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• **Supplementary File** •

Pulse controllability of singular distributed parameter systems

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Appendix A Proof of Theorem 2

Proof. Let $X_2(s)=\int_0^{+\infty}e^{-st}x_2(t)dt$ and $U(s)=\int_0^{+\infty}e^{-st}u(t)dt$ be the Laplace transforms of $x_2(t)$ and $u(t)$ respectively, where the integrals are in the sense of Bochner, and take the Laplace transforms [1] on both sides of system (5), we obtain

$$X_2(s)=(sN-I_2)^{-1}Nx_2(0)+(sN-I_2)^{-1}B_2U(s). \tag{A1}$$

Since $(sN-I_2)^{-1}=-\sum\limits_{i=0}^{h-1}N^is^i$, we have, from (A1)

$$X_2(s)=-\sum_{i=0}^{h-1}N^{i+1}s^ix_2(0)-\sum_{i=0}^{h-1}N^is^iB_2U(s). \tag{A2}$$

Taking the inverse Laplace transforms on both sides of (A2) gives

$$\begin{aligned}x_2(t)&=-\sum_{i=1}^{h-1}N^i\delta^{(i-1)}(t)x_2(0)-\sum_{i=1}^{h-1}N^i\sum_{j=0}^{i-1}\delta^{(j)}(t)B_2u^{(i-j-1)}(0)-\sum_{i=0}^{h-1}N^iB_2u^{(i)}(t)\\&=-\sum_{i=1}^{h-1}N^i[\delta^{(i-1)}(t)x_2(0)+\sum_{j=0}^{i-1}\delta^{(j)}(t)B_2u^{(i-j-1)}(0)]-\sum_{i=0}^{h-1}N^iB_2u^{(i)}(t),\end{aligned} \tag{A3}$$

which can be arranged into the forms of

$$\begin{cases}x_2=x_{2\text{pulse}}(t)+x_{2\text{normal}}(t),\\x_{2\text{pulse}}(t)=-\sum\limits_{i=1}^{h-1}N^i[\delta^{(i-1)}(t)x_2(0)+\sum\limits_{j=0}^{i-1}\delta^{(j)}(t)B_2u^{(i-j-1)}(0)],\\x_{2\text{normal}}(t)=-\sum\limits_{i=0}^{h-1}N^iB_2u^{(i)}(t).\end{cases} \tag{A4}$$

Furthermore, exchanging the order of the double sum and noting that $N^h=0$ in (A3), we have

$$\begin{aligned}&\sum_{i=1}^{h-1}N^iB_2\sum_{j=0}^{i-1}\delta^{(j)}(t)u^{(i-j-1)}(0)=\sum_{i=0}^{h-2}\delta^{(i)}(t)[\sum_{k=i+1}^{h-1}N^kB_2u^{(k-i-1)}(0)]=\sum_{i=0}^{h-2}\delta^{(i)}(t)N^i[\sum_{k=i+1}^{h-1}N^{k-i}B_2u^{(k-i-1)}(0)]\\&=\sum_{i=0}^{h-2}\delta^{(i)}(t)N^i[\sum_{l=1}^{h-i-1}N^lB_2u^{(l-1)}(0)]=\sum_{i=0}^{h-2}\delta^{(i)}(t)N^i[\sum_{l=1}^{h-i-1}N^lB_2u^{(l-1)}(0)+\sum_{l=h-i;i\neq0}^{h-1}N^lB_2u^{(l-1)}(0)]\\&=\sum_{i=0}^{h-2}\delta^{(i)}(t)N^i[\sum_{l=1}^{h-1}N^lB_2u^{(l-1)}(0)]=\sum_{i=0}^{h-2}\delta^{(i)}(t)N^i[\sum_{l=1}^{h}N^lB_2u^{(l-1)}(0)]\\&=\sum_{i=0}^{h-2}\delta^{(i)}(t)N^{i+1}[\sum_{l=1}^{h-1}N^{l-1}B_2u^{(l-1)}(0)]=\sum_{i=1}^{h-1}\delta^{(i-1)}(t)N^i[\sum_{i=0}^{h-1}N^iB_2u^{(i)}(0)].\end{aligned} \tag{A5}$$

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Combing (A3), (A4) and (A5), the following result can be immediately obtained,

$$x_2(t) = x_{2\text{pulse}}(t) + x_{2\text{normal}}(t), \tag{A6}$$

where

$$x_{2\text{pulse}}(t) = -\sum_{i=1}^{h-1} N^i \delta^{(i-1)}(t)[x_{20} + \sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0)],$$

$$x_{2\text{normal}}(t) = -\sum_{i=0}^{h-1} N^i B_2 u^{(i)}(t).$$

By (A4) and (A6), (9) holds. The proof is complete.

Appendix B Proof of Theorem 4

Proof. According to Theorem 3, the solution of the system (2) is given by

$$x(t) = Q \begin{bmatrix} e^{Kt}[I_1 \quad 0]Q^{-1}x_0 + \int_0^t e^{K(t-\tau)} B_1 u(\tau) d\tau \\ -\sum\limits_{i=1}^{h-1} N^i \delta^{(i-1)}(t)[[0 \quad I_2]Q^{-1}x_0 + \sum\limits_{i=0}^{h-1} N^i B_2 u^{(i)}(0)] - \sum\limits_{i=0}^{h-1} N^i B_2 u^{(i)}(t) \end{bmatrix}.$$

Since the solution of system (4) does not contain generalized function, it suffices to consider the solution of the system (5). For the solution of the system (5) corresponding to the initial value $x(0) = x_0$, by letting $x_{20} = [0 \quad I_2]Q^{-1}x_0$, we have

$$x_2(t) = -\sum_{i=1}^{h-1} N^i \delta^{(i-1)}(t)[x_{20} + \sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0)] - \sum_{i=0}^{h-1} N^i B_2 u^{(i)}(t). \tag{B1}$$

Letting $t = 0$ in the above equation gives

$$x_{20}(0) = -\sum_{i=1}^{h-1} N^i \delta^{(i-1)}(0)[x_{20} + \sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0)] - \sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0).$$

When $N \neq 0$, in view of the independency of functions $\delta^{(i)}(t)$, it can be easily observed that for an arbitrary finite value x_{20}, the above equation holds if and only if

$$x_{20}(0) = -\sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0). \tag{B2}$$

When $N = 0$, the above relation becomes

$$x_{20}(0) = -\sum_{i=0}^{h-1} N^i B_2 u^{(i)}(0).$$

which is also in the form of (B2). Therefore, the set of consistent initial conditions is given by S.

Substituting (B2) into (B1) yields

$$x_2(t) = -\sum_{i=0}^{h-1} N^i B_2 u^{(i)}(t).$$

Thus, the classical solution is obtained as in Theorem 4. The proof is complete.

Appendix C Proof of Theorem 5

Proof. According to the definition of PC and (9), the RSDPS (4)-(5) with finite order is pulse controllable if and only if for any initial value vector $x_{20} \in X_2$ there exists an admissible control input vector $u(t) \in C^h$ such that

$$\sum_{i=1}^{h-1} \delta^{(i-1)}(t)[N^i x_{20} + \sum_{k=i}^{h-1} N^k B_2 u^{(k-i)}(0)] = 0. \tag{C1}$$

Since $\delta^{(i)}(t), i = 0, 1, 2, \ldots, h-2,$are linear independent, the equation (C1) is equivalent to

$$N^i x_{20} + \sum_{k=i}^{h-1} N^k B_2 u^{(k-i)}(0) = 0, i = 0, 1, 2, \ldots, h-1. \tag{C2}$$

Moreover, note the nilpotent property of the operator N, it is easy to prove that equations (C2) is equivalent to equation

$$N x_{20} + \sum_{k=0}^{h-2} N^{k+1} B_2 u^k(0) = 0. \tag{C3}$$

Therefore, equation (C3) is equivalent to equation (C1).

In order to complete the proof, we now need only to show that given the values $u^{(k)}(0) = u_0^{(k)}, k = 0, 1, 2, \ldots, h-1$, satisfying (C3), there exists an admissible control input vector $v(t) \in C^h$ such that

$$u^{(k)}(0) = u_0^{(k)}, k = 0, 1, 2, \ldots, h-1 \tag{C4}$$

In fact, let $v(t) = \sum\limits_{k=0}^{h-1} \frac{1}{k!} u_0^{(k)} t^k$. Then $v(t)$ is a polynomial satisfying (C4). This complete the proof.

Appendix D Proof of Theorem 6

In order to prove Theorem 6, first of all, we need to prove the following lemma.

Lemma D Let $F \in L(X_2)$ and $G \in L(U, X_2)$. Then $\mathrm{ran}FG = \mathrm{ran}F$ if and only if $\ker F + \mathrm{ran}G = X_2$.

Proof. Necessity: Since $\mathrm{ran}FG = \mathrm{ran}F$, for any vector $x \in X_2$, there exists vector $y \in U$ such that $Fx = FGy$, which is equivalent to $F(x - Gy) = 0$. Thus $x - Gy \in \ker F$. Noting that $Gy \in \mathrm{ran}G$, we obtain $x = (x - Gy) + Gy \in \ker F + \mathrm{ran}G$. This implies that $\ker F + \mathrm{ran}G = X_2$.

Sufficiency: If $\ker F + \mathrm{ran}G = X_2$, then, for arbitrary vector $x \in X_2$ there exists vector $y \in U$, such that $x - Gy \in \ker F$. This implies that, for any $x \in X_2$, there exists $y \in U$ such that $Fx = FGy$, i.e.,$\mathrm{ran}FG = \mathrm{ran}F$.

Proof of Theorem 6. Conclusion (i) is obvious. Here, we only prove the conclusion (ii).

Proof of (A). According to Theorem 5, the subsystem (5) is pulse controllable if and only if for any initial value vector $x_{20} \in X_2$, there exists an admissible control input vector $u \in C^h$ such that (C3) holds. This is actually equivalent to condition (A).

Proof of the equivalence for (A)and (B). Since

$$\begin{aligned}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2 & 0\end{bmatrix} &= \begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2 & N^hB_2\end{bmatrix}\\ &= N\begin{bmatrix}B_2 & NB_2 & \cdots & N^{h-2}B_2 & N^{h-1}B_2\end{bmatrix},\end{aligned}$$

and

$$\mathrm{ran}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2\end{bmatrix} = \mathrm{ran}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2 & 0\end{bmatrix},$$

we have

$$\mathrm{ran}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2\end{bmatrix} = \mathrm{ran}N\begin{bmatrix}B_2 & NB_2 & \cdots & N^{h-1}B_2\end{bmatrix}.$$

Thus, according to the above relation and Lemma D, we have that

$$\mathrm{ran}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2\end{bmatrix} = \mathrm{ran}N\begin{bmatrix}B_2 & NB_2 & \cdots & N^{h-1}B_2\end{bmatrix} = \mathrm{ran}N$$

if and only if

$$\ker N + \mathrm{ran}\begin{bmatrix}B_2 & NB_2 & \cdots & N^{h-1}B_2\end{bmatrix} = X_2.$$

Therefore (A) and (B) are equivalent.

Proof of the equivalence for (B)and (C). Since

$$\mathrm{ran}\begin{bmatrix}B_2 & NB_2 & \cdots & N^{h-1}B_2\end{bmatrix} = \mathrm{ran}B_2 + \mathrm{ran}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2\end{bmatrix},$$

the condition (B) can be written as

$$\ker N + \mathrm{ran}B_2 + \mathrm{ran}\begin{bmatrix}NB_2 & N^2B_2 & \cdots & N^{h-1}B_2\end{bmatrix} = X_2.$$

Moreover, by using condition (A), we can obtain that condition (B) is equivalent to condition (C). The proof is complete.

Appendix E An illustrative example

In the following, an illustrative example is given, which shows the effectiveness of Theorem 6.

Consider the linear Navier-Stokes equations

$$x_t(\xi, t) = \mu\triangle x(\xi, t) - y(\xi, t) + u(\xi, t), (\xi, t) \in \Omega \times [0, \infty), \tag{E1}$$

boundary condition,

$$x(\xi, t) = 0, (\xi, t) \in \partial\Omega \times [0, \infty), \tag{E2}$$

initial condition,

$$x(\xi, 0) = x_0(\xi), \xi \in \Omega, \tag{E3}$$

$$\nabla \cdot x(\xi, t) = 0, (\xi, t) \in \Omega \times [0, \infty), \tag{E4}$$

where $\mu > 0, \triangle$ is the Laplace operator, $\Omega \subset R^n$ is a bounded domain with boundary $\partial\Omega$ of class C^∞ [2],$y(\xi, t) = \nabla p$ is the pressure gradient, and ∇ is the vector differential operator.

We denote $H^2(\Omega) = \{g : g \in L^2(\Omega), D^\alpha g \in L^2(\Omega), |\alpha| \leqslant 2\}$, where $L^2(\Omega)$ denotes the set of all Lebesgue measurable functions, for any $g \in L^2(\Omega), \int_\Omega \|g(\xi)\|^2 d\xi < \infty; D^\alpha$ has the same sense as in [2].Let

$$\mathrm{H}^2 = (H^2(\Omega))^n, \mathrm{H}^2_0 = \{w : w \in \mathrm{H}^2, w(\xi) = 0, \xi \in \partial\Omega\},$$

$$\mathrm{L}^2 = (L^2(\Omega))^n, \mathrm{L} = \{w : w \in (C_0^\infty(\Omega))^n, \nabla \cdot w = 0\},$$

where $C_0^\infty(\Omega)$ has the same sense in [2],H_σ denotes the closure of the subspace L with respect the norm of the space L^2. This is a Hilbert space with the inner product of the space L^2.L^2 can be decomposed as direct sum $\mathrm{H}_\sigma \bigoplus \mathrm{H}_\pi$, where H_π is the orthogonal complement of H_σ. Let $P_\pi : \mathrm{L}^2 \to \mathrm{H}_\pi$ denote the orthogonal projection corresponding to this decomposition. The restriction of P_π to the space $\mathrm{H}^2_0 \subset \mathrm{L}^2$ is a continuous operator $P_0 : \mathrm{H}^2_0 \to \mathrm{H}^2_0$. Therefore, H^2_0 is the direct sum $\mathrm{H}^2_\sigma \bigoplus \mathrm{H}^2_\pi$, and H^2_σ and H^2_π are dense in H_σ and H_π respectively, where $\mathrm{H}^2_\sigma = \ker P_0, \mathrm{H}^2_\pi = \mathrm{ran}P_0$.

We replace (E4) with a more general equation (E5):

$$P_\pi x(\xi, t) = 0, (\xi, t) \in \partial\Omega \times [0, \infty). \tag{E5}$$

Indeed, if $x(\xi,t)$ is sufficiently smooth, then $P_\pi x(\xi,t)=0$ implies (E4). Otherwise, by (E5),$x(\xi,t)$ is the limit in L^2 of smooth functions satisfying condition (E4).

It is easy to observe that the formula $B=\mathrm{diag}[\triangle,\triangle,\cdots,\triangle]$ determines a bounded linear operator $B:\mathrm{H}_0^2\to\mathrm{L}^2$ with discrete spectrum $\sigma(B)$; this spectrum has finite multiplicity and condenses only at $-\infty$.

Let $B_\sigma=B|_{\mathrm{H}_\sigma^2}, B_\pi=B|_{\mathrm{H}_\pi^2}, \Sigma=I-P_\pi$, then $B_\sigma\in L(\mathrm{H}_\sigma^2,\mathrm{H}_\sigma), B_\pi\in L(\mathrm{H}_\pi^2,\mathrm{H}_\pi)$; let

$$X=\mathrm{H}_\sigma\times\mathrm{H}_\pi\times\mathrm{H}_y, \mathrm{H}_\pi=\mathrm{H}_y, Z=U=\mathrm{H}_\sigma\times\mathrm{H}_\pi\times\mathrm{H}_\pi^2,$$

then for $x=\begin{bmatrix}x_\sigma\\x_\pi\\x_y\end{bmatrix}\in X$, we have that $E=\begin{bmatrix}I_\sigma&0&0\\0&I_\pi&0\\0&0&0\end{bmatrix}:X\to Z$ is a bounded linear operator with $\ker E=\{0\}\times\{0\}\times\mathrm{H}_y$, and $\mathrm{ran}E=\mathrm{H}_\sigma\times\mathrm{H}_\pi\times\{0\}$, $A=\begin{bmatrix}\mu B_\sigma&0&0\\0&\mu B_\pi&-I_\pi\\0&-I_\pi&0\end{bmatrix}:X\to Z$ is a closed and densely defined operator with $\mathrm{dom}A=\mathrm{H}_\sigma^2\times\mathrm{H}_\pi^2\times\mathrm{H}_y$, where I, I_σ and I_π denote the identical operators on $\mathrm{L}^2,\mathrm{H}_\sigma$ and H_π respectively. Let $x(\xi,t)=x(t)(\xi), y(\xi,t)=y(t)(\xi), u(\xi,t)=u(t)(\xi)$,

$$\begin{bmatrix}x_\sigma(t)(\xi)\\x_\pi(t)(\xi)\\x_y(t)(\xi)\end{bmatrix}=\begin{bmatrix}\Sigma x(t)(\xi)\\P_\pi x(t)(\xi)\\y(t)(\xi)\end{bmatrix}, u(t)(\xi)=\begin{bmatrix}u_\sigma(t)(\xi)\\u_\pi(t)(\xi)\\u_y(t)(\xi)\end{bmatrix}.$$

Then (E1)-(E3) and (E5) can be written as

$$\begin{bmatrix}I_\sigma&0&0\\0&I_\pi&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_\sigma(t)\\\dot{x}_\pi(t)\\\dot{x}_y(t)\end{bmatrix}=\begin{bmatrix}\mu B_\sigma&0&0\\0&\mu B_\pi&-I_\pi\\0&-I_\pi&0\end{bmatrix}\begin{bmatrix}x_\sigma(t)\\x_\pi(t)\\x_y(t)\end{bmatrix}+\begin{bmatrix}u_\sigma(t)\\u_\pi(t)\\u_y(t)\end{bmatrix}, \begin{bmatrix}x_\sigma(0)\\x_\pi(0)\\x_y(0)\end{bmatrix}=\begin{bmatrix}\Sigma x(0)\\P_\pi x(0)\\y(0)\end{bmatrix}. \tag{E6}$$

Let

$$P=\begin{bmatrix}I_\sigma&0&0\\0&I_\pi&\mu B_\pi\\0&0&-I_\pi\end{bmatrix}, Q=\begin{bmatrix}I_\sigma&0&0\\0&0&I_\pi\\0&-I_\pi&0\end{bmatrix}, I_1=I_\sigma,$$

$$N=\begin{bmatrix}0&I_\pi\\0&0\end{bmatrix}, K=\mu B_\sigma, I_2=\begin{bmatrix}I_\pi&0\\0&I_\pi\end{bmatrix}, B_2=\begin{bmatrix}0&I_\pi&\mu B_\pi\\0&0&-I_\pi\end{bmatrix}.$$

Then P is injective and Q is bijective,$PEQ=\begin{bmatrix}I_1&0\\0&N\end{bmatrix}$, $PAQ=\begin{bmatrix}K&0\\0&I_2\end{bmatrix}$, N is a nilpotent operator with order 2, K is the generator of the strongly continuous semigroup e^{Kt} [2], and

$$\mathrm{ran}N=\mathrm{ran}\begin{bmatrix}0&I_\pi\\0&0\end{bmatrix}=\mathrm{ran}\begin{bmatrix}0&0&-I_\pi\\0&0&0\end{bmatrix}=\mathrm{ran}[NB_2].$$

Hence linear Navier-Stokes equations (E1)-(E4) are P-controllable by Theorem 6.

Appendix F Analysis of the Latest Research

Controllability is the property of being able to steer between two arbitrary points in the state space. In infinite dimensions, the situation is more complex, and many different types of controllability have been studied in the literature (for example, exact controllability, approximate controllability, exact null controllability, and so on). These properties are very important for studying the infinite dimensional systems, but it is regrettably that none of these results regarding controllability discussed pulsive behavior. In fact, for singular distributed parameter systems, there may be pulse terms in their solutions. In a practical system, the pulse term is generally undesirable in the solutions, since pulse may stop the system from working or even destroy it. Therefore, it requires that we must eliminate these pulse terms by imposing appropriate control input. In view of this fact, in this paper, the concept of pulse controllability of regular singular distributed parameter systems with finite order is considered in Banach space. The impulse observability of regular degenerate evolution systems is discussed in [3]. According to this paper, we can discuss the dual problem of impulse observability of the regular degenerate evolution systems. The solvability of degenerate linear evolution equations with the Riemann-Liouville fractional derivative is discussed in [4]. According to this paper, we can discuss the distributional solution and pulse controllability of degenerate linear evolution equations with the Riemann-Liouville fractional derivative. Hence, the concept of pulse controllability of regular singular distributed parameter systems is the basis for the study of the pulse controllability of other degenerate linear evolution equations.

Appendix G Dirac Function

Dirac function $\delta(t)$ and the $\delta^{(i)}(t)$ (the ith derivative of $\delta(t)$) are the generalized functions. For the details, see [5].

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