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CONTENTS

These supplementary informations give a brief overview of the theoretical calculations and concepts used in the main text. In the first part the low-energy effective theories for two coupled one-dimensional superfluids and their exact solution within the classical-field approximation are discussed. In the second part, we first provide a general introduction to equal-time correlation functions. We furthermore give explicit formulas for the experimentally measured correlation functions and their decomposition into connected and disconnected parts, and discuss the connection to periodic observables used in our previous publications. The last section discusses the connection of the measured correlation functions and quasiparticle interactions.

I. THEORETICAL MODELS

In the first section, we introduce the one-dimensional model of two tunnel-coupled superfluids and discuss the derivation of the sine-Gordon Hamiltonian, proposed as a low-energy effective theory for the relative degrees of freedom.Thereafter, in the second section, we discuss the exact solution within the classical-field approximation, using the transfer matrix formalism.

A. Sine-Gordon model as effective low-energy theory for two tunnel-coupled superfluids

The quantum many-body system we study is an ultracold gas of ^{87}Rb atoms in a double-well (DW) potential on an atom chip as shown in Fig. 1 in the main paper. Each well is tightly confined in the radial direction ($\omega_\perp \simeq 2\pi \times 1.4\,\mathrm{kHz}$) and weakly confined along the longitudinal direction ($\omega_z \simeq 2\pi \times 6.7\,\mathrm{Hz}$). Since both superfluids fulfill the condition of being one-dimensional (1D), $\mu, k_\mathrm{B}T < \hbar\omega_\perp$, dynamics along the radial direction is frozen. However, tunneling through the adjustable DW barrier couples the two superfluids. Integrating over the radial degrees of freedom, reduces the problem to an effective 1D system described by the Hamiltonian

$$H = \sum_{j=1}^{2} \int dz \left[\frac{\hbar^2}{2m} \frac{\partial \psi_j^\dagger}{\partial z} \frac{\partial \psi_j}{\partial z} + \frac{g_{1\mathrm{D}}}{2} \psi_j^\dagger \psi_j^\dagger \psi_j \psi_j + U(z) \psi_j^\dagger \psi_j - \mu \psi_j^\dagger \psi_j \right] - \hbar J \int dz \left[\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1 \right] . \tag{S1}$$

Here m is the atomic mass, $g_{1\mathrm{D}} = 2\hbar a_\mathrm{s} \omega_\perp$ the 1D effective interaction strength, calculated from the s-wave scattering length a_s and the frequency $\omega_\perp$ of the radial confinement. $U(z)$ is the trapping potential in the longitudinal direction, and μ the chemical potential. The field operators fulfill the bosonic commutation relations $[\psi_j(z), \psi_{j'}^\dagger(z')] = \delta_{jj'}\delta(z-z')$. For simplicity, we consider in the following the homogenous case, $U(z) \equiv 0$. In order to derive a low-energy effective theory we use the density-phase representation,

$$\psi_j(z) = \exp[i\theta_j(z)] \sqrt{n_{1\mathrm{D}} + \delta\rho_j(z)} , \tag{S2}$$

with the canonical commutators $[\delta\rho_j(z), \theta_{j'}(z')] = \mathrm{i}\, \delta_{jj'}\delta(z-z')$. We define the symmetric (subscript s) and anti-symmetric (subscript a) degrees of freedom as

$$\delta\rho_s(z) = \delta\rho_1(z) + \delta\rho_2(z) \quad , \qquad \varphi_s(z) = \frac{1}{2}[\theta_1(z) + \theta_2(z)] , \tag{S3}$$

$$\delta\rho_a(z) = \frac{1}{2}[\delta\rho_1(z) - \delta\rho_2(z)] , \qquad \varphi_a(z) = \theta_1(z) - \theta_2(z) . \tag{S4}$$

Evidently these fields fulfil canonical commutation relations as well. Expanding the Hamiltonian (S1) in powers of the density fluctuations $\delta\rho_j$ and phase gradients $\partial_z\varphi_j$ to quadratic order leads to

$$H = H_s[\delta\rho_s, \varphi_s] + \int dz \left[\frac{\hbar^2}{4mn_{1\mathrm{D}}} \left(\frac{\partial \delta\rho_a}{\partial z} \right)^2 + g_{1\mathrm{D}} {\delta\rho_a}^2 + \frac{\hbar^2 n_{1\mathrm{D}}}{4m} \left(\frac{\partial \varphi_a}{\partial z} \right)^2 - 2\hbar J n_{1\mathrm{D}} \cos\varphi_a \right] + \int dz \left[\frac{\hbar J}{n_{1\mathrm{D}}} \delta\rho_a (\cos\varphi_a) \delta\rho_a - \hbar J\, \delta\rho_s \cos\varphi_a \right] , \tag{S5}$$

where the Hamiltonian H_s depends only on the symmetric degrees of freedom. Note that, while phase gradients are expected to be small for all values of J, the phase field φ itself needs to be considered non-perturbatively, leading to the full cosine potentials. The last term couples the symmetric and antisymmetric degrees of freedom and is expected to be significant for, e.g., the non-linear relaxation of the system following a quantum-quench. In, or close to, thermal equilibrium it is presumed that the couplings of density and phase fluctuations are negligible, which leads to a complete decoupling of the symmetric and antisymmetric degrees of freedom. The low-energy effective Hamiltonian describing the relative degrees of freedom takes the form

$$H = \int dz \left[\frac{\hbar^2}{4mn_{1D}} \left(\frac{\partial \delta\rho}{\partial z} \right)^2 + g\delta\rho^2 + \frac{\hbar^2 n_{1D}}{4m} \left(\frac{\partial \varphi}{\partial z} \right)^2 - 2\hbar J n_{1D} \cos\varphi \right], \tag{S6}$$

where we introduced $g = g_{1D} + \hbar J/n_{1D}$ and omitted the subscript a, as we do in the following, and in the main text. For $J = 0$, the Hamiltonian reduces to

$$H = \int dz \left[\frac{\hbar^2}{4mn_{1D}} \left(\frac{\partial \delta\rho}{\partial z} \right)^2 + g\delta\rho^2 + \frac{\hbar^2 n_{1D}}{4m} \left(\frac{\partial \varphi}{\partial z} \right)^2 \right]. \tag{S7}$$

On the other hand, for $\langle \cos(\varphi) \rangle \approx 1$, i.e. for strong tunnel-coupling J, fluctuations of the phase are tightly centered around zero, and the cosine in Eq. (S6) can be expanded to second order leading to

$$H = \int dz \left[\frac{\hbar^2}{4mn_{1D}} \left(\frac{\partial \delta\rho}{\partial z} \right)^2 + g\delta\rho^2 + \frac{\hbar^2 n_{1D}}{4m} \left(\frac{\partial \varphi}{\partial z} \right)^2 + \hbar J n_{1D} \varphi^2 \right]. \tag{S8}$$

Both Hamiltonians, Eq. (S7) and Eq. (S8), are quadratic in the fields and can therefore be diagonalized by a Bogoliubov transformation (see below). The system is described by non-interacting quasi-particles with a gap proportional to $\sqrt{J}$. Note that this remains valid for a non-vanishing external potential $U(z)$, although the explicit form of the dispersion relation changes.

At the low energies considered, density fluctuations are highly suppressed and hence one can neglect the term involving the derivative of the relative density, thereby restricting the spectrum to the phononic regime. For the uncoupled system, Eq. (S7), this leads to the Tomonaga-Luttinger Hamiltonian, whereas Eq. (S6), valid for general couplings J, reduces to the sine-Gordon Hamiltonian,

$$H_{SG} = \int dz \left[g\delta\rho^2 + \frac{\hbar^2 n_{1D}}{4m} \left(\frac{\partial \varphi}{\partial z} \right)^2 - 2\hbar J n_{1D} \cos\varphi \right]. \tag{S9}$$

At the classical level, the equations of motion derived from this Hamiltonian include solitonic and breather solutions. The single soliton/anti-soliton solution is given by

$$\varphi_S(z) = 4 \arctan \left[\pm \exp \frac{z - z_0 - v_S t}{l_J \sqrt{1 - (v_S/c_s)^2}} \right], \tag{S10}$$

where z_0 is the position and v_S the velocity of the soliton, and $c_s = \sqrt{gn_{1D}/m}$ the speed of sound (see, *e.g.* [11]). The width of the soliton is given by the length scale $l_J = \sqrt{\hbar/4mJ}$. Motion of the soliton leads to a contraction of this length scale by the 'Lorentz' factor $\sqrt{1 - (v_S/c_s)^2}$. These topological defects represent a local phase-twist of 2π, connecting adjacent minima of the cosine potential.

The sine-Gordon model represents an exactly-solvable field theory. The sine-Gordon Hamiltonian (S9) can be written in the re-scaled form

$$H_{SG} = \frac{1}{2} \int dz \left[\Pi^2 + (\partial_z \phi)^2 - \Delta \cos\beta\phi \right], \tag{S11}$$

where we set $\hbar = k_{\mathrm{B}} = 1$, rescaled time $t \to c_s t$, and set $c_s = 1$. Furthermore, we define the conjugate momentum $\Pi = \beta\,\delta\rho$, the rescaled phase field $\phi = \varphi/\beta$, as well as the parameters $\beta = \sqrt{2\pi/K}$ and $\Delta = 8Jm/\beta^2$. The Luttinger parameter K is, in the weakly interacting regime ($\gamma \ll 1$), given by $K \approx \pi/\sqrt{\gamma}$, where $\gamma = mg/\hbar^2 n_{1\mathrm{D}}$, characterising the strength of the interaction. For theoretical studies of the sine-Gordon model see *e.g.* [10, 37]. The parameters applying to our experiment correspond to the weakly interacting regime, $K \gg 1$, typically $K = 63\ldots73$, and hence $\beta^2 = 0.1\ldots0.086$.

For completeness, we give a brief overview of the different regimes of the sine-Gordon model, supposing β and Δ as independent parameters. The spectrum of the Hamiltonian, Eq. (S11), depends on the value of β. The system undergoes a Kosterlitz-Thouless transition at the critical point $\beta^2 = 8\pi$. For larger values, $\beta^2 > 8\pi$, the cosine term becomes irrelevant and the system reduces to the Luttinger-Liquid model. As was shown in [6] for $\beta^2 < 8\pi$ the sine-Gordon model is equivalent to the zero-charge sector of the massive Thirring model, describing massive Dirac fermions with local self-interaction. In this regime, the spectrum can be further divided into two distinct sectors, separated by the Luther-Emery point, $\beta^2 = 4\pi$, at which the model describes non-interacting massive Dirac fermions. For $4\pi < \beta^2 < 8\pi$, the system is described by soliton and anti-soliton excitations, whereas for $0 < \beta^2 < 4\pi$, the spectrum contains additional bound states of (anti-)solitons, called *breathers*.

B. Exact results within the classical-field approximation

Within the classical-field approximation, correlation functions of the system at temperature T can be calculated using the transfer-matrix formalism developed in [38, 39]. The harmonic approximation (S8) for two tunnel-coupled superfluids has been analysed in [36]. In particular, the Gaussian fluctuations of the phase along z have been shown to be describable by an Ornstein-Uhlenbeck process.[1] This enables the efficent sampling of the fields, directly from the equilibrium distribution. Here we sketch the extension of these methods for the case of two coupled wave guides described by Eq. (S1) beyond the harmonic approximation.

The system realized in our experiments is a special case of the model described by the (classical) Hamiltonian

$$H = \int dz \Big[\sum_{j=1}^{M} \left(\frac{\hbar^2}{2m} \frac{\partial \psi_j^*}{dz} \frac{\partial \psi_j}{dz} - \mu \psi_j^* \psi_j \right) + V(\psi_M^*, \ldots, \psi_1^*, \psi_1, \ldots, \psi_M) \Big], \tag{S12}$$

for the M-component Bose field $\psi_j(z)$, $j = 1, \ldots, M$, with an arbitrary local, but not necessarily pairwise interaction potential V, conserving the total number of atoms, $N = \int dz \sum_{j=1}^{M} \psi_j^* \psi_j$ (I. Mazets, in preparation). Comparing with Eq. (S1) we get ($M = 2$)

$$V = \frac{g}{2} \left[(\psi_1^* \psi_1)^2 + (\psi_2^* \psi_2)^2 \right] - \hbar J \left[\psi_1^* \psi_2 + \psi_2 \psi_1^* \right], \tag{S13}$$

and the chemical potential $\mu = g n_{1D} - \hbar J$.

The transfer-matrix formalism [38, 39] yields the following expressions for the thermal average and correlation function of operators $\mathcal{O}(z)$:

$$\langle \mathcal{O}(z_1) \rangle = \sum_n \langle 0 | \mathcal{O}(z_1) | 0 \rangle, \tag{S14}$$

$$\langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle = \sum_n \langle 0 | \mathcal{O}(z_2) | n \rangle \langle n | \mathcal{O}(z_1) | 0 \rangle e^{-(\kappa_n - \kappa_0)(z_2 - z_1)} \qquad (z_2 \geq z_1), \tag{S15}$$

[1] Note that we deal with stochastic processes evolving in space, along z, but not in time.

where the matrix elements with respect to the eigenstates $|n\rangle$ of the transfer operator $\hat{K}$ (see below), with eigenvalues κ_n, are defined as:

$$\langle n'|\mathcal{O}(z)|n\rangle = \int_0^\infty dr_1\, r_1 \int_0^\infty dr_2\, r_2 \int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_2\, \Psi^*_{n'} \mathcal{O}(z) \Psi_n\,. \tag{S16}$$

Here, we introduced the density r_j^2, and the phase θ_j is defined, in analogy to the previous discussion, via $\mathrm{Re}(\psi_j) = r_j \cos\theta_j$ and $\mathrm{Im}(\psi_j) = r_j \sin\theta_j$. The observables $\mathcal{O}(z) = \mathcal{O}(r_1, \theta_1, r_2, \theta_2)|_z$ can be arbitrary functions of the classical field provided the integrals exist. The eigenvalues κ_n and orthonormal eigenfunctions $\Psi_n = \Psi_n(r_1, \theta_1, r_2, \theta_2)$ are given by the Hamiltonian-like hermitian operator $\hat{K}$ that arises in the transfer matrix formalism [38, 39]. For our system of two tunnel-coupled superfluids we have

$$\hat{K} = \hat{K}_1^{\mathrm{s}} + \hat{K}_2^{\mathrm{s}} + \frac{\hbar J}{k_{\mathrm{B}}T}(r_1^2 + r_2^2) - \frac{2\hbar J}{k_{\mathrm{B}}T} r_1 r_2 \cos(\theta_1 - \theta_2), \tag{S17}$$

where

$$\hat{K}_j^{\mathrm{s}} = -D\left(\frac{1}{r_j}\frac{\partial}{\partial r_j} r_j \frac{\partial}{\partial r_j} + \frac{1}{r_j^2}\frac{\partial^2}{\partial \theta_j^2}\right) + \frac{g}{2k_{\mathrm{B}}T} r_j^2 (r_j^2 - 2n_{1\mathrm{D}}) \tag{S18}$$

is the auxiliary operator for a single superfluid, and $D = mk_{\mathrm{B}}T/(2\hbar^2)$. The equilibrium distribution of the real classical variables is determined by the ground (lowest-eigenvalue) state of the operator $\hat{K}$ [38, 39], via

$$W_{\mathrm{eq}}(r_1, \theta_1, r_2, \theta_2) = |\Psi_0(r_1, \theta_1, r_2, \theta_2)|^2\,. \tag{S19}$$

It is possible to construct a Fokker-Planck equation for the classical probability distribution $W(r_1, \theta_1, r_2, \theta_2; z)$ that describes the same stochastic process as the transfer-matrix formalism:

$$\frac{\partial}{\partial z} W = \sum_{j=1}^{2N_f} \left[D \frac{\partial^2}{\partial q_j^2} W + \frac{\partial}{\partial q_j}(A_{q_j} W) \right]\,. \tag{S20}$$

To shorten the notation, the variables $\mathrm{Re}(\Psi_{1,2})$ and $\mathrm{Im}(\Psi_{1,2})$ are renamed as q_j ($j = 1, 2, 3, 4$). The stationarity condition of the equilibrium solution $\partial_z W_{\mathrm{eq}} = 0$ determines the drift coefficients A_{q_j}, for which we obtain from Eq. (S20):

$$A_{q_j} \equiv A_{q_j}(q_1, q_2, q_3, q_4) = -D \frac{\partial}{\partial q_j} \ln W_{\mathrm{eq}} = -2D \frac{\partial}{\partial q_j} \ln|\Psi_0|\,. \tag{S21}$$

The last step is to realise, that the Focker-Planck equation is equivalent to a stochastic process described by an Ito equation [16]

$$dq_j = -A_{q_j}\, dz + \sqrt{2D}\, dX_z\,, \tag{S22}$$

where dX_z is a random term obeying Gaussian statistics with zero mean, $\langle dX_z\rangle = 0$, and variance, $\langle dX_z^2\rangle = dz$. Fast sampling of the fields from the full classical equilibrium probability distribution is possible using Eq. (S21) and Eq. (S22), after finding only the ground state Ψ_0 of the auxiliary operator (S17) instead of the whole spectrum as Eq. (S15) requires. Note that the transfer-matrix formalism provides results for the correlation statistics of the unbound phase difference in the limit dominated by thermal fluctuations. This allows us to analyse continuous, unbound phase differences $\Delta\varphi$. Arbitrary correlation functions can therefore easily be calculated by averaging over independently sampled field configurations.

In the limit of vanishing tunnel coupling J, we obtain $A_\varphi \equiv 0$, *i.e.*, the relative phase is described by a diffusion process. In the opposite limit of strong tunnel coupling, we recover the results of Ref. [36]. The sine-Gordon Hamiltonian, Eq. (S9), relevant for intermediate J, is described by the auxiliary operator $\hat{K}$ given in Eq. (S17), neglecting the non-linear coupling between fluctuations of the relative phase $\varphi = \theta_1 - \theta_2$ and the densities $r_{1,2}^2$. In this approximation, the symmetric phase and the densities are determined by the usual Gaussian diffusion and Ornstein-Uhlenbeck processes, respectively. The relative phase needs to be calculated by means of the anharmonic model (S22), Ψ_0 being the lowest-eigenvalue solution of the corresponding Mathieu equation [34].

We compared the results of the direct calculations of the 4^{th} moment, using Eq. (S16) after diagonalising $\hat{K}$, to correlation functions obtained by averaging independently sampled field configurations and found perfect agreement. To explain the experimental observations we consider the latter approach, as it allows to incorporate the finite imaging resolution (see Methods). We furthermore compared the analytical results for the homogenous system to simulations of the stochastic Gross-Pitaevskii equation for harmonically trapped tunnel-coupled superfluids. Thereby, we found good agreement of the correlation functions in the central part of the cloud, for the range of experimental parameters.

II. CORRELATION FUNCTIONS

In this part, we first give a general introduction to equal-time correlation functions, their role in quantum field theory, and their connection to the interaction properties of the system. We further give explicit expressions for the correlation functions used in the experiment and their decomposition into connected and disconnected parts. Thereafter, we discuss their relation to commonly used periodic correlation functions, explaining in detail as to why they are, in general, not suitable to study the interaction properties of our system. In the last section we discuss, how a perturbative approach to the sine-Gordon model readily reveals the connection between equal-time correlation functions of the phase field and N-body quasiparticle interactions.

A. Equal-time correlation functions and their relevance in (quantum) field theory

For a quantum many-body system which is described in terms of a Heisenberg field operator $\mathcal{O}(t,x)$, all physical information is contained in correlation functions like

$$\langle \mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)\cdots\mathcal{O}(t_N,x_N)\rangle \equiv \mathrm{Tr}\left\{\rho_D\,\hat{T}\,\mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)\cdots\mathcal{O}(t_N,x_N)\right\}, \tag{S23}$$

where we consider, for the moment, a real scalar field with a single component. Here ρ_D denotes the density operator specifying the system at a given time, and the trace is taken over the time-ordered product of field operators as indicated by the time-ordering operator $\hat{T}$. For instance, the 2^{nd}-order function

$$G^{(2)}(t_1,x_1;t_2,x_2) \equiv \langle \mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)\rangle \tag{S24}$$

quantifies the correlation between the point x_1 at time t_1 and the point x_2 at time t_2. In the following we assume a vanishing field expectation value, $\langle \mathcal{O}(t,x)\rangle = 0$, as well as vanishing correlations $G^{(N)}$ for odd-integer N. In this case, for a non-zero 4^{th}-order correlation

$$G^{(4)}(t_1,x_1;t_2,x_2;t_3,x_3;t_4,x_4) \equiv \langle \mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)\mathcal{O}(t_3,x_3)\mathcal{O}(t_4,x_4)\rangle\ , \tag{S25}$$

one can identify the following contributions:

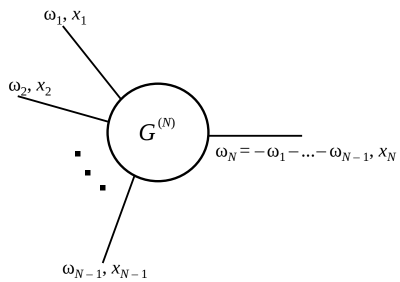
$$\begin{aligned}G^{(4)}(t_1,x_1;t_2,x_2;t_3,x_3;t_4,x_4) = {} & G^{(4)}_{\rm con}(t_1,x_1;t_2,x_2;t_3,x_3;t_4,x_4)\\ & + G^{(2)}(t_1,x_1;t_2,x_2)\,G^{(2)}(t_3,x_3;t_4,x_4)\\ & + G^{(2)}(t_1,x_1;t_3,x_3)\,G^{(2)}(t_2,x_2;t_4,x_4)\\ & + G^{(2)}(t_1,x_1;t_4,x_4)\,G^{(2)}(t_2,x_2;t_3,x_3)\,. \end{aligned} \tag{S26}$$

Here, the connected $4^{\rm th}$-order correlation, $G^{(4)}_{\rm con}$, is obtained from the full correlation by subtracting products of $2^{\rm nd}$-order correlations. In this way, the redundant information that is contained in disconnected lower-order correlations is being eliminated. For any higher $N^{\rm th}$-order correlations, with $N \geq 6$, a similar decomposition into connected and disconnected parts exists, the latter involving products of $G^{(2)}$, $G^{(4)}_{\rm con}$, ..., $G^{(N-2)}_{\rm con}$ that are correspondingly defined. Knowing all connected correlation functions is equivalent to knowing all full correlation functions and therefore sufficient for recovering all information about the system.

If the density operator ρ_D describes thermal equilibrium, then the correlation functions become time-translation invariant. In this case, employing a Fourier transformation with respect to times, one can represent the $N^{\rm th}$-order correlation (S23) as

$$\begin{aligned}\langle \mathcal{O}(t_1,x_1)\cdots\mathcal{O}(t_N,x_N)\rangle = \int \frac{d\omega_1}{2\pi}\cdots\frac{d\omega_N}{2\pi}\, e^{i(\omega_1 t_1+\cdots+\omega_N t_N)}\, 2\pi\,\delta(\omega_1+\cdots+\omega_N)\\ G^{(N)}(\omega_1,\ldots,\omega_{N-1};x_1,\ldots,x_N)\,.\end{aligned} \tag{S27}$$

Here $G^{(N)}(\omega_1,\ldots,\omega_{N-1};x_1,\ldots,x_N)$ denotes the $N^{\rm th}$-order correlation amplitude with external frequencies ω_i at spatial points x_i, for $i = 1,\ldots,N$. Diagrammatically, they can be represented as:



For instance, the $4^{\rm th}$-order amplitude $G^{(4)}(\omega_1,\omega_2,\omega_3;x_1,x_2,x_3,x_4)$ describes all possible quantum processes with $|\omega_i|$ injected ($\omega_i > 0$) or taken out ($\omega_i < 0$) at points x_i for $i = 1,2,3$ such that the total energy is conserved with $-\omega_1-\omega_2-\omega_3$ at x_4. For two-body interactions and the case of a real scalar field, this will involve standard scattering processes with Feynman diagrams having two incoming and two outgoing lines, but also diagrams with one line in and three lines out, the conjugate process (three in, one out), or even all lines in (or all lines out).

In this work, we measure equal-time correlation functions, where $t = t_1 = t_2 = \cdots = t_N$. From the Fourier representation (S27) one observes that an $N^{\rm th}$-order equal-time correlation function represents the sum over all the different processes with N external lines[2]

$$\langle \mathcal{O}(t,x_1)\cdots\mathcal{O}(t,x_N)\rangle = \int \frac{d\omega_1}{2\pi}\cdots\frac{d\omega_{N-1}}{2\pi}\, G^{(N)}(\omega_1,\ldots,\omega_{N-1};x_1,\ldots,x_N)\,. \tag{S28}$$

Measurements of equal-time correlation functions represent, therefore, a powerful tool to quantify the combined effect of all possible quantum processes that contribute to an $N^{\rm th}$-order correlation

— no matter how high the order of a process in terms of powers of Planck's constant h may be, or whether the contribution is of non-perturbative origin.

B. Experimental correlation functions and their decompositions

From the measured phase field $\varphi(z)$ we determine equal-time N^{th}-order correlation functions

$$G^{(N)}(\mathbf{z},\mathbf{z}')=\langle\triangle\varphi(z_1,z_1')\dots\triangle\varphi(z_N,z_N')\rangle\ , \tag{S29}$$

where $\triangle\varphi(z_i,z_i')=\varphi(z_i)-\varphi(z_i')$ are unambiguous phase differences of the unbound phase at different spatial points z_i, z_i', and we suppressed the common time label t to shorten the notation. In our experiment, the first-order correlation function vanishes by symmetry, $\langle\triangle\varphi(z_i,z_i')\rangle=0$, as well as all other correlation functions where N is an odd positive integer. While all information is contained in the N^{th}-order correlation functions, it is more enlightening to measure connected correlations, since the redundant information of lower-order correlations is eliminated as explained in the previous section. In the following we give explicit expressions for the decomposition

$$G^{(N)}(\mathbf{z},\mathbf{z}')=G^{(N)}_{\text{con}}(\mathbf{z},\mathbf{z}')+G^{(N)}_{\text{dis}}(\mathbf{z},\mathbf{z}') \tag{S30}$$

of the experimental correlations defined in Eq. (S29). The general formula for the connected part [32] is

$$G^{(N)}_{\text{con}}(\mathbf{z},\mathbf{z}')=\sum_{\pi}\ (|\pi|-1)!\ (-1)^{|\pi|-1}\prod_{B\in\pi}\left\langle\prod_{i\in B}\triangle\varphi(z_i,z_i')\right\rangle. \tag{S31}$$

Here, the sum runs over all possible partitions π of $\{1,\dots,N\}$, the first product runs over all blocks B of the partition and the second product over all elements i of the block. Since, in our system, all correlation functions where N is an odd positive integer vanish by symmetry, we get $G^{(2)}_{\text{con}}(\mathbf{z},\mathbf{z}')=G^{(2)}(\mathbf{z},\mathbf{z}')$ and, e.g., for the 4^{th}-order connected correlation function:

$$\begin{aligned}G^{(4)}_{\text{con}}(\mathbf{z},\mathbf{z}')=G^{(4)}(\mathbf{z},\mathbf{z}')&-\langle\triangle\varphi(z_1,z_1')\triangle\varphi(z_2,z_2')\rangle\,\langle\triangle\varphi(z_3,z_3')\triangle\varphi(z_4,z_4')\rangle\\&-\langle\triangle\varphi(z_1,z_1')\triangle\varphi(z_3,z_3')\rangle\,\langle\triangle\varphi(z_2,z_2')\triangle\varphi(z_4,z_4')\rangle\\&-\langle\triangle\varphi(z_1,z_1')\triangle\varphi(z_4,z_4')\rangle\,\langle\triangle\varphi(z_2,z_2')\triangle\varphi(z_3,z_3')\rangle\ .\end{aligned} \tag{S32}$$

In case of a Gaussian state, all connected parts of correlation functions ($N>2$) vanish, i.e. $G^{(N>2)}_{\text{con}}(\mathbf{z},\mathbf{z}')\equiv 0$. Hence, all correlation functions factorise and one recovers Wick's theorem [3] stating that, for a Gaussian state, all correlation functions with ($N>2$) are determined by second-order correlation functions. Explicitely, the Wick decomposition is given by

$$G^{(N)}_{\text{wick}}(\mathbf{z},\mathbf{z}')=\sum_{\pi_2}\left[\prod_{B\in\pi_2}\Big\langle[\varphi(z_{B_1})-\varphi(z'_{B_1})][\varphi(z_{B_2})-\varphi(z'_{B_2})]\Big\rangle\right]. \tag{S33}$$

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[2] More precisely, equal-time correlation functions for bosonic fields measure the symmetrized (anti-commutator) part of the time-ordered correlators (S23). For the real scalar field operator considered, this can be directly observed from the definition of the time-ordering operator, as e.g. for the 2^{nd}-order correlation:

$$\begin{aligned}\langle\hat{T}\mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)\rangle&=\langle\mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)\rangle\Theta(t_1-t_2)+\langle\mathcal{O}(t_2,x_2)\mathcal{O}(t_1,x_1)\rangle\Theta(t_2-t_1)\\&=\frac{1}{2}\langle\{\mathcal{O}(t_1,x_1),\mathcal{O}(t_2,x_2)\}\rangle+\frac{1}{2}\langle[\mathcal{O}(t_1,x_1),\mathcal{O}(t_2,x_2)]\rangle\,\text{sgn}(t_1-t_2)\,.\end{aligned}$$

Here the step function is defined by $\Theta(t>0)=1$ and $\Theta(t<0)=0$ and the sign function is $\text{sgn}(t)\equiv\Theta(t)-\Theta(-t)$. Since the equal-time commutator vanishes, $[\mathcal{O}(t,x_1),\mathcal{O}(t,x_2)]=0$ for the real scalar field operator, the symmetrized part is given by the anti-commutator $\{\mathcal{O}(t_1,x_1),\mathcal{O}(t_2,x_2)\}\equiv\mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)+\mathcal{O}(t_2,x_2)\mathcal{O}(t_1,x_1)$ at equal times $t_1=t_2$.

Here the sum runs over all possible partitions π_2 of $\{1, \dots, N\}$ into blocks of size 2. The product again runs over all blocks B of the partition (see [32]).

The relevance of the connected part $G_{\mathrm{con}}^{(N)}$ can be quantified by the measure

$$M^{(N)} = \frac{\sum_{\mathbf{z}} \left| G_{\mathrm{con}}^{(N)}(\mathbf{z}, 0) \right|}{\sum_{\mathbf{z}} \left| G^{(N)}(\mathbf{z}, 0) \right|} \tag{S34}$$

with $M^{(N)} \in [0, 1]$. For a Gaussian state, $M^{(N)} \equiv 0$ for all $N > 2$.

Choosing coordinates $z_1 = z_2 = \ldots = z_N$ and $z'_1 = z'_2 = \ldots = z'_N$, the above formulas simplify and the N^{th}-order connected correlation function can be determined by the recursion formula

$$G_{\mathrm{con}}^{(N)}(\mathbf{z_1}, \mathbf{z}'_1) = G^{(N)}(\mathbf{z}_1, \mathbf{z}'_1) - \sum_{m=1}^{N-1} \binom{N-1}{m-1} G_{\mathrm{con}}^{(m)}(\mathbf{z}_1, \mathbf{z}'_1) G^{(N-m)}(\mathbf{z}_1, \mathbf{z}'_1) \,. \tag{S35}$$

Specifically for the lowest orders we get

$$\begin{aligned} G_{\mathrm{con}}^{(2)}(\mathbf{z}_1, \mathbf{z}'_1) &= \langle \triangle\varphi^2 \rangle \,, \\ G_{\mathrm{con}}^{(4)}(\mathbf{z}_1, \mathbf{z}'_1) &= \langle \triangle\varphi^4 \rangle - 3 \langle \triangle\varphi^2 \rangle^2 \,, \\ G_{\mathrm{con}}^{(6)}(\mathbf{z}_1, \mathbf{z}'_1) &= \langle \triangle\varphi^6 \rangle - 15 \langle \triangle\varphi^4 \rangle \langle \triangle\varphi^2 \rangle + 30 \langle \triangle\varphi^2 \rangle^3 \,, \end{aligned} \tag{S36}$$

with $\triangle\varphi = \triangle\varphi(z_1, z'_1)$. For a Gaussian state, we get from Wick's theorem

$$G^{(N)}(\mathbf{z}, \mathbf{z_1'}) \overset{\text{Gaussian}}{=} \langle \triangle\varphi^2 \rangle^{N/2} (N-1)!! \,, \tag{S37}$$

where $(\ldots)!!$ is the double factorial. These simplified formulas will be helpful in the next section, where we discuss the factorisation properties of commonly used periodic observables.

C. Connected versus periodic correlation functions

The periodic observables used in our previous experiments [21],

$$\mathcal{C}(\mathbf{z}, \mathbf{z}') \approx \langle e^{i \sum_n \triangle\varphi(z_n, z'_n)} \rangle \,, \tag{S38}$$

are, by neglecting the density fluctuations (suppressed due to atomic repulsion), related to correlations of the bosonic fields $\psi_{1,2}$ via

$$\mathcal{C}(\mathbf{z}, \mathbf{z}') := \frac{\langle \psi_1(z_1) \psi_2^\dagger(z_1) \psi_1^\dagger(z'_1) \psi_2(z'_1) \ldots \psi_1(z_N) \psi_2^\dagger(z_N) \psi_1^\dagger(z'_N) \psi_2(z'_N) \rangle}{\sum_{n=1}^{N} \sqrt{\langle |\psi_1(z_n)|^2 \rangle \langle |\psi_2(z_n)|^2 \rangle \langle |\psi_1(z'_n)|^2 \rangle \langle |\psi_2(z'_n)|^2 \rangle}} \,. \tag{S39}$$

These correlations are not suitable for the present analysis as even the second-order correlation function $\mathcal{C}(z_1, z'_1) = \langle e^{i\triangle\varphi(z_1, z'_1)} \rangle$ contains all higher cumulants of the phase. This can be directly seen by expanding the expression $\log \langle e^{i\lambda\triangle\varphi(z_1, z'_1)} \rangle$ (usually called the cumulant generating function) in powers of λ resulting in

$$\begin{aligned} \log \langle e^{i\lambda\triangle\varphi} \rangle &= i\lambda \langle \triangle\varphi \rangle - \frac{\lambda^2}{2} \left\{ \langle \triangle\varphi^2 \rangle - \langle \Delta\varphi \rangle^2 \right\} + \frac{(i\lambda)^3}{3!} \left\{ \langle \triangle\varphi^3 \rangle - 3 \langle \triangle\varphi \rangle \langle \triangle\varphi^2 \rangle + 2 \langle \triangle\varphi \rangle^3 \right\} \\ &\quad - \frac{\lambda^4}{4!} \left\{ \langle \triangle\varphi \rangle^4 - 4 \langle \triangle\varphi \rangle \langle \triangle\varphi^3 \rangle - 3 \langle \triangle\varphi^2 \rangle^2 + 12 \langle \triangle\varphi^2 \rangle \langle \triangle\varphi \rangle^2 - 6 \langle \triangle\varphi \rangle^4 \right\} + \mathcal{O}(\lambda^5) \\ &= \exp\left[\sum_{m=1}^{\infty} \frac{(-\lambda)^m}{(2m)!} G_{\mathrm{con}}^{(2m)}(z_1, z'_1) \right] \,, \end{aligned} \tag{S40}$$

where, again, $\triangle\varphi = \triangle\varphi(z_1, z_1')$. For $\lambda = 1$ one recovers the second-order correlation function

$$\mathcal{C}(z_1, z_1') = \exp\left[\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!} G_{\text{con}}^{(2m)}(z_1, z_1')\right] . \tag{S41}$$

This shows that it is experimentally not possible to extract the individual connected correlation functions $G_{\text{con}}^{(N)}$, i.e. information about N-particle interactions (see the following section), from the periodic correlations (S38). In case of Gaussian fluctuations of the phase, Eq. (S41) reduces to

$$\mathcal{C}(z_1, z_1') = \exp\left[-\frac{1}{2}\left\langle[\varphi(z_1) - \varphi(z_1')]\right\rangle^2\right] , \tag{S42}$$

with the correct factorisation, e.g. for the 4$^{\text{th}}$-order correlation function, given by

$$\mathcal{C}(z_1, z_2, z_1', z_2') = \frac{\mathcal{C}(z_1, z_1')\mathcal{C}(z_2, z_2')\mathcal{C}(z_1', z_2)\mathcal{C}(z_2', z_1))}{\mathcal{C}(z_1, z_2)\mathcal{C}(z_1', z_2')} . \tag{S43}$$

This form of the factorisation, determined from Gaussian fluctuations of the phase, is due to the periodicity and the resulting restricted (finite) domain of these correlation functions.

D. Relation to quasiparticle interactions

In this section, we give a brief explanation as to how higher-order connected correlation functions are related to quasiparticle interactions, i.e. fully connected diagrams. Be aware that we do not anticipate to solve the problem using perturbation theory in any way, and hence do not concern ourselves with the inevitable problems of divergencies occurring in the perturbative expansion, and their solutions using well-established field-theoretical tools as resummation, renormalisation, and summation of divergent series. For details of the presented methods see any book on (statistical) field theory, e.g. [3].

In thermal equilibrium the equal-time correlation functions defined in Eq. (S29) are, due to the linearity of the trace, determined by correlations of the form

$$\left\langle \prod_{i=1}^{N} \varphi(z_i) \right\rangle \equiv Z^{-1}\text{Tr}\left[\text{e}^{-\beta H}\prod_{i=1}^{N}\varphi(z_i)\right] , \tag{S44}$$

where $Z = \text{Tr}[\text{e}^{-\beta H}]$ is the partition function, and the trace is defined as $\text{Tr}[\dots] = \sum_n \langle n|\dots|n\rangle$, with $|n\rangle$ being a complete, orthonormal basis of the Hilbert space. While being exact this equation is in general not solvable without further approximations. First, we will approximate the system through its low-energy effective theory, discussed in Sect. I A, and therefore determined by the sine-Gordon Hamiltonian (S9). Expanding the cosine we write H_{SG} as

$$\begin{aligned} H &= \int dz : \left[g\delta\rho^2 + \frac{\hbar^2 n_{\text{1D}}}{4m}\left(\frac{\partial\varphi}{\partial z}\right)^2 + \hbar\tilde{J}n_{\text{1D}}\varphi^2\right] : - \int dz : \left[2\hbar\tilde{J}n_{\text{1D}}\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\varphi^{2n}\right] : \\ &= H_0 + V , \end{aligned} \tag{S45}$$

where we split the Hamiltonian into a free part H_0, quadratic in the fields, and an interaction part $V = V_4 + V_6 + \dots$, containing all higher-order terms. We wrote the Hamiltonian in its normal-ordered form (where all creation operators are to the left, denoted by : :) which leads to a multiplicative renormalisation of the coupling $J \to \tilde{J}$, and we dropped an irrelevant constant (see e.g. Ref. [6]).

Diagonalisation of the quadratic Hamiltonian H_0 defines the quasi-particle basis through the Bogoliubov expansion

$$B(z) = \sum_k \left[u_k(z)\, b_k + v_k^*(z)\, b_k^\dagger\right] \tag{S46}$$

of the quadrature field $B(z) = \varphi(z)\sqrt{n_{1\mathrm{D}}/2} - \mathrm{i}\delta\rho(z)/\sqrt{2n_{1\mathrm{D}}}$. The mode functions $(u_k,\, v_k)$ are eigenmodes of the Bogoliubov operator with eigenvalues ϵ_k. They ensure cancellation of all non-diagonal quadratic terms in the Hamiltonian and are normalized as $\int \mathrm{d}z[|u_k(z)|^2 - |v_k(z)|^2] = 1$. Finally, the quadratic Hamiltonian takes the form

$$H_0 = \sum_k \epsilon_k b_k^\dagger b_k \ , \tag{S47}$$

describing non-interacting quasiparticles. The mode expansion of the fields is given by

$$\delta\rho(z) = \sum_k \delta\rho_k(z)\, b_k + \delta\rho_k^*(z)\, b_k^\dagger \tag{S48}$$

$$\varphi(z) = \sum_k \varphi_k(z)\, b_k + \varphi_k^*(z)\, b_k^\dagger \ , \tag{S49}$$

where $\delta\rho_k(z) = [u_k(z) + v_k(z)]\sqrt{n_{1\mathrm{D}}}$ and $\varphi_k(z) = [u_k(z) - v_k(z)]/(2\mathrm{i}\sqrt{n_{1\mathrm{D}}})$. In the limiting cases of zero or very strong tunnel coupling the interaction potential V vanishes, for the latter due to smallness of the fluctuations φ. Inserting the mode expansion into Eq. (S44), a direct calculation of the trace in the quasiparticle Fock basis leads to the observed factorisation of correlations, as expected for a free theory. This can easily be generalised to arbitrary order by use of Wick's theorem for thermal states. Factorisation of equal-time phase correlation functions according to Wick's theorem, as was determined in the experiment for the uncoupled and the strongly coupled system, therefore shows the absence of quasiparticle interactions in the theory. Note that this is by far not a trivial result, even for vanishing coupling J, as we neglected an infinite number of higher order terms by replacing the full Hamiltonian H by the low-energy effective model H_{SG}.

In case of a non-vanishing interaction potential V the equations become increasingly more complicated due to the non-vanishing commutator $[H_0, V]$. In thermal equilibrium, the correlation functions of the phase can be calculated in perturbation theory in the imaginary-time (Matsubara) formalism. One defines the time-ordered correlation functions in imaginary time τ (Matsubara Green's functions)

$$\langle \hat{T}\varphi_H(\tau_1, z_1)\ldots\varphi_H(\tau_N, z_N)\rangle \equiv \frac{\mathrm{Tr}\left[\mathrm{e}^{-\beta H_0}\,\hat{T}\,\mathcal{U}(\beta,0)\varphi_I(\tau_1,z_1)\ldots\varphi_I(\tau_N,z_N)\right]}{\mathrm{Tr}\left[\mathrm{e}^{-\beta H_0}\,\mathcal{U}(\beta,0)\right]} \ , \tag{S50}$$

where $\varphi_H(\tau, z) = \mathrm{e}^{\tau H}\varphi(z)\mathrm{e}^{-\tau H}$ are the Heisenberg field operators in imaginary time τ, and $\varphi_I(\tau, z) = \mathrm{e}^{\tau H_0}\varphi(z)\mathrm{e}^{-\tau H_0}$ are the fields in the interaction picture (denoted by the subscript I), evolving in imaginary time with the free Hamiltonian H_0. The time evolution operator $\mathcal{U}(\beta, 0)$ fulfills

$$\partial_\tau \mathcal{U}(\tau, 0) = -V_I(\tau)\mathcal{U}(\tau, 0) \ . \tag{S51}$$

It can be written as the Dyson series

$$\mathcal{U}(\tau, \tau') = \hat{T}\mathrm{e}^{-\int_{\tau'}^{\tau}\mathrm{d}\tau'' V_I(\tau'')} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\tau'}^{\tau}\mathrm{d}\tau_1\ldots\int_{\tau'}^{\tau}\mathrm{d}\tau_n\,\hat{T}\,V_I(\tau_1)\ldots V_I(\tau_n) \ , \tag{S52}$$

which allows to express the correlation functions (S50), up to any order in V_I, as a diagrammatic expansion in Feynman diagrams. The sine-Gordon Hamiltonian, Eq. (S9), represents a scalar field theory with an infinite number of polynomial interaction terms. Standard results of quantum field theory allow to distinguish between three distinct types of diagrams. First, all diagrams in which the fields of the interaction potential V are contracted among themselves and are otherwise disconnected (vacuum diagrams). These vacuum diagrams are exactly canceled by the denominator in Eq. (S50) to all orders in the perturbative expansion. Second, all diagrams which are not fully connected only contribute to the disconnected part of the correlation function, and can be factorised into full, lower-order correlation functions. Third, the fully connected diagrams describe genuine N-body quasiparticle interactions and constitute the connected part of the correlation function.

Note that the above time-ordered imaginary-time correlation functions are only related to physical observables for equal times $\tau_1 = \cdots = \tau_N$, for which they coincide with the experimentally measured correlation functions.[3] However, the Matsubara Green's functions may be analytically continued to the real-time axis to determine the physically relevant retarded Green's functions. This continuation immediately allows to infer the effect of N-particle interactions in the theory. As explained in Sect. II A, the N^{th}-order equal-time connected correlation function represents the sum over all these fully connected processes with N external lines. Since the experimentally measured phase fields are linear in the quasiparticle creation/annihilation operators, N^{th}-order correlation functions are a direct measure for the combined effect of the N-body quasiparticle interaction (to all orders in the coupling). This is in contrast to the periodic observables, discussed in the previous section, which sum over all possible quasiparticle interactions (for all values of N). Measurements of higher-order correlation functions therefore allow for a direct comparison to highly non-trivial field-theoretical calculations, and give valuable information about the convergence of the perturbative expansion, the validity of non-perturbative theoretical methods, and the summation of divergent series.

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[3] Note that, in general, the limit $\tau_1, \ldots, \tau_N \to \tau$ need to be taken with care as the time-ordered Matsubara Greens-functions due to non-vanishing commutators might be discontinuous at equal times (see also footnote[2]). Since, for the correlations considered here, the equal-time commutator vanishes no further problems occur in taking the equal-time limit.