# Time-Varying Univariate and Bivariate Frequency Analysis of Nonstationary Extreme Sea Level for New York City: Supplementary Materials

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This is a supplementary document to the corresponding paper submitted to the Water Resources Management. section 1 contains the details of the univariate and partial Mann-Kendal test of stationarity. section 2 is details of 120 different models of univariate GEV distribution for water level with precipitation as covariate. section 3 contains copula function and Kendall's $\tau$ for four copula families which are used in the main article. section 4 is details of AIC and BIC for univariate and bivariate parameter estimation. section 5 contains more details of univariate and bivariate return periods and return levels in both stattionary and time-varying cases. section 6 contains the methods of estimating copula parameters. section 7 contains details of the structures of dynamic copulas and section 8 is the details of a goodness-of-fit test for static and dynamic copula.

## 1 Mann-Kendall test

### 1.1 Univariate Mann–Kendall test

The test statistic is

$$S = \sum_{i<j} \mathrm{sgn}(x_j - x_i), \tag{1}$$

where $\mathrm{sgn}(t)$ is the sign of $t$ which is -1, 0, and 1 whenever $t$ is negative, 0, or positive, respectively. Under $H_0$, i.e., when there is no trend in the data, $S$ is distributed as a normal distribution with mean zero. If there is no tie in the data, the variance of S is approximate:

$$Var(S) = \frac{n(n-1)(2n+5)}{18}. \tag{2}$$

If there are no ties in the observations and no trend is present in the data, the test statistic is a normal distribution with zero mean and variance $Var(S) = n(n-1)(2n+5)/18$.

### 1.2 Partial Mann-Kendall test

Let $X$ be a target variable and Y be a covariate and let $S_X$ and $S_Y$ be the univariate Mann-Kendall test statistic (1), for $X$ and $Y$, respectively. Then the bivariate distribution of the random vector $S = (S_X, S_Y)$ is approximately normal with mean vector 0, and variances define in (2) and estimated covariance:

$$\hat{\sigma}_{xy} = \frac{1}{3}\left\{\sum_{i<j}\operatorname{sgn}(X_j - X_i)\operatorname{sgn}(Y_j - Y_i) + \sum_{i<j<k}\operatorname{sgn}(X_k - X_i)\operatorname{sgn}(Y_k - Y_j)\right\}.$$

Hence, the conditional distribution of $S_X|S_Y = s$ can be approximated by:

$$S_X|S_Y = s \sim \mathrm{N}\left(\hat{\mu}_s = \frac{\hat{\sigma}_{xy}}{\sigma^2}s, \hat{\sigma}_s^2 = \sigma^2 - \frac{\hat{\sigma}_{xy}^2}{\sigma^2}\right).$$

where $\sigma^2$ is the common variance of $S_X$ and $S_Y$ defined in (2). The hypothesis $H_0$ can be tested by standardizing the conditional distribution of $S_X|S_y = s$, and the p-value would be

$$2P\left[Z > \hat{\sigma}_{S_Y}^{-1}\left|S_X - \hat{\mu}_{S_Y}\right|\right].$$

## 2 Format of 120 models for water level distribution with TMIN and TMAX as covariate

As mentioned in the main article, for each target variables there are 120 different models; These 120 models for water level are shown in Table 1. Replacing WL with PREC gives the 120 models for precipitation.

**Table 1** 120 distinct models of GEV distribution for target variable $WL$ with covariates $TMIN$ and $TMAX$.

| Model | Location ($\mu_W L$) | Scale ($\sigma_W L$) | Shape ($\xi_W L$) | |
|---|---|---|---|---|
| $M_1$ | $\mu_0$ | $\sigma_0$ | $\xi_0$ | (a) |
| $M_2$ | | | $\xi_0 + \xi_1 \ln t$ | (b) |
| $M_3$ | | | $\xi_0 + \xi_1 + \xi_1 TMIN$ | (c) |
| $M_4$ | | | $\xi_0 + \xi_1 \ln t + \xi_2 TMIN$ | (d) |
| $M_5$ | | | $\xi_0 + \xi_1 TMAX$ | (e) |
| $M_6$ | | | $\xi_0 + \xi_1 \ln t + \xi_2 TMAX$ | (f) |
| $M_7$–$M_{12}$ | | $\sigma_0 + \sigma_1 \ln t$ | (a)-(f) | |
| $M_{13}$–$M_{16}$ | | $\sigma_0 + \sigma_1 TMIN$ | (a)-(d) | |
| $M_{17}$–$M_{20}$ | | $\sigma_0 + \sigma_1 \ln t + \sigma_2 TMIN$ | (a)-(d) | |
| $M_{21}$–$M_{24}$ | | $\sigma_0 + \sigma_1 TMAX$ | (a),(b),(e),(f) | |
| $M_{25}$–$M_{28}$ | | $\sigma_0 + \sigma_1 \ln t + \sigma_2 TMAX$ | (a),(b),(e),(f) | |
| $M_{29}$–$M_{56}$ | $\mu_0 + \mu_1 \ln t$ | same as models $M_1$–$M_{28}$ | | |
| $M_{57}$–$M_{72}$ | $\mu_0 + \mu_1 TMIN$ | all models $M_1$–$M_{28}$ except those which $\sigma$ or $\xi$ depends on $TMAX$ (16 models) | | |
| $M_{73}$–$M_{88}$ | $\mu_0 + \mu_1 \ln t + \mu_2 TMIN$ | same as models $M_{57}$–$M_{72}$ | | |
| $M_{89}$–$M_{104}$ | $\mu_0 + \mu_1 TMAX$ | all models $M_1$–$M_{28}$ except those which $\sigma$ or $\xi$ depends on $TMIN$ (16 models) | | |
| $M_{105}$–$M_{120}$ | $\mu_0 + \mu_1 \ln t + \mu_2 TMAX$ | same as models $M_{89}$–$M_{104}$ | | |

## 3 Details of 4 copulas which are used in the main article

Four famous copulas are used in the main article: Clayton, Gumbel, Frank, and Normal copula. The copula functions as well as the Kendall $\tau$'s are given in Table 2.

**Table 2** Copula function, parameter space, and Kendall's $\tau$ for families of bivariate copulas ($D_1$ is the Debye function and $\Phi$ is the distribution function of standard normal distribution

| Family | Copula function: $C(u, v;\, \theta)$ | Range of $\theta$ | Kendall's $\tau$ | $\lambda_L$ | $\lambda_U$ |
|---|---|---|---|---|---|
| Clayton | $\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}$ | $\theta > 0$ | $\frac{\theta}{2+\theta}$ | $2^{-1/\theta}$ | 0 |
| Frank | $-\frac{1}{\theta}\ln\left[1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{\left(e^{-\theta}-1\right)}\right]$ | $\theta \neq 0$ | $1+\frac{4}{\theta}\left(D_1(\theta)-1\right)$ | 0 | 0 |
| Gumbel | $\exp\left\{-\left[(-\ln u)^{\theta}+(-\ln v)^{\theta}\right]^{1/\theta}\right\}$ | $\theta \geq 1$ | $1-\frac{1}{\theta}$ | 0 | $2-2^{1/\theta}$ |
| Normal | $\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-\theta^2}}\exp\left\{-\frac{x^2-2\theta xy+y^2}{2(1-\theta^2)}\right\}\mathrm{d}y\,\mathrm{d}x$ | $-1 \leq \theta \leq 1$ | $\frac{2}{\pi}\arcsin\theta$ | 0 | 0 |

## 4 More about AIC and BIC

More generally, in a parameter estimation model, the Akaike information criteria (AIC) and Bayesian information criteria (BIC) are defined as below:

$$AIC = -2\ell(\hat{\theta}) + 2p, \tag{3}$$

$$BIC = -2\ell(\hat{\theta}) + \ln np, \tag{4}$$

where $n$ is the number of observations, $\hat{\theta}$ is the vector of maximum likelihood estimates of the parameters, $\ell(\theta) = \sum_{i=1}^{n} \ln f_t\,(x_t;\theta)$ is the likelihood function at point $\theta$, and $p$ indicates the number of the model parameters. Since the number of parameters are not equal in all models, the AIC may overestimate the fitness of the model, hence using BIC is more appropriate in this case.

Usually, the models with lower AIC/BIC are better fitted since they have greater Likelihood value. AIC/BIC can be used for univariate FA as well as copula FA; also, they can be used for both static and time-varying models. For computing AIC/BIC for copula, the density function $f_t$ in (3) and (4) should be replaced by $c(u_t,\, v_t|\theta)$ (or $c(u_t, v_t|\theta_t)$ for time-varying copula).

## 5 More about return levels and return periods for time-varying univariate distribution and copula

Return periods are defined as the inverse of the upper quantile of the estimated distributions. For simplicity, $\mathscr{L}$ and $\mathscr{P}$ are used to show return level and return period, respectively. When working with static univariate distributions, for each return level, $\mathscr{L}$, there is a unique return period defined as:

$$\mathscr{P}(\mathscr{L}) = \frac{1}{\Pr[X \geq \mathscr{L}]} = \frac{1}{1-F(\mathscr{L})} = \frac{1}{\bar{F}(\mathscr{L})}.$$

In time-varying univariate models, at each time epoch, $t$, a distinct distribution is estimated; therefore, given a single return level, $\mathscr{L}$, there is a return period function which for each time gives a distinct return period; more precisely:

$$\mathscr{P}_t(\mathscr{L}) = \frac{1}{\Pr[X_t \geq \mathscr{L}]} = \frac{1}{1-F_t(\mathscr{L})}.$$

In case of stationary/time-varying univariate models, given $F$ or $F_t$, there is a one-to-one relationship between return period and return level, because $F$ and $F_t$ are strictly increasing continuous functions and consequently invertible; the return level can be characterized uniquely for a given return period; more precisely:

$$\mathscr{L}(\mathscr{P}) = F^{-1}(1-1/\mathscr{P}) \qquad \text{and} \qquad \mathscr{L}_t(\mathscr{P}) = F_t^{-1}(1-1/\mathscr{P}).$$

For multivariate models, two approaches can be used to obtain the upper quantiles. If the marginal distribution functions, bivariate distribution function, and copula of pair $(X,Y)$, are denoted by $F$, $G$, $H$, and $C$, respectively, then two types of upper quantiles could be defined as:

$$\bar{H}(x,y) - \Pr[X \geq x_n \cap Y \geq y] = 1 - F(x) - G(y) + H(x,y) = 1 - u - v + C(u,v) = \bar{C}(u,v),$$

and

$$\tilde{H}(x,y) = \Pr[X \geq x \cup Y \geq y] = 1 - H(x,y) = 1 - C(u,v) = \tilde{C}(u,v),$$

where $u = F(x)$ and $v = G(y)$. $\bar{C}$ is called the "survival function associated with $C$". Based on these types of upper quantiles, two types of return periods can be defined:

$$\mathscr{P}^{\wedge}(\mathscr{L}_1,\mathscr{L}_2) = \frac{1}{\bar{H}(\mathscr{L}_1,\mathscr{L}_2)} = \frac{1}{\bar{C}(F(\mathscr{L}_1),G(\mathscr{L}_2))}, \tag{5}$$

and

$$\mathscr{P}^{\vee}(\mathscr{L}_1,\mathscr{L}_2) = \frac{1}{\tilde{H}(\mathscr{L}_1,\mathscr{L}_2)} = \frac{1}{\tilde{C}(F(\mathscr{L}_1),G(\mathscr{L}_2))}. \tag{6}$$

$\mathscr{P}^{\wedge}$ and $\mathscr{P}^{\vee}$ are called "AND" and "OR" return periods, respectively. Regardless of the type of return periods, determining the return period for a given return level is the same as for the univariate case: for every (bivariate) return level $(\mathscr{L}_1,\mathscr{L}_2)$ there is a single associated return period which can be derived by equations (5) and (6). However, since the bivariate functions $\bar{H}$ and $\tilde{H}$ are not one-to-one, for each return period $\mathscr{P}$ there is a set of bivariate return levels associated with $\mathscr{P}$:

$$\mathscr{L}^{\wedge}(\mathscr{P}) = \{(x,y): \quad \bar{H}(x,y) = 1 - 1/\mathscr{P}\} = \left\{(x,y): \quad \bar{C}(F(x),G(y)) = 1 - 1/\mathscr{P}\right\},$$

and

$$\mathscr{L}^{\vee}(\mathscr{P}) = \left\{(x,y): \quad \tilde{H}(x,y) = 1 - 1/\mathscr{P}\right\} = \left\{(x,y): \quad \tilde{C}(F(x),G(y)) = 1 - 1/\mathscr{P}\right\}.$$

The set of return levels associated with a single return period forms a contour of $\bar{H}$ or $\tilde{H}$, which is usually a smooth curve in plane $\Re^2$.

In fact, "AND" return period at level $\mathscr{L} = (\mathscr{L}_1,\mathscr{L}_2)$ is the expected time of both $X$ and $Y$ exceeding their critical values: $X > \mathscr{L}_1$ and $Y > \mathscr{L}_2$. It means that bivariate "critical status" is occurred when both variables are at their high level. In contrast, the "OR" return period is the expected time for the occurrence of critical value of either $X$ or $Y$: $X > \mathscr{L}_1$ or $Y > \mathscr{L}_2$. In this case, the "critical status" is happened whenever at least one of the variables achieves their critical value. The focus of this paper is on the "AND" return period Salvadori et al. (2007, 2016). Like the univariate case, when the copula is time-varying, for each time epoch, $t$, different return levels and return periods can be found. More precisely, for a given time epoch, $t$, the "AND" return period is defined as:

$$\mathscr{P}_t^{\wedge}(\mathscr{L}_1,\mathscr{L}_2) = \frac{1}{\bar{H}_t(\mathscr{L}_1,\mathscr{L}_2)} = \frac{1}{\bar{C}_t(F(\mathscr{L}_1),G(\mathscr{L}_2))}.$$

Time-varying "OR" return period and "AND" and "OR" time-varying return levels are defined in similar way. Note that regardless of copula being time-varying or stable, the marginal distributions can be stable or time-varying. Whenever copula or one of the marginal distributions is time-varying, the bivariate return level (period) will be time-varying.

## 6 Estimation of the copula model parameter

### 6.1 Maximum Likelihood Estimation (MLE)

When the observations come from a similar distribution, the log-likelihood function is expressed as follows:

$$\ell(\theta) = \sum_{t=1}^{n} \ln f(x_t;\theta),$$

where $x_1,...,x_n$ are observations; $f$ is the common probability density of $X_1,...,X_n$; $n$ is the number of observations, and $\theta$ indicates the (vector of) parameter(s). The maximum likelihood estimate (MLE) of $\theta$ is the value of $\theta$ which maximizes $\ell(\theta)$. Note that for copula estimation, the observation $x_t$ is replaced by $(u_t, v_t)$ and the density function f is replaced by $c$. When the observations are not identically distributed, i.e., when the model is dynamic, based on the form of the dynamic model, the form of the likelihood function is changed. For instance, if the shape of distribution remains unchanged and only the parameter is changed over time, e.g., when $\theta_t = \alpha_1 + \alpha_2 t$, then the vector of parameters would be $\alpha = (\alpha_1, \alpha_2)$ and the log-likelihood function can be expressed as:

$$\ell(\alpha) = \sum_{t=1}^{n} \ln f(x_t;\alpha_1 + \alpha_2 t).$$

### 6.2 Pseudo Maximum Likelihood Estimation (PMLE)

For fitting a copula on a data containing n bivariate observations $(u_i, v_i)$, the univariate marginals should be standard uniform. In practice, the marginal distribution of underlying variables rarely become standard uniform, so, before fitting a copula, the data should be transformed into a bivariate "pseudo" observation with the standard uniform marginal distribution. The two well-known methods for transformations are: (i) fitting suitable marginal distributions on $x_i$'s and $y_i$'s, separately, and putting $\hat{u}_t = \hat{F}_X(x_t)$ and $\hat{v}_t = \hat{G}_Y(y_t)$; (ii) putting $\hat{u}_t = \frac{\#\{j: x_j \leq x_t\}}{n+1}$ and $\hat{v}_t = \frac{\#\{j: y_j \leq y_t\}}{n+1}$. Note that when the marginal distributions are not static, the second method may lead to wrong results (Shih and Louis, 1995). In this paper, a two-step pseudo maximum likelihood estimation was adopted for parameter estimation. First, the marginal distribution is fitted by the MLE method, then the pseudo-observations for copula are derived with estimated distribution functions, and, finally, the copula is fitted on the pseudo-observations. For the first step, the log-likelihood function is as follows:

$$\ell_X(\theta^x) = \sum_{t=1}^{n} \ln f(x_t;\theta^x) \quad , \quad \ell_Y(\theta^y) = \sum_{t=1}^{n} \ln g(y_t;\theta^y)$$

Estimated marginal distribution functions are $\hat{F}_X(x;\hat{\theta}^x)$ and $\hat{G}_Y(y;\hat{\theta}^y)$. Hence, pseudo observations are $\hat{u}_t = \hat{F}_X(x_t;\hat{\theta}^x)$ and $\hat{v}_t = \hat{G}_Y(y_t;\hat{\theta}^y)$ and pseudo-log-likelihood function is as follows:

$$\ell_C(\theta^c) = \sum_{t=1}^{n} \ln c\left(\hat{u}_t, \hat{v}_t; \theta^C\right), \tag{7}$$

where $c(u,v;\theta)$ indicates the density of the copula function at $(u,v)$ with parameter $\theta$. The parameter of copula can be estimated by minimizing (7). In this paper, several packages of R software such as "base", "stats4", "mle", "extRemes" are used for maximizing the stable/time-varying univariate and copula log-likelihood functions (Joe, 2014).

## 7 Bivariate time-varying copulas

### 7.1 Dynamic Frank copula

Bivariate distribution of static Frank copula with parameter $\theta$ is: (Manner and Reznikova, 2012; Bender et al., 2014; van den Goorbergh et al., 2003; Golian et al., 2020)

$$C_{Frank}(u,v;\theta) = -\frac{1}{\theta}\ln\left[1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{\left(e^{-\theta}-1\right)}\right],$$

and its bivariate density function is as follows:

$$c_{Frank}(u,v;\theta) = \frac{\partial^2 C_{Frank}}{\partial u \partial v} = \frac{\theta\left(1-e^{-\theta}\right)e^{-\theta(u+v)}}{\left(e^{-\theta}-1+\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)\right)^2},$$

in which $\theta \neq 0$. For time-varying Frank copula, the parameter $\theta_t$ is considered as an $ARMA(1,m)$ time series. The error term in this ARMA model is considered $\varepsilon_t = |u_t - v_t|$. More precisely,

$$\theta_t = \alpha + \beta\theta_{t-1} + \frac{\gamma}{10}\sum_{i=1}^{m}|u_{t-i}-v_{t-i}|, \tag{8}$$

Note that in the case of the time-varying copula, the vector of parameters is $\theta = (\alpha,\beta,\gamma)$. Patton (2012) considered the model (8) for Kendall's $\tau_t$ instead of the parameter itself. Since Kendall's $\tau$ should be lie between $[0,1]$, Patton applied a transformation function on the model; more precisely, he considered the following model for time-varying Kendall's *tau*:

$$\tau_t = \Lambda\left(\alpha + \beta\Lambda^{-1}(\tau_{t-1}) + \frac{\gamma}{m}\sum_{i=1}^{m}|u_{t-i}-v_{t-i}|\right),$$

where $\Lambda(x) = \Lambda_\tau(x) = \frac{1-\exp(-x)}{1+\exp(-x)}$ is the transformation function. Note that since the range of parameters of Frank copula has no limitation, there is no need to use a transformation function in (8).

### 7.2 Dynamic Gumbel copula

Bivariate distribution of static Gumbel copula with parameter $\theta$ is:

$$C_Gumbel(u,v;\theta) = \exp\left[-\left\{(\ln u)^\theta + (\ln v)^\theta\right\}^{\frac{1}{\theta}}\right],$$

and its bivariate density function is as follows:

$$\begin{aligned} c_{Gumbel}(u,v;\theta) = \frac{\partial^2 C_{Gumbel}}{\partial u \partial v} =& C_{Gumbel}(u,v;\theta)\times(uv)^{-1}\left\{(-\ln u)^\theta + (-\ln v)^\theta\right\}^{-2+\frac{2}{\theta}}(\ln u \dot{\ln} v)^{\theta-1} \\ &\times\left\{1+(\theta-1)\left((-\ln u)^\theta(-\ln v)^\theta\right)^{-\frac{1}{\theta}}\right\} \end{aligned}$$

where $\theta \in (1,\infty)$.
Similar to the case of time-varying Frank copula, the parameter of time-varying Gumbel copula is considered

as a $ARMA(1,m)$ time series. Again, the error term is $\varepsilon_t = |u_t - v_t|$. Similar to Patton's model, a transformation is applied to the model which holds the parameter $\theta_t$ in its acceptable range. More precisely,

$$\theta_t = \Lambda\left(\alpha + \beta\Lambda^{-1}(\theta_{t-1}) + \frac{\gamma}{m}\sum_{i=1}^{m}|u_{t-i} - v_{t-i}|\right), \tag{9}$$

where

$$\Lambda(x) = \Lambda_{Gumbel}(x) = \begin{cases} 1+\exp(x), & x < 0, \\ 2+\ln(x+1), & x \geq 0. \end{cases}$$

This transformation function is smooth in the sense that it is continuous and differentiable. It varies smoother than Patton's transformation function, and, in practice, it covers a wide range of parameter values. Note that in the case of the time-varying copula, the vector of parameters is $\theta = (\alpha, \beta, \gamma)$.

### 7.3 Dynamic Clayton copula

Bivariate distribution and density function of Clayton copula are as follows:

$$C_{Clayton}(u,v;\theta) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{1}{\theta}},$$

and

$$c_{Clayton}(u,v;\theta) = \frac{\partial^2 C_{Clayton}}{\partial u \partial v} = (1+\theta)(uv)^{-1-\theta}\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}-2},$$

where $\theta > 0$. Parameter of time-varying Clayton copula is similar to the Gumbel time-varying copula in (9) with transformation function:

$$\Lambda(x) = \Lambda_{Clayton}(x) = \begin{cases} \exp(x), & x < 0, \\ 1+\ln(x+1), & x \geq 0. \end{cases}$$

### 7.4 Dynamic Gaussian (normal) copula

Bivariate distribution and density function of Gaussian copula are defined in the terms of density and distribution function of univariate standard normal distributions. The univariate density function of the standard normal distribution is as follows: (Almeida and Czado, 2012)

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right).$$

The distribution function of the standard normal distribution cannot be evaluated analytically, and should be derived numerically as the following definite integrals:

$$\Phi(z) = \int_{-\infty}^{z}\phi(w)dw.$$

Now, the bivariate distribution function of Gaussian copula can be expressed as the following multiple definite integral:

$$C_{Normal}(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{\sqrt{2\pi(1-\theta^2)}} \exp\left(-\frac{1}{2(1-\theta^2)}\left(x^2+y^2-2\theta xy\right)\right) dydx,$$

and the bivariate density function of Gaussian copula is:

$$c_{Normal}(u,v;\theta) = \frac{\partial^2 C_{Normal}}{\partial u \partial v} = \frac{1}{\sqrt{2\pi(1-\theta^2)}\phi\left(\Phi^{-1}(u)\right)\phi\left(\Phi^{-1}(v)\right)} \times \exp\left[-\frac{1}{2(1-\theta^2)}\left\{\left(\Phi^{-1}(u)\right)^2+\left(\Phi^{-1}(v)\right)^2-2\theta\Phi^{-1}(u)\Phi^{-1}(v)\right\}\right].$$

where $\theta \in (-1,1)$ and $\Phi^{-1}$ is the quantile function of standard normal distribution. The parameter of time-varying Gaussian copula is modeled as a $ARMA(1,m)$ time series with error term $\varepsilon_t = \left|\Phi^{-1}(u_t) - \Phi^{-1}(v_t)\right|$ and similar to the case of Gumbel and Clayton, a transformation function is applied:

$$\theta_t = \Lambda\left(\alpha + \beta\Lambda^{-1}(\theta_{t-1}) + \frac{\gamma}{m}\sum_{i=1}^{m}\left|\Phi^{-1}(u_t) - \Phi^{-1}(v_t)\right|\right), \tag{10}$$

where the transformation function is:

$$\Lambda(x) = \Lambda_{Normal}(x) = \frac{1-\exp(-x)}{1+\exp(-x)} = \tanh\left(\frac{x}{2}\right).$$

## 8 Goodness-of-fit test for static and time-varying copula

AIC/BIC can be used only for comparing several models and they do not provide information about fitting a distribution on a single dataset; i.e., they cannot suggest accepting or rejecting the null hypothesis $H_0: \quad F = F_0$. To do this, for univariate goodness-of-fit test there are three famous test statistics: Kolmogorov-Smirnov, Anderson-Darling, Cramer-Von-Mises; these test statics are as follows (Hofert et al., 2018; Manner and Reznikova, 2012; Deheuvels, 1980; Genest et al., 2009):

$$T_{KS} = \sup_x |\ldots F_n(x) - F_0(x)|,$$
$$T_{AD} = n\int_{-\infty}^{\infty} \frac{(\mathbb{F}_n(x) - F_0(x))}{F_0(x)(1-F_0(x))} \mathrm{d}F_0(x),$$
$$T_{CM} = n\int_{-\infty}^{\infty} (\mathbb{F}_n(x) - F_0(x))^2 \mathrm{d}F_0(x),$$

where $\mathbb{F}_n$ is the empirical distribution of $x_1,\ldots,x_n$; i.e., $\mathbb{F}_n(x) = \frac{1}{n}\sum_{t=1}^{n} I_{\{x_t \le x\}}$. All three test statistics tend to be small if $H_0$ is true. The critical values are computed numerically.
For copula goodness-of-fit test, suppose the copula of random vector $(U,V)$ be $C(u,v)$. Then the conditional distribution function of $U|V = v$ is $C(u|v) = \frac{\partial C(u,v)}{\partial v}$. Hence, the random variable $W = C(U|V)$ is distributed as standard uniform distribution. This can be used to obtain a method for testing $H_0: C = C_0$: compute $w_t = C_0(u_t|v_t)$ and by a well-known univariate method mentioned above, test whether $H_0': W \sim U(0,1)$. When

the copula is time-varying with time-varying parameter $\theta_t$ since $w_t = C_0(u_t|v_t;\theta_t)$, $t = 1,2,\ldots,n$ are not independent and identically distributed, hence, the critical values of well-known univariate goodness-of-fit test cannot be applied. Manner and Reznikova (2012) proposed a parametric bootstrap method for approximating $p$-values.
Their algorithm for computing approximate $p$-value is as follows; let $\theta_t$ be the $ML$ estimator of $\theta_t$.

1. Compute $T_{KS}$, $T_{AD}$, and $T_{CM}$ for the main sample $(u_t, v_t)$, $t = 1,2,\ldots,n$.
2. Simulate synthetic observation $(u_t^b, v_t^b)$ from copula $C$ with parameter $\hat{\theta}_t$, $t = 1,2,\ldots,n$.
3. Estimate the time-varying $\hat{\theta}_t^b$ using the bootstrap sample, $(u_t^b, v_t^b)$.
4. Compute $w_t^b = C_0\left(u_t^b|v_t^b;\hat{\theta}_t\right)$.
5. Compute the $KS$, $AD$, and $CM$ test statistic $T_{KS}^b$, $T_{AD}^b$, and $T_{CM}^b$.
6. Repeat Steps 1 to 4 many times and compute the bootstrap $p$-values as the fraction of times $T_{KS} < T_{KS}^b$, $T_{AD} < T_{AD}^b$, and $T_{CM} < T_{CM}^b$ (Fermanian, 2005).

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