Supplementary Material

Article Title:
Theoretical Development and Critical Analysis of Burst Frequency Equations for Passive Valves on Centrifugal Microfluidic Platforms

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This document presents the mathematical proofs for the equations for Capillary Flow, and Stage 1, 2, and 3 Pressures.

Where presented in the paper, the difference between the proofs shown and the equations from the literature is indicated in boxed sections.

The sequence of the equations presented here may not be the same as that presented in the paper:

[eq#] refers to the sequence of equation in this supplementary document
(#) refers to the sequence of equations in the paper

Calculation of the Burst Frequency for Capillary Valves on Centrifugal Microfluidic Platforms

On a centrifugal microfluidic CD with capillary channels, there are two pressures present: The centrifugal pressure, and the capillary pressure

The centrifugal pressure is due to the rotation of the CD:

$$P_{centrifugal} = \rho\omega^2 \Delta r \bar{r} \quad \text{[eq1]} \quad (1)$$

where ρ is the density of the liquid
ω is the rotational speed of the CD in rounds per minute (rpm)
Δr is the difference between the top and bottom of the liquid levels at rest with respect to the centre of the CD
$\bar{r}$ is the average distance of the liquid from the centre of the CD

The pressure that is due to the liquid-air surface tension, contact angle of the liquid, and the hydraulic diameter of the capillary channel is known as:

$$P_{cap} = \frac{4\gamma_{al}\cos\theta_c}{D_h} \quad \text{[eq2]}$$

where γ_{al} is the liquid-air surface tension
θ_c is the contact angle of the liquid with respect to the solid wall of the channel
D_h is the hydraulic diameter of the channel given as

$$D_h = \frac{4 \times Area}{Perimeter} = \frac{4wh}{w+w+h+h} = \frac{2wh}{w+h} \quad (13)$$

where w is the width, and h is the height of the channel

For the liquid to move towards the edge of a CD, the centrifugal pressure must be greater than the capillary pressure. This allows us to calculate the burst frequency:

From [eq1] and [eq2]

$$P_{centrifugal} = \rho\omega^2 \Delta r \bar{r} = P_{cap}$$

Rearranging yields:

$$\omega = \sqrt{\frac{P_{cap}}{\rho \Delta r \bar{r}}}$$

Alternatively:

$$rpm = \omega \times \frac{30}{\pi} = \sqrt{\frac{P_{cap}}{\rho \Delta r \bar{r}}}\left(\frac{30}{\pi}\right) \quad \text{[eq3]} \quad (7)$$

Fluid Flow within an Infinitely Long Channel (Capillary Flow):

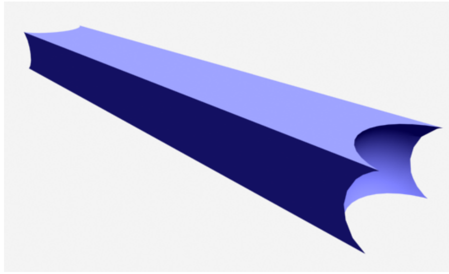

Figure 1: Model of Rectangular Channel

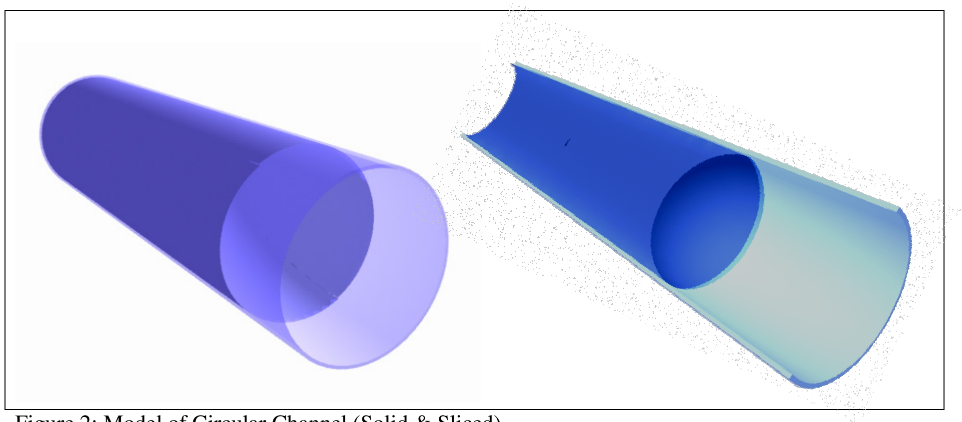

Figure 2: Model of Circular Channel (Solid & Sliced)

The total interfacial energy within a solid-liquid-air system is expressed as

$$U_T = A_{sl}\gamma_{sl} + A_{sa}\gamma_{sa} + A_{la}\gamma_{la} \quad \text{[eq4]} \quad (3)$$

where A_{sl}, A_{sa}, A_{la} are the solid-liquid, solid-air, and liquid-air interface areas,
$\gamma_{sl}, \gamma_{sa}, \gamma_{la}$ are the corresponding surface energies per unit area. A

The surface energies per unit area are related by Young's equation

$$\gamma_{sl} = \gamma_{sa} - \gamma_{la}\cos\theta_c \quad \text{[eq5]}$$

Substituting Young's equation into [eq4] gives us

$$U_T = A_{sl}(\gamma_{sa} - \gamma_{la}\cos\theta_c) + A_{sa}\gamma_{sa} + A_{la}\gamma_{la}$$
$$U_T = (A_{sl} + A_{sa})\gamma_{sa} + (A_{la} - A_{sl}\cos\theta_c)\gamma_{la}$$

As all the terms in $(A_{sl} + A_{sa})\gamma_{sa}$ are constants, we can denote it as U_0

$$U_T = U_0 + (A_{la} - A_{sl}\cos\theta_c)\gamma_{la} \quad \text{[eq6]} \quad (4)$$

The pressure in the capillary can be derived from the change of total interfacial energy of the solid-liquid-air system with respect to the injected liquid volume:

$$P_{cap} = -\frac{dU_T}{dV} \quad \text{[eq7]} \quad (2)$$
$$P_{cap} = -\left[\frac{d}{dV}U_0 + \left(\frac{d}{dV}A_{la} - \frac{d}{dV}A_{sl}\cos\theta_c\right)\gamma_{la}\right] \quad \text{[eq8]} \quad (5)$$

Since U_0 is a constant, $\frac{d}{dV}U_0 = 0$. This simplifies [eq8] to

$$P_{cap} = \gamma_{la}\left(\frac{d}{dV}A_{sl}\cos\theta_c - \frac{d}{dV}A_{la}\right) \quad \text{[eq9]}$$

If we further apply $\frac{d}{dV}A_{la} = 0$ (as A_{la} which is the surface boundary between liquid and air in a capillary opening does not change just before reaching the end of the channel we can simplify [eq9] to

$$P_{cap} = \frac{d}{dV}A_{sl}\cos\theta_c\gamma_{la}$$
$$P_{cap} = \frac{dA_{sl}}{dV}\cos\theta_c\gamma_{la} \quad \text{[eq10]} \quad (6)$$

We can now consider A_{sl} for various capillary shapes

For a circular capillary

The surface boundary between the solid and liquid can be found by

$$A_{sl} = \pi\ D\ x \quad (8)$$

where D is the diameter of a circular capillary
x is the length of expansion along the liquid's axis of movement

The volume of the liquid can be found as

$$V = \frac{\pi D^2\ x}{4} \quad (9)$$

Hence we can determine for a circular capillary

$$\frac{dA_{sl}}{dV} = \frac{\pi\ D\ dx}{\frac{\pi D^2}{4}dx} = \frac{4}{D} \quad \text{[eq11]}$$

For a rectangular capillary

The surface boundary can be determined as

$$A_{sl} = 2(w+h)\ x \quad (10)$$

where w, h are the width and height of the capillary
x is the length of expansion along the liquid's axis of movement

The volume of the liquid can be found as

$$V = w\ h\ x \quad (11)$$

Hence we can determine for a rectangular capillary

$$\frac{dA_{sl}}{dV_l} = \frac{2(w+h)\ dx}{w\ h\ dx} = \frac{4}{D_h} \quad \text{[eq12]} \quad (12)$$

Alternatively, manipulating [eq12] gives

$$\frac{dA_{sl}}{dV_l} = 2\frac{(w+h)}{w\ h}$$
$$\frac{dA_{sl}}{dV_l} = 2\left(\frac{1}{h} + \frac{1}{w}\right) \quad \text{[eq13]}$$

Hence, for both Circular and Rectangular Capillary

Substituting [eq11] into [eq10] yields

$$P_{cap} = \frac{4\cos\theta_c \gamma_{la}}{D} \quad \text{for circular capillary} \qquad \text{[eq14]}$$

Substituting [eq12] into [eq10] yields

$$P_{cap} = \frac{4\cos\theta_c \gamma_{la}}{D_h} \quad \text{for rectangular capillary} \qquad \text{[eq15] (14)}$$

Different from Zeng *et al* (2000) eq (2)

$$P_{cb} = \frac{dU_T}{dV_l} = \frac{4\gamma_{la}\sin\theta_c}{D} \qquad (2)$$

This is different

Different from Chen *et al* (2008) eq (2)

$$\Delta p_b = 4\gamma_{la}\sin\theta_c / D_h \qquad (2)$$

Substituting [eq13] into [eq10] yields

$$P_{cap} = 2\cos\theta_c \gamma_{la}\left(\frac{1}{h} + \frac{1}{w}\right) \qquad \text{[eq16] (15)}$$

Substituting either [eq14] or [eq15] into [eq3] yields

$$rpm = \sqrt{\frac{4\gamma_{al}\cos\theta_c}{\rho\Delta r\bar{r}D_h}}\left(\frac{30}{\pi}\right) \qquad \text{[eq17]}$$

Stage 1:

The liquid is in a capillary channel, stopped at an opening with a concave meniscus (the meniscus changes from concave to flat)

According to [eq6] and [eq8], we need to determine A_{la} and A_{sl} for $U_T = U_0 + (A_{la} - A_{sl}\cos\theta_c)\gamma_{la}$, and we also need to determine the volume of the liquid, V.

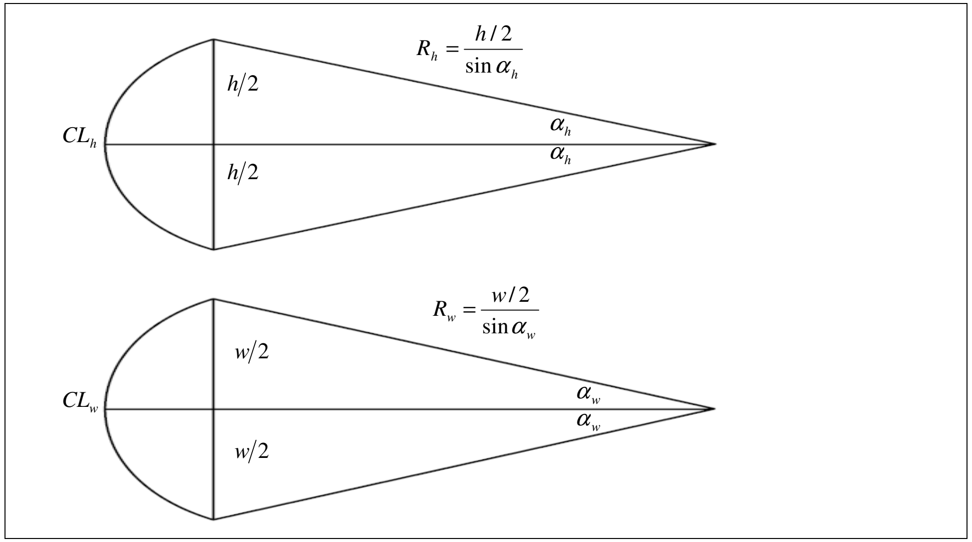

Figure 3: Mathematical Model for Stage 1 Meniscus

A_{la} can be found by multiplying the curve length of height, CL_h, and the curved length of width, CL_w:

$$A_{la} = CL_h \times CL_w \qquad \text{[eq18]}$$

Let the curved lengths have angles of $2\alpha_w$ and $2\alpha_h$, and curve radii of R_w and R_h respectively.
We can derive the following from the figure:

$$R_h = \frac{h/2}{\sin\alpha_h} \quad \text{- see Figure 3} \quad [eq19]$$

$$R_w = \frac{w/2}{\sin\alpha_w} \quad \text{- see Figure 3} \quad [eq20]$$

$$CL_h = R_h \times 2\alpha_h = \frac{h\alpha_h}{\sin\alpha_h} \quad \text{- see Figure 3} \quad [eq21]$$

$$CL_w = R_w \times 2\alpha_w = \frac{w\alpha_w}{\sin\alpha_w} \quad \text{- see Figure 3} \quad [eq22]$$

Hence substituting [eq21] and [eq22] into [eq18] yields,

$$A_{la} = \frac{hw\alpha_h\alpha_w}{\sin\alpha_h \sin\alpha_w} \quad [eq23]$$

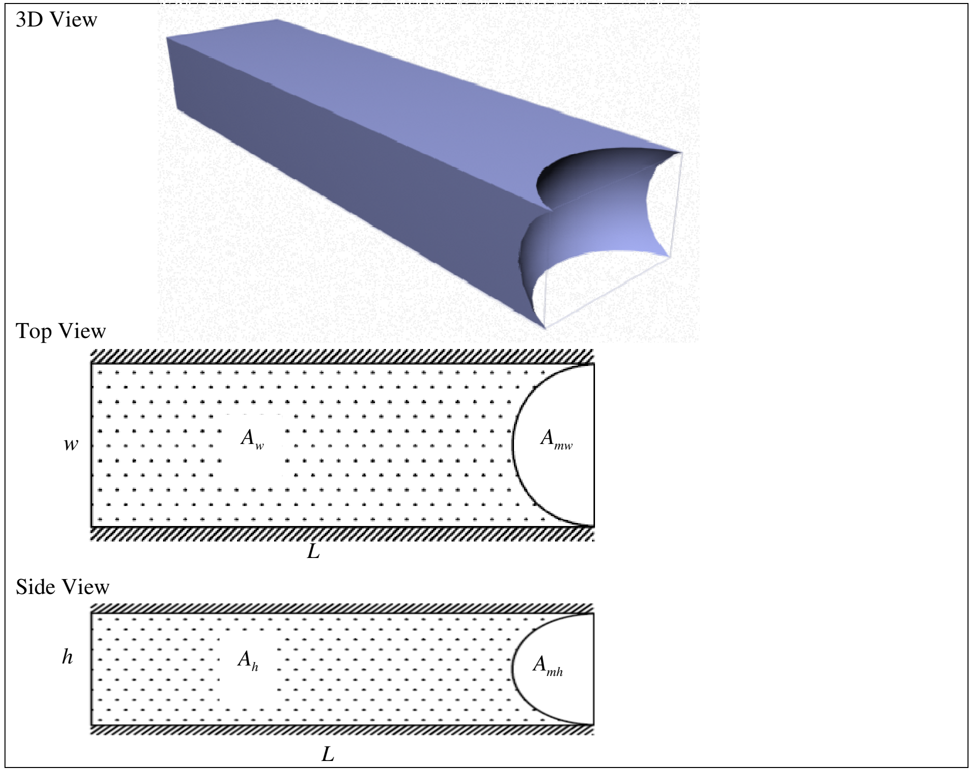

Figure 4: Model for Stage 1 Fluid Flow

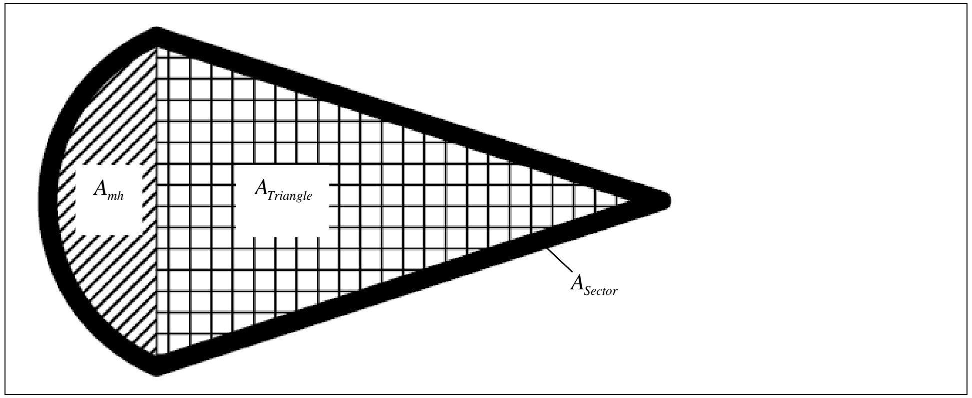

Figure 5: Mathematical Model for Stage 1 Liquid Surface

A_{sl} can be found by taking the surface of solid-liquid within the capillary if there was no meniscus (A_h and A_w) and <u>subtracting</u> the surface of liquid-air due to the meniscus (A_{mh} and A_{mw}). We assume the liquid to progress up to length L from the point of reference (right up to the opening).

$$A_{sl} = 2\left(A_h + A_w - A_{mh} - A_{mw}\right) \quad \text{- see Figure 4} \quad [eq24]$$

Where $A_h = hL$ - see Figure 4 [eq25]

$A_w = wL$ - see Figure 4 [eq26]

$A_{mh} = A_{Sector} - A_{Triangle}$ - see Figure 5 [eq27]

Where $A_{Sector} = angle\ of\ curvature \times radius\ of\ curvature$ - see Figure 3

$$A_{Sector} = \frac{2\alpha_h}{2\pi}(\pi)\times R_h^{\,2} = \alpha_h R_h^{\,2}$$

$$A_{Sector} = \alpha_h\left(\frac{h/2}{\sin\alpha_h}\right)^2 = \frac{h^2\alpha_h}{4\sin\alpha_h} \quad [eq28]$$

And $A_{Triangle} = \frac{1}{2}\times height \times base$ - see Figure 3

$${}_{Triangle} = \frac{1}{2}\times R_h\cos\alpha_h \times h = \frac{h}{2}\times\frac{h/2}{\sin\alpha_h}\cos\alpha_h$$

$$A_{Triangle} = \frac{h^2\cos\alpha_h}{4\sin\alpha_h} \quad [eq29]$$

$$A_{mh} = \frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h} \quad [eq30]$$

Similarly

$$A_{mw} = \frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w} \quad [eq31]$$

Hence substituting [eq25], [eq26], [eq30], and [eq31] into [eq24] yields,

$$A_{sl} = 2\left(hL + wL - \frac{h^2\alpha_h}{4\sin^2\alpha_h} + \frac{h^2\cos\alpha_h}{4\sin\alpha_h} - \frac{w^2\alpha_w}{4\sin^2\alpha_w} + \frac{w^2\cos\alpha_w}{4\sin\alpha_w} \right)$$

$$A_{sl} = 2\left[(h+w)L - \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) - \frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) \right] \qquad \text{[eq32]}$$

Now, substituting [eq23] and [eq32] into [eq6] yields,

$$U_T = U_0 + (A_{la} - A_{sl}\cos\theta_c)\gamma_{la} \qquad \text{[eq33]}$$

$$U_T = U_0 + \left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h\sin\alpha_w} - 2\left[\begin{matrix} (h+w)L - \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \\ - \frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) \end{matrix} \right]\cos\theta_c \right)\gamma_{la}$$

$$U_T = U_0 - 2\cos\theta_c\gamma_{la}\left[\begin{matrix} (h+w)L - \frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) \\ - \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \end{matrix} \right] + \gamma_{la}\left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h\sin\alpha_w}\right) \qquad \text{[eq34] (16)}$$

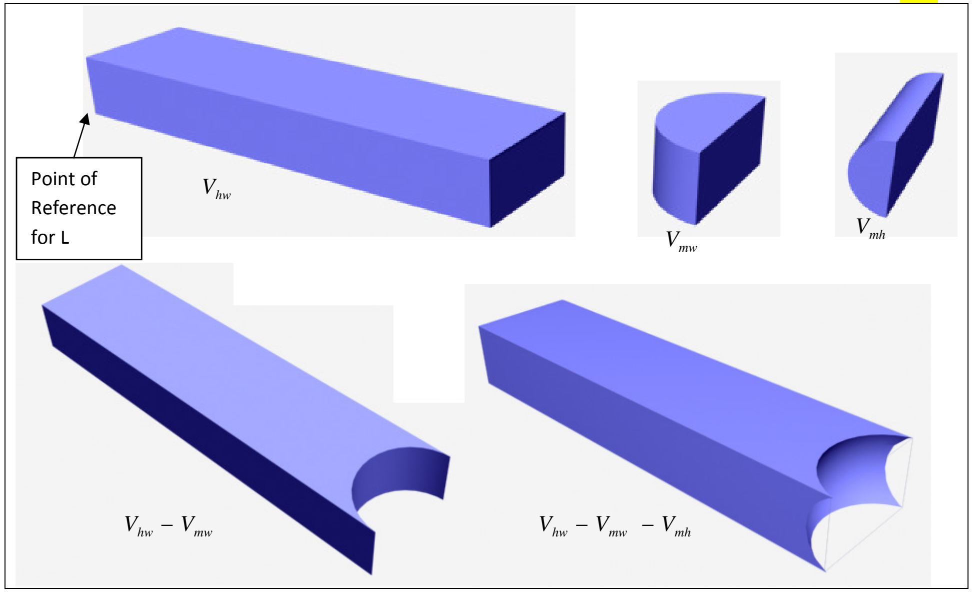

Figure 6: Mathematical Model for Stage 1 Liquid Volume

The volume V can be found by taking the volume of liquid within the capillary if there was no meniscus (V_{hw}) and <u>subtracting</u> the volume of displaced liquid due to the meniscus (V_{mw} and V_{mh}). We assume the liquid to progress up to length L from the point of reference:

$$V = V_{hw} - V_{mw} - V_{mh} \qquad \text{- see Figure 6} \qquad \text{[eq35]}$$

where $V_{wh} = hwL$ [eq36]

$$V_{mw} = A_{mw} \times h = \left(\frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w} \right)h \qquad \text{- see Figure 3, 4, 5, 6} \quad \text{[eq37]}$$

$$V_{mh} = A_{mh} \times CL_w = \left(\frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h} \right)\frac{w\alpha_w}{\sin\alpha_w} \qquad \text{- see Figure 3, 4, 5, 6} \quad \text{[eq38]}$$

Hence substituting [eq36], [eq37], and [eq38] into [eq35] yields

$$V = hwL - \left(\frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w} \right)h - \left(\frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h} \right)\frac{w\alpha_w}{\sin\alpha_w}$$

$$V = hwL - \frac{hw^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) - \frac{h^2w\alpha_w}{4\sin\alpha_h\sin\alpha_w}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \qquad \text{[eq39] (17)}$$

Finally, we can apply [eq34] and [eq39] into:

$$P_{Stage1} = -\frac{dU_T}{dV} \qquad \text{[eq40]}$$

To simplify the equation, as A_{la}, A_{sl} and V are functions of α_w and α_h, we can obtain the burst pressure by letting *f* to be a function of α_w and α_h. Hence

$$P_{Stage1} = -\frac{dU_T}{dV} = -\frac{dU_T/df}{dV/df} \qquad \text{[eq41]}$$

$$P_{Stage1} = -\frac{\left(\frac{\partial U_T/\partial\alpha_w}{\partial f/\partial\alpha_w}\right) + \left(\frac{\partial U_T/\partial\alpha_h}{\partial f/\partial\alpha_h}\right)}{\left(\frac{\partial V/\partial\alpha_w}{\partial f/\partial\alpha_w}\right) + \left(\frac{\partial V/\partial\alpha_h}{\partial f/\partial\alpha_h}\right)} = -\frac{\left(\partial U_T/\partial\alpha_w\right) + \left(\frac{\partial U_T/\partial\alpha_h}{\partial f/\partial\alpha_h}\right)\left(\partial f/\partial\alpha_w\right)}{\left(\partial V/\partial\alpha_w\right) + \left(\frac{\partial V/\partial\alpha_h}{\partial f/\partial\alpha_h}\right)\left(\partial f/\partial\alpha_w\right)}$$

$$P_{Stage1} = -\frac{\left(\partial U_T/\partial\alpha_w\right) + \left(\partial U_T/\partial\alpha_h\right)\left[\left(\partial f/\partial\alpha_w\right)\left(\partial f/\partial\alpha_h\right)^{-1}\right]}{\left(\partial V/\partial\alpha_w\right) + \left(\partial V/\partial\alpha_h\right)\left[\left(\partial f/\partial\alpha_w\right)\left(\partial f/\partial\alpha_h\right)^{-1}\right]} \qquad \text{[eq42]}$$

Now from [eq34], we can derive $\frac{\partial U_T}{\partial \alpha_w}$ and $\frac{\partial U_T}{\partial \alpha_h}$:

$$\frac{\partial U_T}{\partial \alpha_w} = \frac{\partial}{\partial \alpha_w}\left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h \sin\alpha_w}\right)\gamma_{la} - 2\cos\theta_c \gamma_{la} \frac{\partial}{\partial \alpha_w}\left[-\frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right]$$

$$\frac{\partial U_T}{\partial \alpha_w} = \frac{hw\alpha_h}{\sin\alpha_h}\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}\right)\gamma_{la} + \frac{\cos\theta_c \gamma_{la} w^2}{2}\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin^2\alpha_w} - \frac{\cos\alpha_w}{\sin\alpha_w}\right)$$

$$\frac{\partial U_T}{\partial \alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}\right) + \gamma_{la}\frac{w^2\cos\theta_c}{2}\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin^2\alpha_w} - \cot\alpha_w\right)$$

$$\frac{\partial U_T}{\partial \alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) + \gamma_{la}\frac{w^2\cos\theta_c}{2}\left(\frac{\sin^2\alpha_w - \alpha_w 2\sin\alpha_w\cos\alpha_w}{\sin^4\alpha_w} + \csc^2\alpha_w\right)$$

$$\frac{\partial U_T}{\partial \alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) + \gamma_{la}\frac{w^2\cos\theta_c}{2}\left(\frac{1}{\sin^2\alpha_w} - \frac{\alpha_w 2\cos\alpha_w}{\sin^3\alpha_w} + \frac{1}{\sin^2\alpha_w}\right)$$

$$\frac{\partial U_T}{\partial \alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) + \gamma_{la}\frac{w^2\cos\theta_c}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial U_T}{\partial \alpha_w} = \gamma_{la}\left(\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) \qquad \text{[eq43]}$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{\partial}{\partial \alpha_h}\left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h \sin\alpha_w}\right)\gamma_{la} - 2\cos\theta_c \gamma_{la} \frac{\partial}{\partial \alpha_h}\left[-\frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w}{\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}\right)\gamma_{la} - \frac{h^2\cos\theta_c \gamma_{la}}{2}\frac{\partial}{\partial \alpha_h}\left[-\left(\frac{\alpha_h}{\sin^2\alpha_h} - \frac{\cos\alpha_h}{\sin\alpha_h}\right)\right]$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w}{\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}\right)\gamma_{la} + \frac{h^2\cos\theta_c \gamma_{la}}{2}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin^2\alpha_h} - \cot\alpha_h\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{\sin\alpha_h - \alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) + \frac{h^2\cos\theta_c\gamma_{la}}{2}\left(\frac{\sin^2\alpha_h - \alpha_h 2\sin\alpha_h\cos\alpha_h}{\sin^4\alpha_h} + ccs^2\alpha_h\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) + \frac{h^2\cos\theta_c\gamma_{la}}{2}\left(\frac{1}{\sin^2\alpha_h} - \frac{\alpha_h 2\cos\alpha_h}{\sin^3\alpha_h} + \frac{1}{\sin^2\alpha_h}\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) + \frac{h^2\cos\theta_c\gamma_{la}}{2}\left(\frac{2}{\sin^2\alpha_h} - \frac{\alpha_h 2\cos\alpha_h}{\sin^3\alpha_h}\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) + \frac{h^2\cos\theta_c\gamma_{la}}{\sin^2\alpha_h}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \gamma_{la}\left(\frac{h^2\cos\theta_c}{\sin\alpha_h} + \frac{hw\alpha_w}{\sin\alpha_w}\right)\left(\frac{\sin\alpha_h - \alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) \qquad \text{[eq44]}$$

Now from [eq39], we can derive $\frac{\partial V}{\partial \alpha_w}$ and $\frac{\partial V}{\partial \alpha_h}$:

$$\frac{\partial V}{\partial \alpha_w} = -\frac{\partial}{\partial \alpha_w}\frac{hw^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) - \frac{\partial}{\partial \alpha_w}\frac{h^2w\alpha_w}{4\sin\alpha_h\sin\alpha_w}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)$$

$$\frac{\partial V}{\partial \alpha_w} = -\frac{hw^2}{4}\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin^2\alpha_w} - \cot\alpha_w\right) - \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = -\frac{hw^2}{4}\left(\frac{\sin^2\alpha_w - \alpha_w 2\sin\alpha_w\cos\alpha_w}{\sin^4\alpha_w} + ccs^2\alpha_w\right) - \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = -\frac{hw^2}{4}\left(\frac{1}{\sin^2\alpha_w} - \frac{\alpha_w 2\cos\alpha_w}{\sin^3\alpha_w} + \frac{1}{\sin^2\alpha_w}\right) - \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = -\frac{hw^2}{4}\left(\frac{2}{\sin^2\alpha_w} - \frac{\alpha_w 2\cos\alpha_w}{\sin^3\alpha_w}\right) - \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = -\frac{hw^2}{2\sin\alpha_w}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) - \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = -\left[\frac{hw^2}{2\sin\alpha_w} + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) \qquad \text{[eq45]}$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{4\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left[\frac{1}{\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{4\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin^2\alpha_h} - \frac{\cos\alpha_h}{\sin\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{4\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin^2\alpha_h} - \cot\alpha_h\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{4\sin\alpha_w}\left(\frac{\sin^2\alpha_h - \alpha_h 2\sin\alpha_h\cos\alpha_h}{\sin^4\alpha_h} + \csc^2\alpha_h\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{4\sin\alpha_w}\left(\frac{1}{\sin^2\alpha_h} - \frac{\alpha_h 2\sin\alpha_h\cos\alpha_h}{\sin^4\alpha_h} + \frac{1}{\sin^2\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{4\sin\alpha_w}\left(\frac{2}{\sin^2\alpha_h} - \frac{\alpha_h 2\cos\alpha_h}{\sin^3\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{2\sin\alpha_w}\left(\frac{\sin\alpha_h}{\sin^3\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^4\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{2\sin\alpha_w\sin\alpha_h}\left(\frac{\sin\alpha_h}{\sin^2\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = -\frac{h^2w\alpha_w}{2\sin\alpha_h\sin\alpha_w}\left(\frac{\sin\alpha_h - \alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) \qquad \text{[eq46]}$$

Now if we assume that α_w is changing in such a way that the meniscus now starts to changes from concave to flat (on its way to becoming convex); while α_h remains constant, we can simplify [eq42] to:

$$P_{Stage1}\Big|_{\Delta\alpha_h=0} = -\frac{\left(\partial U_T / \partial \alpha_w\right)}{\left(\partial V / \partial \alpha_w\right)} \qquad \text{[eq47]}$$

Now, substituting [eq43] and [eq45] into [eq47] yields

$$P_{Stage1} = -\frac{\gamma_{la}\left(\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)}{-\left[\frac{hw^2}{2\sin\alpha_w} + \frac{h^2 w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)}$$

$$P_{Stage1} = \frac{\gamma_{la}\left(\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)}{\frac{hw^2}{2\sin\alpha_w} + \frac{h^2 w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)} \qquad \text{[eq48]}$$

Now, if we assume that α_h approaches 0, then we will have

$$P_{Stage1}\Big|_{\alpha_h\to 0} = \frac{\gamma_{la}\left(\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)}{\frac{hw^2}{2\sin\alpha_w} + \frac{h^2 w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)}\Bigg|_{\alpha h\to 0} = \frac{\gamma_{la}\left(\frac{w^2\cos\theta_c}{\sin\alpha_w} + hw\right)}{\left(\frac{hw^2}{2\sin\alpha_w}\right)}$$

$$P_{Stage1}\Big|_{\alpha_h\to 0} = \gamma_{la}\left(\frac{w^2\cos\theta_c}{\sin\alpha_w} + hw\right)\left(\frac{2\sin\alpha_w}{hw^2}\right)$$

$$P_{Stage1}\Big|_{\alpha_h\to 0} = \frac{2\gamma_{la}}{w}\left(\frac{w}{h}\cos\theta_c + \sin\alpha_w\right) \qquad \text{[eq49]} \ (20)$$

Note: $\frac{\alpha_h}{\sin\alpha_h}\Big|_{\alpha_h\to 0} = 1$

Stage 2:

The liquid is in the capillary channel, stopped at the opening. The meniscus becomes convex from the flat transitory stage and starts to expand beyond the opening with the convex profile

The derivative of the equations for this is similar to that of Stage 1. The differences are as such:
For Stage 1: we subtract away the meniscus Area and Volume
For Stage 2: we add on the meniscus Area and Volume

A_{la} is the same as from Stage 1:

$$A_{la} = \frac{hw\alpha_h\alpha_w}{\sin\alpha_h\sin\alpha_w} \qquad \text{[eq50]}$$

A_{sl} can be found by taking the surface of solid-liquid within the capillary if there was no meniscus (A_h and A_w) and <u>adding</u> the surface of liquid-air due to the meniscus (A_{mh} and A_{mw}). We assume the liquid to progress up to length L from the point of reference (right up to the opening).

$$A_{sl} = 2(A_h + A_w + A_{mh} + A_{mw}) \qquad \text{- see Figure 4} \qquad \text{[eq51]}$$

Where $A_h = hL$ - see Figure 4 [eq52]

$A_w = wL$ - see Figure 4 [eq53]

$$A_{mh} = \frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h} \qquad \text{- see Figure 4} \qquad \text{[eq54]}$$

$$A_{mw} = \frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w} \qquad \text{- see Figure 4} \qquad \text{[eq55]}$$

Hence substituting [eq52], [eq53], [eq54], and [eq55] into [eq51] yields,

$$A_{sl} = 2\left(hL + wL + \frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h} + \frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w}\right)$$

$$A_{sl} = 2\left[(h+w)L + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) + \frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right] \qquad \text{[eq56]}$$

Now, substituting [eq50] and [eq56] into [eq6] yields

$$U_T = U_0 + \left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h \sin\alpha_w} - 2\begin{bmatrix} (h+w)L + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \\ + \frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) \end{bmatrix}\cos\theta_c \right)\gamma_{la}$$

$$U_T = U_0 - 2\cos\theta_c\gamma_{la}\begin{bmatrix} (h+w)L + \frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) \\ + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \end{bmatrix} + \gamma_{la}\left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h \sin\alpha_w}\right) \quad \text{[eq57]}$$

The volume V can be found by taking the volume of liquid within the capillary if there was no meniscus (V_{hw}) and <u>adding</u> the volume of displaced liquid due to the meniscus (V_{mw} and V_{mh}). We assume the liquid to progress up to length L from the point of reference:

$$V = V_{hw} + V_{mw} + V_{mh} \quad \text{[eq58]}$$

where $V_{wh} = hwL$ [eq59]

$$V_{mw} = A_{mw} \times h = \left(\frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w}\right)h \quad \text{[eq60]}$$

$$V_{mh} = A_{mh} \times CL_w = \left(\frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h}\right)\frac{w\alpha_w}{\sin\alpha_w} \quad \text{[eq61]}$$

Hence substituting [eq59], [eq60], and [eq61] into [eq58] yields

$$V = hwL + \left(\frac{w^2\alpha_w}{4\sin^2\alpha_w} - \frac{w^2\cos\alpha_w}{4\sin\alpha_w}\right)h + \left(\frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h}\right)\frac{w\alpha_w}{\sin\alpha_w}$$

$$V = hwL + \frac{hw^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) + \frac{h^2 w\alpha_w}{4\sin\alpha_h\sin\alpha_w}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \quad \text{[eq62]}$$

Finally, we can apply [eq57] and [62] into

$$P_{Stage2} = -\frac{dU_T}{dV} \quad \text{[eq63]}$$

To simplify the equation, as A_{la}, A_{sl} and V are functions of α_w and α_h, we can obtain the burst pressure by letting *f* to be a function of α_w and α_h.

Hence

$$P_{Stage2} = -\frac{dU_T}{dV} = -\frac{dU_T/df}{dV/df} \quad \text{[eq64]}$$

$$P_{Stage2} = -\frac{\left(\partial U_T/\partial\alpha_w\right) + \left(\partial U_T/\partial\alpha_h\right)\left[\left(\partial f/\partial\alpha_w\right)\left(\partial f/\partial\alpha_h\right)^{-1}\right]}{\left(\partial V/\partial\alpha_w\right) + \left(\partial V/\partial\alpha_h\right)\left[\left(\partial f/\partial\alpha_w\right)\left(\partial f/\partial\alpha_h\right)^{-1}\right]} \quad \text{[eq65]}$$

Now from [eq57], we can derive $\frac{\partial U_T}{\partial\alpha_w}$ and $\frac{\partial U_T}{\partial\alpha_h}$:

$$\frac{\partial U_T}{\partial\alpha_w} = \frac{\partial}{\partial\alpha_w}\left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h\sin\alpha_w}\right)\gamma_{la} - 2\cos\theta_c\gamma_{la}\frac{\partial}{\partial\alpha_w}\left[+\frac{w^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right]$$

$$\frac{\partial U_T}{\partial\alpha_w} = \frac{hw\alpha_h}{\sin\alpha_h}\frac{\partial}{\partial\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}\right)\gamma_{la} - \frac{\cos\theta_c\gamma_{la}w^2}{2}\frac{\partial}{\partial\alpha_w}\left(\frac{\alpha_w}{\sin^2\alpha_w} - \frac{\cos\alpha_w}{\sin\alpha_w}\right)$$

$$\frac{\partial U_T}{\partial\alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\frac{\partial}{\partial\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}\right) - \gamma_{la}\frac{w^2\cos\theta_c}{2}\frac{\partial}{\partial\alpha_w}\left(\frac{\alpha_w}{\sin^2\alpha_w} - \cot\alpha_w\right)$$

$$\frac{\partial U_T}{\partial\alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) - \gamma_{la}\frac{w^2\cos\theta_c}{2}\left(\frac{\sin^2\alpha_w - \alpha_w 2\sin\alpha_w\cos\alpha_w}{\sin^4\alpha_w} + \csc^2\alpha_w\right)$$

$$\frac{\partial U_T}{\partial\alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) - \gamma_{la}\frac{w^2\cos\theta_c}{2}\left(\frac{1}{\sin^2\alpha_w} - \frac{\alpha_w 2\cos\alpha_w}{\sin^3\alpha_w} + \frac{1}{\sin^2\alpha_w}\right)$$

$$\frac{\partial U_T}{\partial\alpha_w} = \gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) - \gamma_{la}\frac{w^2\cos\theta_c}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial U_T}{\partial\alpha_w} = \gamma_{la}\left(-\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) \quad \text{[eq66]}$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{\partial}{\partial \alpha_h}\left(\frac{hw\alpha_h\alpha_w}{\sin\alpha_h \sin\alpha_w}\right)\gamma_{la} - 2\cos\theta_c\gamma_{la}\frac{\partial}{\partial \alpha_h}\left[\frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w}{\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}\right)\gamma_{la} - \frac{h^2\cos\theta_c\gamma_{la}}{2}\frac{\partial}{\partial \alpha_h}\left[\left(\frac{\alpha_h}{\sin^2\alpha_h} - \frac{\cos\alpha_h}{\sin\alpha_h}\right)\right]$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w}{\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}\right)\gamma_{la} - \frac{h^2\cos\theta_c\gamma_{la}}{2}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin^2\alpha_h} - \cot\alpha_h\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{\sin\alpha_h - \alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) - \frac{h^2\cos\theta_c\gamma_{la}}{2}\left(\frac{\sin^2\alpha_h - \alpha_h 2\sin\alpha_h\cos\alpha_h}{\sin^4\alpha_h} + ccs^2\alpha_h\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) - \frac{h^2\cos\theta_c\gamma_{la}}{2}\left(\frac{1}{\sin^2\alpha_h} - \frac{\alpha_h 2\cos\alpha_h}{\sin^3\alpha_h} + \frac{1}{\sin^2\alpha_h}\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) - \frac{h^2\cos\theta_c\gamma_{la}}{2}\left(\frac{2}{\sin^2\alpha_h} - \frac{\alpha_h 2\cos\alpha_h}{\sin^3\alpha_h}\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \frac{hw\alpha_w\gamma_{la}}{\sin\alpha_w}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) - \frac{h^2\cos\theta_c\gamma_{la}}{\sin^2\alpha_h}\left(\frac{1}{\sin\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right)$$

$$\frac{\partial U_T}{\partial \alpha_h} = \gamma_{la}\left(-\frac{h^2\cos\theta_c}{\sin\alpha_h} + \frac{hw\alpha_w}{\sin\alpha_w}\right)\left(\frac{\sin\alpha_h - \alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) \qquad \text{[eq67]}$$

Now from [eq62], we can derive $\frac{\partial V}{\partial \alpha_w}$ and $\frac{\partial V}{\partial \alpha_h}$:

$$\frac{\partial V}{\partial \alpha_w} = \frac{\partial}{\partial \alpha_w}\frac{hw^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right) + \frac{\partial}{\partial \alpha_w}\frac{h^2w\alpha_w}{4\sin\alpha_h\sin\alpha_w}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)$$

$$\frac{\partial V}{\partial \alpha_w} = \frac{hw^2}{4}\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin^2\alpha_w} - \cot\alpha_w\right) + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{\partial}{\partial \alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = \frac{hw^2}{4}\left(\frac{\sin^2\alpha_w - \alpha_w 2\sin\alpha_w\cos\alpha_w}{\sin^4\alpha_w} + ccs^2\alpha_w\right) + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = \frac{hw^2}{4}\left(\frac{1}{\sin^2\alpha_w} - \frac{\alpha_w 2\cos\alpha_w}{\sin^3\alpha_w} + \frac{1}{\sin^2\alpha_w}\right) + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = \frac{hw^2}{4}\left(\frac{2}{\sin^2\alpha_w} - \frac{\alpha_w 2\cos\alpha_w}{\sin^3\alpha_w}\right) + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = \frac{hw^2}{2\sin\alpha_w}\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left(\frac{1}{\sin\alpha_w} - \frac{\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)$$

$$\frac{\partial V}{\partial \alpha_w} = \left[\frac{hw^2}{2\sin\alpha_w} + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right) \qquad \text{[eq68]}$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{4\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left[\frac{1}{\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{4\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin^2\alpha_h} - \frac{\cos\alpha_h}{\sin\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{4\sin\alpha_w}\frac{\partial}{\partial \alpha_h}\left(\frac{\alpha_h}{\sin^2\alpha_h} - \cot\alpha_h\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{4\sin\alpha_w}\left(\frac{\sin^2\alpha_h - \alpha_h 2\sin\alpha_h\cos\alpha_h}{\sin^4\alpha_h} + \csc^2\alpha_h\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{4\sin\alpha_w}\left(\frac{1}{\sin^2\alpha_h} - \frac{\alpha_h 2\sin\alpha_h\cos\alpha_h}{\sin^4\alpha_h} + \frac{1}{\sin^2\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{4\sin\alpha_w}\left(\frac{2}{\sin^2\alpha_h} - \frac{\alpha_h 2\cos\alpha_h}{\sin^3\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{2\sin\alpha_w}\left(\frac{\sin\alpha_h}{\sin^3\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^4\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{2\sin\alpha_w\sin\alpha_h}\left(\frac{\sin\alpha_h}{\sin^2\alpha_h} - \frac{\alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right)$$

$$\frac{\partial V}{\partial \alpha_h} = \frac{h^2w\alpha_w}{2\sin\alpha_h\sin\alpha_w}\left(\frac{\sin\alpha_h - \alpha_h\cos\alpha_h}{\sin^2\alpha_h}\right) \qquad \text{[eq69]}$$

Now if we assume that α_w is changing in such a way that the meniscus changes from almost flat to convex, while α_h remains constant, we can simplify [eq65] to:

$$P_{Stage2}\Big|_{\Delta\alpha_h=0} = -\frac{\left(\partial U_T / \partial\alpha_w\right)}{\left(\partial V / \partial\alpha_w\right)} \qquad \text{[eq70]}$$

Now, substituting [eq66] and [eq68] into [eq70] yields

$$P_{Stage2} = -\frac{\gamma_{la}\left(-\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)}{\left[\frac{hw^2}{2\sin\alpha_w} + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\right]\left(\frac{\sin\alpha_w - \alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)}$$

$$P_{Stage2} = -\frac{\gamma_{la}\left(-\frac{w^2\cos\theta_c}{\sin\alpha_w} + \frac{hw\alpha_h}{\sin\alpha_h}\right)}{\frac{hw^2}{2\sin\alpha_w} + \frac{h^2w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)} \qquad \text{[eq71]}$$

Now, if we assume that α_h approaches 0, then we will have

$$P_{Stage2}\Big|_{\alpha_h\to 0} = -\frac{\gamma_{la}\left(-\frac{w^2\cos\theta_c}{\sin\alpha_w}+\frac{hw\alpha_h}{\sin\alpha_h}\right)}{\frac{hw^2}{2\sin\alpha_w}+\frac{h^2 w}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}-\cos\alpha_h\right)}\Bigg|_{\alpha h\to 0} = -\frac{\gamma_{la}\left(-\frac{w^2\cos\theta_c}{\sin\alpha_w}+hw\right)}{\left(\frac{hw^2}{2\sin\alpha_w}\right)}$$

$$P_{Stage2}\Big|_{\alpha_h\to 0} = -\gamma_{la}\left(-\frac{w^2\cos\theta_c}{\sin\alpha_w}+hw\right)\left(\frac{2\sin\alpha_w}{hw^2}\right)$$

$$P_{Stage2}\Big|_{\alpha_h\to 0} = -\frac{2\gamma_{la}}{w}\left(-\frac{w}{h}\cos\theta_c+\sin\alpha_w\right)$$

$$P_{Stage2}\Big|_{\alpha_h\to 0} = \frac{2\gamma_{la}}{w}\left(\frac{w}{h}\cos\theta_c-\sin\alpha_w\right)$$ [eq72] (22)

Mistake in Chen *et al* (2008) eq (15)

$$p = -\left(\frac{\partial U_T}{\partial\alpha_w}\right)\left(\frac{\partial V_l}{\partial\alpha_w}\right)^{-1}_{\alpha_h=0} = -\left\{\begin{array}{l}\gamma_{la}\frac{w^2\cos\theta_c}{\sin\alpha_w}\left(\frac{\sin\alpha_w-\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)\\ +\gamma_{la}\frac{hw\alpha_h}{\sin\alpha_h}\left(\frac{\sin\alpha_w-\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)\end{array}\right\}\left\{-\frac{hw^2}{2\sin\alpha_w}\left(\frac{\sin\alpha_w-\alpha_w\cos\alpha_w}{\sin^2\alpha_w}\right)\right\}^{-1} \quad (15)$$

$$= \frac{2\gamma_{la}}{w}\left[\left(\frac{w}{h}\right)\cos\theta_c+\sin\alpha_w\right]$$

Note: $\frac{\alpha_h}{\sin\alpha_h}\Big|_{\alpha_h\to 0} = 1$

Stage 3:

The liquid has expanded beyond the border of the opening (opening with an angle β), with a convex meniscus

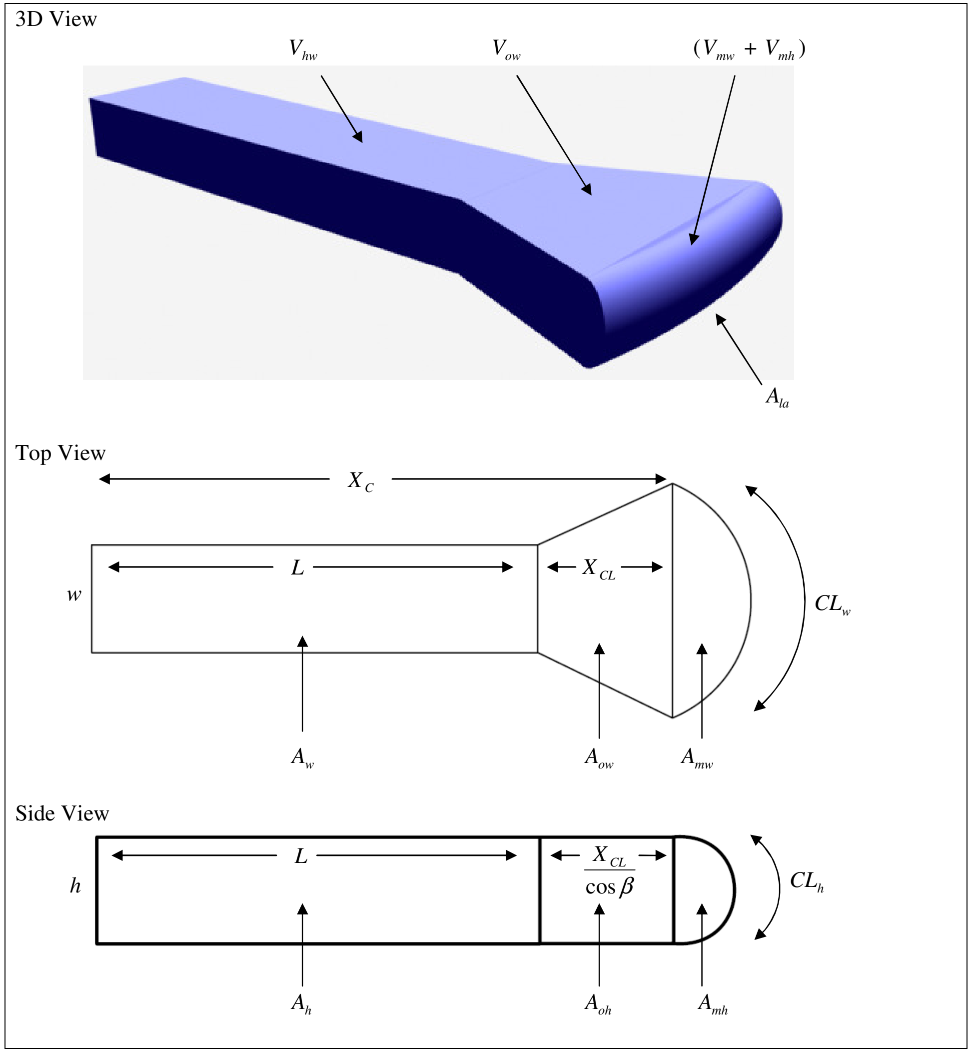

Figure 7: Model for Stage 2

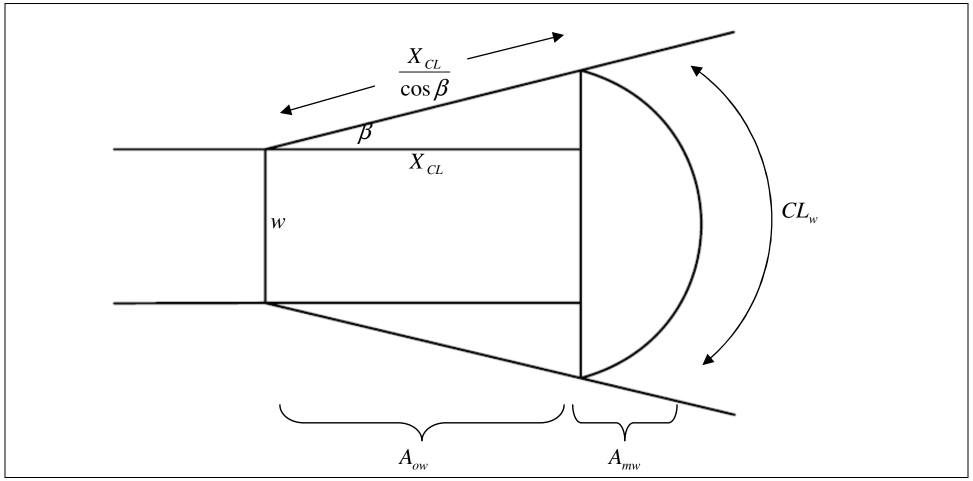

Figure 8: Model for Stage 2 Meniscus

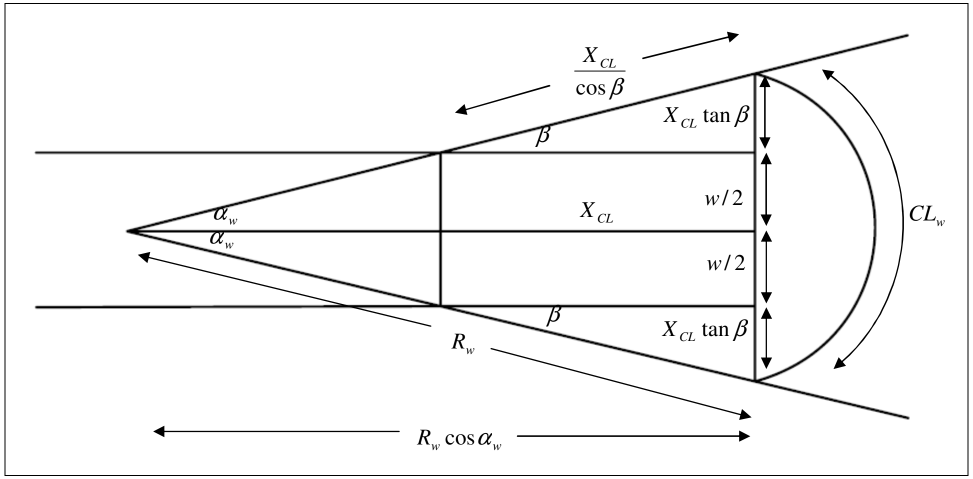

Figure 9: Mathematical Model for Stage 2 Meniscus

According to [eq6] and [eq8], we need to determine A_{la} and A_{sl} for $U_T = U_0 + (A_{la} - A_{sl}\cos\theta_c)\gamma_{la}$, and we also need to determine the volume of the liquid, *V*.

A_{sl} can be found by taking the surface of solid-liquid within the capillary if there was no meniscus (A_h and A_w) and adding the surface of solid-liquid due to the opening (A_{oh} and A_{ow}), and due to the meniscus (A_{mh} and A_{mw}).

$$A_{sl} = 2(A_h + A_w + A_{oh} + A_{ow} + A_{mh} + A_{mw}) \quad \text{- see Figure 7} \quad \text{[eq73]}$$

Where $A_h = hL$ - see Figure 7 [eq74]

$A_w = wL$ - see Figure 7 [eq75]

$A_{oh} = \left(\frac{X_{CL}}{\cos\beta}\right)\times h$ - see Figure 7, 8 [eq76]

$A_{ow} = (X_{CL})\times w + X_{CL}\times(X_{CL}\tan\beta)$ - see Figure 7, 8 [eq77]

$$A_{mh} = A_{Sector} - A_{Triangle} \quad \text{[eq78]}$$

Where $A_{Sector} = \textit{angle of curvature} \times \textit{radius of curvature}$

$$A_{Sector} = \frac{2\alpha_h}{2\pi}(\pi)\times R_h^{\ 2} = \alpha_h R_h^{\ 2}$$

$$A_{Sector} = \alpha_h\left(\frac{h/2}{\sin\alpha_h}\right)^2 = \frac{h^2\alpha_h}{4\sin\alpha_h} \quad \text{[eq79]}$$

And $A_{Triangle} = \frac{1}{2}\times \textit{height} \times \textit{base}$

$$A_{Triangle} = \frac{1}{2}\times R_h\cos\alpha_h \times h = \frac{h}{2}\times\frac{h/2}{\sin\alpha_h}\cos\alpha_h$$

$$A_{Triangle} = \frac{h^2\cos\alpha_h}{4\sin\alpha_h} \quad \text{[eq31g]}$$

$$A_{mh} = \frac{h^2\alpha_h}{4\sin^2\alpha_h} - \frac{h^2\cos\alpha_h}{4\sin\alpha_h} = \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right) \quad \text{[eq80]}$$

$$A_{mw} = A_{Sector} - A_{Triangle} \quad \text{[eq81]}$$

Where $A_{Sector} = \textit{angle of curvature} \times \textit{radius of curvature}$

$A_{Sector} = \frac{2\alpha_w}{2\pi}(\pi)\times R_w^{\ 2} = \alpha_w R_w^{\ 2}$ - see Figure 9

$$A_{Sector} = \alpha_w\left(\frac{\left(\frac{w}{2}\right) + X_{CL}\tan\beta}{\sin\alpha_w}\right)^2 = \alpha_w\left(\frac{w + 2X_{CL}\tan\beta}{2\sin\alpha_w}\right)^2$$

$$A_{Sector} = \frac{(w + 2X_{CL}\tan\beta)^2\alpha_w}{2\sin\alpha_w} \quad \text{[eq82]}$$

And $$A_{Triangle} = \frac{1}{2} \times height \times base \qquad \text{[eq83]}$$

$$A_{Triangle} = \frac{1}{2} \times \left(R_w \cos \alpha_w\right) \times 2\left(\left(w/2\right) + X_{CL} \tan \beta\right) \qquad \text{- see Figure 7}$$

$$A_{Triangle} = \left(\frac{\left(w/2\right) + X_{CL} \tan \beta}{\sin \alpha_w}\right) \cos \alpha_w \times \left(\left(w/2\right) + X_{CL} \tan \beta\right)$$

$$A_{Triangle} = \left(\frac{w + 2X_{CL} \tan \beta}{2 \sin \alpha_w}\right) \cos \alpha_w \times \left(\frac{w + 2X_{CL} \tan \beta}{2}\right)$$

$$A_{Triangle} = \frac{\left(w + 2X_{CL} \tan \beta\right)^2 \cos \alpha_w}{4 \sin \alpha_w} \qquad \text{[eq84]}$$

$$A_{mw} = \frac{\left(w + 2X_{CL} \tan \beta\right)^2 \alpha_w}{4 \sin^2 \alpha_w} - \frac{\left(w + 2X_{CL} \tan \beta\right)^2 \cos \alpha_w}{4 \sin \alpha_w}$$

$$A_{mw} = \frac{\left(w + 2X_{CL} \tan \beta\right)^2}{4 \sin \alpha_w} \left(\frac{\alpha_w}{\sin \alpha_w} - \cos \alpha_w\right) \qquad \text{[eq85]}$$

Hence substituting [eq74], [eq75], [eq76], [eq77], [eq80], and [eq85] into [eq73] yields:

$$A_{sl} = 2\left(\begin{array}{l} hL + wL + \dfrac{X_{CL}}{\cos \beta} h + X_{CL} w + X_{CL}{}^2 \tan \beta + \dfrac{h^2}{4 \sin \alpha_h}\left(\dfrac{\alpha_h}{\sin \alpha_h} - \cos \alpha_h\right) \\ + \dfrac{\left(w + 2X_{CL} \tan \beta\right)^2}{4 \sin \alpha_w}\left(\dfrac{\alpha_w}{\sin \alpha_w} - \cos \alpha_w\right) \end{array}\right)$$

$$A_{sl} = 2\left(\begin{array}{l} L(h+w) + X_{CL}\left(\dfrac{h}{\cos \beta} + w + X_{CL} \tan \beta\right) + \dfrac{\left(w + 2X_{CL} \tan \beta\right)^2}{4 \sin \alpha_w}\left(\dfrac{\alpha_w}{\sin \alpha_w} - \cos \alpha_w\right) \\ + \dfrac{h^2}{4 \sin \alpha_h}\left(\dfrac{\alpha_h}{\sin \alpha_h} - \cos \alpha_h\right) \end{array}\right) \qquad \text{[eq86]}$$

A_{la} can be found by multiplying the curve length of height, CL_h, and the curved length of width, CL_w:

$$A_{la} = CL_h \times CL_w \qquad \text{- see Figure 7} \qquad \text{[eq87]}$$

Note: this is the same method as in [eq18]

Let the curved lengths have angles of $2\alpha_w$ and $2\alpha_h$, and curve radii of R_w and R_h respectively.
We can derive the following:

$$R_h = \frac{h/2}{\sin \alpha_h} \qquad \text{[eq88]}$$

$$R_w = \frac{\left(w/2\right) + X_{CL} \tan \beta}{\sin \alpha_w} = \frac{\left(w + 2X_{CL} \tan \beta\right)}{2 \sin \alpha_w} \qquad \text{- see Figure 9} \qquad \text{[eq89]}$$

$$CL_h = R_h \times 2\alpha_h = \frac{h\alpha_h}{\sin \alpha_h} \qquad \text{[eq90]}$$

$$CL_w = R_w \times 2\alpha_w = \frac{\left(w + 2X_{CL} \tan \beta\right)\alpha_w}{\sin \alpha_w} \qquad \text{- see Figure 9} \qquad \text{[eq91]}$$

Substituting [eq90] and [eq91] into [eq87] yields:

$$A_{la} = \frac{h\left(w + 2X_{CL} \tan \beta\right)\alpha_h \alpha_w}{\sin \alpha_h \sin \alpha_w} \qquad \text{[eq92]}$$

Note: The derivation of [eq89] and [eq91] are similar to that of [eq20] and [eq22], except with different representation of "w".

Now, substituting [eq86] and [eq92] into [eq6] yields,

$$U_T = U_0 + \left[\frac{h\left(w + 2X_{CL} \tan \beta\right)\alpha_h \alpha_w}{\sin \alpha_h \sin \alpha_w} - 2\left(\begin{array}{l} L(h+w) + X_{CL}\left(\dfrac{h}{\cos \beta} + w + X_{CL} \tan \beta\right) \\ + \dfrac{\left(w + 2X_{CL} \tan \beta\right)^2}{4 \sin \alpha_w}\left(\dfrac{\alpha_w}{\sin \alpha_w} - \cos \alpha_w\right) \\ + \dfrac{h^2}{4 \sin \alpha_h}\left(\dfrac{\alpha_h}{\sin \alpha_h} - \cos \alpha_h\right) \end{array}\right) \cos \theta_c\right] \gamma_{la}$$

$$U_T = U_0 + \gamma_{la}\left[\left(w + 2X_{CL} \tan \beta\right)\frac{h\alpha_h \alpha_w}{\sin \alpha_h \sin \alpha_w}\right] - 2\gamma_{la} \cos \theta_c \left(\begin{array}{l} L(h+w) + X_{CL}\left(\dfrac{h}{\cos \beta} + w + X_{CL} \tan \beta\right) \\ + \dfrac{\left(w + 2X_{CL} \tan \beta\right)^2}{4 \sin \alpha_w}\left(\dfrac{\alpha_w}{\sin \alpha_w} - \cos \alpha_w\right) \\ + \dfrac{h^2}{4 \sin \alpha_h}\left(\dfrac{\alpha_h}{\sin \alpha_h} - \cos \alpha_h\right) \end{array}\right)$$

[eq93] (24)

Mistake in Chen et al (2008) eq (16)

$$U_T = U_0 - 2\gamma_{la} \cos \theta_c \times \left[\begin{array}{l} L(h+w) + x_{cL}\left(\dfrac{h}{\cos \beta} + w + x_{cL} \tan \beta\right) \\ - \dfrac{\left(w + 2x_{cL} \tan \beta\right)^2}{4 \sin \alpha_w}\left(\dfrac{\alpha_w}{\sin \alpha_w} - \cos \alpha_w\right) \\ - \dfrac{h^2}{4 \sin \alpha_h}\left(\dfrac{\alpha_h}{\sin \alpha_h} - \cos \alpha_h\right) \end{array}\right] + \gamma_{la}\left[\left(w + 2x_{cL} \tan \beta\right)\frac{h\alpha_w \alpha_h}{\sin \alpha_w \sin \alpha_h}\right] \qquad (16)$$

The volume V can be found by taking the volume of liquid within the capillary (V_{hw}) and adding the volume of liquid at the opening (V_{ow}) and the volume of the liquid in the meniscus (V_{mw} and V_{mh}). We assume the liquid to progress up to length Xc from the point of reference (and $X_{CL} = X_C - L$):

$$V = V_{hw} + V_{ow} + (V_{mw} + V_{mh}) \quad \text{- see Figure 5} \qquad \text{[eq94]}$$

Where $V_{wh} = hwL$ - see Figure 5, 6, 7 [eq95]

$$V_{ow} = A_{ow}h = \left(X_{CL}w + X_{CL}{}^2 \tan\beta\right)h \quad \text{- see Figure 5, 6, 7} \qquad \text{[eq96]}$$

$$V_{mw} = A_{mw} \times h = \frac{(w + 2X_{CL}\tan\beta)^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)h \quad \text{- see Figure 5, 6, 7} \qquad \text{[eq97]}$$

$$V_{mh} = A_{mh} \times CL_w = \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{(w + 2X_{CL}\tan\beta)\alpha_w}{\sin\alpha_w} \quad \text{- see Figure 5, 6, 7} \qquad \text{[eq98]}$$

Hence substituting [eq95], [eq96], [eq97], and [eq98] into [eq94] yields:

$$V = \left(\begin{aligned} & hwL + \left(X_{CL}w + X_{CL}{}^2\tan\beta\right)h + \frac{(w+2X_{CL}\tan\beta)^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)h \\ & + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{(w+2X_{CL}\tan\beta)\alpha_w}{\sin\alpha_w} \end{aligned}\right)$$

$$V = \left(\begin{aligned} & h\left[wL + X_{CL}w + X_{CL}{}^2\tan\beta + \frac{(w+2X_{CL}\tan\beta)^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right] \\ & + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{(w+2X_{CL}\tan\beta)\alpha_w}{\sin\alpha_w} \end{aligned}\right) \qquad \text{[eq99] (25)}$$

Mistake in Chen *et al* (2008) eq (17)

$$V_l = h\left[wL + wx_{cL} + x_{cL}^2\tan\beta - \frac{(w+2x_{cL}\tan\beta)^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right] - \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\left[(w + 2x_{cL}\tan\beta)\frac{\alpha_w}{\sin\alpha_w}\right] \qquad (17)$$

To simplify [eq7] for our purpose, as A_{la}, A_{sl} and V are functions of X_{CL}, we can obtain the burst pressure as follows:

$$P_{burst} = -\frac{dU_T}{dV} = -\frac{\partial U_T / \partial X_{CL}}{\partial V / \partial X_{CL}} \qquad \text{[eq100]}$$

Now from [eq93], we can derive $\frac{\partial U}{\partial X_{CL}}$:

$$\frac{\partial U_T}{\partial X_{CL}} = \frac{\partial}{\partial X_{CL}}\left(\gamma_{la}\left[(2X_{CL}\tan\beta)\frac{h\alpha_h\alpha_w}{\sin\alpha_h\sin\alpha_w}\right] - 2\gamma_{la}\cos\theta_c\left(\begin{aligned} & X_{CL}\left(\frac{h}{\cos\beta} + w + X_{CL}\tan\beta\right) \\ & + \frac{(w+2X_{CL}\tan\beta)^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\end{aligned}\right)\right)$$

$$\frac{\partial U_T}{\partial X_{CL}} = \gamma_{la}(2\tan\beta)\frac{h\alpha_h\alpha_w}{\sin\alpha_h\sin\alpha_w} - 2\gamma_{la}\cos\theta_c\left(\begin{aligned} & \frac{h}{\cos\beta} + w + 2X_{CL}\tan\beta \\ & + \frac{2(w+2X_{CL}\tan\beta)}{4\sin\alpha_w}(2\tan\beta)\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\end{aligned}\right)$$

$$\frac{\partial U_T}{\partial X_{CL}} = hw\left[\frac{2\gamma_{la}}{w}\frac{\alpha_h\alpha_w\tan\beta}{\sin\alpha_h\sin\alpha_w} - \frac{2\gamma_{la}}{w}\cos\theta_c\left(\begin{aligned} & \frac{1}{\cos\beta} + \frac{w}{h} + \frac{2X_{CL}}{h}\tan\beta \\ & + \frac{\tan\beta}{\sin\alpha_w}\left(\frac{w}{h} + \frac{2X_{CL}}{h}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\end{aligned}\right)\right] \qquad \text{[eq101]}$$

Now from [eq99], we can derive $\frac{\partial V}{\partial X_{CL}}$:

$$\frac{\partial V}{\partial X_{CL}} = \frac{\partial}{\partial X_{CL}}\left(\begin{aligned} & h\left[wL + X_{CL}w + X_{CL}{}^2\tan\beta + \frac{(w+2X_{CL}\tan\beta)^2}{4\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right] \\ & + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{(2X_{CL}\tan\beta)\alpha_w}{\sin\alpha_w}\end{aligned}\right)$$

$$\frac{\partial V}{\partial X_{CL}} = \left(\begin{aligned} & h\left[w + 2X_{CL}\tan\beta + \frac{2(w+2X_{CL}\tan\beta)}{4\sin\alpha_w}(2\tan\beta)\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right] \\ & + \frac{h^2}{4\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\frac{(2\tan\beta)\alpha_w}{\sin\alpha_w}\end{aligned}\right)$$

$$\frac{\partial V}{\partial X_{CL}} = hw\left(\begin{aligned} & \left[1 + \frac{2X_{CL}}{w}\tan\beta + \frac{\tan\beta}{\sin\alpha_w}\left(1 + \frac{2X_{CL}}{w}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right] \\ & + \frac{h}{2w}\frac{\alpha_w\tan\beta}{\sin\alpha_w\sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h} - \cos\alpha_h\right)\end{aligned}\right) \qquad \text{[eq102]}$$

Substituting [eq101] and [eq102] into [eq100] yields:

$$P_{burst} = -\frac{dU_T}{dV} = -\frac{hw\left[\frac{2\gamma_{la}}{w}\frac{\alpha_h\alpha_w \tan\beta}{\sin\alpha_h \sin\alpha_w} - \frac{2\gamma_{la}}{w}\cos\theta_c\left(\begin{array}{l}\frac{1}{\cos\beta}+\frac{w}{h}+\frac{2X_{CL}}{h}\tan\beta \\ +\frac{\tan\beta}{\sin\alpha_w}\left(\frac{w}{h}+\frac{2X_{CL}}{h}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\end{array}\right)\right]}{hw\left(\begin{array}{l}\left[1+\frac{2X_{CL}}{w}\tan\beta+\frac{\tan\beta}{\sin\alpha_w}\left(1+\frac{2X_{CL}}{w}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right] \\ +\frac{h}{2w}\frac{\alpha_w \tan\beta}{\sin\alpha_w \sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}-\cos\alpha_h\right)\end{array}\right)}$$

$$P_{burst} = -\left[\begin{array}{l}\frac{2\gamma_{la}}{w}\frac{\alpha_h\alpha_w \tan\beta}{\sin\alpha_h \sin\alpha_w} \\ -\frac{2\gamma_{la}}{w}\cos\theta_c\left(\frac{1}{\cos\beta}+\frac{w}{h}+\frac{2X_{CL}}{h}\tan\beta+\frac{\tan\beta}{\sin\alpha_w}\left(\frac{w}{h}+\frac{2X_{CL}}{h}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right)\end{array}\right]$$
$$\times\left(\left[1+\frac{2X_{CL}}{w}\tan\beta+\frac{\tan\beta}{\sin\alpha_w}\left(1+\frac{2X_{CL}}{w}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right]+\frac{h}{2w}\frac{\alpha_w \tan\beta}{\sin\alpha_w \sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}-\cos\alpha_h\right)\right)^{-1}$$

[eq103] (26)

Similar to Chen *et al* (2008) eq (18)

$$p = -\left(\frac{\partial U_T}{\partial x_{cL}}\right)\left(\frac{\partial V_1}{\partial x_{cL}}\right)^{-1}$$
$$= \left\{\begin{array}{l}\frac{-2\gamma_{la}}{w}\cos\theta_c\left[\frac{1}{\cos\beta}+\frac{w}{h}+\frac{2x_{cL}}{h}\tan\beta-\frac{\tan\beta}{\sin\alpha_w}\left(\frac{w}{h}+\frac{2x_{cL}}{h}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right] \\ +\frac{2\gamma_{la}}{w}\frac{\alpha_w\alpha_h \tan\beta}{\sin\alpha_w \sin\alpha_h}\end{array}\right\} \quad (18)$$
$$\times\left[1+\frac{2x_{cL}}{w}\tan\beta-\frac{\tan\beta}{\sin\alpha_w}\left(1+\frac{2x_{cL}}{w}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)-\frac{h}{2w}\frac{\alpha_w \tan\beta}{\sin\alpha_w \sin\alpha_h}\left(\frac{\alpha_h}{\sin\alpha_h}-\cos\alpha_h\right)\right]^{-1}$$

If we consider the pressure when X_{CL} and α_h both approach 0, [eq103] then reduces to:
(Note: Also See *Alternative*)

$$P_{burst} = -\left[\frac{2\gamma_{la}}{w}\frac{\alpha_w \tan\beta}{\sin\alpha_w} - \frac{2\gamma_{la}}{w}\cos\theta_c\left(\begin{array}{l}\frac{1}{\cos\beta}+\frac{w}{h}+ \\ +\frac{\tan\beta}{\sin\alpha_w}\left(\frac{w}{h}\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\end{array}\right)\right]\times\left(\left[1+\frac{\tan\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right]\right)^{-1}$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[\frac{\frac{\alpha_w \sin\beta}{\sin\alpha_w \cos\beta}-\cos\theta_c\left(\frac{1}{\cos\beta}+\frac{w}{h}+\frac{\sin\beta}{\sin\alpha_w \cos\beta}\left(\frac{w}{h}\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right)}{\left(\left[\frac{\cos\beta}{\cos\beta}+\frac{\sin\beta}{\sin\alpha_w \cos\beta}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right]\right)}\right]$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[\frac{\frac{\alpha_w \sin\beta}{\sin\alpha_w}-\cos\theta_c\left(1+\frac{w}{h}\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{w}{h}\right)\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right)}{\left(\left[\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right]\right)}\right] \quad \textit{multiplied with } \frac{\cos\beta}{\cos\beta}$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[\frac{\frac{\alpha_w \sin\beta}{\sin\alpha_w}-\cos\theta_c-\frac{w}{h}\cos\theta_c\left(\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right)}{\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)}\right]$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[\frac{\frac{\alpha_w \sin\beta}{\sin\alpha_w}-\cos\theta_c}{\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)}-\frac{w}{h}\cos\theta_c\right]$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c+\frac{\cos\theta_c-\frac{\alpha_w \sin\beta}{\sin\alpha_w}}{-\left[\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)\right]}\right]$$

[eq104] (27)

Mistake in Chen et al (2008) eq (20)

$$\Delta p_b = \frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c+\frac{\cos\theta_c-\frac{\alpha_w \sin\beta}{\sin\alpha_w}}{-\cos\beta+\frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w}-\cos\alpha_w\right)}\right] \quad (20)$$

Should the aspect ratio of w/h approaches 0 (h >> w), the equation then reduces to

$$P_{burst} = \frac{2\gamma_{la}}{w}\left[\frac{\cos\theta_c - \dfrac{\alpha_w \sin\beta}{\sin\alpha_w}}{\left[\cos\beta + \dfrac{\sin\beta}{\sin\alpha_w}\left(\dfrac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right]}\right]$$ [eq105] (28)

Alternative

If we consider the case as α_h approaches 0, [eq103] then reduces to**:**

$$P_{burst} = -\left[-\frac{2\gamma_{la}}{w}\cos\theta_c\left(\frac{1}{\cos\beta} + \frac{w}{h} + \frac{2X_{CL}}{h}\tan\beta + \frac{\tan\beta}{\sin\alpha_w}\left(\frac{w}{h} + \frac{2X_{CL}}{h}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right)\right]$$
$$\times\left(\left[1 + \frac{2X_{CL}}{w}\tan\beta + \frac{\tan\beta}{\sin\alpha_w}\left(1 + \frac{2X_{CL}}{w}\tan\beta\right)\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)\right]\right)^{-1}$$

[eq106]

Restriction:

Special Note (see Figure 9):

As [eq104] (and subsequently [eq105]) is derived based on the hidden assumption that $\beta = \alpha_w$, applying $\beta = \alpha_w$ to [eq104] yields:

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c + \frac{\cos\theta_c - \dfrac{\beta\sin\beta}{\sin\beta}}{-\left[\cos\beta + \dfrac{\sin\beta}{\sin\beta}\left(\dfrac{\beta}{\sin\beta} - \cos\beta\right)\right]}\right]$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c + \frac{\cos\theta_c - \beta}{-\left[\cos\beta + \left(\dfrac{\beta}{\sin\beta} - \cos\beta\right)\right]}\right]$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c + \frac{\cos\theta_c - \beta}{-\left[\dfrac{\beta}{\sin\beta}\right]}\right]$$

$$P_{burst} = -\frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c - \left(\frac{\cos\theta_c}{\beta} - 1\right)\sin\beta\right]$$

$$P_{burst} = \frac{2\gamma_{la}}{w}\left[\frac{w}{h}\cos\theta_c + \left(\frac{\cos\theta_c}{\beta} - 1\right)\sin\beta\right]$$ [eq107]

$$P_{burst} = \frac{2\gamma_{la}}{w}\left[\left(\frac{w}{h} + \frac{\sin\beta}{\beta}\right)\cos\theta_c - \sin\beta\right]$$ [eq108] (29)

[eq107] and [eq108] differs from to Chen *et al* (2008) eq (20)

$$\Delta p_b = \frac{2\gamma_{la}}{w}\left[-\frac{w}{h}\cos\theta_c + \frac{\cos\theta_c - \frac{\alpha_w \sin\beta}{\sin\alpha_w}}{-\cos\beta + \frac{\sin\beta}{\sin\alpha_w}\left(\frac{\alpha_w}{\sin\alpha_w} - \cos\alpha_w\right)}\right] \qquad (20)$$

Should the aspect ratio of w/h approaches 0 (h >> w), [eq107] equation then reduces to

$$P_{burst} = \frac{2\gamma_{la}}{w}\left(\frac{\cos\theta_c}{\beta} - 1\right)\sin\beta$$ [eq109]

$$P_{burst} = \frac{2\gamma_{la}}{w}\left[\left(\frac{\sin\beta}{\beta}\right)\cos\theta_c - \sin\beta\right]$$ [eq110] (30)

[eq109] and [eq110] differs from Man *et al* (1998) eq (13)

$$\Delta P = \frac{2\gamma_{la}}{h_n}\left(\frac{\cos\theta_c - \frac{\alpha}{\sin\alpha}\sin\beta}{\cos\beta + \frac{\sin\beta}{\sin\alpha}\left[\frac{\alpha}{\sin\alpha} - \cos\alpha\right]}\right) \qquad (13)$$

Special Consideration of Meniscus Angle Conditions

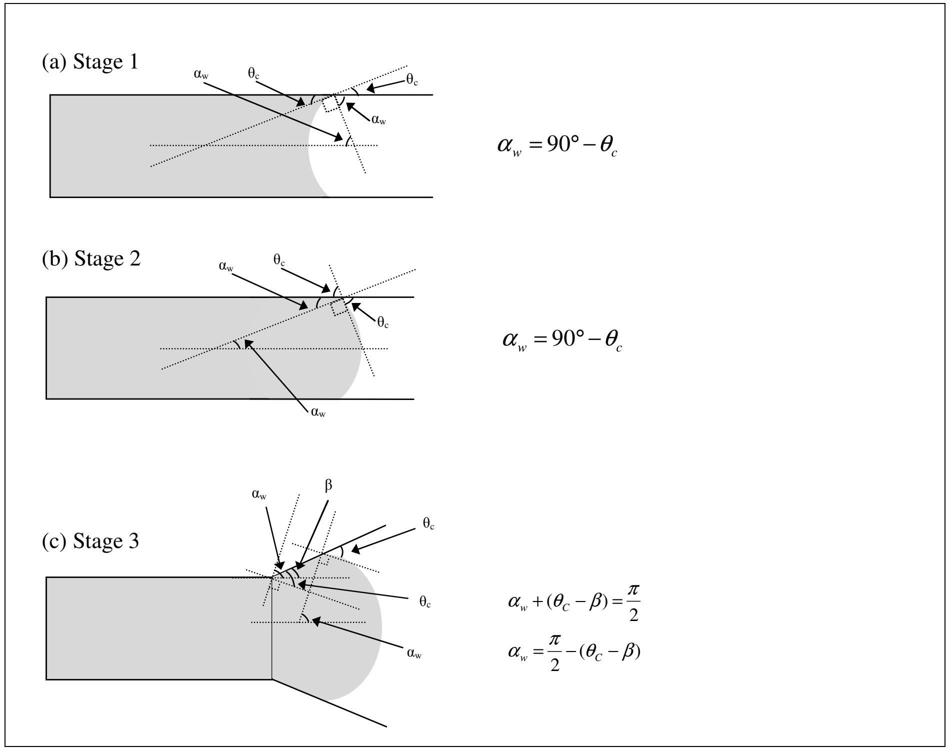

Figure 10 Geometry of meniscus angle for various stages

Applying the Meniscus angle of Stage 1 from Figure 10 into [eq19] and [eq20] yields

$$R_h = \frac{h}{2\cos\theta_C} \qquad \text{[eq111] (18)}$$

$$R_w = \frac{w}{2\cos\theta_C} \qquad \text{[eq112] (19)}$$

Applying the Meniscus angle of Stage 1 from Figure 10 into [eq49] yields

$$P_{Stage1}\Big|_{\alpha_h \to 0} = \frac{2\gamma_{la}}{w}\left(\frac{w}{h} + 1\right)\cos\theta_c \qquad \text{[eq113] (21)}$$

Applying the Meniscus angle of Stage 2 from Figure 10 into [eq72] yields

$$P_{Stage2}\Big|_{\alpha_h \to 0} = \frac{2\gamma_{la}}{w}\left(\frac{w}{h} - 1\right)\cos\theta_c \qquad \text{[eq114] (23)}$$