}\right) & n=2k-1\\ \mathbb{P}_{n-1}\cup\left(\mathbb{P}_{n-2}+\dfrac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^{*}\right) & n=2k\end{cases}\quad (k=1,2,3,\cdots). \tag{a5}$$

Expanding eq. (a5) recursively, the direct expression of $\mathbb{P}_n$ is obtained as

$$\mathbb{P}_n=\begin{cases}\mathbb{P}_0\cup\mathbb{P}_1\cup\left(\mathbb{P}_0+\dfrac{1}{3^2}D_{\mathrm{e}}^{*}\right)\cup\left(\mathbb{P}_1+\dfrac{1}{3^2}D_{\mathrm{o}}^{*}\right)\cup\cdots\cup\left(\mathbb{P}_{n-2}+\dfrac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^{*}\right) & n=2k-1,\\ \mathbb{P}_0\cup\mathbb{P}_1\cup\left(\mathbb{P}_0+\dfrac{1}{3^2}D_{\mathrm{e}}^{*}\right)\cup\left(\mathbb{P}_1+\dfrac{1}{3^2}D_{\mathrm{o}}^{*}\right)\cup\cdots\cup\left(\mathbb{P}_{n-2}+\dfrac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^{*}\right) & n=2k.\end{cases} \tag{a6}$$

Eq. (a6) shows that cell centers at $\mathbb{P}_n$ can be denoted in turn by the accumulation of $\frac{1}{3}D_{\mathrm{o}}^{*}\cup\{0\}$, $\frac{1}{3^2}D_{\mathrm{e}}^{*}\cup\{0\}$, $\frac{1}{3^3}D_{\mathrm{o}}^{*}\cup\{0\}$, …, $\frac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^{*}\cup\{0\}$ ($\frac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^{*}\cup\{0\}$). That is,

$$\mathrm{e}_i\in\begin{cases}\dfrac{1}{3^{\frac{i+1}{2}}}D_{\mathrm{o}}^{*}\cup\{0\} & i=2k-1\\ \dfrac{1}{3^{\frac{i+2}{2}}}D_{\mathrm{e}}^{*}\cup\{0\} & i=2k\end{cases}\quad (k=1,2,3,\cdots)$$

exists, which makes $\beta=\sum_{i=1}^{n}\mathrm{e}_i\in\mathbb{P}_n$.

Similarly, let S_n be the set of all cell centers satisfying $\mathrm{e}_i=0$ or $\mathrm{e}_{i+1}=0$, and let it be of the form $\beta=\sum_{i=1}^{n}\mathrm{e}_i$. According to eq. (a5),

$$S_n=\begin{cases}S_{n-1}\cup\left(S_{n-2}+\dfrac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^{*}\right) & n=2k-1\\ S_{n-1}\cup\left(S_{n-2}+\dfrac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^{*}\right) & n=2k\end{cases}\quad (k=1,2,3,\cdots).$$

Here, S_n comprises S_{n-1} together with $S_{n-2}+\frac{1}{3^{\frac{n+1}{2}}}D_{\mathrm{o}}^{*}$ or $S_{n-2}+\frac{1}{3^{\frac{n+2}{2}}}D_{\mathrm{e}}^{*}$. Cell centers at S_{n-1} can be denoted by $\beta=\sum_{i=1}^{n-1}\mathrm{e}_i$, and in terms of the form of S_n, can also be denoted by $\beta=\sum_{i=1}^{n}\mathrm{e}_i=\sum_{i=1}^{n-1}\mathrm{e}_i+\mathrm{e}_n$, hence $\mathrm{e}_n=0$. Similarly, cell centers at $S_{n-2}+\frac{1}{3^{\frac{n+1}{2}}}D_{\rm o}^{*}$ or $S_{n-2}+\frac{1}{3^{\frac{n+2}{2}}}D_{\rm e}^{*}$ can be denoted by $\beta=\sum_{i=1}^{n-2}{\rm e}_i+{\rm e}_n$, where ${\rm e}_n\in\frac{1}{3^{\frac{n+1}{2}}}D_{\rm o}^{*}\cup\{0\}$ or ${\rm e}_n\in\frac{1}{3^{\frac{n+2}{2}}}D_{\rm e}^{*}\cup\{0\}$, and in the form of S_n, $\beta=\sum_{i=1}^{n}{\rm e}_i=\sum_{i=1}^{n-2}{\rm e}_i+{\rm e}_{n-1}+{\rm e}_n$, hence ${\rm e}_{n-1}=0$. It is easy to verify that $S_n=\mathbb{P}_n$ when *n*=0,1, and, according to the recursive relation, for all *n*, $S_n=\mathbb{P}_n$.

Now we prove the uniqueness of representation of β. For *n*=0,1, $S_n=\mathbb{P}_n$ the uniqueness is easy to verify. Suppose that the representation is not unique, that is, that $\sum_{i=1}^{n}{\rm e}_i=\sum_{i=1}^{n}{\rm e}_i'$. If ${\rm e}_n={\rm e}_n'$, then $\sum_{i=1}^{n-1}{\rm e}_i=\sum_{i=1}^{n-1}{\rm e}_i'$, and hence ${\rm e}_i={\rm e}_i'$. If ${\rm e}_n\neq{\rm e}_n'$, two cases exist. The first case is that either ${\rm e}_n$ or ${\rm e}_n'$ is equal to 0. Let ${\rm e}_n'=0$, so ${\rm e}_n=\sum_{i=1}^{n-1}({\rm e}_i'-{\rm e}_i)\in\mathbb{P}_{n-1}$ because $\sum_{i=1}^{n}{\rm e}_i=\sum_{i=1}^{n}{\rm e}_i'$, which contradicts ${\rm e}_n\notin\mathbb{P}_{n-1}$. The second case is that neither ${\rm e}_n$ nor ${\rm e}_n'$ is equal to 0. Because ${\rm e}_i=0$ or ${\rm e}_{i-1}=0$, ${\rm e}_{n-1}=0$. The form of cell center becomes $S_{n-2}+\frac{1}{3^{\frac{n+1}{2}}}D_{\rm o}^{*}$ or $S_{n-2}+\frac{1}{3^{\frac{n+2}{2}}}D_{\rm e}^{*}$, so ${\rm e}_n-{\rm e}_n'=\sum_{i=1}^{n-2}({\rm e}_i'-{\rm e}_i)\in\mathbb{P}_{n-2}$; this contradicts ${\rm e}_n-{\rm e}_n'\notin\mathbb{P}_{n-2}$. The two cases above show that the representation $\beta=\sum_{i=1}^{n}{\rm e}_i$ is unique.

Therefore, Theorem 2 holds.