Supplementary methods for 'Space-efficient binary optimization for variational quantum computing'

Adam Glos*[1], Aleksandra Krawiec[1], and Zoltán Zimborás[2,3]

[1]Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, ul. Bałtycka 5, 44-100 Gliwice, Poland
[2]Wigner Research Centre for Physics, H-1525, P.O.Box 49, Budapest, Hungary
[3]BME-MTA Lendület Quantum Information Theory Research Group, Budapest, Hungary

In this Supplementary Information we will provide a detailed analysis of resources required for each model, and we will show the correctness of the $H_{\text{valid}}^{\text{HOBO}}$ Hamiltonian.

1 Supplementary methods

1.1 QUBO

Let us recall that the QUBO model takes the form

$$H^{\text{QUBO}}(b) = A_1 \sum_{t=0}^{N-1} \left(1 - \sum_{i=0}^{N-1} b_{ti}\right)^2 + A_2 \sum_{i=0}^{N-1} \left(1 - \sum_{t=0}^{N-1} b_{ti}\right)^2 + B \sum_{\substack{i,j=0 \\ i\neq j}}^{N-1} W_{ij} \sum_{t=0}^{N-1} b_{ti} b_{t+1,j}. \tag{1}$$

Number of qubits The model requires N^2 qubits.

Number of terms The number of terms can be determined as follows. First we note that we have N^2 1-local terms. Secondly, from the first addend for each t we have $\binom{N}{2}$ 2-local terms, similarly for the second. Finally, for the last part for each $i \neq j$ we have additional N 2-local terms. Note that each 2-local term is present only in one part, which makes our calculation tight. Finally we have

$$\#\text{terms} = N^2 + 1 + 2N\binom{N}{2} + NN(N-1) = 2N^3 - N^2 + 1. \tag{2}$$

Depth of the circuit Following the reasoning presented in the Method section, we can conclude that the 1-local terms can be implemented with the circuit of depth 1. The first addend from Supplementary Eq. (1) can be implemented with the circuit of depth N for even N, and $N-1$ for odd N, counting Z_iZ_j gates. We can apply the same strategy for the second addend.

For last addend we can independently consider parts $\sum_{i\neq j} W_{ij} b_{ti} b_{t+1,j}$ for even t, and then for odd t, which will double the circuit depth comparing to fixed t (in case of odd t we have to implement $(t, t+1) = (1, N)$ separately, which requires tripling the circuit depth). Let us fix t. We can implement terms $\mathcal{Z}_{t,k} = \{Z_{t,i}Z_{t+1,i+k} | 0 \leq i < N\}$ with

*aglos@iitis.pl

circuit of depth 1. Since $1 \le k < N$, we can simulate the last addend with the circuit of depth $2N-2$ for even N (and $3N-3$ for odd N). Thus in total our circuit has depth $4N-1$ for even N (and $5N-4$ for odd N), counting Z_iZ_j gates. In order to count total number of controlled NOT and 1-qubit gates, we need to triple the depth.

Since we have N^2 qubits, we can simulate at most $N^2/2$ 2-local terms independently. We have $\sim 2N^3$ terms and N^2 qubits. The minimum circuit depth for simulating 2-local terms is $\sim 2N^3/(N^2/2) = 4N$, which shows that our analysis is tight.

The calculations were done in terms of gates $\exp(-\mathrm{i}tZ)$ and $\exp(-\mathrm{i}tZ_iZ_j)$. Since the latter requires 2 CNOTs and a single rotation gate, we have to triple the circuit depth implementing 2-local gates, which finally gives us $12N-5$ for even N and $15N-14$ for odd N.

Number of measurements For the sake of simplicity we assume that $A_1, A_2 \le C \max_{i\neq j} W_{ij}$ and $B = 1$. Note that for each t, the expression $\left(1 - \sum_{i=0}^{N-1} b_{ti}\right)^2$ can be bounded from above by $(N-1)^2$. We can similarly upperbound the next addend. For the last part we have

$$\sum_{\substack{i,j=0\\ i\neq j}}^{N-1} W_{ij} \sum_{t=0}^{N-1} b_{ti}b_{t+1,j} \le N \sum_{\substack{i,j=0\\ i\neq j}}^{N-1} W_{ij} \le N\binom{N}{2} \max_{i\neq j} W_{ij}. \tag{3}$$

Finally we have

$$\begin{aligned} H^{\mathrm{QUBO}}(b) &\le A_1 N(N-1)^2 + A_2 N(N-1)^2 + BN\binom{N}{2} \max_{i\neq j} W_{ij} \\ &\le CN^3 \max_{i\neq j} W_{ij} + CN^3 \max_{i\neq j} W_{ij} + N^3 \max_{i\neq j} W_{ij} \\ &= (2C+1)N^3 \max_{i\neq j} W_{ij}. \end{aligned} \tag{4}$$

Note that the results is tight in order of N, which can be shown using $b_{ti} \equiv 1$ assignment.

1.2 HOBO

Let us recall that the model takes the form

$$\begin{aligned} H^{\mathrm{HOBO}}(b) = A_1 \sum_{t=0}^{N-1} H^{\mathrm{HOBO}}_{\mathrm{valid}}(b_t) + A_2 \sum_{t=0}^{N-1} \sum_{t'=t+1}^{N-1} H^{\mathrm{HOBO}}_{\neq}(b_t, b_{t'}) \\ + B \sum_{\substack{i,j=0\\ i\neq j}}^{N-1} W_{ij} \sum_{t=0}^{N-1} H^{\mathrm{HOBO}}_{\delta}(b_t, i) H^{\mathrm{HOBO}}_{\delta}(b_{t+1}, j). \end{aligned} \tag{5}$$

Provided that $\tilde{b}_{K-1} \dots \tilde{b}_0$ is a binary representation of $N-1$, we define

$$H^{\mathrm{HOBO}}_{\mathrm{valid}}(b_t) \coloneqq \sum_{k_0 \in K_0} b_{t,k_0} \prod_{k=k_0+1}^{K-1} (1 - (b_{t,k} - \tilde{b}_k)^2), \tag{6}$$

$$\begin{aligned} H^{\mathrm{HOBO}}_{\neq}(b, b') \coloneqq H^{\mathrm{HOBO}}_{\delta}(b, b') &\coloneqq \prod_{k=0}^{K-1} (1 - (b_k - b'_k)^2) = \prod_{k=0}^{K-1} \left(1 - \frac{1}{4}(Z_k - Z'_k)^2\right) = \prod_{k=0}^{K-1} \left(1 - \frac{1}{2}(1 - Z_kZ'_k)\right) \\ &= \frac{1}{2^K} \prod_{k=0}^{K-1} (1 + Z_kZ'_k), \end{aligned} \tag{7}$$

where $k_0 \in K_0$ are such indices that $\tilde{b}_{k_0} = 0$.

The proof that $H^{\mathrm{HOBO}}_{\mathrm{valid}}$ is a valid Hamiltonian for this encoding will be presented in Supplementary Section 1.4. For the sake of convenience, we will assume $K = \lceil \log(N) \rceil$, which is at the same time the number of bits in b_t for any t. We assume $K = \log N + \varepsilon$ for some $0 \leq \varepsilon < 1$. We also define $C \coloneqq 2^{\varepsilon} \in [1, 2)$ for convenience.

Number of qubits The required number of qubits is $NK + \frac{N}{2} \sim N \log(N)$. The $\frac{N}{2}$ part comes from the Gray code technique presented in the main text.

Number of terms Let us first consider the terms contributing to both b_t and b_{t+1}. We can see Hamiltonians $H^{\mathrm{HOBO}}_{\delta}(b_t, i)$ and $H^{\mathrm{HOBO}}_{\delta}(b_{t+j}, j)$ are Ising model consisting of all Pauli configurations. The function $H^{\mathrm{HOBO}}_{\delta}(b_t, i) \cdot H^{\mathrm{HOBO}}_{\delta}(b_{t+1}, j)$ is an Ising model defined over all variables from b_t and b_{t+1}, thus having $2^{2K} = 2^{2\log N + 2\varepsilon} = 4^{\varepsilon} N^2$ spins. We need to sum all operators, and exclude the part related to b_t only, as they were counted twice for pairs $(t-1, t)$ and $(t, t+1)$. The total number of Pauli operators is

$$N 2^{2K} - N 2^{K} = N 2^{2\log N + 2\varepsilon} - N 2^{\log N + \varepsilon} = 4^{\varepsilon} N^3 + \mathcal{O}\big(N^2\big). \tag{8}$$

Note that all terms which are part of the Hamiltonian $H^{\mathrm{HOBO}}_{\mathrm{valid}}(b_t)$ were already included.

Let us continue with $H_{\neq}$. All Pauli terms which appear in there consist of product of spins of the form $Z_k Z_{k'}$, thus we have exactly $2^K = 2^{\varepsilon} N$. For consecutive times $t, t+1$, the Pauli terms were already included. Thus, the total number of terms is

$$\left(\binom{N}{2} - N \right) 2^K = \frac{N(N-1) - 2N}{2} 2^{\varepsilon} N = 2^{\varepsilon - 1} N^3 + \mathcal{O}\big(N^2\big). \tag{9}$$

Taking all of the considerations into account we obtain

$$\#\text{terms} = 4^{\varepsilon} N^3 + \mathcal{O}\big(N^2\big) + 2^{\varepsilon - 1} N^3 + \mathcal{O}\big(N^2\big) = (4^{\varepsilon} - 2^{\varepsilon - 1}) N^3 + \mathcal{O}\big(N^2\big) = (C^2 - C/2) N^3 + \mathcal{O}\big(N^2\big). \tag{10}$$

Depth of the circuit Let us first implement the objective Hamiltonian. For even N, we first implement $(t, t+1)$ for even t, then for odd t. For odd N we need extra level for $(t, t+1) = (N-1, 0)$. The total depth will be the 2 times (3 times for N odd) the cost of implementing $\sum_{\substack{i,j=0 \\ i \neq j}}^{N-1} W_{ij} H^{\mathrm{HOBO}}_{\delta}(b_t, i) H^{\mathrm{HOBO}}_{\delta}(b_{t+1}, j)$ for arbitrary t. Note that for each t we have only $4^{\varepsilon} N^2$ Pauli terms, which can be implemented within depth 2 terms choosing the Gray order as in Fig. **??** d). Thus the total depth will be $\sim 2 \cdot 2 \cdot 4^{\varepsilon} N^2 = 4^{1+\varepsilon} N^2$ ($6 \cdot 4^{\varepsilon} N^2$ for odd N). Note that Pauli terms from H_{valid} can be already implemented together with the objective Hamiltonian.

Let us now consider the $H_{\neq}$ related Hamiltonian. We can apply round-robin schedule on registers b_t so that the total depth will be $N + \mathcal{O}(1)$ times the single implementation of $H_{\neq}$ Hamiltonian. Let us now consider the depth of $H_{\neq}$ treating each pair $Z_k Z_{k'}$ as a separate wire. We can use Gray ordering again, however now instead of 2 gates, we will need 2 CNOTs and single Z rotation. Thus the depth for single $H_{\neq}$ is $\sim 3 \cdot 2^K \sim 3 \cdot 2^{\varepsilon} N$, and for the whole constraint it will be $\sim 3 \cdot 2^{\varepsilon} N^2$.

The total depth will be at most

$$\sim 4^{1+\varepsilon} N^2 + 3 \cdot 2^{\varepsilon} N^2 = (4C^2 + 3C) N^2 \tag{11}$$

for even N and

$$\sim 6 \cdot 4^{\varepsilon} N^2 + 3 \cdot 2^{\varepsilon} N^2 = (6C^2 + 3C) N^2 \tag{12}$$

for odd N.

It is not obvious to provide lower bound on the circuit's depth. Since most of the factors are of order $\log N$, one could consider that applying each term requires the depth of the same order as well. However using Gray code ordering it is clear that only two qubits may be needed for applying higher-local terms. For this reason we will assume that only finite-depth circuit is required to implement each term. This gives us the lower bound $\sim (C^2 - C/2) N^3 / (N \log(N)) = \Theta(N^2 / \log N)$ which shows that our approach is tight up to $\log(N)$ factor.

Number of measurements For simplicity, we will assume that $A_1, A_2 \leq C \max_{i\neq j} W_{ij}$ and $B = 1$. Note that $H^{\mathrm{HOBO}}_{\mathrm{valid}}$ is a sum of at most $K-1$ elements, each giving the value either 0 or 1. Hence, for each t we have $H^{\mathrm{HOBO}}_{\mathrm{valid}}(b_t) \leq K-1$.

Note that $H^{\mathrm{HOBO}}_{\neq} \equiv H^{\mathrm{HOBO}}_{\delta}$ and $H^{\mathrm{HOBO}}_{\delta}(\cdot,\cdot) \in \{0,1\}$. Furthermore, since b_t can decode a single number only, $H_\delta(b_t, i) = 1$ only for a single i. Thus

$$\begin{aligned}\sum_{\substack{i,j=0\\ i\neq j}}^{N-1} W_{ij} \sum_{t=0}^{N-1} H^{\mathrm{HOBO}}_{\delta}(b_t, i) H^{\mathrm{HOBO}}_{\delta}(b_{t+1}, j) &= \sum_{t=0}^{N-1} \sum_{\substack{i,j=0\\ i\neq j}}^{N-1} W_{ij} H^{\mathrm{HOBO}}_{\delta}(b_t, i) H^{\mathrm{HOBO}}_{\delta}(b_{t+1}, j) \\ &= \sum_{t=0}^{N-1} W_{b_t, b_{t+1}} \leq N \max_{i\neq j} W_{ij}.\end{aligned} \tag{13}$$

and we can upper bound the energy by

$$\begin{aligned}H^{\mathrm{HOBO}}(b) &\leq A_1 N(K-1) + A_2 \binom{N}{2} + BN \max_{i\neq j} W_{ij} \\ &\leq CN \log N \max_{i\neq j} W_{ij} + C\frac{N^2}{2} \max_{i\neq j} W_{ij} + N \max_{i\neq j} W_{ij} \\ &\leq C' N^2 \max_{i\neq j} W_{ij}.\end{aligned} \tag{14}$$

Note that the results is tight in order of N, which can be shown using $b_{ti} \equiv 1$ assignment.

1.3 Mixed approach

The Hamiltonian takes the form

$$\begin{aligned}H^{\mathrm{MIX}}(b) = A_1 \sum_{t=0}^{N-1} H^{\mathrm{MIX}}_{\mathrm{valid}}(b_t; \xi_t) + A_2 \sum_{t=0}^{N-1} \sum_{t'=t+1}^{N-1} H^{\mathrm{MIX}}_{\neq}(b_t, b_{t'}) \\ + B \sum_{\substack{i,j=0\\ i\neq j}}^{N-1} W_{ij} \sum_{t=0}^{N-1} H^{\mathrm{MIX}}_{\delta}(b_t, i) H^{\mathrm{MIX}}_{\delta}(b_{t+1}, j),\end{aligned} \tag{15}$$

where ξ_t are slack variables required to implement $H^{\mathrm{MIX}}_{\mathrm{valid}}$ and

$$H^{\mathrm{MIX}}_{\mathrm{valid}}(b_t) \coloneqq \left(-\sum_{l=0}^{L-1} \sum_{k=0}^{K-1} b_{tlk} + 1 + \sum_{i=0}^{\lceil \log(KL) \rceil} 2^i \xi_{t,i} \right)^2 + \sum_{l=0}^{L-1} \left(\sum_{k=0}^{K-1} b_{tlk} \right) \left(\sum_{\substack{l'=0\\ l'\neq l}}^{L-1} \sum_{k=0}^{K-1} b_{tlk} \right) \tag{16}$$

$$\begin{aligned}H^{\mathrm{MIX}}_{\neq}(b_t, b_{t'}) &\coloneqq \sum_{l=0}^{L-1} \left(\sum_{k=0}^{K-1} (b_{tlk} + b_{t',l,k}) \right) \prod_{k=0}^{K-1} (1 - (b_{t,l,k} - b_{t',l,k})^2) = \\ &= \sum_{l=0}^{L-1} \left(\sum_{k=0}^{K-1} \frac{1}{2}(2 - Z_{tlk} - Z_{t',l,k}) \right) \frac{1}{2^K} \prod_{k=0}^{K-1} (1 + Z_{t,l,k} Z_{t',l,k})\end{aligned} \tag{17}$$

$$H_\delta(b_t, i)^{\mathrm{MIX}} \coloneqq \prod_{k=0}^{K-1} (1 - (b_{t',\bar{l}(i),k} - b^i_k)^2) = \frac{1}{2^K} \prod_{k=0}^{K-1} (1 + Z_{t',\bar{l}(i),k} Z^i_k). \tag{18}$$

For a general choice of $\alpha \in (0,1)$ it is hardly possible that $\alpha \log N$ will be an integer number. Hence for fixed α let $K \coloneqq \lceil \alpha \log N \rceil$. Note that $K \sim \alpha \log N$. On the other hand, we will encounter elements of the form 2^K and 2^{2K}, for which such an equivalence is not always valid, as

$$2^K = 2^{\lceil \alpha \log N \rceil} = 2^{\alpha \log N + \varepsilon_\alpha(N)} = C_\alpha(N) N^\alpha, \tag{19}$$

where $C_\alpha(N) \coloneqq 2^{\varepsilon_\alpha(N)}$ depends on the choice of α and N, but always $1 \leq C_\alpha(N) \leq 2$. Similarly

$$2^{2K} = 2^{2\lceil \alpha \log N \rceil} = 2^{2\alpha \log N + 2\varepsilon_\alpha(N)} = C_\alpha^2(N) N^{2\alpha}. \tag{20}$$

Furthermore

$$L \coloneqq \left\lceil \frac{N}{2^K - 1} \right\rceil \sim \left\lceil \frac{N}{C_\alpha(N) N^\alpha} \right\rceil \sim \frac{1}{C_\alpha(N)} N^{1-\alpha} \tag{21}$$

Note that if $N \neq (2^K - 1)L$, then we have to add a separate Hamiltonian of the form similar to $H^{\mathrm{HOBO}}_{\mathrm{valid}}$, as this encoding will not a encode valid city. This does not change the estimations derived in next paragraphs, as

- it does not require additional qubits,
- it does not produce new terms (they are already included in $H^{\mathrm{HOBO}}_{\mathrm{valid}}$), and by this it does not change the depth of the circuit,
- it has negligible impact on the energy upperbound, since for each t the mentioned Hamiltonian will increase energy by at most K.

Number of qubits The Hamiltonian requires

$$\begin{aligned} NKL + \left\lfloor \frac{N}{2} \right\rfloor L + N(\lceil \log(KL) + 1 \rceil) &\sim \frac{\alpha}{C_\alpha(N)} N N^{1-\alpha} \log N + \frac{1}{2C_\alpha(N)} N N^{1-\alpha} + N \left(1 + \log\left(\frac{\alpha}{C_\alpha(N)} N^{1-\alpha} \log N\right)\right) \\ &= \frac{\alpha}{C_\alpha(N)} N^{2-\alpha} \log N + \frac{1}{2C_\alpha(N)} N^{2-\alpha} + N \operatorname{poly}(\log(N)) \end{aligned} \tag{22}$$

qubits. The $\lfloor \frac{N}{2} \rfloor L$ is required for implementing the Gray code scheduling, while $N\lceil \log(KL) \rceil + 1$ qubits are needed for ξ variables.

Number of terms Let us start by calculating the terms generated from the objective Hamiltonian. Each $H_\delta(b_t, i)$ is a full Ising model defined over K spins, thus having 2^K spins. Note that if $\bar{l}(i) = \bar{l}(j)$, then we receive different Ising models defined over the same qubits. Thus only registers for different $l(i)$ matter. We have L^2 different registers, which gives $L^2 2^K$ terms. However, each Pauli term defined only over single $b_{t,l}$ were calculated L times, so we need to substract $(L-1)2^K$ terms. Finally, the same terms were considered L times for consecutive time points $(t, t+1), (t+1, t+2)$. So the final number of Pauli terms for the objective function is

$$NL^2 2^K - N(2L-1)2^K \sim N \frac{1}{C_\alpha^2(N)} N^{2-2\alpha} C_\alpha(N) N^\alpha = \frac{1}{C_\alpha(N)} N^{3-\alpha}. \tag{23}$$

Let us now consider $H_{\neq}$ related term. We only need to consider terms for nonconsecutive timepoints, and there are $\binom{N}{2} - N \sim \frac{1}{2}N^2$ of them. Let us fix l, t, t'. From the product $\prod_{k=0}^{K-1}(1 + Z_{t',l,k} Z_{t,l,k})$ we have 2^K even-local Pauli terms. However multiplying it by the sum on the left, we necessarily make all of the odd-local Pauli terms. All these Pauli terms are of the following form: they are of product of $Z_{t,l,k} Z_{t,l,k}$ terms, except of single variable without match. Such Pauli term was created in two ways: either by adding it, or by removing it's match. Since there are $2K2^K$

combinations of $H_{\neq}$, we have $K2^K$ terms. Such Hamiltonian occurs for each l and non-consecutive t, t', and for the whole constraints we have

$$\left(\binom{N}{2} - N\right) LK2^K \sim \frac{1}{2}N^2 \frac{1}{C_\alpha(N)} N^{1-\alpha}\alpha \log N C_\alpha(N) N^\alpha = \frac{\alpha}{2}N^3 \log N. \tag{24}$$

Note that resulting 1-local terms were counter several times, and were considered in the objective Hamiltonian, but their number is negligible and has no effect in our derivation.

Finally, let us consider $H^{\mathrm{MIX}}_{\mathrm{valid}}$. First note that the Hamiltonian is QUBO, and the former part consists of all possible second order interactions. Thus we can omit the latter part, and the total number of terms will be

$$\binom{KL + \lceil \log(KL) \rceil + 1}{2} \sim \frac{1}{2}(KL + \lceil \log(KL) \rceil + 1)^2 \sim \frac{1}{2}K^2L^2 \sim \frac{\alpha^2}{2C_\alpha^2(N)} N^{2-2\alpha} \log^2(N). \tag{25}$$

The determined number is negligible compared to number of terms for objective function and $H_{\neq}$.

Taking all numbers above we see that

$$\#\text{terms} \sim \frac{\alpha}{2}N^3 \log N. \tag{26}$$

Depth of the circuit For the depth we apply the same strategy as we did for the HOBO approach. First we implement objective Hamiltonian, then with round-robin method we apply $H_{\neq}$ related terms. At the end we implement H_{valid}.

We first implement the Hamiltonian defined over b_t, b_{t+1} for even t, and then odd t for even N. For odd N the case $t = N-1$ needs to be implemented separately. For each bunch $l = 0, \ldots, L-1$ the Hamiltonian can be implemented independently, and each of these Hamiltonians has 2^K Pauli terms. Each Pauli term (because it has pairs of spin) requires 3 gates. Thus for even N the depth is $6 \cdot 2^K = 2N^\alpha$, while for odd N it is $9N^\alpha$.

Let us now consider $H_{\neq}$ related Hamiltonian, which we will implement with round-robin schedule on b_t registers. This will require N rounds each of depth equal to implementing single $H_{\neq}$. Single $H_{\neq}$ consists of parts defined for different $l = 0, \ldots, L-1$, which can be implemented independently. Finally, for each fixed l we have $K2^K$ Pauli terms, and implementing them one by one using the decomposition as in Fig. 3a) from the main paper we can implement them with at most $(2K-1)K2^K \sim 2K^2 2^K$ depth. So the total depth is at most

$$\sim N2K^2 2^K = 2\alpha^2 C_\alpha(N) N^{1+\alpha} \log^2 N. \tag{27}$$

Finally, the H_{valid} is a QUBO defined over $\sim LK$ qubits. Using round-robin scheme, one can implement it with the depth $\sim LK = \frac{\alpha}{C_\alpha(N)} N^{1-\alpha} \log N$, which is negligible compared to the formulas derived before.

Based on the derivation above we can see that the total depth is at most

$$\sim 2\alpha^2 C_\alpha(N) N^{1+\alpha} \log^2 N. \tag{28}$$

Similarly as it was done for HOBO, we can compute minimal depth as

$$\sim \frac{\alpha}{2}N^3 \log N / \left(\frac{\alpha}{C_\alpha(N)} N^{2-\alpha} \log N\right) = \frac{C_\alpha(N)}{2} N^{1+\alpha} \tag{29}$$

which shows our derivation is tight up to logarithmic factor.

Number of measurements For the sake simplicity, we will assume that $A_1, A_2 \leq C \max_{i \neq j} W_{ij}$ and $B = 1$. For general b we have

$$\begin{aligned} H(b) &\leq A_1 N(2LK-1)^2 + A_1 N \cdot L \cdot K \cdot LK + A_2 \binom{N}{2} L \cdot 2K + BN \max_{i \neq j} W_{ij} \\ &\leq \left(2CNL^2K^2 + CN^2LK + N\right) \max_{i \neq j} W_{ij} \\ &\sim \left(\frac{2C\alpha^2}{C_\alpha^2(N)} N^{3-2\alpha} \log^2 N + CN^{3-\alpha} \log N + N\right) \max_{i \neq j} W_{ij}. \end{aligned} \tag{30}$$

By this we conclude that $H(b) = \mathcal{O}\big(N^{3-\alpha}\log N\big)\max_{i\neq j} W_{ij}$. Note that the bound is achievable when taking $b_{tlk} \equiv 1$.

1.4 Proof for $H^{\mathrm{HOBO}}_{\mathrm{valid}}$

Theorem 1. *Let $N > 0$ and K satisfies $2^{K-1} \leq N < 2^K$. Let $\tilde{b} = \tilde{b}_{K-1}\dots\tilde{b}_0$ is a binary representation of $N-1$. Let $K_0 \subseteq \{0,\dots,K-1\}$ be indices such that $k_0 \in K_0$ iff $\tilde{b}_{k_0} = 0$ Let*

$$H(b) \coloneqq \sum_{k_0\in K_0} b_{k_0} \prod_{k=k_0+1}^{K-1} (1-(b_k-\tilde{b}_k)^2) \tag{31}$$

and $b = b_{K-1}\dots b_0$ be a vector of bits encoding some number $n \in \{0,\dots,2^K-1\}$. Then $H(b) \geq 0$, with equality iff $n < N$.

Proof. Note that $(1-(b_k-\tilde{b}_k)^2)$ is nonnegative, hence $H(b) \geq 0$ independently on b. Let $n = N-1$, which means $b = \tilde{b}$. Then

$$H(\tilde{b}) = \sum_{k_0\in K_0} \tilde{b}_{k_0} \prod_{k=k_0+1}^{K-1} (1-(\tilde{b}_k-\tilde{b}_k)^2) = \sum_{k_0\in K_0} 0\cdot \prod_{k=k_0+1}^{K-1} (1-(\tilde{b}_k-\tilde{b}_k)^2) = 0. \tag{32}$$

Let $n < N-1$. Then there exists a unique $k' \in \{0,\dots,K-1\}\setminus K_0$ such that for all $k > k'$ we have $b_k = \tilde{b}_k$, $b_{k'} = 0$. In other words, there exists a bit, which for $N-1$ is one, and for n is 0. It is the first one starting from most significant one. Then we have

$$\begin{aligned}
H(b) &= \sum_{k_0\in K_0} b_{k_0} \prod_{k=k_0+1}^{K-1} (1-(b_k-\tilde{b}_k)^2) \\
&= \sum_{\substack{k_0\in K_0\\ k_0>k'}} b_{k_0} \prod_{k=k_0+1}^{K-1} (1-(b_k-\tilde{b}_k)^2) + \sum_{\substack{k_0\in K_0\\ k_0<k'}} b_{k_0} \prod_{k=k_0+1}^{K-1} (1-(b_k-\tilde{b}_k)^2) \\
&= \sum_{\substack{k_0\in K_0\\ k_0>k'}} \tilde{b}_{k_0} \prod_{k=k_0+1}^{K-1} (1-(\tilde{b}_k-\tilde{b}_k)^2) + \sum_{\substack{k_0\in K_0\\ k_0<k'}} b_{k_0} \prod_{k=k_0+1}^{K-1} (1-(b_k-\tilde{b}_k)^2) \\
&= \sum_{\substack{k_0\in K_0\\ k_0>k'}} 0\cdot \prod_{k=k_0+1}^{K-1} (1-(\tilde{b}_k-\tilde{b}_k)^2) + \sum_{\substack{k_0\in K_0\\ k_0<k'}} b_{k_0}(1-(b_{k'}-\tilde{b}_{k'})^2) \prod_{\substack{k=k_0+1\\ k\neq k'}}^{K-1} (1-(b_k-\tilde{b}_k)^2) \\
&= 0 + \sum_{\substack{k_0\in K_0\\ k_0<k'}} b_{k_0}(1-(0-1)^2) \prod_{\substack{k=k_0+1\\ k\neq k'}}^{K-1} (1-(b_k-\tilde{b}_k)^2) = 0.
\end{aligned} \tag{33}$$

Let $n > N$. Then there exists a unique $k' \in K_0$ such that for all $k > k'$ we have $b_k = \tilde{b}_k$ and $b_{k'} = 1$. In other words, there exists a bit, which for bit from $N-1$ is zero, and for bit from n is one It is the first one starting from most significant one. Then, taking the addend to $k_0 = k'$ we have

$$b_{k'} \prod_{k=k'+1}^{K-1} (1-(b_k-\tilde{b}_k)^2) = 1 \prod_{k=k'+1}^{K-1} (1-(\tilde{b}_k-\tilde{b}_k)^2) = 1, \tag{34}$$

which is enough to prove that $H(b) > 0$ as each addend is nonnegative. □