Supplementary Notes

In this Supplementary Notes, we further elaborate our analysis by explaining some technical aspects of the Letter

Below, we first present the concept of extensivity, which allows to precisely discriminate between the regime where biodiversity is stable (i.e. maintained) and the situation where it is unstable and the system settles in one of the uniform (absorbing) states. Then, we present details on the stochastic differential equations as well as the complex Ginzburg-Landau equation used to analyse the system.

We also show how the spirals' wavelength λ is related to the mobility M. We explain that such a relation has allowed to derive the functional dependence $M_c(\mu)$ of the critical mobility on the reproduction rate. Finally, the main findings reported in the Letter are revisited in a supplementary discussion centered on the information conveyed by the movies.

Extensivity

Even if coexistence appears stable, as observed for low mobilities, there is a certain probability that two species go extinct due to possible large yet rare fluctuations. Indeed, the only absorbing states where no reactions (and therefore no fluctuations) occur, are the uniform configurations where only one species survives. For this reason, these are the only stable states in the long run. However, the typical waiting time $T(N)$ until extinction occurs is generally very long when the system size N is large. This suggests to consider the dependence of the waiting time $T(N)$ on N. Quantitatively, we discriminate between stable and unstable coexistence by using the concept of *extensivity*, adapted from statistical physics. If the ratio T/N tends to infinity ($T/N \to \infty$) in the asymptotic limit of large systems ($N \to \infty$), the typical waiting time strongly prolongs with N (typically with an exponential dependence). This scenario is called *super-extensive* or stable. On the other hand, if $T/N \to O(1)$ (i.e. the ratio approaches a finite non-zero value) that is referred to as the *extensive* case, which has been shown to correspond to marginal (or neutral) stability[12]. Instability of the coexistence state (towards a uniform one) is encountered when $T/N \to 0$ (*sub-extensive* scenario), where the waiting time is short as compared to the system size. These definitions of stability and instability (with neutral stability separating the two

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cases) in the presence of absorbing states are intimately related to the concept of transients. In fact, Hastings[22] suggested to study not only the ultimate fate of a system, but also to consider the behaviour at smaller (and probably ecologically more relevant) time scales. According to the definition introduced above, stable or neutrally stable coexistence implies coevolution of the populations for a very large number of generations. This corresponds to the existence of extremely long-lived persistent transients[22] (super-persistent). It is also worth noticing that transients lasting for several generations can occur even in the case of unstable coexistence. This typically happens when the number of individuals N is large.

In the situation of Fig. 2, we have considered the extinction probability P_{ext} that, starting from random initial conditions (i.e. spatially homogeneous configurations, with equal concentrations of each species), the system has reached a uniform state after a time t proportional to the system size, i.e. $t \sim N$. In the asymptotic limit $N \to \infty$, three distinct cases arise. In a first regime, the extinction probability tends to zero with the system size N. In Fig. 2, this occurs when $M < M_c$. This scenario corresponds to the superextensive situation (i.e. $T/N \to \infty$, with $N \to \infty$) where the coexistence of all populations is stable. As a second case, the extinction probability approaches a finite value between 0 and 1, i.e. $T/N \to O(1)$, and we recover neutral stability. In Fig. 2, such a behaviour arises exclusively at the vicinity of the critical mobility M_c. In a third regime, the extinction probability does reach the value 1, which means that $T/N \to 0$. This is the subextensive scenario where the coexistence is unstable and biodiversity is lost. In Fig. 2, this happens above the critical mobility, i.e. when $M > M_c$.

Stochastic partial differential equations

Within the theory of stochastic processes[27], the dynamics of the stochastic lattice system is described by a master equation. In the limit of large systems, using a Kramers-Moyal expansion, the latter allows for the derivation of a proper Fokker-Planck equation, which in turn is equivalent to a set of stochastic partial differential equations. The latter consist of a mobility term, nonlinear terms describing the deterministic dynamics of the nonspatial model (May-Leonard equations), and noise terms. For the noise terms, we have found that contributions stemming from selection and reproduction events scale as $N^{-1/2}$, while fluctuations originating from exchanges (mobility) decay as N^{-1}; the latter may therefore

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be ignored. What remains, is (multiplicative) white noise whose strength scales as $N^{-1/2}$.

Complex Ginzburg-Landau Equation (CGLE)

Ignoring the noise terms in the stochastic differential equations describing the system, the resulting partial differential equations fall into the class of the Poincaré-Andronov-Hopf bifurcation, known from the mathematics literature[28]. Applying the theory of center manifolds and normal forms developed there, we have been able to cast the deterministic equations into the form of the complex Ginzburg-Landau equation:

$$\partial_t z = M\partial_r^2 z + c_1 z - (1 - ic_3)|z|^2 z \; , \tag{1}$$

where z is a complex variable and c_1, c_3 are constants depending on the rates σ and μ. This equation leads to the formation of dynamic spirals and allows to derive analytic results for their wavelength and frequency, see e.g. the review by Aranson and Kramer[29].

Scaling relation and critical mobility

An important question is to understand what is the mechanism driving the transition from a stable coexistence to extinction at the critical mobility M_c. To address this issue, we first note that varying the mobility induces a scaling effect, as illustrated in Fig. 2. In fact, increasing the mobility rate M results in zooming into the system. As discussed above (see the main text and Methods), the system's dynamics is described by a set of suitable stochastic partial differential equations (SPDE) (T.R., M.M., and E.F., in preparation) whose basic properties help rationalize this scaling relation. In fact, the mobility enters the stochastic equations through a diffusive term $M\Delta$, where Δ is the Laplace operator involving second-order spatial derivatives. Such a term is left invariant when M is multiplied by a factor α while the spatial coordinates are rescaled by $\sqrt{\alpha}$. It follows from this reasoning that varying M into αM translates in a magnification of the system's characteristic size by a factor $\sqrt{\alpha}$ (say $\alpha > 1$). This implies that the spirals' wavelength λ is proportional to $\sqrt{M}$ (i.e. $\lambda \sim \sqrt{M}$) up to the critical M_c .

When the spirals have a critical wavelength λ_c, associated with the mobility M_c, these rotating patterns outgrow the system size which results in the loss of biodiversity (see the

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main text). In the "natural units" (length is measured in lattice size units and the time-scale is set by keeping $\sigma = 1$), we have numerically computed $\lambda_c = 0.8 \pm 0.05$. This quantity has been found to be universal, i.e. its value remains constant upon varying the rates σ and μ. However, this is not the case of the critical mobility M_c, which depends on the parameters of the system. Below the critical threshold M_c, the dynamics is characterized by the formation of spirals of wavelength $\lambda(\mu, M) \sim \sqrt{M}$. This relation, together with the universal character of λ_c, leads to the following equation:

$$M_c(\mu) = \left(\frac{\lambda_c}{\lambda(\mu, M)} \right)^2 M \ , \tag{2}$$

which gives the functional dependence of the critical mobility upon the system's parameter. To obtain the phase diagram reported in Fig. 4 we have used Eq. (2) together with values of $\lambda(\mu, M)$ obtained from numerical simulations. For computational convenience, we have measured $\lambda(\mu, M)$ by carrying out a careful analysis of the SPDE's solutions. The results are reported as black dots in Fig. 4. We have also confirmed these results through lattice simulations for systems with different sizes and the results are shown as blue dots in Fig. 2. Finally, we have taken advantage of the analytical expression (up to a constant prefactor, taken into account in Fig. 2) of $\lambda(\mu, M)$ derived from the complex Ginzburg-Landau equation (CGLE) associated with the system's dynamics (see Methods): with Eq. (2), we have obtained the red curve displayed in Fig. 2. This figure corroborates the validity of the various approaches (SPDE, lattice simulations and CGLE), which all lead to the same phase diagram where the biodiverse and the uniform phases are identified.

Supplementary Video Legend 1

In the first movie, the dynamics of individuals of species A, B and C follows the reactions illustrated in Fig. 1 with rates $\mu = 1$ (reproduction), $\sigma = 1$ (selection) and $\epsilon = 2.4$ (exchange rate). In Movie 1, individuals of each species are indicated in different colours (empty sites are shown as black dots). The dynamics takes place on a square lattice of $N = 400 \times 400$ sites, such that there are up to 1.6×10^5 individuals in the system. This set of parameters corresponds to a mobility rate $M = 2\epsilon/N = 3 \times 10^{-5}$ well below the critical threshold $M_c \approx 4.5 \pm 0.5 \times 10^{-4}$ (see Figs. 2, 4 and text). Initially the system is in a well-mixed configuration with equal density of individuals of each species and empty sites. As time increases and since $M < M_c$, biodiversity is maintained and complex dynamical patterns

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form in the course of the temporal development resulting in a rich entanglement of spiral waves.

Supplementary Video Legend 2

In the second movie, the mobility of the individuals has been increased. In fact, the dynamics of individuals of all species still follows the reactions illustrated in Fig. 1 with rates $\mu = 1$ (reproduction) and $\sigma = 1$ (selection), but the exchange rate is now $\epsilon = 6$. In Movie 2, individuals of each species are still indicated in different colours (empty sites are shown as black dots). The dynamics takes place on a square lattice of $N = 200 \times 200$ sites, allowing up to 4×10^4 individuals in the system. This set of parameters corresponds to a mobility rate $M = 3 \times 10^{-4}$ relatively close to the critical threshold $M_c \approx 4.5 \pm 0.5 \times 10^{-4}$ (see Figs. 2, 4 and text). Initially the system is in a well-mixed state with equal density of individuals of each species and empty sites. As time increases and since $M < M_c$, biodiversity is still maintained and patterns form in the course of the time development. However, as compared to the first movie, one notices that the size of the patterns has increased and one now only distinguishes one pair of antirotating spirals.

Supplementary Discussion

The supplementary movies illustrate the system's time development in the coexistence phase, i.e. the emergence of dynamical complex patterns deep in that phase (Movie 1) and close to (yet below) the threshold M_c (Movie 2, see text and Fig. 3).
Starting from initially homogeneous (well-mixed) configurations, after a short transient regime, spiral waves rapidly emerge and characterize the long-time behaviour of the system which settles in a reactive steady state (*super-extensive case*, see text). The wavefronts, merging to form entanglement of spirals, propagate with spreading speed v^* and wavelength λ. In Movies 1 and 2, it appears clearly that by rising the individuals' mobility, one increases the wavefronts propagation velocity and the wavelength of the resulting dynamical patterns, as well as the size of each spiral. From PDE associated with the system's dynamics, we can rationalize this discussion and estimate these quantities for the cases illustrated in Movies 1 and 2. Namely, for the spreading speed, we have obtained $v^* \approx 3.5 \times 10^{-3}$ (lattice-size

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units per time-step, Movie 1) and $v^* \approx 1.1 \times 10^{-2}$ (Movie 2). Similarly, the wavelength were found to be $\lambda \approx 0.21$ (lattice-size units, Movie 1) and $\lambda \approx 0.65$ (Movie 2). Here, rising the rate M from 3×10^{-5} (Movie 1) to 3×10^{-4} (Movie 2) results in the enhancement of v^* and λ by a factor $\sqrt{10} \approx 3.16$. In Movie 2, the size of the spirals can also be estimated to have been magnified by the same factor $\sqrt{10} \approx 3.16$ with respect to those of Movie 1. As explained in the text, this scaling property of the system can be understood by considering the stochastic partial differential equation describing the dynamics, which were obtained from the underlying master equation through a system size expansion (see Methods).

By rising the individuals' mobility, one therefore increases the size of the spiralling patterns (whose wavelength is proportional to $\sqrt{M}$) and for sufficiently large value of the exchange rate (i.e. of M), as in Movie 2, just a few spirals nearly cover the entire lattice. This happens up to the critical value $\lambda_c \approx 0.8$, found to be universal. In fact, when $\lambda \geq \lambda_c$ the whole system is covered with one single ("giant") spiral which cannot fit within the lattice. This effectively results in the extinction of two species and the loss of biodiversity. As explained in the text, by exploiting the fact that $\lambda \propto \sqrt{M}$ and the universal character of λ_c, one can infer the existence of the critical mobility rate $M_c = M_c(\mu)$ [see Eq. (2)], as illustrated in Fig. 4. This allows to discuss the fate of the system (i.e. biodiversity versus extinction) in terms of the reaction and mobility rates μ and M, respectively: For given reaction rates μ and ϵ (without loss of generality σ is set to unity, see text) and system size L, we obtain a critical value $M_c(\mu)$ of the mobility rate. In fact, a reactive steady state is reached (and biodiversity maintained) only if $M < M_c(\mu)$. When the individuals' mobility is too fast, i.e. when $M > M_c(\mu)$, the system can be considered to be *well-mixed* and its dynamics therefore can be aptly described in terms of homogeneous rate equations which predicts the extinction of two species (see Methods).

In the cases illustrated in Movies 1 and 2, $M_c \approx 4.5 \pm 0.5 \times 10^{-4}$ and the wavefronts propagate with $\lambda < \lambda_c$, so that biodiversity is always preserved. However, we notice that the resulting spatio-temporal patterns are quite different: while one finds a rich entanglement of spirals deep in the coexistence phase (i.e. for $M \approx 3 \times 10^{-5} \ll M_c$, Movie 1), only one pair of antirotating spirals fill the system when one approaches the critical value M_c (Movie 2).

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