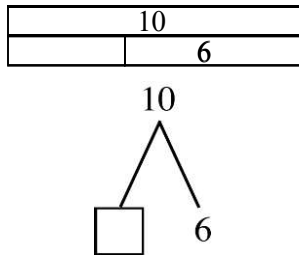
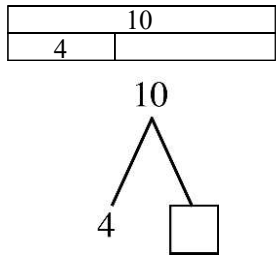
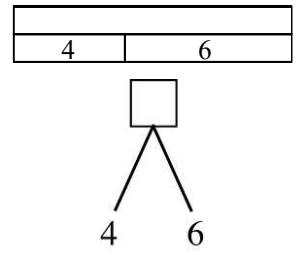
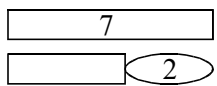
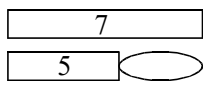
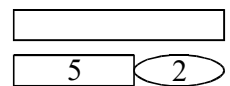


Appendix A: Addition and Subtraction Problem Types (Carpenter & Fennema, 1992; adapted from Clements & Sarama, 2014; NGA/CCSSO, 2010).

	Unknown		
Situation	Start/Part/Smaller	Change/Part/Difference	Result/Whole/Larger
<p>Add-To (Join)</p> <p>An action of adding increases the number in a set.</p>	<p><i>Start Unknown</i></p> $\square + 6 = 11$ Al had some balls. Then he got 6 more. Now he has 11 balls. How many did he start with?	<p><i>Change Unknown</i></p> $5 + \square = 11$ Al had 5 balls. He bought some more. Now he has 11. How many did he buy?	<p><i>Result Unknown</i></p> $5 + 6 = \square$ Al had 5 balls and gets 6 more. How many does he have in all?
<p>Take-From (Separate)</p> <p>An action of taking away decreases the number in a set.</p>	<p><i>Start Unknown</i></p> $\square - 5 = 4$ Al had some balls. He gave 5 to Barb. Now he has 4. How many did he have to start with?	<p><i>Change Unknown</i></p> $9 - \square = 4$ Al had 9 balls. He gave some to Barb. Now he has 4. How many did he give to Barb?	<p><i>Result Unknown</i></p> $9 - 5 = \square$ Al had 9 balls and gave 5 to Barb. How many does he have left?
<p>Part-Part-Whole (Put Together/Take Apart)</p> <p>Two parts make a whole, but there is no action—the situation is static.</p>	<p><i>Part Unknown</i></p>  <p>Al has 10 balls. Some are blue, 6 are red. How many are blue?</p>	<p><i>Part Unknown</i></p>  <p>Al has 10 balls; 4 are blue and the rest are red. How many are red?</p>	<p><i>Whole Unknown</i></p>  <p>Al has 4 red balls and 6 blue balls. How many balls does he have in all?</p>
<p>Compare</p> <p>The numbers of objects in two sets are compared.</p>	<p><i>Smaller Unknown</i></p>  <p>Al has 7 balls. Barb has 2 fewer balls than Al. How many balls does Barb have?</p>	<p><i>Difference Unknown</i></p>  <p>Al has 7 blocks. Barb has 5. How many more does Al have than Barb?</p>	<p><i>Unknown</i></p>  <p>Al has 5 marbles. Barb has 2 more than Al. How many balls does Barb have?</p>

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Appendix B: Skip Condition Children’s Pretest and Posttest

Adding/Subtracting LT Level and Assigned Instruction Plan by Problem

Type and Problem Size

Child	Adding/Subtracting Pretest LT Level ($x = 0$)	Instruction A: Problem Type Problem Size ($n + x$)	Instruction B: Problem Type Problem Size ($n + x$)	Instruction C: Problem Type Problem Size ($n + x$)	Adding/Subtracting Posttest LT Level
14	Developing <i>1–Small Number</i> +/-	Start/Part Unknown 20-100 $n + 3$	Start/Part Unknown 0-20 $n + 3$	Change/Difference Unknown 0-10 $n + 2$	Developing <i>1–Small Number</i> +/-
15	Developing <i>1–Small Number</i> +/-	Start/Part Unknown 20-100 $n + 3$	Change/Difference Unknown 20-100 $n + 2$	Result/Whole Unknown 20-100 $n + 1$	Developing <i>1–Small Number</i> +/-
16	1–Small Number +/-	Start/Part Unknown 20-100 $n + 3$	Change/Difference Unknown 20-100 $n + 2$	Result/Whole Unknown 20-100 $n + 1$	Developing <i>1–Small Number</i> +/-
17	Developing <i>1–Small Number</i> +/-	Start/Part Unknown 20-100 $n + 3$	Start/Part Unknown 0-10 $n + 3$	Change/Difference Unknown 20-100 $n + 2$	None
19	2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 0-10 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	Developing 2–Find Result +/-
20	None	Start/Part Unknown 20-100 $n + 4$	Change/Difference Unknown 20-100 $n + 3$	Change/Difference Unknown 0-10 $n + 2$	None
21	Developing 2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 20-100 $n + 2$	Result2Whole Unknown 20-100 $n + 1$	Developing 2–Find Result +/-
22	1–Small Number +/-	Start/Part Unknown 20-100 $n + 3$	Change/Difference Unknown 0-10 $n + 2$	Change/Difference Unknown 0-10 $n + 2$	Developing <i>1–Small Number</i> +/-
23	Developing 2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 20-100 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	2–Find Result +/-

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24	Developing 2-Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 0-10 $n + 2$	Change/Difference Unknown 0-10 $n + 1$	Developing 2-Find Result +/-
25	2-Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	Change/Difference Unknown 0-10 $n + 1$	Developing 2-Find Result +/-
26	Developing 2-Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 0-10 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	1-Small Number +/-

Appendix C: Each Child’s Pretest and Posttest LT Levels, Anticipated Level

Transition, and Incremental Improvement within Levels

Child	Pretest LT Level	Anticipated Posttest LT Level based on Session Performance	Posttest LT Level	Level Change*	Incremental Improvement
LT Treatment					
1	Developing 1– <i>Small Number</i> +/-	5–Part-Whole +/-	Developing 2– Find Result +/-	+	1–Small Number +/- 3–Find Change +/- 4–Counting Strategies +/- 5–Part-Whole +/-
2	Developing 2– Find Result +/-	3–Find Change +/-	Developing 2– Find Result +/-		7–Deriver +/-
3	None	2–Find Result +/-	Developing 1– <i>Small Number</i> +/-	+	1–Small Number +/-
4	Developing 2– Find Result +/-	3–Find Change +/-	Developing 2– Find Result +/-		3–Find Change +/-
5	1–Small Number +/-	6–Numbers-in-Numbers +/-	1– <i>Small Number</i> +/-		2–Find Result +/- 4–Counting Strategies +/- 5–Part Whole +/-
6	Developing 1– <i>Small Number</i> +/-	3–Find Change +/-	Developing 2– Find Result +/-	+	1–Small Number +/- 2–Find Result +/- 5–Part-Whole +/- 6–Numbers-in-Numbers +/-
7	None	4–Counting Strategies +/-	Developing 2– Find Result +/-	++	1–Small Number +/- 3–Find Change +/-
8	Developing 2– Find Result +/-	3–Find Change +/-	Developing 2– Find Result +/-		1–Small Number +/- 2–Find Result +/- 3–Find Change +/-
9	Developing 2– Find Result +/-	8–Problem Solver +/-	Developing 2– Find Result +/-		3–Find Change +/- 5–Part-Whole +/- 7–Deriver +/-
10	Developing 1– <i>Small Number</i> +/-	5–Part-Whole +/-	Developing 3– Find Change +/-	++	1–Small Number +/- 2–Find Result +/- 4–Counting Strategies +/-
11	1–Small Number +/-	3–Find Change +/-	2–Find Result +/-	+	2–Find Result +/- 4–Counting Strategies +/-
12	Developing 2– Find Result +/-	7–Deriver +/-	Developing 2– Find Result +/-		3–Find Change +/- 4–Counting Strategies +/- 6–Numbers-in-Numbers +/- 7–Deriver +/-
13	Developing 2– Find Result +/-	6–Numbers-in-Numbers +/-	Developing 2– Find Result +/-		3–Find Change +/- 7–Deriver +/-

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Skip Treatment					
14	Developing 1– <i>Small Number</i> +/-	Developing 1– <i>Small Number</i> +/-	Developing 1– <i>Small Number</i> +/-		7–Deriver +/- 8–Problem Solver +/-
15	Developing 1– <i>Small Number</i> +/-	Developing 1– <i>Small Number</i> +/-	Developing 1– <i>Small Number</i> +/-		2–Find Result +/- 4–Counting Strategies +/- 5–Part-Whole +/- 6–Numbers-in-Numbers +/-
16	1–Small Number +/-	1–Small Number +/-	Developing 1– <i>Small Number</i> +/-	(–)	2–Find Result +/- 3–Find Change +/-
17	Developing 1– <i>Small Number</i> +/-	Developing 1– <i>Small Number</i> +/-	None	(–)	2–Find Result +/- 7–Deriver +/-
19	2–Find Result +/-	2–Find Result +/-	Developing 2– Find Result +/-	(–)	2–Find Result +/- 8–Problem Solver +/-
20	None	None	None		5–Part-Whole +/- 8–Problem Solver +/-
21	Developing 2– Find Result +/-	Developing 2– Find Result +/-	Developing 2– Find Result +/-		2–Find Result +/- 7–Deriver +/-
22	1–Small Number +/-	1–Small Number +/-	Developing 1– <i>Small Number</i> +/-	(–)	1–Small Number +/- 2–Find Result +/-
23	Developing 2– Find Result +/-	Developing 2– Find Result +/-	2–Find Result +/-	(+)	2–Find Result +/- 3–Find Change +/- 5–Part-Whole +/- 6–Numbers-in-Numbers +/- 7–Deriver +/-
24	Developing 2– Find Result +/-	Developing 2– Find Result +/-	Developing 2– Find Result +/-		4–Counting Strategies +/- 5–Part-Whole +/-
25	2–Find Result +/-	2–Find Result +/-	Developing 2– Find Result +/-	(–)	2–Find Result +/- 3–Find Change +/- 4–Counting Strategies +/-
26	Developing 2– Find Result +/-	Developing 2– Find Result +/-	Developing 1– <i>Small Number</i> +/-	–	5–Part-Whole +/-

* + indicates level increase; – indicates level decrease; (–) indicates partial level decrease.

Appendix D: Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group, Separated by Amount of Level Change (note that some children participated in 9 or 10 sessions)

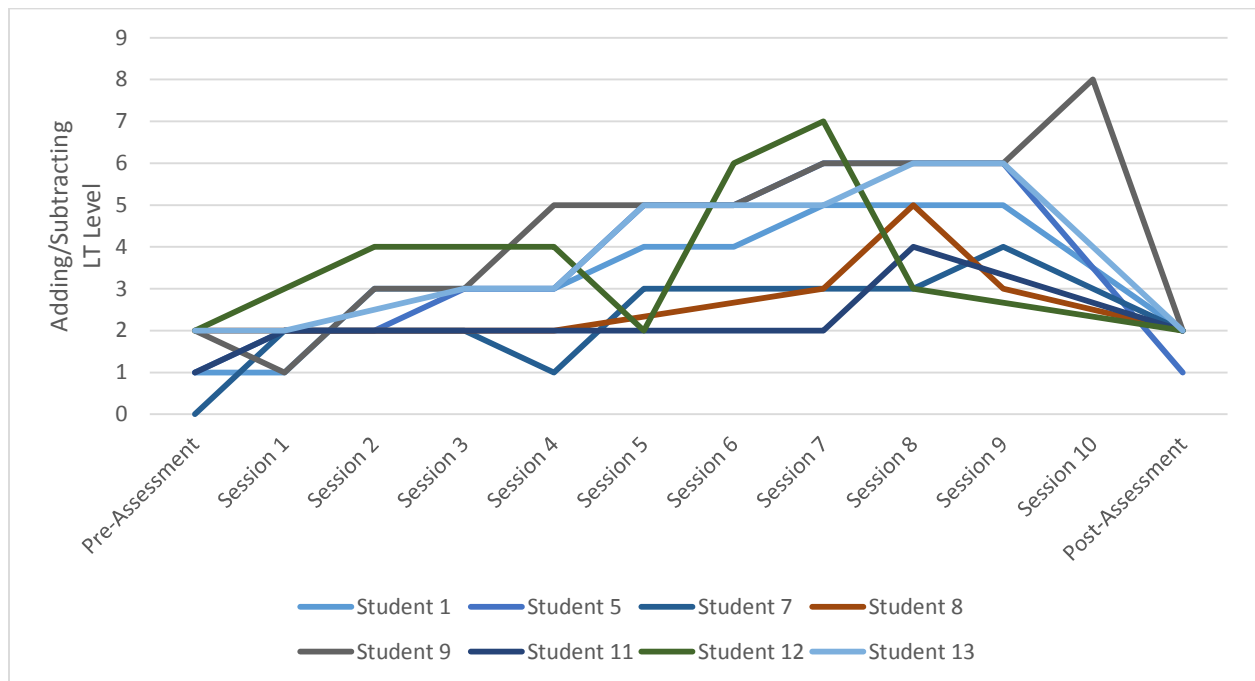


Figure D-1. For children who advanced at least 3 levels at some point in instruction, Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group (note that some children participated in 9 or 10 sessions). Levels, defined in Figure 1, are 1–Small Number +/-, 2–Find Result +/-, 3–Find Change +/-, 4–Counting Strategies +/-, 5–Part-Whole +/-, 6–Numbers-in-Numbers +/-, 7–Deriver +/-, and 8–Problem Solver +/-.

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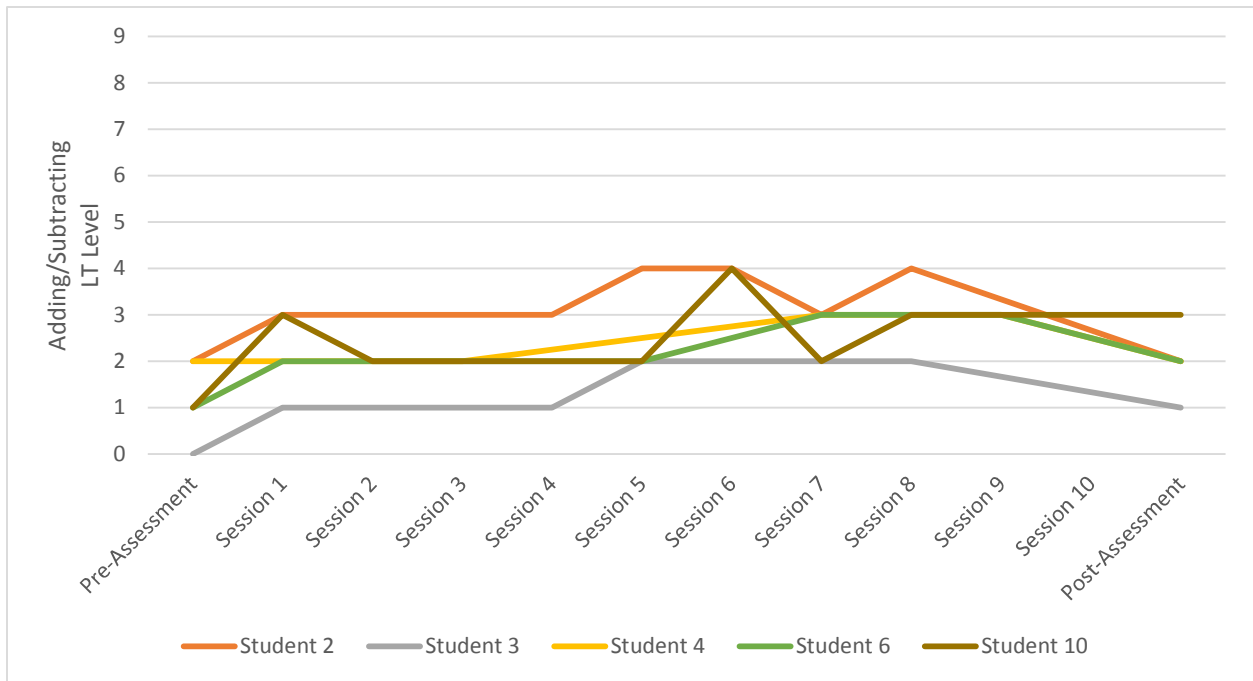


Figure D-2. For children who advanced only one or two levels at some point in instruction, Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group (note that some children participated in 9 or 10 sessions). Levels, defined in Figure 1, are 1–Small Number +/-, 2–Find Result +/-, 3–Find Change +/-, 4–Counting Strategies +/-, 5–Part-Whole +/-, 6–Numbers-in-Numbers +/-, 7–Deriver +/-, and 8–Problem Solver +/-.