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A quantum network of clocks

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I. GHZ CASCADE IN A NETWORK OF K CLOCKS

Here, we discuss the details of using quantum correlated states constructed out of $N' = Kn$ qubits, equally distributed among K clocks, namely the GHZ state of the form

$$[|00\dots0\rangle + e^{i\chi}|11\dots1\rangle]/\sqrt{2}, \tag{1}$$

where $|qq\dots q\rangle = |q\rangle^{\otimes N'}$, $q \in \{0,1\}$. Entanglement has two effects here: First, it makes the phase of such a GHZ state, χ, sensitive to the accumulated phase of the *center-of-mass* of all the K independent local oscillators, (each located at one of the clocks) $\Phi_{\text{COM}} = \sum_{j=1}^{K}\Phi^{(j)}/K$, where $\Phi^{(j)} = \int_0^T dt\,(\omega^{(j)}(t) - \omega_0)$ is the accumulated phase of the LO at clock j, during the interrogation time T, here $\omega^{(j)}(t)$ is the instantaneous frequency of the LO, while ω_0 is the transition frequency of the clock qubit. Second, it increases the sensitivity, since the relative phase in the state

$$\left(\prod_{j}^{K}\prod_{i}^{N'/K}\hat{U}_{i,j}\right)\left[|\mathbf{0}\rangle + e^{i\chi}|\mathbf{1}\rangle\right]/\sqrt{2} = \\ = [|\mathbf{0}\rangle + e^{i(\chi + N'\Phi_{\text{COM}})}|\mathbf{1}\rangle]/\sqrt{2}, \tag{2}$$

grows N' times faster. Here $\hat{U}_{i,j} = |0\rangle\langle 0| + e^{i\Phi^{(j)}}|1\rangle\langle 1|$ is the time evolution operator during the interrogation time, acting on the ith qubit at clock j, and $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ are product states of all qubits being in $|0\rangle$ or $|1\rangle$, respectively.

A. Parity measurement

By setting the initial phase of the GHZ state, χ, to 0 and $\pi/2$ in two parallel instances, we effectively measure the real and imaginary part of $e^{iN'\Phi_{\text{COM}}}$, and thus get an estimate on the value of $N'\Phi_{\text{COM}}$ up to 2π phase shifts. The most cost-effective way to do this is to measure all qubits in the local x-basis. In this basis, the state from Eq. (2) can be written as

$$\frac{1}{\sqrt{2}}\left[\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)^{\otimes N'} + e^{i\phi}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)^{\otimes N'}\right], \tag{3}$$

where $\phi = \chi + N'\Phi_{\text{COM}}$, and $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$. The above state can be expanded in a sum:

$$\frac{1}{2^{(N'+1)/2}}\sum_{\mathbf{q}\in\{+,-\}^{\times N'}}\left[\left(\prod_{j=1}^{N'}q_j\right) + e^{i\phi}\right]|q_1, q_2, \dots q_{N'}\rangle, \tag{4}$$

where we labeled all qubits with $k \in \{1, 2, \dots N'\}$, irrespective of which clock they belong to. The probability of a certain outcome $\mathbf{q} = (q_1, q_2, \dots q_{N'})$, $(q_j \in \{+,-\})$, is

$$\mathcal{P}(\mathbf{q}) = \frac{1}{2^{N'+1}}|1 + p(\mathbf{q})e^{i\phi}|^2, \tag{5}$$

where $p(\mathbf{q}) = \prod_{j=1}^{N'} q_j$ is the parity of the sum of all measurement bits. Now, the clocks send their measurement bits to the center node, which evaluates p. This parity is the global observable that is sensitive to the accumulated phase, since its distribution is

$$\mathcal{P}(p = \pm) = \frac{1 \pm \cos(\phi)}{2}. \tag{6}$$

The above procedure is identical to the parity measurement scheme described in [1].

B. Cascaded GHZ scheme

Here we perform an analysis very similar to the analysis of the scheme in [2], using local GHZ cascade.

Provided with N qubits distributed equally among K clocks, we imagine that each clock separates its qubits into $M+1$ different groups. The 0th group contains n_1/K uncorrelated qubits, and the ith group ($i = 1, 2 \dots M$) contains n_0 independent instances of 2^{i-1} qubits that are entangled with the other groups of 2^{i-1} qubits in each clock. In other words, there are n_0 independent copies of GHZ states with a total of $2^{i-1}K$ qubits entangled on

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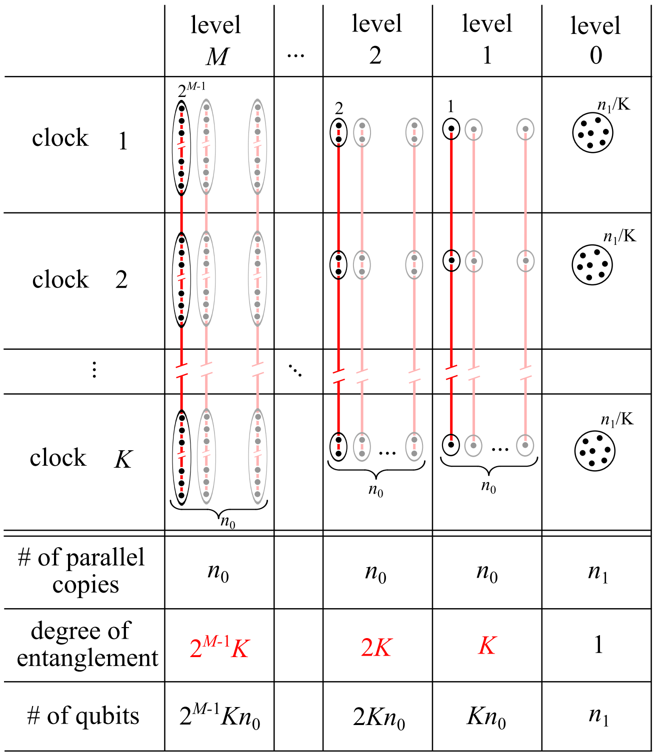

FIG. 1. GHZ cascade protocol for K clocks. Each allocates qubits for different levels of the protocol: In level 0, n_1/K qubits are put into an uncorrelated ensemble. In level i, $(i = 1, 2 \ldots M)$, each clock allocates $n_0 2^{i-1}$ qubits for creating n_0 parallel instances of GHZ states with $2^{i-1}K$ entangled qubits. Due to the exponential scaling of the degree of entanglement, most of the total available qubits are used in higher levels of the cascade. This is a necessary condition to achieve Heisenberg scaling, up to logarithmic factors.

the ith level of the cascade $(i \geq 1)$ (See Fig. 1). This way the total number of qubits can be written as

$$N = n_1 + n_0 \sum_{i=1}^{M} 2^{i-1} K \approx n_0 2^M K \tag{7}$$

where we assumed $n_1 \ll N$.

The purpose of this cascaded scheme is to directly assess the digits Y_1 and $\{Z_j : j = 2, 3, \ldots\}$ in the binary fraction representation of the phase

$$\Phi_{\rm LO} \mod [-\pi, \pi] = \frac{2\pi}{K} \left[Y_1 + \sum_{i=1}^{\infty} Z_{i+1}/2^i \right] - \pi, \tag{8}$$

where $x \mod [-\pi, \pi] = (x + \pi) \mod 2\pi - \pi$, $Y_1 \in \{0, 1, 2 \ldots K - 1\}$ and $Z_i \in \{0, 1\}$, and $\Phi_{\rm LO} = \Phi_{\rm COM}$. The 0th level of the cascade estimates $\Phi_0 = \sum_{j=1}^{K} \left(\Phi^{(j)} \mod [-\pi, \pi]\right)/K$, and every ith level after that estimates $\Phi_i = K 2^{i-1} \Phi_{\rm LO} \mod [-\pi, \pi]$. In other words, every level of the cascade is sensitive to a different multiple of the LO phase $\Phi_{\rm LO}$ (mod 2π). After the separate measurements of the levels, each provide an estimate to Φ_i which can be combined to obtain the best estimate for $\Phi_{\rm LO}$. Here we describe this using the digits Y_1 and Z_i as intermediate quantities, which are calculated from Φ_i the following way,

$$Y_1 = [K(\Phi_0 + \pi) - (\Phi_1 + \pi)]/(2\pi), \tag{9}$$
$$Z_i = [2(\Phi_{i-1} + \pi) - (\Phi_i + \pi)]/(2\pi), \tag{10}$$

for $i = 2, 3, \ldots M$.

The last group $(i = M)$ contains GHZ states with the most entangled qubits. These are the ones with the fastest evolving phase, and therefore they provide the best resolution on $\Phi_{\rm LO}$. Since there are n_0 independent instances, their phase $\Phi_M = 2\pi \sum_{i=1}^{\infty} Z_{M+i}/2^i$ is known up to the uncertainty, $\langle \Delta\Phi_M^2 \rangle_{\rm pr} = \frac{1}{n_0}$,

Assuming that all lower digits $\{Y_1, Z_j | j = 2 \ldots M\}$ have been determined correctly, this results in the total measurement uncertainty for $\Phi_{\rm LO}$:

$$\langle \Delta\Phi_{\rm LO}^2 \rangle_{\rm pr} = \frac{\langle \Delta\Phi_M^2 \rangle_{\rm pr}}{(2^{M-1}K)^2} = \frac{4n_0}{N^2}, \tag{11}$$

where, for the moment, we neglected individual qubit noise and assumed $\Phi_{\rm LO} \in [-\pi, \pi]$. However, in general, the estimation of the lower digits will not be perfect. In the following section we investigate the effect of these *rounding errors* on the final measurement accuracy. From this analysis we find the optimal number of copies n_0 and n_1.

C. Rounding errors

Whenever $|\Phi_0^{\rm est} - \Phi_0| > \pi/K$, or $|\Phi_i^{\rm est} - \Phi_i| > \pi/2$ (for $i \geq 1$), we make a mistake by under- or overestimating the number of phase slips Y_1 or Z_{i+1}, respectively. To minimize the effect of this error, we need to optimize how the total of N qubits are distributed among the various levels of the cascade. In other words we need to find $n_{0,\rm opt}$ and $n_{1,\rm opt}$.

The probability that a rounding error occurs during the estimation of Z_{i+1} is

$$\mathcal{P}_{i,\rm re} = 2 \int_{\pi/2}^{\infty} d\phi\, \rho_i(\phi + \Phi_i) \leq 2 \int_{\pi/2}^{\infty} d\phi\, \frac{1}{s_i^3} \exp\left[-\frac{\phi^2}{2s_i^2}\right] \tag{12}$$

where $\phi = \Phi_i^{\rm est} - \Phi_i$, and ρ_i is the conditional density function of $\Phi_i^{\rm est}$ for a given real Φ_i, and $s_i^2 = {\rm Var}(\Phi_i^{\rm est} - \Phi_i) = 1/n_0$ for $i \geq 1$, and $s_0^2 = \langle \Delta\Phi_0^2 \rangle_{\rm pr} = \frac{1}{K^2} \sum_{j=1}^{K} \langle (\Delta\Phi^{(j)})^2 \rangle_{\rm pr} = 1/n_1$, since $\langle (\Delta\Phi^{(j)})^2 \rangle_{\rm pr} = \frac{K}{n_1}$ for all j. The upper bound for ρ_i is obtained by using the following upper bound for any binomial distribution: $\binom{n}{k} p^k (1-p)^{n-k} \leq \exp\left[-n\left(\frac{k}{n} - p\right)^2\right]$. (For details, see Supplementary Materials of [2].) The resulting probabilities, after dropping the higher order terms in the asymptotic expansions, are

$$\mathcal{P}_{0,\mathrm{re}} \approx \frac{2K}{\pi} n_1^{1/2} \exp\left[-\frac{n_1\pi^2}{2K^2}\right], \tag{13}$$

$$\mathcal{P}_{i,\mathrm{re}} \approx \frac{4}{\pi} n_0^{1/2} \exp\left[-\frac{n_0\pi^2}{8}\right] \qquad (i \geq 1). \tag{14}$$

These approximate formulas are valid for $n_0 \geq 5$, as we have checked numerically.

The phase shift imposed on the estimate of Φ_{LO} by a manifested rounding error of Y_1 is $2\pi/K$ and of Z_i is $2\pi/(K2^{i-1})$, for $i = 2, 3 \ldots M$. This results in the total variance contribution,

$$\langle \Delta\Phi_{\mathrm{LO}}^2 \rangle_{\mathrm{re}} = = \left(\frac{2\pi}{K}\right)^2 \left[\mathcal{P}_{0,\mathrm{re}} + \sum_{i=2}^{M} \mathcal{P}_{i-1,\mathrm{re}} (2^{-i+1})^2\right] \tag{15}$$

$$\approx \left(\frac{2\pi}{K}\right)^2 \left[\mathcal{P}_{0,\mathrm{re}} + \frac{1}{3}\mathcal{P}_{i-1,\mathrm{re}}\right]. \tag{16}$$

We simplify this expression by choosing n_1 so that $\mathcal{P}_{0,\mathrm{re}} \approx \frac{2}{3}\mathcal{P}_{i,\mathrm{re}}$:

$$n_1 = \alpha K^2 n_0, \tag{17}$$

where $\alpha \approx \max\left\{1\,,\ \frac{2}{\pi^2 n_0}\log\left(3K^2\frac{\sqrt{8}}{\pi n_0^{1/2}}\right)\right\} \ll n_0, K$. With this choice, we can write the rounding error contribution as

$$\langle \Delta\Phi_{\mathrm{LO}}^2 \rangle_{\mathrm{re}} \approx \frac{16\pi}{K^2} n_0^{1/2} \exp\left[-\frac{n_0\pi^2}{8}\right]. \tag{18}$$

We note that the amount of extra resources needed for the 0th level is marginally small, since the total qubit number can be expressed as

$$N = n_1 + n_0 K \sum_{i=1}^{M} 2^{i-1} = n_0 K(\alpha K + 2^M - 2) \approx n_0 K 2^M, \tag{19}$$

under the assumption $K \ll 2^M$.

By adding the two error contributions from Eq. (11) and Eq. (18), we obtain the corresponding Allan-variance in the stationary noise approximation,

$$\sigma_y^2(\tau) = \frac{1}{\omega_0^2 \tau T}\langle \Delta\Phi_{\mathrm{LO}}^2 \rangle =: \frac{1}{\omega_0^2 \tau}\left[\Gamma_1 + \Gamma_2\right] = \tag{20}$$

$$= \frac{1}{\omega_0^2 \tau}\left[\frac{4n_0}{N^2 T} + \frac{16\pi}{K^2 T} n_0^{1/2} \exp\left[-\frac{n_0\pi^2}{8}\right]\right] \tag{21}$$

Now, let us find the optimal value of n_0. We write $\Gamma_1 + \Gamma_2$, using the new variable $x = \frac{8}{\pi^2}\frac{1}{n_0}$, as

$$\Gamma_1 + \Gamma_2 = \frac{4}{T}\left(\frac{8}{\pi^2}\frac{1}{xN^2} + \frac{\sqrt{32}}{K^2}\frac{1}{x^{1/2}}\exp\left[-\frac{1}{x}\right]\right). \tag{22}$$

Taking the derivative with respect to x and equating it with 0, while using the assumption $x \ll 1$ results in $\Gamma_2 \approx x_{\mathrm{opt}}\Gamma_1 \ll \Gamma_1$, which can be written as the following transcendental equation for the optimal value, x_{opt},

$$x_{\mathrm{opt}}^{1/2} \approx \frac{\pi^2 N^2}{\sqrt{8}K^2}\exp\left[-\frac{1}{x_{\mathrm{opt}}}\right]. \tag{23}$$

The general solution of any equation of the form $x^\nu = A\exp[-1/x]$, in the limit of $A \gg 1$ and $x \ll 1$, is $x = [\log(A)]^{-1}$. (For details, see the Supplementary Materials of [2].) Using this result we can write

$$x_{\mathrm{opt}} \approx \left[\log\left(\frac{\pi^2}{\sqrt{8}}\frac{N^2}{K^2}\right)\right]^{-1} \sim \left[2\log(N/K)\right]^{-1} \tag{24}$$

$$n_{0,\mathrm{opt}} \approx \frac{8}{\pi^2}\frac{1}{x_{\mathrm{opt}}} \sim \left(\frac{4}{\pi}\right)^2 \log\left(N/K\right). \tag{25}$$

For the realistic case of $N/K \gg 1$, indeed $x_{\mathrm{opt}} \ll 1$, and the corresponding minimal value of $\Gamma_1 + \Gamma_2$ is

$$\left[\Gamma_1 + \Gamma_2\right]_{\min} \approx \Gamma_1(x_{\mathrm{opt}}) = \left(\frac{8}{\pi}\right)^2 \frac{\log(N/K)}{N^2 T}. \tag{26}$$

This result indicates that, in terms of qubit number, only a logarithmic extra cost is required to achieve the Heisenberg limit.

D. Phase slip errors

Although the cascade is designed to detect phase slips of all levels $i = 1, 2 \ldots M$, a possible phase wrap of level $i = 0$ remains undetected. Since the qubits at different clocks are interrogated independently on the 0th level, each of them estimates the phase of the corresponding LO, $\Phi_0^{(j)}$ $(j = 1, 2, \ldots K)$, and not Φ_{LO}, the phase accumulated by the local oscillator. The probability of $\Phi_0^{(j)}$ falling outside the interval $[-\pi, \pi]$ at least once during the total measurement time τ is

$$\mathcal{P}_{j,\mathrm{slip}} = 2\frac{\tau}{T}\int_{\pi}^{\infty} d\phi \, \frac{1}{\sqrt{2\pi\gamma_{\mathrm{LO}}T}}\exp\left[-\frac{\phi^2}{2\gamma_{\mathrm{LO}}T}\right] \approx \approx \frac{\tau}{T}\frac{\sqrt{2}}{\pi^{3/2}}\sqrt{\gamma_{\mathrm{LO}}T}\exp\left[-\frac{\pi^2}{2\gamma_{\mathrm{LO}}T}\right], \tag{27}$$

where γ_{LO} is the linewidth of the local oscillator at clock j, corresponding to a white noise spectrum, resulting in a constant phase diffusion over the interrogation time T, (which is assumed to be approximately equal to the cycle time). The approximate form above is obtained by neglecting the higher order terms in the asymptotic series expansion under the assumption $\gamma_{\mathrm{LO}}T \ll 1$. (The white noise assumption can be relaxed to include more realistic LO noise spectra. However, numerical simulations have shown that, for a LO subject to a feedback loop, the low frequency noise is essentially white even though the LO may have more complicated noise spectrum, such as 1/f noise. Therefore the results derived for white noise applies up to a constant prefactor. [3] We proceed with the white noise assumption for simplicity.) Once such a phase slip happens, it introduces a 2π phase shift in $\Phi_0^{(j)}$, and therefore contributes to its overall uncertainty with $\langle(\Delta\Phi_0^{(j)})^2\rangle = (2\pi)^2\mathcal{P}_{j,\mathrm{slip}}$. Physically Φ_0 is the phase of the COM signal, that the center can obtain after averaging the frequencies of all K local oscillators with equal weights, $\Phi_0 = \Phi_{\mathrm{COM}} = \sum_{j=1}^{K}\Phi_0^{(j)}/K$, therefore $\langle\Delta\Phi_0^2\rangle = \frac{1}{K^2}\sum_{j=1}^{K}\langle(\Delta\Phi_0^{(j)})^2\rangle = \frac{1}{K}\langle(\Delta\Phi_0^{(j)})^2\rangle$, where we assumed that the LOs are independent but they have the same linewidth, γ_{LO}. Therefore the noise of the COM phase is reduced compared to the individual LOs. Since $\Phi_0 = \Phi_{\mathrm{LO}}$, the above results in the following variance contribution

$$\langle\Delta\Phi_{\mathrm{LO}}^2\rangle_{\mathrm{slip}} = \sqrt{32\pi}\frac{\tau\gamma_{\mathrm{LO}}^{1/2}}{T^{1/2}K}\exp\left[-\frac{\pi^2}{2\gamma_{\mathrm{LO}}T}\right]. \quad (28)$$

After adding this error to the previously minimized projection and rounding error terms (from Eq. (26)), we obtain the corresponding Allan-variance, $\sigma_y^2(\tau) = \frac{1}{\omega_0^2\tau}\left([\Gamma_1+\Gamma_2]_{\min}+\Gamma_3\right)$, where

$$[\Gamma_1+\Gamma_2]_{\min}+\Gamma_3 = \quad (29)$$
$$= \left(\frac{8}{\pi}\right)^2\frac{\log(N/K)}{N^2}\frac{2\gamma_{\mathrm{LO}}}{\pi^2}\frac{1}{y}+\frac{16}{\pi^{5/2}}\frac{\tau\gamma_{\mathrm{LO}}^2}{K}\frac{1}{y^{3/2}}\exp\left[-\frac{1}{y}\right],$$

using the variable $y = \frac{2}{\pi^2}\gamma_{\mathrm{LO}}T$.

Now, let us find the optimal Ramsey time T_{opt}, under the assumption that τ is sufficiently long. After taking the derivative with respect to y and equating it with zero, the assumption $y_{\mathrm{opt}} \ll 1$ results in the $\Gamma_3 \approx y_{\mathrm{opt}}[\Gamma_1+\Gamma_2]_{\min} \ll [\Gamma_1+\Gamma_2]_{\min}$ which can be written as the following transcendental equation,

$$y_{\mathrm{opt}}^{3/2} \approx \frac{\pi^{3/2}}{8}\frac{\tau\gamma_{\mathrm{LO}}}{K}\frac{N^2}{\log(N/K)}\exp\left[-\frac{1}{y}\right]. \quad (30)$$

The asymptotic solution in case of $y_{\mathrm{opt}} \ll 1$ is (see Supplementary of [2])

$$y_{\mathrm{opt}} \approx \left[\log\left(\frac{\pi^{3/2}}{8}\frac{\tau\gamma_{\mathrm{LO}}}{K}\frac{N^2}{\log(N/K)}\right)\right]^{-1}, \quad (31)$$

$$T_{\mathrm{opt}} \approx \frac{\pi^2}{2}\frac{y_{\mathrm{opt}}}{\gamma_{\mathrm{LO}}} \sim \frac{\pi^2}{2\gamma_{\mathrm{LO}}}\left[\log(\tau\gamma_{\mathrm{LO}}N^2/K)\right]^{-1} \quad (32)$$

in the realistic limit of $\gamma_{\mathrm{LO}}\tau N^2/K \gg 1$. The corresponding minimal Allan-variance is

$$\sigma_y^2(\tau) = \frac{1}{\omega_0^2\tau}\Big[[\Gamma_1+\Gamma_2]_{\min}+\Gamma_3\Big]_{\min} \approx \frac{1}{\omega_0^2}\frac{L\gamma_{\mathrm{LO}}}{N^2\tau}, \quad (33)$$

where $L = \frac{128}{\pi^4}\log(N/K)\log(\tau\gamma_{\mathrm{LO}}N^2/K)$.

For short τ averaging times, the optimal Ramsey time is $T_{\mathrm{opt}} = \tau$, instead of Eq. (32). This makes Γ_3 negligible compared to $[\Gamma_1+\Gamma_2]_{\min}$, resulting in a $1/\tau^2$ scaling:

$$\sigma_y^2(\tau) = \frac{1}{\omega_0^2\tau}[\Gamma_1+\Gamma_2]_{\min}^{T=\tau} = \frac{1}{\omega_0^2}\frac{L'}{N^2\tau^2}. \quad (34)$$

where $L' = \left(\frac{8}{\pi}\right)^2\log(N/K)$. This scaling is more favorable, but it applies to higher τ values up to $\tau \sim \gamma_{\mathrm{LO}}^{-1}$, where it switches to the $1/\tau$ behavior according to Eq. (33).

E. Pre-narrowing the linewidth

We can minimize the limiting effect of γ_{LO} by narrowing the effective linewidth of the local oscillators beforehand. We imagine using N^* qubits to locally pre-narrow the linewidth of all LOs down to an effective linewidth $\gamma_{\mathrm{eff}} \sim \gamma_{\mathrm{ind}}N$, before using the remaining $N-N^*$ qubits in the GHZ cascade. This $\gamma_{\mathrm{eff}} \ll \gamma_{\mathrm{LO}}$ allows the optimal Ramsey time going above the previous limit, set by $\sim \gamma_{\mathrm{LO}}^{-1}$ in Eq. (32). This step-by-step linewidth narrowing procedure, using uncorrelated ensembles in every step, is introduced in [3, 4], and further analyzed in [2]. Working under the small N^* assumption, one can obtain γ_{eff} as

$$\gamma_{\mathrm{eff}} \approx \gamma_{\mathrm{LO}}\left[\frac{2}{\pi^2}\frac{\log(\gamma_{\mathrm{LO}}\tau n)}{n}\right]^{N^*/n}, \quad (35)$$

where we imagine using n qubits in each narrowing step. We find the optimal value of n to be

$$n_{\mathrm{opt}} \approx \frac{2e}{\pi^2}\log(\gamma_{\mathrm{LO}}\tau), \quad (36)$$

by minimizing γ_{eff}, which yields

$$[\gamma_{\mathrm{eff}}]_{\min} \sim \gamma_{\mathrm{LO}}\exp\left[-\frac{N^*\pi^2}{2e\log(\gamma_{\mathrm{LO}}\tau)}\right]. \quad (37)$$

For a given τ, we can always imagine carrying out this pre-narrowing, so that $\gamma_{\mathrm{eff}} < \tau^{-1}$, and therefore Eq. (34) remains valid with the substitution $N \mapsto N - N^*$ for $\tau > \gamma_{\mathrm{LO}}^{-1}$ as well. The required number of qubits, N^*, is

$$N^* \sim \frac{2e}{\pi^2}\log(\gamma_{\mathrm{LO}}\tau)\log\left(\frac{\gamma_{\mathrm{LO}}}{\gamma_{\mathrm{ind}}N}\right) \ll N. \quad (38)$$

due to the exponential dependence in Eq. (37).

F. Individual qubit dephasing noise

Our scheme, as well as any scheme, is eventually limited by individual qubit noise. Such a noise dephases GHZ states at an increased rate, compared to uncorrelated qubits, due to the entanglement, giving the corresponding variance contribution for the phase of the GHZ states in the Mth group, $\langle\Delta\Phi_M^2\rangle_{\mathrm{dephasing}} = \frac{2^{M-1}K\gamma_{\mathrm{ind}}T}{n_0}$, after averaging over the n_0 independent copies of the GHZ states, each containing $2^{M-1}K$ entangled qubits. The resulting variance contribution for Φ_{LO} is

$$\langle\Delta\Phi_{\mathrm{LO}}^2\rangle_{\mathrm{dephasing}} = \frac{\gamma_{\mathrm{ind}}T}{n_0 2^{M-1}K} = \frac{2\gamma_{\mathrm{ind}}T}{N}. \quad (39)$$

This term represents a noise floor, which we add to Eq. (34) and obtain our final result for the minimal achievable Allan-variance,

$$\sigma_y^2(\tau) = \frac{1}{\omega_0^2}\left[\frac{L'}{N^2\tau^2} + \frac{2\gamma_{\text{ind}}}{N\tau}\right]. \tag{40}$$

For long τ times, the ultimate limit, set by the standard quantum limit, $\sigma_y^2(\tau) = \frac{1}{\omega_0^2}\frac{\gamma_{\text{ind}}}{N\tau}$, can be reached by changing the base of the cascade. Instead of entangling 2-times as many qubits in each level of the cascade than in the previous level, we imagine changing it to a base number D. Carrying out the same calculation results in our final result for the achievable Allan-variance:

$$\sigma_y^2(\tau) = \frac{1}{\omega_0^2}\left[\left(\frac{D}{2}\right)^2\frac{L'}{N^2\tau^2} + \frac{D}{D-1}\frac{\gamma_{\text{ind}}}{N\tau}\right], \tag{41}$$

where $L' = \left(\frac{8}{\pi}\right)^2\log(N/K)$. (See Supplementary of [2] for details.) The optimal value of D depends on τ. For small τ, $D_{\text{opt}} = 2$, however for large τ one can gain a factor of 2 by choosing $D_{\text{opt}} = D_{\max}$. Due to natural constraints, $D_{\max} \sim \sqrt{N}$, in which regime, the protocol consists of only two cascade levels, an uncorrelated 0th level, with $\sim \sqrt{N}$ qubits and an entangled 1st level with $\sim N$ qubits.

II. SECURITY COUNTERMEASURES

A. Sabotage

In order to detect sabotage, the center can occasionally perform assessment tests of the different nodes by teleporting an uncorrelated qubit state $[|0\rangle + e^{i\chi}|1\rangle]/\sqrt{2}$, where χ is a randomly chosen phase known only to the center. A properly operating node creates a local GHZ state $[|\mathbf{0}\rangle + e^{i\chi}|\mathbf{1}\rangle]/\sqrt{2}$ from the sent qubit, measures the parity of the GHZ state, and sends it to the center. The measured parity holds information on the phase $\phi' = \chi + \phi$, where ϕ is the accumulated phase of the LO at the node. Due to the random shift χ, this appears to be random to the node, and therefore indistinguishable from the result of a regular (non-testing) cycle. On the other hand, the center can subtract χ, and recover ϕ from the same measurement results. In the last step, the center verifies ϕ by comparing it with the classically determined phase ϕ_{cl} of the sent LO signal with respect to the COM signal. The expected statistical deviation of ϕ from ϕ_{cl} is $\Delta(\phi - \phi_{\text{cl}}) \sim \sqrt{\frac{K}{N}}$, while the accuracy of the COM phase $\Delta(\phi_{\text{COM}} - T\omega_0) \sim \sqrt{\frac{K}{(K-K_t)N}}$ is much smaller, where K_t is the number of simultaneously tested nodes. In the likely case of $K_t \ll K$, this method is precise enough for the center to discriminate between healthy and unhealthy nodes by setting an acceptance range, $|\phi - \phi_{\text{cl}}| \leq \Lambda\sqrt{\frac{K}{N}}$. E.g. the choice of $\Lambda = 4$ results in a "4σ confidence level", meaning only 0.0063% chance for false positives (healthy node detected as unhealthy), and similarly small chance for false negatives (unhealthy node being undetected) ($\sim \Lambda\frac{\Delta\phi'}{2\pi} \propto 1/\sqrt{N}$) due to the high precision with which ϕ' is measured. The fact, that the teleported qubit can be measured only once, also prevents the nodes from discovering that it is being tested. In fact the center performs a very simple blind quantum computing task using the resources of the other clocks, see [5–7].

B. Eavesdropping

Provided that the threat of sabotage is effectively eliminated, we turn our attention to the problem of eavesdropping. Eavesdroppers would try to intercept the sent LO signals, and synthesize the stabilized ν_{COM} for themselves. Our protocol minimizes the attainable information of this strategy by prescribing that only the *non-stabilized* LO signals are sent through classical channels. This requires the feedback to be applied to the LO signal after some of it has been split off by a beam splitter, and the center to integrate the generated feedback in time. Alternatively, eavesdroppers could try intercepting the LO signals *and* the feedback signals, and gain access to the same information, the center has. This can be prevented by encoding the radio frequency feedback signal with phase modulation according to a shared secret key. Since such a key can be shared securely with quantum key distribution, this protocol keeps the feedback signal hidden from outsiders. As a result, even the hardest-working eavesdropper, who intercepts all LO signals, is able to access only the non-stabilized COM signal, and the stabilized COM signal remains accessible exclusively to parties involved in the collaboration.

C. Rotating center role

Since the center works as a hub for all information, ensuring its security has the highest priority. In a scenario, where a small number of nodes (without knowing which) cannot be trusted enough to play the permanent role of the center, a rotating stage scheme can be used. By passing the role of the center around, the potential vulnerability of the network due to one untrustworthy site is substantially lowered. This requires a fully connected network and a global scheme for assigning the role of the center.

III. NETWORK OPERATION

A. Different degree of feedback

Apart from the full feedback, described in the main text, alternatively, the center can be operated to provide restricted feedback information to the nodes. If the center sends the averaged error signal $\tilde{\delta}_{\rm COM}$ only, the LOs at the nodes will not benefit from the enhanced stability and only the center can access the stabilized signal. Of course the LO at each node will have its own local feedback to keep it within a reasonable frequency range around the clock transition. Such a 'safe' operational mode allows the center node to use the resources of other nodes, while keeping the world time signal hidden from them. Such an asymmetric deal can be incentivized by monetary compensation, and allow the inclusion of nodes that cannot be trusted to keep the time signal secret.

As an intermediate possibility, the center can choose to send regionally averaged feedback signals $\tilde{\delta}_{\rm COM} + \sum_{j\in R}(\nu_j - \nu_{\rm COM})/|R|$, uniformly for all $j \in R$ nodes, where R is a set of nodes, ie. a region. Such a feedback scheme creates the incentive of cooperation for the nodes in region R. By properly sharing their LO signals with each other, the nodes can synthesize the regional COM frequency, $\sum_{j\in R}(\nu_j)/|R|$, and steer it with the feedback, received from the center.

B. Timing

Proper timing of local qubit operations is necessary to ensure that every qubit in the network is subject to the same T free evolution time. The finite propagation time of light signals introduces delays in the quantum links and classical channels. Similarly, during the entangling step, the finite time required to do local entangling operations make the free evolution start at slightly different times for different qubits. Since both the initialization and the measurement are local operations, we can resolve the issue of delay by prescribing that the measurement of qubit i_j (ith qubit at node j) takes place exactly T time after its initialization. Occasional waiting times of known length can be echoed out with a π-pulse at half time.

In extreme cases, this might cause some qubits to be measured before others are initialized. However, this is not a problem, since the portion of the GHZ state that is alive during the time in question is constantly accumulating the ϕ_j phases from the qubits it consists of. This results in the phenomenon that the total time of phase accumulation can be much longer than the length of individual phase accumulations, provided that the said interrogations overlap.

C. Dick effect

Classical communication between the clocks, separated by large geographical distances, takes considerable time compared to the interrogation time. The requirement to communicate the outcome of the Bell measurement in each teleportation step results in a waiting time in the teleportation protocol. During this waiting time (or dark time) the qubits are not interrogated, and therefore the local oscillator runs uncontrollably. If not countered, this effect (Dick effect) will deteriorate the overall stability of the clock network. Essentially the problem arises from the duty cycle of the clock being less than 100 percent. By employing two parallel realizations of the network scheme (supported by the same clock stations, but using different qubits) whose cycles are offset by half a period in time, we can cover the entire cycle time with at least one copy of the clock network being interrogated at all times. As a result, we can cancel the Dick effect with only a constant factor (~ 2) increase in the required resources, if the required time to prepare and measure the state is not longer than the free evolution.

D. More general architectures

So far, we focused on the simplest network structure with one center initiating every Ramsey cycle and nodes with equal number of clock qubits.

In a more general setup, node j has N_j clock qubits. If N_j is different for different j, then the nodes will contribute the global GHZ states unequally, resulting in entangled states which consists of different N'_j number of qubits from each site j. Such a state picks up the phase

$$\Phi = \sum_j N'_j \phi_j, \tag{42}$$

where ϕ_j is the phase of the LO at site j relative to the atomic frequency. As a result, the clock network measures the following collective LO frequency

$$\nu_{\rm LO} = \frac{\sum_j N'_j \nu_j}{\sum_j N'_j}. \tag{43}$$

This represents only a different definition of the world time (a weighted average of the times at the locations of the nodes, instead of a uniform average), but it does not affect the overall stability.

The initial laser linewidths of the nodes $\gamma^j_{\rm LO}$ can also be different. The stability achievable in this case is bounded by the stability obtained for a uniform linewidth $\gamma_{\rm LO} = \max_j \gamma^j_{\rm LO}$. If linewidths are known, the center can devise the best estimation method which uses linewidth dependent weights in the LO frequency averaging step.

Although it is simple to demonstrate the important network operational concepts with the architecture with one center, this structure is not a necessary. The quantum channels, connecting different nodes, can form a sparse (but still connected) graph, and the entanglement global entanglement can still be achieved by intermediate nodes acting as repeater stations. This way entanglement can be passed along by these intermediate nodes. Moreover, the center can be eliminated from the entangling procedure by making the nodes generate local GHZ states, and connect them with their neighbors by both measuring their shared EPR qubit with one of the qubits form the local GHZ state in the Bell-basis. After communicating the measurement result via classical channels, and performing the required single qubit operations, a global GHZ state is formed.

E. Accuracy

We define the accuracy of a time signal as the total uncertainty of its frequency with respect to the fundamental time reference. This includes the precision (characterized by the Allan deviation) and the uncertainty of frequency shifts due to systematic effects. So far we analyzed only the precision.

If two clocks have different levels of uncertainty for the local systematic shifts, then their individual accuracy is different. Our scheme requires the clocks to work together coherently, and thereby it averages out these differences in accuracy sub-optimally. In order to determine the individual accuracies, the clock network has to disentangle one of the clocks from the others. Next, by comparing the time signal from the entangled network with the one from the excluded clock, it can determine if the said clock has higher or lower accuracy than the average. By rotating this scheme, the systematic shifts of the worse clock can be measured more precisely, and, on the long run, this results in an overall improvement of the accuracy.

Apart from the local systematic effects (such as blackbody shift, second order Zeeman shift, etc.), a quantum network of clocks is subject to gravitational redshifts, which affect clocks at different positions differently. In order to ensure precise measurement of this systematic shift, the position of the clocks has to be tracked with high precision. This can be done similarly to the way ground stations track GPS satellites. The time signal generated by the quantum clock network can be used to set up a more precise tracking system, which can measure this systematics at a higher accuracy.

F. Efficient use of qubits

In the main text, during the generation of the initial global GHZ state, we assumed that the center makes use of $2(K-1)$ additional ancilla qubits (a_j, b_j). We chose this to present the idea at its simplest, however the qubits used as ancillae do not have to be sitting uselessly during the interrogation time. The center can use $2(K-1)$ out of its N/K clock qubits to perform the local generation and teleportation steps. This relabeling goes as follows $\{(b_j = j_1,\ a_j = (K-1+j)_1)\ :\ j = 2 \dots K\}$. After it is done, the qubits used as ancillae can be reinitialized, the center can entangle the $2(K-1)$ qubits (alongside with the others) with 1_1, and interrogate them without any obstacles. As a result, no clock qubits are wasted in the protocol.

G. Required EPR generation rate

In the main text, we mainly ignored the time required to generate pairwise entanglement. In this section we investigate the required EPR generation rate to achieve the desired performance. Before every cycle of operation $n[\log(N/K)/\log(2)+1](K-1)$ EPR pairs have to be shared between the center and the other clocks, where n is the number of parallel copies, N is the total number of qubits in a single copy and K is the number of clocks. Since $n_{\text{opt}} \sim (4/\pi)^2 \log(N/K)$, and approximately K quantum channels are used, the number of EPR pair per channel is about $(\log(N))^2$. We can assume that entangled qubits can be stored in degenerate memory states until they are used in the beginning of the next cycle. Therefore each channel has T time to initialize the $(\log(N))^2$ number of pairs. The optimal value for T is the available averaging time τ, which we choose to be $\sim 0.1\gamma_i^{-1}$ for order of magnitude estimation. (Note that $\tau < \gamma_i^{-1}$ is required to benefit from quantum enhancement.) For $N = 10^4$ atoms and $\gamma_i = 2\pi \times 1$ mHz, the EPR generation rate per channel has to be at least $(\log(N))^2\gamma_i \sim 1$ Hz. Such a rate is within reach of technology currently under intensive research [8].

H. Threshold fidelity

In this section we estimate the fidelity of the collective entangling operations required to keep the benefit from quantum enhancement. In the GHZ state generation step of every cycle, the n copies of the cascaded network containing K clocks are initialized. This requires K collective entangling operations to be performed per level per copy, one at each clock. Among other unitary operations, this is likely to be the bottleneck. If one collective entangling operation (creating a GHZ state of $N/(Kn)$ qubits) can be performed with fidelity $F = \exp[-\epsilon]$ $(\epsilon \ll 1)$, then K repetitions succeed with fidelity $F_{\text{total}} = F^K = \exp[-K\epsilon]$.

Whenever a copy fails on level i, its measurement result ϕ_i becomes completely random. This happens with probability $(1-F_{\text{total}})$. In the meantime, with probability F_{total}, the result is consistent with $\phi_i = \phi_{\text{real}}$, where ϕ_{real} is the actual value of ϕ_i. Out of the n copies nF_{total} contributes to a peak centered at ϕ_{real}, with width of $1/\sqrt{F_{\text{total}}n}$ and weight F_{total}, while the rest contributes to a uniform distribution with weight $(1-F_{\text{total}})$: The expectation value of ϕ_i is still

$$\langle \phi_i \rangle = \phi_{\text{real}}, \tag{44}$$

but the variance is

$$\text{Var}(\phi_i) \approx \frac{1}{n} + \frac{\pi^2}{3}(1-F_{\text{total}}). \tag{45}$$

The threshold fidelity F_{th} is defined by the criteria that if $F \geq F_{\text{th}}$, then $\text{Var}(\phi_i) \approx 1/n$. If this is satisfied, then losing the information from some of the copies does not deteriorate the precision significantly with which ϕ_i can be determined .

This requires $1 - F_{\text{total}} < \frac{3}{\pi^2 n}$, where the optimal value for n is $n_{\text{opt}} \sim \left(\frac{4}{\pi}\right)^2 \log(N/K)$, while $F_{\text{total}} = \exp[-\epsilon K]$ from above. From these, we conclude that $F_{\text{th}} = \exp[-\epsilon_{\text{th}}]$, where

$$\epsilon_{\text{th}} \approx \frac{3}{16K\log(N/K)} \sim \frac{1}{K\log(N)}. \tag{46}$$

Using $N = 10^4$ qubits, distributed in 10 clocks therefore requires $F_{\text{th}} \approx 0.99$ fidelity level for the local entangling operation. Using a multi-qubit entangling gate [9], this operation can be realized with current technology for fidelity ~ 0.95 and for small number of ions (~ 5). [10] The errors in such operations increases with N, making their realization more challenging. Namely, a 2-ion GHZ state was produced with 0.993 fidelity in a trapped ion system [11]. Currently the largest GHZ state of 14 ions was produced with 0.51 fidelity [10].

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