Supplementary Material: A Colour Image Segmentation Method and Its Application to Medical Images

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1. Fractional Fourier spectral method

In this section, we will fractional Fourier spectral method for our time discretized scheme (11). The fractional time discretized scheme with $0 < \zeta \leq 2$ is,

$$\frac{U_i^{(k+1)} - U_i^{(k)}}{\Delta t} + C_1(-\Delta)^{\zeta} U_i^{(k+1)} + C_1(-\Delta)^{\frac{\zeta}{2}} U_i^{(k+1)} + C_2 U_i^{(k+1)}$$
$$= C_1(-\Delta)^{\zeta} U_i^{(k)} + C_1(-\Delta)^{\frac{\zeta}{2}} U_i^{(k)} + \epsilon(-\Delta)^{\frac{\zeta}{2}} \left(\nabla \cdot \left(\frac{\nabla U i^{(k)}}{|\nabla U_i^{(k)}|_{\delta}}\right)\right)$$
$$+ C_2 U_i^{(k)} - \frac{1}{\epsilon}(-\Delta)^{\frac{\zeta}{2}} W_1'(U_i^{(k)}) + \lambda(f_i - U_i^{(k)}) \tag{1}$$

and we assume the Neumann boundary condition

$$\nabla U_i \cdot \mathbf{n} = \nabla(\Delta U_i) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{2}$$

where $\mathbf{n}$ is outward normal to the boundary.

To define fractional Laplacian we have used the eigenfunction expansion like in [1, 2]. For Laplace eigenvalue problem with zero Neumann boundary

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condition in $\Omega = [0,a]\times[0,b]$ has eigenvalues

$$\lambda_{m,n} = \pi^2\Big(\frac{(m-1)^2}{a^2} + \frac{(n-1)^2}{b^2}\Big)$$

and the corresponding eigenfunctions

$$\varphi_{m,n} = \frac{1}{\sqrt{ab}} cos\Big(\frac{(m-1)\pi x}{a}\Big) cos\Big(\frac{(n-1)\pi y}{a}\Big).$$

Let us define the space [2],

$$\mathcal{U}_\zeta := \Big\{u = \sum_{m,n=1}^{\infty} \hat{u}_{m,n}\varphi_{m,n}, \hat{u}_{m,n} = \langle u, \varphi_{m,n}\rangle_{L^2} : \sum_{m,n=1}^{\infty} |\hat{u}_{m,n}|^2 |\lambda_{m,n}|^{\frac{\zeta}{2}} < \infty, 0 < \zeta \leq 2\Big\} \tag{3}$$

Then for any $u \in \mathcal{U}_\zeta$, the fractional Laplace operator can be defined by

$$(-\Delta)^{\frac{\zeta}{2}} u = \sum_{m,n=1}^{\infty} \hat{u}_{m,n}\lambda_{m,n}^{\frac{\zeta}{2}}\varphi_{m,n}. \tag{4}$$

Hence $(-\Delta)^{\frac{\zeta}{2}}$ has the same interpretation as $-\Delta$ in terms of its spectral decomposition. Fourier spectral methods represent the truncated series expansion when a finite number of orthonormal eigenfunction $\{\varphi_{m,n}\}$ is considered. Taking Fourier transform of (1) and using the spectral decomposition of Laplacian (4) we get

$$\widehat{U}_i^{(k+1)}(j,l) = \hat{U}_i^{(k)}(j,l) + \frac{\lambda\widehat{(f - U_i^{(k)})}(j,l) + \epsilon\lambda_{j,l}^{\frac{\alpha}{2}}\hat{\kappa}(j,l) - \frac{1}{\epsilon}\lambda_{j,l}^{\frac{\alpha}{2}}\widehat{W_1'(U_i^{(k)})}(j,l)}{\frac{1}{\Delta t} + C_1\lambda_{j,l}^{\alpha} + C_1\lambda_{j,l}^{\frac{\alpha}{2}} + C_2} \tag{5}$$

where $\kappa = \nabla\cdot\Big(\frac{\nabla U_i^{(k)}}{\sqrt{|\nabla U_i^{(k)}|^2+\delta^2}}\Big)$. In every time step we will calculate the $\widehat{U_k}$ and the real part of the inverse Fourier transform will be the solution U_k. The fractional scheme gives the flexibility to choose the fractional power ζ which gives better result than the integral power.

2. Results using fractional Fourier Spectral methods

In this subsection we have tested our model using the fractional version of the proposed model (5). For this case also we have fixed all the parameters as earlier except λ of the first stage and the α in the second stage. First we

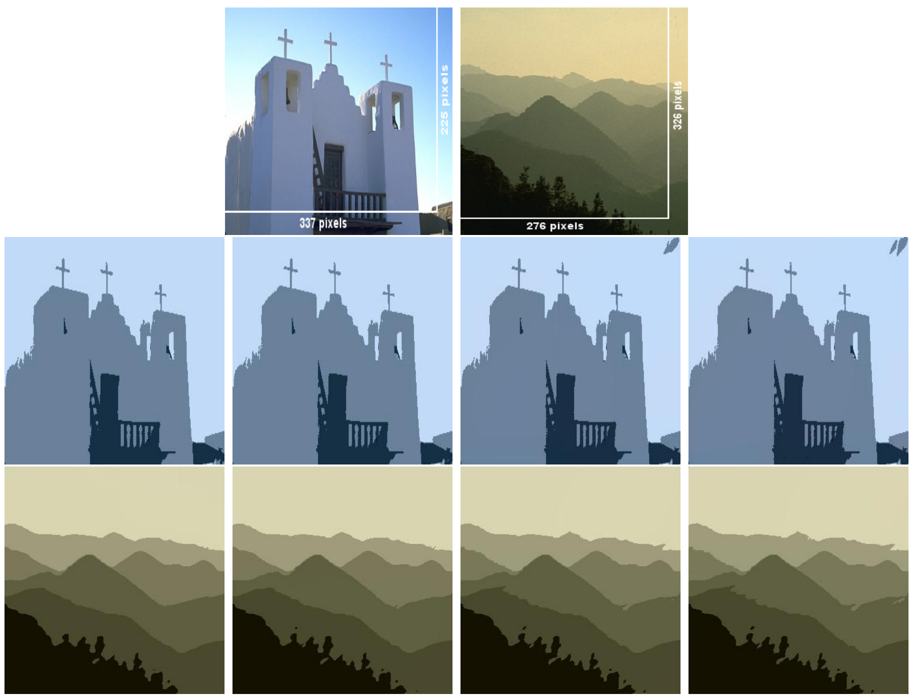

Figure 1: Segmentation of Real World images by our Fractional model with four different fractional power $\zeta = 1.2, 1.4, 1.8, 2$ respectively.

Figure 2: Comparison of Segmentation result obtained by our fractional model with the result of model [3].

have shown two results with different fractional power ζ and observed that the model is giving better result than the case $\zeta = 2$ which corresponds to the actual discretisation. After that, we have compared our results with the result of model [3] only. Also we have used the fractional power $\zeta = 1.2$ for all our experiment of this subsection.

In Fig 1, we have taken two images from from BDS300 collection and segmented by using our fractional model with different fractions ζ. We have shown results corresponding to $\zeta = 1.2, 1.4, 1.8, 2$ for both the images. For both the images the parameters are $\lambda = 5 \times 10^2$ and $\alpha = 0.003$. From the figure one can see that the results corresponding to $\zeta = 1.2$ and 1.4 are better. For the other two values of ζ segmentation results have some error. Like in the upper right corner of the Church image some parts are appearing with intencity different from rest of the sky part although in the original image in that corner intencity of blue colour is more. Similarly the peak of the hills are not sharp on the results

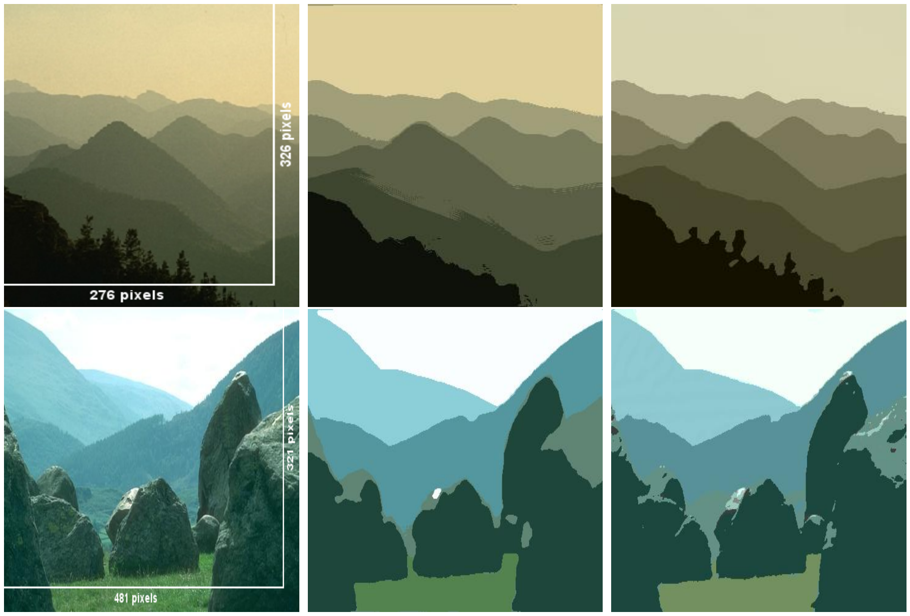

Figure 3: Comparison of Segmentation result obtained by our fractional model with the result of model [3] on Hill images.

corresponding to $\zeta = 1.8$ and 2.

In Fig 2, we have shown segmented result of two images taken from from BDS300 collection. In the first column we have included the original image and the results of [3] and our model are reported in second and third column respectively. The first image contains two birds image flying in the sky. In this case we have chosen the parameters $\lambda = 5 \times 10^2$ in 1st stage and $\alpha = 0.001$ in second stage also we have use the fraction. For the first image we can see that our model is giving better result than that of [3] model which is evident if we look at the wings of the bird. The second image is the image of church which consists of three segments. In this case the parameters are chosen as $\lambda = 5 \times 10^2$ and $\alpha = 0.003$. From the figure it is clear that our model gives better result than the Slat model [3] by looking at the cross symbol and the stairs of the image. Also note that for slat model we have to give number of segments as input whereas our model automatically gives the number of segments.

Then we have taken two more natural images from BDS300 collection and segmented them by our fractional model and compared the with the result of [3] in Fig. 3. For the first image of the Fig. 3 we have set the parameters as $\lambda = 5 \times 10^2$ and $\alpha = 0.003$ and those for the second image are $\lambda = 10^2$ and $\alpha = 0.001$. Both the images has 6 segments. For the second image both the model gives almost similar result but for the first model our model gives better result than [3] model which is evident if we look at the segments of the plants present in the original image.

References

[1] A. Bueno-Orovio, D. Kay, and K. Burrage, Fourier spectral methods for fractional-in-space reaction- diffusion equations, BIT, 54 (2014), pp. 937-954.

[2] J. Bosch and M. Stoll, A Fractional Inpainting Model Based on the Vector-Valued Cahn-Hilliard Equation, SIAM Journal on Imaging Sciences, vol. 8(4), pp. 2352-2382, 2015.

[3] X. Cai, R. Chan, M. Nikolova, T. Zeng, A Three-Stage Approach for Segmenting Degraded Color Images: Smoothing, Lifting and Thresholding (SLaT), Journal of scientific computing, 2017, DOI 10.1007/s10915-017-0402-2.