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Exponentially convergent angular velocity estimator design for rigid body motion: a singular perturbation approach

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Appendix A Notions and Mathematical preliminaries

The following notions are adopted throughout this work. $\|x\|$ denotes the Euclidean norm of a vector x. Let I_p denote the identity matrix of $p \times p$. $x \in \mathbb{S}^2$ represents $x \in \mathbb{R}^3$ with unit Euclidean norm. $\mathrm{diag}(x_i)$ represents a block-diagonal matrix with $x_i, i = 1, \cdots, n$ on the diagonal. For a matrix X, $\mathrm{tr}(X)$ denotes the trace of the matrix X. Let $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the minimum and maximum eigenvalues of a symmetric matrix.

For a free rigid body which moves in three dimensional space, the attitude rotation matrix of body-fixed frame $\mathcal{F}_B$ relative to the inertial frame $\mathcal{F}_I$ is globally and uniquely described by the Special Orthogonal group which is given as

$$SO(3) := \{R \in \mathbb{R}^{3\times3} | R^T R = RR^T = I_3, \det(R) = 1\}$$

Based on the Rodrigues' formula [1], the rotation matrix $R \in SO(3)$ can be expressed by unique rotational axis $n \in \mathbb{S}^2$ and angle $|\theta| \leqslant \pi$, such that

$$R = \exp(\hat{n}\theta) = I_3 + \sin(\theta)\hat{n} + (1 - \cos(\theta))\hat{n}^2$$

The hat operator $(\cdot)^\wedge : R^3 \to \mathfrak{so}(3)$ is a linear map that transforms a vector $x \in \mathbb{R}^3$ to the 3×3 skew-symmetric matrices $\hat{x} \in \mathfrak{so}(3)$, where $\mathfrak{so}(3) = \{\hat{x} \in \mathbb{R}^{3\times3} | \hat{x}^T = -\hat{x}\}$. For any vector $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$, for example, it follows that

$$\hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The inverse operator that corresponds to the hat $(\cdot)^\wedge$ is the vee operator $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. For any $x, y \in \mathbb{R}^3$, $z \in \mathbb{S}^2$ and $R \in SO(3)$, the following properties hold [3].

$$\hat{x}y = x \times y = -y \times x = -\hat{y}x, \quad \text{(A1)}$$

$$\mathrm{tr}(R\hat{x}) = \mathrm{tr}(\hat{x}R) = -x^T(R - R^T)^\vee, \quad \text{(A2)}$$

$$\hat{x}R + R^T\hat{x} = ([\mathrm{tr}(R)I_3 - R]x)^\wedge, \quad \text{(A3)}$$

$$R\hat{x}R^T = (Rx)^\wedge, \quad \text{(A4)}$$

$$(\hat{z}^T - \hat{z})^\vee((\hat{z}^T - \hat{z})^\vee)^T = 4zz^T. \quad \text{(A5)}$$

Appendix B The proof of Proposition 1

Proof. Based on the Rodrigues' formula, the rotation error matrix $E \in SO(3)$ can be expressed by unique rotational axis $n_e \in \mathbb{S}^2$ and angle $|\theta_e| < \pi$, such that

$$E = \exp(\hat{n}_e\theta_e) = I_3 + \sin(\theta_e)\hat{n}_e + (1 - \cos(\theta_e))\hat{n}_e^2 \quad \text{(B1)}$$

Then, it leads to that $1 + \mathrm{tr}(E) = 4\cos^2(\frac{\theta_e}{2})$ and $\|(E - E^T)^\vee\|_2^2 = 4\sin^2(\theta_e)$. Thus, it follows that $\|E_e\|_2^2 = \frac{1}{4}\tan^2(\frac{\theta_e}{2})$. This shows (i). Using the property of $(A5)$, $E_eE_e^T$ is given by

$$E_eE_e^T = (\frac{1}{2(\mathrm{tr}(E) + 1)})^2(E - E^T)^\vee((E - E^T)^\vee)^T = \frac{1}{4}\tan^2(\frac{\theta_e}{2})(n_en_e^T) \quad \text{(B2)}$$

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which shows (ii). Using equations $\dot{R} = R\hat{\Omega}$ and $\dot{\tilde{R}} = \tilde{R}(\tilde{\Omega} - k_2 E_e)^\wedge$, the time derivative of E is given as

$$\dot{E} = -\tilde{R}^T\dot{\tilde{R}}\tilde{R}^T R + \tilde{R}^T\dot{R} = -(\tilde{\Omega} - k_2E_e)^\wedge E + E\hat{\Omega} \tag{B3}$$

where the property of the hat map $(A4)$ is used. By the property of the hat map $(A3)$, it follows

$$(\dot{E} - \dot{E}^T)^\vee = -[\mathrm{tr}(E)I_3 - E](\tilde{\Omega} - k_2E_e) + [\mathrm{tr}(E)I_3 - E^T]\Omega \tag{B4}$$

From $(B3)$ $(B4)$ and properties of $(A2)$ $(A3)$, the time derivative of the attitude error vector E_e is given by

$$\begin{aligned}
\dot{E}_e =& \frac{\mathrm{tr}(\dot{E})(E - E^T)^\vee}{2(1+\mathrm{tr}(E))^2} - \frac{(\dot{E} - \dot{E}^T)^\vee}{2(1+\mathrm{tr}(E))} \\
=& \frac{(\tilde{\Omega} - k_2E_e)^T(E - E^T)^\vee}{2(1+\mathrm{tr}(E))^2}(E - E^T)^\vee - \frac{\Omega^T(E - E^T)^\vee}{2(1+\mathrm{tr}(E))^2}(E - E^T)^\vee \\
&+ \frac{[\mathrm{tr}(E)I_3 - E](\tilde{\Omega} - k_2E_e)}{2(1+\mathrm{tr}(E))} - \frac{[\mathrm{tr}(E)I_3 - E^T]\Omega}{2(1+\mathrm{tr}(E))} \\
=& 2E_eE_e^T(\tilde{\Omega} - k_2E_e) - 2E_eE_e^T\Omega \\
&+ \frac{[\mathrm{tr}(E)I_3 - E](\tilde{\Omega} - k_2E_e)}{2(1+\mathrm{tr}(E))} - \frac{[\mathrm{tr}(E)I_3 - E^T]\Omega}{2(1+\mathrm{tr}(E))} \\
=& -\Phi_e^T\Omega + \Phi_e(\tilde{\Omega} - k_2E_e)
\end{aligned} \tag{B5}$$

This shows (iii). Equation $(B1)$ shows that $4(1 + \mathrm{tr}(E))E_eE_e^T + [\mathrm{tr}(E)I_3 - E]$ can be represented by $E^T + I_3$, and Φ_e can be rewritten as

$$\Phi_e = \frac{E^T + I_3}{8\cos^2(\frac{\theta_e}{2})} \tag{B6}$$

where equation (B2) is used. Using equation $(B1)$, $E^T + E + 2I_3$ can be written as

$$E^T + E + 2I_3 = 4I_3 + 2(1 - \cos(\theta_e))\hat{n}_e^2$$

Since that the spectrum of symmetric matrix $\hat{n}_e^2$ is the set $\{0, -1, -1\}$ [4], there exists an invertible matrix $P \in \mathbb{R}^{3\times3}$ such that

$$P^{-1}(4I_3{+}2(1 - \cos(\theta_e))\hat{n}_e^2)P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 + 2\cos(\theta_e) & 0 \\ 0 & 0 & 2 + 2\cos(\theta_e) \end{bmatrix}$$

Thus, the spectrum of symmetric matrix $\Phi_e^T + \Phi_e$ is located in the set $\{\frac{1}{2}, \frac{1}{2}, 1/(2\cos^2(\frac{\theta_e}{2}))\}$ and $\lambda_{min}(\Phi_e^T + \Phi_e) = \frac{1}{2}$ if $E(t) \in D$ for $t \geqslant 0$. This shows (iv).

Appendix C The proof of Theorem 1

Proof. Consider the following Lyapunov candidate such that $V = \Psi + W$, where $\Psi = \ln(2) - \frac{1}{2}\ln(1 + \mathrm{tr}(E))$, and $W = \frac{\gamma}{2}\Omega_e^T\Omega_e$, γ is a positive constant to be determined. Denote two subsets $D_c = \{E|\Psi < c < 2\}$ and $D_b = \{E|\Psi < b < c < 2\}$ for the slow system (i.e.,$\dot{E} = -(\tilde{\Omega} - k_2E_e)^\wedge E + E\hat{\Omega}$). Denote the set $\Gamma = \{\Omega_e|W \leqslant \rho\varepsilon^2\}$ for the fast system (i.e., $\dot{\Omega}_e = -k_1\Phi_e^T\Omega_e + f(\tilde{\Omega}) - f(\Omega)$), where $\rho > 0$ is some constant. Then, the singular perturbation approach is adopted to analyze the stability of the overall error system, which consists of three steps [5].

Firstly, for a certain constant $\rho > 0$, it will be shown that there exists a small constant $\varepsilon_1^* > 0$ to ensure that the set $D_c \times \Gamma$ is positively invariant if $\varepsilon \in (0, \varepsilon_1^*]$. Secondly, it will be proved that there exists another small constant $\varepsilon_2^* > 0$ such that the trajectories start from $E(t_0) \in D_b, \Omega_e(t_0) \notin \Gamma$ will enter the set $D_c \times \Gamma$ in a finite time if $\varepsilon \in (0, \varepsilon_2^*]$. Finally, we shall prove the property of exponential convergence for the closed loop system.

Firstly, the time derivative of W and Ψ are given by

$$\dot{\Psi} = \Omega_e^TE_e - k_2\|E_e\|^2 \leqslant -k_2\|E_e\|^2 + \|E_e\|_{max}\sqrt{\rho}\varepsilon \tag{C1}$$

$$\dot{W} = -k_1\gamma\Omega_e^T\Phi_e^T\Omega_e + \gamma\Omega_e^T(f(\tilde{\Omega}) - f(\Omega)) \tag{C2}$$

where $\|E_e\|_{max}$ is the upper bound of E_e on the set $D_c \times \Gamma$. If ε is chosen to be in the interval $(0, \varepsilon_1^*]$, then it follows that $\dot{\Psi} \leqslant 0$ on the set $\partial D_c \times \Gamma$. Denote $\varepsilon_1^* = k_2\kappa/(2\sqrt{\rho}\|E_e\|_{max})$, where $\kappa = \min_{\partial D_c}\|E_e\|_2^2$ which is the norm minimum on the set boundary ∂D_c. Based on the result of Proposition 1 in [6], we have the inequality $\|f(\tilde{\Omega}) - f(\Omega)\| \leqslant d\|\Omega_e\|^2 + \sqrt{2}d\Omega_{max}\|\Omega_e\|$. Then, Equation $(C2)$ can be rewritten as

$$\begin{aligned}
\dot{W} &= -\frac{k_1}{2}\gamma\Omega_e^T(\Phi_e^T + \Phi_e)\Omega_e + \gamma\Omega_e^T(f(\tilde{\Omega}) - f(\Omega)) \\
&\leqslant -\frac{k_1}{2}\gamma\lambda_{min}(\Phi_e^T + \Phi_e)\Omega_e^T\Omega_e + \gamma\|\Omega_e\|\|f(\Omega) - f(\tilde{\Omega})\| \\
&\leqslant -(\frac{k_1}{4} - d\|\Omega_e\| - \sqrt{2}d\Omega_{max})\gamma\|\Omega_e\|^2
\end{aligned} \tag{C3}$$

If there exists a gain k_1 satisfies equation $\frac{k_1}{4} - d((\sqrt{2}+1)\Omega_{max} + k_1\|E_e(t_0)\|) \geqslant \frac{1}{\varepsilon}$, this shows that $\dot{W}(t_0) \leqslant -\frac{\gamma}{\varepsilon}\|\Omega_e(t_0)\|^2$ which implies that evolution of $\|\Omega_e\|$ is the exponential convergence, where $\|\Omega_e(t_0)\| \leqslant \|\tilde{\Omega}(t_0)\| + \Omega_{max} \leqslant \|k_1 E_e(t_0)\| + \Omega_{max}$ is used. At the next moment, it leads to that $\|\Omega_e(t)\| \leqslant \|\Omega_e(t_0)\|$, and $\dot{W}(t) \leqslant -(\frac{k_1}{4} - d\|\Omega_e(t)\| - \sqrt{2}d\Omega_{max})\gamma\|\Omega_e(t)\|^2 \leqslant 0$, where $0 \leqslant (\frac{k_1}{4} - d\|\Omega_e(t_0)\| - \sqrt{2}d\Omega_{max}) \leqslant \frac{k_1}{4} - d\|\Omega_e(t)\| - \sqrt{2}d\Omega_{max}$ is used. Thus, for $t \geqslant 0$, inequality $\dot{W} \leqslant -\frac{\gamma}{\varepsilon}\|\Omega_e\|^2 \leqslant 0$ always holds in the set $D_c \times \Gamma$. For a certain constant $\rho > 0$, $(C2)$ and $(C1)$ can be proved to have the property that the set $D_c \times \Gamma$ is positively invariant if ε is chosen to be in the interval $(0, \varepsilon_1^*]$.

Secondly, consider the initial states $E(t_0) \in D_b$, $\Omega_e(t_0) \notin \Gamma$, where $\|\Omega_e(t_0)\|$ is bounded. Since D_c is an open set, and the attitude error function Ψ and the dynamics of E are continuous, there exists a finite time T_1, independent of ε, such that $E(t) \in D_c$ for all $t \in (0, T_1]$. During the interval $(0, T_1]$, the trajectory stays in the set D_c which implies that $\Omega_e^T(\Phi_e^T + \Phi_e)\Omega_e > 0$ for all nonzero vector $\Omega_e(t)$. Based on gain equation $\frac{k_1}{4} - d((\sqrt{2}+1)\Omega_{max} + k_1\|E_e(t_0)\|) \geqslant \frac{1}{\varepsilon}$, the equation $(C2)$ can be rewritten as

$$\dot{W} \leqslant -\frac{\gamma}{\varepsilon}\Omega_e^T\Omega_e = -\frac{2}{\varepsilon}W \tag{C4}$$

Then, one has that

$$W(t_0) \leqslant W(t_0)e^{-2(t-t_0)/\varepsilon} \tag{C5}$$

Since $\|\Omega_e(t_0)\|$ is bounded, there exists a constant $\delta > 0$ such that $W(t_0) \leqslant \delta$. It follows from inequality $(C5)$ that $\Omega_e(t)$ reaches the set Γ within the time interval $[t_0, t_0 + T_\varepsilon]$, where

$$T_\varepsilon = \frac{\varepsilon}{2}\mathrm{In}(\frac{\delta}{\rho\varepsilon^2}) \tag{C6}$$

Since δ, ρ are positive constants, equation $(C6)$ indicates that $\lim_{\varepsilon\to 0} T_\varepsilon = 0$. Thus, there exists a sufficiently small constant ε_2^* such that $T_{\varepsilon_2^*} \leqslant aT_1$, where $0 < a < 1$. Choosing $\varepsilon \in (0, \varepsilon^*]$, $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*\}$, all the trajectories that start from $E(t_0) \in D_b$, $\Omega_e(t_0) \notin \Gamma$ will enter the set $D_c \times \Gamma$ in a finite time $T_1 > 0$, and stay in this set for the time $t \geqslant T_1$. Furthermore, it shows that the convergence rate of estimated angular velocity mainly depends on k_1.

Finally, the exponentially convergence of overall system is proved. Based on the above analysis, all trajectories enter the set $D_c \times \Gamma$ after a finite time $T_1 > 0$, and stay in this set during the time interval $[T_1, \infty)$. Thus, the time derivative of the Lyapunov function V is

$$\begin{aligned}
\dot{V} =& \Omega_e^T E_e - k_2\|E_e\|^2 - k_1\gamma\Omega_e^T\Phi_e^T\Omega_e + \gamma\Omega_e^T(f(\tilde{\Omega}) - f(\Omega)) \\
\leqslant& \Omega_e^T E_e - k_2\|E_e\|^2 - \frac{\gamma}{\varepsilon}\|\Omega_e\|^2 \\
\leqslant& -\frac{k_2}{2}\|E_e\|^2 - (\frac{2}{\varepsilon}\gamma - \frac{1}{2k_2})\|\Omega_e\|^2 \\
\leqslant& -\frac{k_2}{2}\|E_e\|^2 - \|\Omega_e\|^2
\end{aligned} \tag{C7}$$

where the inequality is obtained by using Young's Inequality $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$ for any two nonnegative real numbers a, b, and p, q are real numbers that satisfy the equation $\frac{1}{p} + \frac{1}{q} = 1$, and γ is chosen as $\gamma \geqslant \frac{\varepsilon(1+2k_2)}{2k_2}$. Then, $V \geqslant 0$ and $\dot{V} \leqslant 0$ indicate the boundness of the Lyapunov function V, thus all sates variables of the closed-loop system are bounded, i.e. $E_e, \Omega_e \in \mathcal{L}_\infty$. By integrating both sides of the inequality $(C7)$, it follows that

$$V(t_0) - V(\infty) \geqslant \int_0^\infty \{\frac{k_2}{2}\|E_e\|^2 + \|\Omega_e\|^2\}$$

which shows that $E_e, \Omega_e \in \mathcal{L}_2$. On the other hand, according to the overall error system, we have $\dot{E}_e, \dot{\Omega}_e \in \mathcal{L}_\infty$. By using the Barbalat's Lemma, it follows that $\lim_{t\to\infty} E_e = 0$, and $\lim_{t\to\infty} \Omega_e = 0$.

By virtue of inequality $b_1\|E_e\|_2^2 \leqslant \Psi \leqslant b_2\|E_e\|_2^2$ [2], we can obtain that $\lambda_m(M_{11})\|h\|^2 \leqslant V \leqslant \lambda_M(M_{12})\|h\|^2$, where $b_1 = \frac{4c}{e^{2c}-1}$ with $0 < c < \infty$, $b_2 = 2$, $h = [\|E_e\| \ \ \|\Omega_e\|]^T \in \mathbb{R}^2$, and the matrices $M_{11} = [b_1, 0; 0, \frac{\gamma}{2}]$, $M_{12} = [b_2, 0; 0, \frac{\gamma}{2}] \in \mathbb{R}^{2\times 2}$. Recalling that all the trajectory always stay in the set D_c, inequality $(C7)$ can be rewritten as $\dot{V} \leqslant -\min\{\frac{k_2}{4}, \frac{2}{\gamma}\}V$. Thus, we obtain that

$$\|h(t)\|^2 \leqslant \frac{\lambda_M(M_{12})}{\lambda_m(M_{11})}\exp\left(-\min\{\frac{k_2}{4}, \frac{2}{\gamma}\}t\right)\|h(t_0)\|^2 \tag{C8}$$

Therefore, the trajectories E_e and Ω_e converge to zero exponentially, implying $E \to I_3, \tilde{\Omega} \to \Omega$ exponentially. Other unstable equilibriums do not exist since the trajectory E always stays in the set D_c.

Appendix D Simulation results

In this section, the numerical simulation is presented to validate the performance of the proposed novel estimator in Theorem 1 and the classical estimator presented in Remark 2. The states data are generated by the dynamics $\dot{R} = R\hat{\Omega}, \dot{\Omega} = J^{-1}((J\Omega)^\wedge\Omega + \tau)$, and the initial attitude and angular velocity are chosen as $R(t_0) = \exp(\frac{\pi}{2}\cdot\hat{e}_3)\exp(\frac{-\pi}{4}\cdot\hat{e}_2)\exp(\frac{2\pi}{3}\cdot\hat{e}_1)$ and $\Omega(0) = [0.8, -0.4, 0.6]^T$, where $e_1 = [1, 0, 0]^T$, $(\frac{2\pi}{3}, \frac{-\pi}{4}, \frac{\pi}{2})$ is the Euler angles. The control input is given as $\tau = [0.1\sin(0.2\pi t), 0.15\sin(0.2\pi t), -0.2\sin(0.2\pi t)]^T$, and the moment of inertia matrix is chosen as $J = \mathrm{diag}\{3, 2, 1\}$. The initial attitude estimation is given as $\tilde{R}(t_0) = \exp(\frac{-\pi}{4}\cdot\hat{e}_3)\exp(\frac{\pi}{6}\cdot\hat{e}_2)\exp(\frac{-\pi}{3}\cdot\hat{e}_1)$. Two set control gains are selected as $\{k_1 = \alpha_1 = 3, k_2 = \alpha_2 = 1.2\}$, and $\{k_1 = \alpha_1 = 15, k_2 = \alpha_2 = 1.2\}$. The performance comparison results of two types of estimators are shown in Fig.$D1$ and Fig.$D2$. Both the evolution of attitude estimation error $(E - E^T)^\vee$ and the angular velocity estimation error Ω_e converge to zero, and this shows the attitude and the angular velocity are perfectly estimated. It can be seen that the estimation error converges to zero within 8s, and it is slightly faster than convergence speed of the classical estimator presented in Remark 2. Due to intermediate variable, there exists a large initial estimation error of angular velocity which is shown in Fig.$D1$. The evolution of attitude error vector $(E-E^T)^\vee$ of proposed estimator also has the desired convergence rate, and this shows that the fast angular velocity convergence rate could speed up the convergence rate of attitude estimation. Fig.$D3$, which shows the transient performance of estimated angular velocity error more clearly, illustrates that the convergence rate of estimated angular velocity can be mainly controlled by the gain k_1. The simulation illustrates that the estimator proposed in this literature can be applied to the controller design of the vehicles which are only equipped with the direction sensors such as accelerometer, magnetometer or Sun sensors.

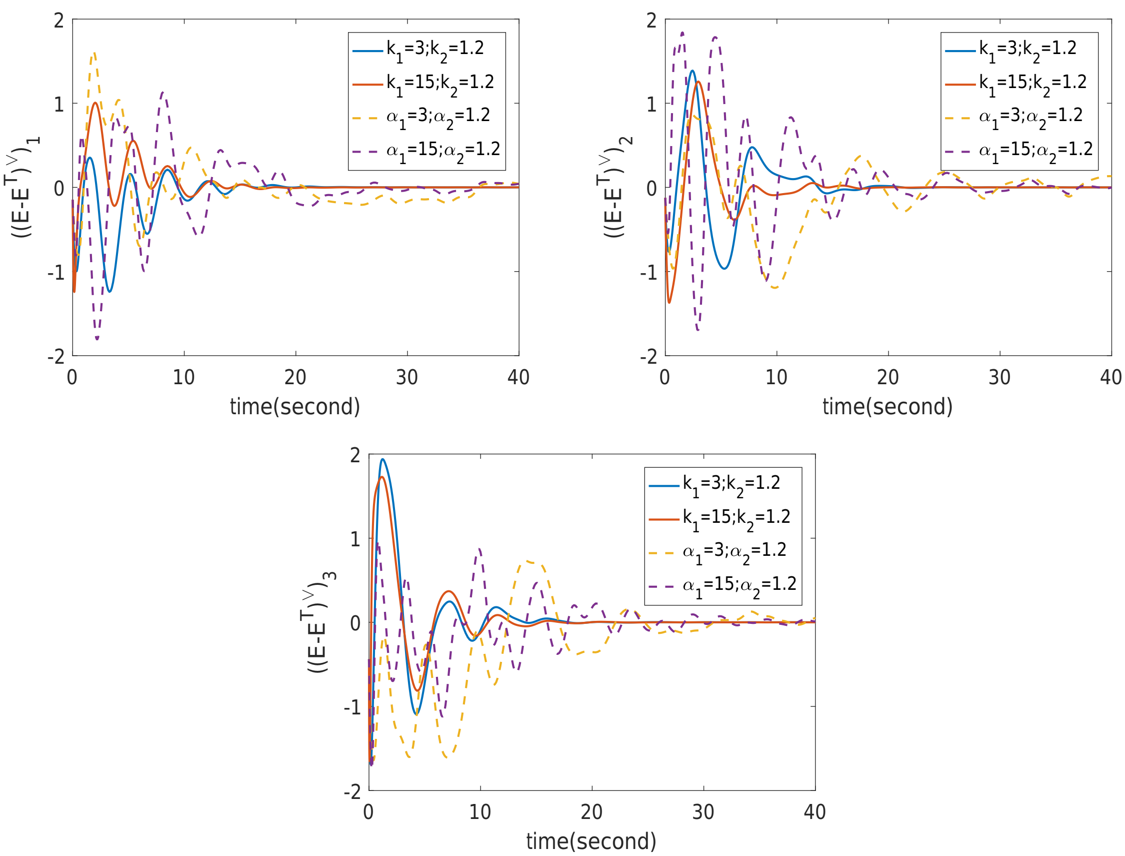

Figure D1 The evolution of estimated attitude error $(E-E^T)^\vee$ with total estimation data

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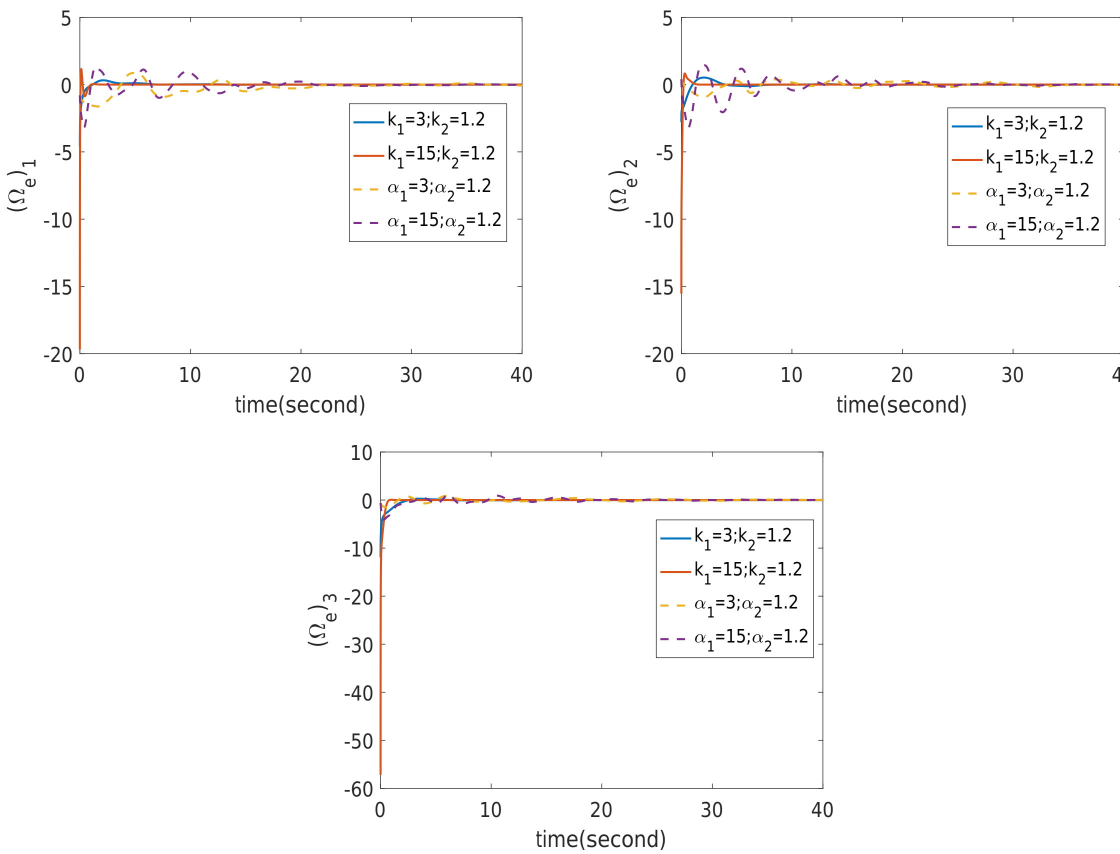

Figure D2 The evolution of estimated angular velocity error Ω_e with total estimation data

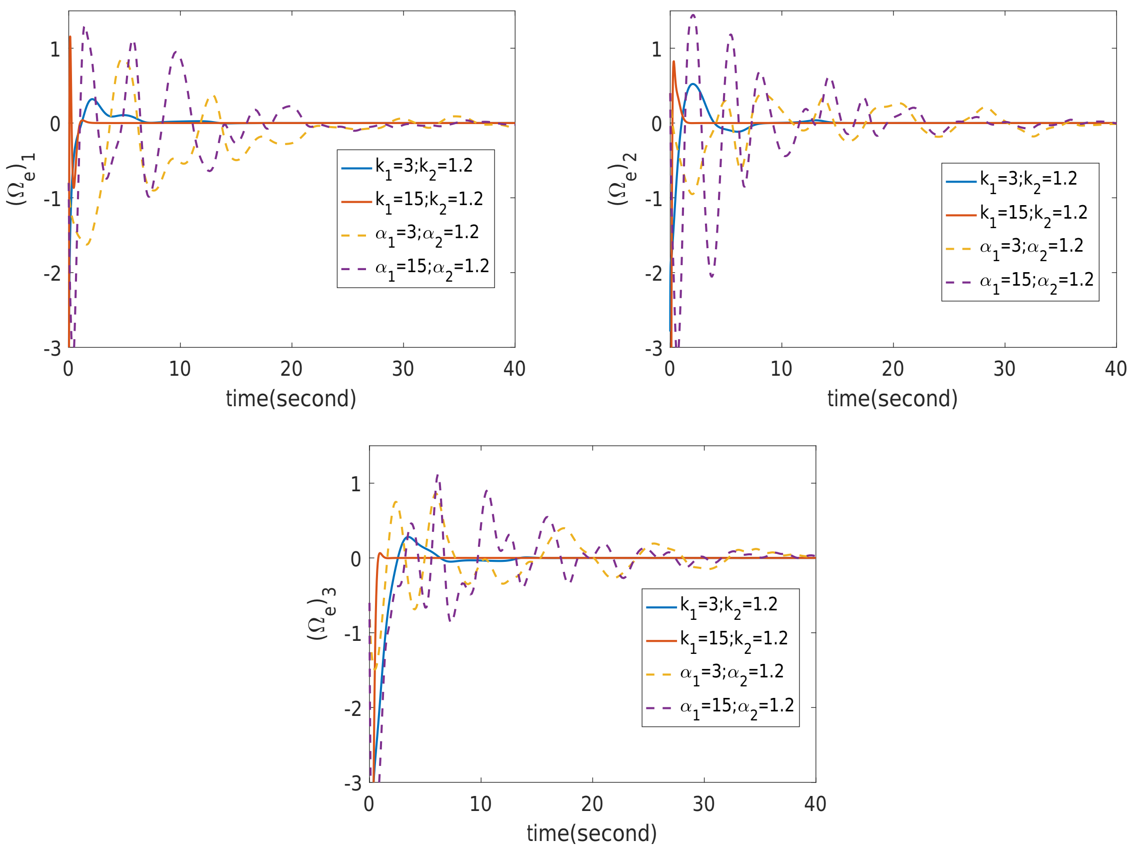

Figure D3 The transient performance of estimated angular velocity error Ω_e