# Supplementary materials for "A general class of shock models with dependent inter-arrival times"


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**Proof of Lemma 3.3:** Let $\epsilon = u + iv$, where $u > 0$ and $i = \sqrt{-1}$. Then

$$\begin{aligned}
\int_0^\infty \exp\{-(\epsilon x+1)q\}q^{n-1}dq &= \int_0^\infty q^{n-1}\exp\{-(1+ux)q\}\exp\{-ivxq\}dq \\
&= \frac{1}{(-2\pi i)^{n-1}}\mathcal{F}((-2\pi iq)^{n-1}\exp\{-(1+ux)q\}I_0(q))(vx/2\pi) \\
&= \frac{1}{(-2\pi i)^{n-1}}\,\frac{d^{n-1}}{ds^{n-1}}\mathcal{F}(\exp\{-(1+ux)q\}I_0(q))(s)\Big|_{s=vx/2\pi} \\
&= \frac{1}{(-2\pi i)^{n-1}}\,\frac{d^{n-1}}{ds^{n-1}}\left(\frac{1}{2\pi is+(1+ux)}\right)\Big|_{s=vx/2\pi} \\
&= \Gamma(n)\left(\frac{1}{2\pi is+(1+ux)}\right)^n\Big|_{s=vx/2\pi} = \Gamma(n)\left(\frac{1}{ivx+(1+ux)}\right)^n \\
&= \Gamma(n)\left(\frac{1}{1+\epsilon x}\right)^n,
\end{aligned}$$

where the second equality follows from Definition 3.1 and Lemma 3.2 $(ii)$; the third equality follows from Lemma 3.2 $(iii)$, and the fourth equality follows from Lemma 3.2 $(i)$. Moreover, the function $I_0(\cdot)$ is the same as in Lemma 3.2 $(i)$. □

**Proof of Theorem 3.1 $(i)$:** Consider

$$\bar{F}_L(t) = P\left(\sum_{i=1}^M X_i > t\right) = \int_0^\infty P\left(\sum_{i=1}^M X_i > t|Q=q\right) f_Q(q)dq, \tag{A1}$$


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where $Q$ is the mixing distribution of the HPGGP. Note that, for given $Q = q$, the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$ is the same as the HPP with intensity $q\lambda$. Then, from Lemma 3.1, we get

$$P\left(\sum_{i=1}^{M} X_i > t | Q = q\right) = \boldsymbol{a} \exp\{-q\lambda t(\boldsymbol{I} - \boldsymbol{A})\}\boldsymbol{e}. \tag{A2}$$

By using (A2) and (2.1) in (A1), we get

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \int_0^\infty (\boldsymbol{a} \exp\{-q\lambda t(\boldsymbol{I} - \boldsymbol{A})\}\boldsymbol{e}) \frac{q^{k-1} \exp\{-\alpha q\}}{(q+l)^\nu} dq. \tag{A3}$$

It is given that $\boldsymbol{I} - \boldsymbol{A}$ is in the Jordan normal form, i.e., $\boldsymbol{I} - \boldsymbol{A} = \boldsymbol{P} diag(\boldsymbol{J_1}, \boldsymbol{J_2}, \ldots, \boldsymbol{J_r})\boldsymbol{P^{-1}}$, for some $r$, with

$$\boldsymbol{J_i} = \begin{pmatrix} \epsilon_i & 1 & 0 & \ldots & 0 \\ 0 & \epsilon_i & 1 & \ldots & 0 \\ 0 & 0 & \epsilon_i & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & \epsilon_i \end{pmatrix}_{d_i \times d_i}, \quad \epsilon_i > 0, i = 1, 2, \ldots, r.$$

Moreover, the dimension of $d_i$ is equal to the multiplicity of the eigenvalue $\epsilon_i$ that belongs to this particular Jordan block, and $\sum_{i=1}^r d_i = d$. Therefore,

$$\exp\{-q\lambda t(\boldsymbol{I} - \boldsymbol{A})\} = \boldsymbol{P} diag(\exp\{-q\lambda t\boldsymbol{J_1}\}, \exp\{-q\lambda t\boldsymbol{J_2}\}, \ldots, \exp\{-q\lambda t\boldsymbol{J_r}\})\boldsymbol{P^{-1}}, \tag{A4}$$

where

$$\exp\{-q\lambda t\boldsymbol{J_i}\} = \begin{pmatrix} \exp\{-q\lambda t\epsilon_i\} & -q\lambda t \exp\{-q\lambda t\epsilon_i\} & \ldots & \frac{(-1)^{d_i-1}(q\lambda t)^{d_i-1}}{(d_i-1)!} \exp\{-q\lambda t\epsilon_i\} \\ 0 & \exp\{-q\lambda t\epsilon_i\} & \ldots & \frac{(-1)^{d_i-2}(q\lambda t)^{d_i-2}}{(d_i-2)!} \exp\{-q\lambda t\epsilon_i\} \\ 0 & 0 & \ldots & \frac{(-1)^{d_i-3}(q\lambda t)^{d_i-3}}{(d_i-3)!} \exp\{-q\lambda t\epsilon_i\} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \exp\{-q\lambda t\epsilon_i\} \end{pmatrix}_{d_i \times d_i},$$

for $i = 1, 2, \ldots, r$. Again, from (2.2), we can write

$$\int_0^\infty \exp\{-q(\lambda t\epsilon_i + \alpha)\} \frac{q^{n+k-1}}{(q+l)^\nu} dq = \frac{\Gamma_\nu(n+k, (\lambda t\epsilon_i + \alpha)l)}{(\lambda t\epsilon_i + \alpha)^{k+n-\nu}},$$

for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N} \cup \{0\}$. Consequently,

$$\begin{aligned}
&\int_0^\infty \exp\{-q\lambda t \boldsymbol{J_i}\} \exp\{-q\alpha\} \frac{q^{k-1}}{(q+l)^\nu} dq \\
&= \begin{pmatrix} \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} & \frac{-\lambda t\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-1}\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}} \\ 0 & \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-2}\Gamma_\nu(k+d_i-2,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-2-\nu}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} \end{pmatrix}_{d_i\times d_i} \\
&= \boldsymbol{D_i(t)}, \quad i = 1, 2, \ldots, r. \qquad \text{(A5)}
\end{aligned}$$

Now, by using (A4) and (A5) in (A3), we get

$$\bar{F}_L(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right) \boldsymbol{aP} diag(\boldsymbol{D_1(t)}, \boldsymbol{D_2(t)}, \ldots, \boldsymbol{D_r(t)})\boldsymbol{P^{-1}e}$$

and hence, the result is proved. □

**Proof of Theorem 3.1** (*ii*)**:** By proceeding in the same line as in part (*i*), we get, from Remark 2.2 (*b*), that

$$\bar{F}_L(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty (\boldsymbol{a} \exp\{-(qt/b)(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e})\, q^{\beta-1} \exp\{-q\} dq, \qquad \text{(A6)}$$

where

$$\exp\{-(qt/b)(\boldsymbol{I}-\boldsymbol{A})\} = \boldsymbol{P} diag(\exp\{-(qt/b)\boldsymbol{J_1}\}, \exp\{-(qt/b)\boldsymbol{J_2}\}, \ldots, \exp\{-(qt/b)\boldsymbol{J_r}\})\boldsymbol{P^{-1}}$$

and

$$\exp\{-(qt/b)\boldsymbol{J_i}\} = \begin{pmatrix} \exp\{-(qt/b)\epsilon_i\} & -(qt/b)\exp\{-(qt/b)\epsilon_i\} & \cdots & \frac{(-1)^{d_i-1}(qt/b)^{d_i-1}}{(d_i-1)!}\exp\{-(qt/b)\epsilon_i\} \\ 0 & \exp\{-(qt/b)\epsilon_i\} & \cdots & \frac{(-1)^{d_i-2}(qt/b)^{d_i-2}}{(d_i-2)!}\exp\{-(qt/b)\epsilon_i\} \\ 0 & 0 & \cdots & \frac{(-1)^{d_i-3}(qt/b)^{d_i-3}}{(d_i-3)!}\exp\{-(qt/b)\epsilon_i\} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp\{-(qt/b)\epsilon_i\} \end{pmatrix}_{d_i\times d_i},$$

for $i = 1, 2, \ldots, r$. Again, if $\beta \in \mathbb{N}$, then we get, from Lemma 3.3, that

$$\int_0^\infty \exp\{-(\epsilon_i(t/b)+1)q\} q^{n+\beta-1} dq = \Gamma(\beta+n)\left(\frac{b}{b+\epsilon_i t}\right)^{\beta+n},$$

for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N} \cup \{0\}$. Consequently, we can write

$$\begin{aligned}
&\int_0^\infty \exp\{-(qt/b)\boldsymbol{J_i}\}\exp\{-q\}q^{\beta-1}dq \\
&= \begin{pmatrix}
\Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^\beta & -\Gamma(\beta+1)\left(\frac{t}{t\epsilon_i+b}\right)\left(\frac{b}{t\epsilon_i+b}\right)^\beta & \cdots & \frac{\Gamma(\beta+d_i-1)}{(d_i-1)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-1}\left(\frac{b}{t\epsilon_i+b}\right)^\beta \\
0 & \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^\beta & \cdots & \frac{\Gamma(\beta+d_i-2)}{(d_i-2)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-2}\left(\frac{b}{t\epsilon_i+b}\right)^\beta \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^\beta
\end{pmatrix}_{d_i\times d_i}, \\
&= \boldsymbol{\widetilde{D}_i(t)}, \quad i = 1, 2, \ldots, r.
\end{aligned}$$

On using the above equality in (A6), we get

$$\bar{F}_L(t) \quad = \quad \frac{1}{\Gamma(\beta)}\boldsymbol{P}diag(\boldsymbol{\widetilde{D}_1(t)}, \boldsymbol{\widetilde{D}_2(t)}, \ldots, \boldsymbol{\widetilde{D}_r(t)})\boldsymbol{P^{-1}e}$$

and hence, the result is proved for $\beta \in \mathbb{N}$. Now, if $Im(\epsilon_i) = 0$, then $\epsilon_i$ is real and positive for all $i = 1, 2, \ldots, r$. Then, by using part $(i)$ and Remark 2.2 $(b)$, we get the required result. □

**Proof of Theorem 3.2:** Let $\boldsymbol{C_1}, \boldsymbol{C_2}, \ldots, \boldsymbol{C_d}$ be the column vectors of the matrix $\boldsymbol{P}$, and $\boldsymbol{R_1}, \boldsymbol{R_2}, \ldots, \boldsymbol{R_d}$ be the row vectors of the matrix $\boldsymbol{P^{-1}}$. Then we can write

$$\boldsymbol{P} = \begin{pmatrix}\boldsymbol{C_1} & \boldsymbol{C_2} & \ldots & \boldsymbol{C_d}\end{pmatrix} \text{ and } \boldsymbol{P^{-1}} = \begin{pmatrix}\boldsymbol{R_1} \\ \boldsymbol{R_2} \\ \vdots \\ \boldsymbol{R_d}\end{pmatrix}.$$

Now,

$$\begin{aligned}
\boldsymbol{aPD(t)P^{-1}e} &= \boldsymbol{a}\begin{pmatrix}\boldsymbol{C_1} & \boldsymbol{C_2} & \ldots & \boldsymbol{C_d}\end{pmatrix}\boldsymbol{D(t)}\begin{pmatrix}\boldsymbol{R_1} \\ \boldsymbol{R_2} \\ \vdots \\ \boldsymbol{R_d}\end{pmatrix}\boldsymbol{e} \\
&= \begin{pmatrix}\boldsymbol{aC_1} & \ldots & \boldsymbol{aC_d}\end{pmatrix} diag\left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_1+\alpha)l)}{(\lambda t\epsilon_1+\alpha)^{k-\nu}}, \ldots, \frac{\Gamma_\nu(k, (\lambda t\epsilon_d+\alpha)l)}{(\lambda t\epsilon_d+\alpha)^{k-\nu}}\right)\begin{pmatrix}\boldsymbol{R_1e} \\ \vdots \\ \boldsymbol{R_de}\end{pmatrix} \\
&= \sum_{i=1}^d \left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}}\right)(\boldsymbol{aC_iR_ie}) = \sum_{i=1}^d c_i\left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}}\right),
\end{aligned}$$

where $c_i = \boldsymbol{aC_iR_ie}$. Then, from Corollary 3.1, we get

$$\bar{F}_L(t) = \sum_{i=1}^d c_i\left(\frac{\alpha}{\lambda t\epsilon_i+\alpha}\right)^{k-\nu}\frac{\Gamma_\nu(k, (\lambda t\epsilon_i+\alpha)l)}{\Gamma_\nu(k, \alpha l)}.$$

Since all eigenvalues of $\boldsymbol{I}-\boldsymbol{A}$ are real, all entries of $\boldsymbol{P}$ are also real. Hence, $c_i$ is a real constant, for all $i=1,2,3,\ldots,d$. Further,

$$\sum_{i=1}^{d} c_i = \sum_{i=1}^{d} \boldsymbol{aC_iR_ie} = \boldsymbol{a}\begin{pmatrix}\boldsymbol{C_1} & \boldsymbol{C_2} & \ldots & \boldsymbol{C_d}\end{pmatrix}\begin{pmatrix}\boldsymbol{R_1}\\ \boldsymbol{R_2}\\ \vdots \\ \boldsymbol{R_d}\end{pmatrix}\boldsymbol{e} = \boldsymbol{aPP^{-1}e} = 1.$$

Hence, the result is proved. □

**Proof of Theorem 3.3:** From (3.1), we get

$$f_L(t) = -\left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\boldsymbol{aP}diag(\boldsymbol{D_1'(t)},\boldsymbol{D_2'(t)},\ldots,\boldsymbol{D_r'(t)})\boldsymbol{P^{-1}e},$$

where $\boldsymbol{D_i'(t)} = \frac{d}{dt}(\boldsymbol{D_i(t)})$, for all $i=1,2,\ldots,r$. By using (2.2), we get, for every $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}\frac{d}{dt}\left(\frac{(\lambda t)^n\Gamma_\nu(k+n,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}}\right) &= \frac{d}{dt}\left((\lambda t)^n\int_0^\infty \frac{q^{k+n-1}\exp\{-(\lambda t\epsilon_i+\alpha)q\}}{(q+l)^\nu}dq\right)\\ &= -\lambda\epsilon_i(\lambda t)^n\int_0^\infty\frac{q^{k+n}\exp\{-(\lambda t\epsilon_i+\alpha)q\}}{(q+l)^\nu}dq\\ &\quad +(n\lambda)\frac{\Gamma_\nu(k+n,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}}(\lambda t)^{n-1}\\ &= -\lambda\epsilon_i(\lambda t)^n\frac{\Gamma_\nu(k+n+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n+1-\nu}}\\ &\quad +(n\lambda)\frac{\Gamma_\nu(k+n,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}}(\lambda t)^{n-1}.\end{aligned}$$

Consequently,

$$\boldsymbol{D_i'(t)} = -\lambda(\boldsymbol{E_i(t)}-\boldsymbol{S_i(t)}),$$

where

$$\boldsymbol{E_i(t)} = \begin{pmatrix} \frac{\epsilon_i\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \frac{-\lambda t\epsilon_i\Gamma_\nu(k+2,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+2-\nu}} & \ldots & \frac{(-\lambda t)^{d_i-1}\epsilon_i\Gamma_\nu(k+d_i,(\lambda t\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda t\epsilon_i+\alpha)^{k+d_i-\nu}}\\ 0 & \frac{\epsilon_i\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \ldots & \frac{(-\lambda t)^{d_i-2}\epsilon_i\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}}\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \frac{\epsilon_i\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}}\end{pmatrix}$$

and

$$\boldsymbol{S_i(t)} = \begin{pmatrix} 0 & \frac{-\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \ldots & \frac{-(-\lambda t)^{d_i-2}\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}}\\ 0 & 0 & \ldots & \frac{-(-\lambda t)^{d_i-3}\Gamma_\nu(k+d_i-2,(\lambda t\epsilon_i+\alpha)l)}{(d_i-3)!(\lambda t\epsilon_i+\alpha)^{k+d_i-2-\nu}}\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$i = 1, 2, \ldots, r$. Finally, the result follows from the fact that $r_L(t) = f_L(t)/\bar{F}_L(t)$. Hence, the result is proved. □

**Proof of Theorem 3.4:** From Theorem 3.2 $(ii)$, we get

$$\begin{aligned} r_L(t) = \beta \frac{\sum_{i=1}^{d} g_i \epsilon_i \left(\frac{1}{t\epsilon_i + b}\right)^{\beta+1}}{\sum_{i=1}^{d} g_i \left(\frac{1}{t\epsilon_i+b}\right)^{\beta}} &= \beta \sum_{i=1}^{d} g_i \epsilon_i \left( \frac{\left(\frac{1}{t\epsilon_i+b}\right)^{\beta+1}}{\sum_{j=1}^{d} g_j \left(\frac{1}{t\epsilon_j+b}\right)^{\beta}} \right) \\ &= \beta \sum_{i=1}^{d} g_i \epsilon_i \left( \frac{1}{\sum_{j=1}^{d} g_j \left(\frac{t\epsilon_i+b}{t\epsilon_j+b}\right)^{\beta} (t\epsilon_i + b)} \right). \end{aligned}$$

Taking limit $t \to \infty$ on both sides of the above expression, we get $\lim_{t\to\infty} r_L(t) = 0$. □

**Proof of Theorem 3.5:** We have

$$E(L - t | L > t) = \frac{1}{\bar{F}_L(t)} \int_t^{\infty} \bar{F}_L(x) dx.$$

For any $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} &\int_t^{\infty} \frac{(\lambda x)^n \Gamma_\nu(k+n, (\alpha + \lambda x \epsilon_i) l)}{(\alpha + \lambda x \epsilon_i)^{n+k-\nu}} dx \\ &= \int_t^{\infty} \int_0^{\infty} \frac{(\lambda x)^n q^{k+n-1} \exp\{-(\lambda x \epsilon_i + \alpha) q\}}{(q+l)^\nu} dq dx \\ &= \int_0^{\infty} \frac{\lambda^n q^{k+n-1} \exp\{-\alpha q\}}{(q+l)^\nu} \left( \int_t^{\infty} x^n \exp\{-\lambda x \epsilon_i q\} dx \right) dq \\ &= \frac{(\lambda t)^n}{\lambda \epsilon_i} \int_0^{\infty} \frac{q^{k+n-2} \exp\{-(\lambda t \epsilon_i + \alpha) q\}}{(q+l)^\nu} dq \\ &\quad + \frac{(\lambda t)^n}{\lambda \epsilon_i} \sum_{i=1}^{n} \frac{n(n-1)\ldots(n-i+1)}{(\lambda t \epsilon_i)^i} \int_0^{\infty} \frac{q^{k+n-i-2} \exp\{-(\lambda t \epsilon_i + \alpha) q\}}{(q+l)^\nu} dq \\ &= \frac{(\lambda t)^n \Gamma_\nu(k+n-1, (\alpha + \lambda t \epsilon_i) l)}{\lambda \epsilon_i (\alpha + \lambda t \epsilon_i)^{n+k-\nu-1}} \\ &\quad + \frac{(\lambda t)^n}{\lambda \epsilon_i} \sum_{i=1}^{n} \frac{n(n-1)\ldots(n-i+1)}{(\lambda t \epsilon_i)^i} \frac{\Gamma_\nu(n+k-i-1, (\alpha + \lambda t \epsilon_i) l)}{(\alpha + \lambda t \epsilon_i)^{n+k-\nu-i-1}} \\ &= \phi(n, t, \epsilon_i), \end{aligned} \tag{A7}$$

where the first equality follows from (2.2), and the third equality follows from the fact that

$$\int_t^{\infty} x^n \exp\{-\lambda x \epsilon_i q\} dx = \frac{\exp\{-t \epsilon_i q \lambda\}}{(\lambda \epsilon_i q)^{n+1}} \left[ (t \lambda \epsilon_i q)^n + \sum_{i=1}^{n} n(n-1)\ldots(n-i+1)(t \lambda \epsilon_i q)^{n-i} \right].$$

On using (A7), we get

$$\int_t^{\infty} \bar{F}_L(x) dx = \left( \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \right) \boldsymbol{aP} \left( \int_t^{\infty} diag(\boldsymbol{D_1(t)}, \boldsymbol{D_2(t)}, \ldots, \boldsymbol{D_r(t)}) dx \right) \boldsymbol{P^{-1} e}$$

$$= \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right) \boldsymbol{aP} diag(\boldsymbol{R_1(t)}, \boldsymbol{R_2(t)}, \ldots, \boldsymbol{R_r(t)})\boldsymbol{P^{-1}e},$$

where

$$\begin{aligned}
\boldsymbol{R_i(t)} &= \begin{pmatrix} \int_t^\infty \left(\frac{\Gamma_\nu(k,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k-\nu}}\right)dx & \int_t^\infty \left(\frac{-\lambda x\Gamma_\nu(k+1,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k+1-\nu}}\right)dx & \ldots & \int_t^\infty \left(\frac{(-\lambda x)^{d_i-1}\Gamma_\nu(k+d_i-1,(\lambda x\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda x\epsilon_i+\alpha)^{k+d_i-1-\nu}}\right)dx \\ 0 & \int_t^\infty \left(\frac{\Gamma_\nu(k,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k-\nu}}\right)dx & \ldots & \int_t^\infty \left(\frac{(-\lambda x)^{d_i-2}\Gamma_\nu(k+d_i-2,(\lambda x\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda x\epsilon_i+\alpha)^{k+d_i-2-\nu}}\right)dx \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \int_t^\infty \left(\frac{\Gamma_\nu(k,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k-\nu}}\right)dx \end{pmatrix}_{d_i\times d_i} \\
&= \begin{pmatrix} \phi(0,t,\epsilon_i) & -\phi(1,t,\epsilon_i) & \ldots & \frac{(-1)^{d_i-1}\phi(d_i-1,t,\epsilon_i)}{(d_i-1)!} \\ 0 & \phi(0,t,\epsilon_i) & \ldots & \frac{(-1)^{d_i-2}\phi(d_i-2,t,\epsilon_i)}{(d_i-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \phi(0,t,\epsilon_i) \end{pmatrix}_{d_i\times d_i},
\end{aligned}$$

$i = 1, 2, \ldots, r$. Hence, the result is proved. □

**Proof of Theorem 3.6:** We have

$$E(L) = E(E(L|Q)),$$

where $Q$ is a mixing random variable of the HPGGP with parameter set $\{\lambda, \nu, k, \alpha, l\}$, $k > 1$. Now, for given $Q = q$, $X_i$'s are i.i.d. random variables and follow the exponential distribution with parameter $q\lambda$. Then

$$E(L|Q=q) = E\left(\sum_{i=1}^{M} X_i | Q = q\right) = E(M)E(X_1|Q=q) = (\boldsymbol{a}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{e})\frac{1}{q\lambda}$$

and consequently,

$$\begin{aligned}
E(L) &= \frac{(\boldsymbol{a}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{e})}{\lambda} E\left(\frac{1}{Q}\right) \\
&= \frac{(\boldsymbol{a}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{e})}{\lambda} \int_0^\infty \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \frac{q^{k-2}\exp\{-\alpha q\}}{(q+l)^\nu} dq \\
&= \frac{\alpha(\boldsymbol{a}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{e})}{\lambda} \frac{\Gamma_\nu(k-1,\alpha l)}{\Gamma_\nu(k,\alpha l)}.
\end{aligned}$$

Hence, the result is proved. □

**Proof of Theorem 3.7:** Let $\boldsymbol{\gamma_i} = (\theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_d^{(i)})$, $i = 1, 2$. Then from Theorem 3.2 $(i)$, we have

$$\bar{F}_{L_1}(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right) \sum_{j=1}^{d} \frac{\theta_j^{(1)}\Gamma_\nu(k,(\lambda t\epsilon_j^{(1)}+\alpha)l)}{(\lambda t\epsilon_j^{(1)}+\alpha)^{k-\nu}}$$

and

$$\bar{F}_{L_2}(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\sum_{i=1}^{d}\frac{\theta_j^{(2)}\Gamma_\nu(k,(\lambda t\epsilon_j^{(2)}+\alpha)l)}{(\lambda t\epsilon_j^{(2)}+\alpha)^{k-\nu}}.$$

Clearly, both $L_1$ and $L_2$ are mixtures of $d$ random variables with the survival functions given by

$$\begin{aligned}\bar{F}_j^{(i)}(t) &= \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\frac{\Gamma_\nu(k,(\lambda t\epsilon_j^{(i)}+\alpha)l)}{(\lambda t\epsilon_j^{(i)}+\alpha)^{k-\nu}}\\ &= \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\int_0^\infty\frac{q^{k-1}\exp\{-(\lambda t\epsilon_j^{(i)}+\alpha)q\}}{(q+l)^\nu}dq, \quad j=1,2,\ldots,d,\end{aligned}$$

where the second equality follows from (2.2). Since $\epsilon_1^{(i)} \le \epsilon_2^{(i)} \le \cdots \le \epsilon_d^{(i)}$, we get $\bar{F}_1^{(i)}(t) \ge \bar{F}_2^{(i)}(t)\cdots \ge \bar{F}_d^{(i)}(t)$, $i=1,2$. Again, $\epsilon_j^{(1)} \ge \epsilon_j^{(2)}$ implies $\bar{F}_j^{(1)}(t) \le \bar{F}_j^{(2)}(t)$, for all $j=1,2,\ldots,d$. Moreover, we have that $\boldsymbol{\gamma_1} \ge_{st} \boldsymbol{\gamma_2}$. Then the result follows from Theorem 4.6 of Shojaee et al. [30]. □

**Proof of Theorem 3.8:** Now, for $j=1,2$, we have

$$\bar{F}_{L_j}(t) = P\left(\sum_{i=1}^{M}X_i^{(j)} > t\right) = \int_0^\infty P\left(\sum_{i=1}^{M}X_i^{(j)} > t|Q_j=q_j\right)f_{Q_j}(q_j)dq_j, \tag{A8}$$

where $Q_j$ is the mixing random variable of the HPGGP with parameter set $\{\lambda_j,\nu_j,k_j,\alpha_j,l_j\}$, $j=1,2$. Note that, for given $Q_j=q_j$, the HPGGP with parameter set $\{\lambda_j,\nu_j,k_j,\alpha_j,l_j\}$ is the same as the HPP with intensity $q_j\lambda_j$, $j=1,2$. Then, from Corollary 3.1, we get

$$P\left(\sum_{i=1}^{M}X_i^{(j)} > t|Q_j=q_j\right) = \boldsymbol{a}\exp\{-q_j\lambda_j t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}.$$

By using the above equality in (A8), we get

$$\bar{F}_{L_j}(t) = E(\boldsymbol{a}\exp\{-Q_j\lambda_j t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}), \quad j=1,2.$$

Again, note that $\boldsymbol{a}\exp\{-x(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}$ is decreasing in $x>0$ and hence, for $\lambda_1 \le \lambda_2$, we get

$$\begin{aligned}\bar{F}_{L_1}(t) &= E(\boldsymbol{a}\exp\{-Q_1\lambda_1 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e})\\ &\ge E(\boldsymbol{a}\exp\{-Q_1\lambda_2 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}).\end{aligned} \tag{A9}$$

Now, if $Q_2 \ge_{st} Q_1$ holds then $\boldsymbol{a}\exp\{-Q_1\lambda_2 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e} \ge_{st} \boldsymbol{a}\exp\{-Q_2\lambda_2 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}$ (see Theorem 1.A.3 ($a$) in Shaked and Shanthikumar [27]) and hence, $E(\boldsymbol{a}\exp\{-Q_1\lambda_2 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}) \ge \boldsymbol{a}\exp\{-Q_2\lambda_2 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}$ . By combining this and (A9), we get

$$\bar{F}_{L_1}(t) \ge E(\boldsymbol{a}\exp\{-Q_2\lambda_2 t(\boldsymbol{I}-\boldsymbol{A})\}\boldsymbol{e}) = \bar{F}_{L_2}(t)$$

provided that $Q_2 \ge_{st} Q_1$ holds. Again, $Q_2 \ge_{st} Q_1$ follows provided that one of the following conditions holds (see the proof of Proposition 5 in Cha and Mercier [7]).

- $\alpha_1 = \alpha_2$, $k_2 \geq k_1$ and $k_2 - k_1 \geq \nu_2 - \nu_1$;
- $\alpha_2 < \alpha_1$ and $(\alpha_1 - \alpha_2 + k_2 - k_1 + \nu_1 - \nu_2)^2 - 4(\alpha_1 - \alpha_2)(k_2 - k_1) \leq 0$;
- $\alpha_2 < \alpha_1$ and $\alpha_1 - \alpha_2 + k_2 - k_1 + \nu_1 - \nu_2 \geq 0$

Note that these conditions are given in the assumption and hence, the result is proved. $\square$