Supplementary Notes

Noisy Qudit vs Multiple Qubits: Conditions on Gate Efficiency for Enhancing Fidelity

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Supplementary Notes 1 : Complementary Derivations

Supplementary Notes 1a : Fluctuation-dissipation relation

Substituting, from the article, (3) and (2) into (6) and (5) leads to

$$\mathscr{E}(\rho^*) = \gamma t \left(\frac{1}{2} \mathrm{Tr} \left(\rho^* \left\{ L^\dagger L, \rho^* \right\} \right) - \mathrm{Tr} \left(\rho^* L \rho^* L^\dagger \right) \right) + \mathscr{O}((\gamma t)^2). \quad (1)$$

The trace being invariant by cyclic permutations and ${\rho^*}^2 = \rho^*$ leads to the simplification

$$\frac{1}{2} \mathrm{Tr} \left(\rho^* \left\{ L^\dagger L, \rho^* \right\} \right) = \mathrm{Tr} \left(\rho^* L^\dagger L \right) \equiv \langle L^\dagger L \rangle_* \, . \quad (2)$$

Moreover

$$\mathrm{Tr}\left(\rho^* L \rho^* L^\dagger\right) = \mathrm{Tr}\left(|\varphi^*\rangle \langle\varphi^*|L|\varphi^*\rangle \langle\varphi^*|L^\dagger\right) \tag{3}$$

$$= \langle L\rangle_* \mathrm{Tr}\left(\rho^* L^\dagger\right) \tag{4}$$

$$= \langle L^\dagger\rangle_* \langle L\rangle_* \ . \tag{5}$$

Accounting for the above results, one finally obtains

$$\mathscr{E}(\rho^*) = \gamma t \left(\langle L^\dagger L\rangle_* - \langle L^\dagger\rangle_* \langle L\rangle_*\right) + \mathscr{O}((\gamma t)^2), \tag{6}$$

which can be rewritten as (7) from the article.

Supplementary Notes 1b : Average Gate Infidelity for the Pure Dephasing Channel of one qudit

In a trivial way we have $\mathrm{Tr}(E_1) \propto \mathrm{Tr}(J_z) = 0$, and

$$\begin{aligned}
\mathrm{Tr}(E_0) &= d - \frac{\gamma t}{2}\mathrm{Tr}\left(J_z^2\right)\\
&= d - \frac{\gamma t}{2}\sum_{k=0}^{d-1}\left(\frac{d-1-2k}{2}\right)^2\\
&= d - \frac{\gamma t}{8}\left[d(d-1)^2 - 4(d-1)\sum_{k=0}^{d-1}k + 4\sum_{k=0}^{d-1}k^2\right]\\
&= d - \frac{\gamma t}{8}\left[d(d-1)^2 - 2d(d-1)^2 + 4\frac{d(d-1)(2d-1)}{6}\right]\\
&= d - \frac{\gamma t}{24}d(d^2-1),
\end{aligned}$$

which results in

$$|\mathrm{Tr}(E_0)|^2 = d^2 - \frac{\gamma t}{12}d^2(d^2-1) + \mathscr{O}((\gamma t)^2)\ . \tag{7}$$

Therefore

$$\overline{\mathscr{F}}(\mathscr{E}_z) = \frac{d + d^2 - \frac{\gamma t}{12}d^2(d^2-1)}{d(d+1)} + \mathscr{O}((\gamma t)^2) = 1 - \frac{\gamma t}{12}d(d-1) + \mathscr{O}((\gamma t)^2)\ , \tag{8}$$

which leads to the simplified expression (13) from the article.

Moreover, this is the origin of the factor $\frac{1}{12}$ in (14) from the article whose ratio with the $\frac{1}{4}$ obtained in (20) from the article and (10) leads to the non-trivial factor $\frac{1}{3}$ in (22) from the article.

Supplementary Notes 1c : Average Gate Infidelity for the pure dephasing channel of n qubits

Since $\mathrm{Tr}(E_k) = 0\,\forall k \neq 0$, only $\mathrm{Tr}(E_0)$ is left and is given by

$$\begin{aligned} \mathrm{Tr}(E_0) &= 2^n - \frac{\gamma t}{2}\sum_{k=1}^{n}\left[\mathrm{Tr}\Big(S_z^{2\,(k)}\Big)\prod_{j\neq k}\mathrm{Tr}\Big(\mathbb{1}_2^{2\,(j)}\Big)\right] \\ &= 2^n - n\frac{\gamma t}{8}2^n, \end{aligned} \tag{9}$$

leading to

$$|\mathrm{Tr}(E_0)|^2 = 2^{2n} - \frac{\gamma t}{4} n 2^{2n} + \mathscr{O}((\gamma t)^2)\;, \tag{10}$$

which allows to obtain (20) from the article using (10) from the article.

Supplementary Notes 1d : Average over the Fubini-Study measure of the uncertainty of L

First, rewriting $\int \mathrm{d}\rho\, \mathrm{Tr}\big(\rho M^\dagger M\big)$ yields

$$\int dU\; \mathrm{Tr}\Big(U\rho U^\dagger M^\dagger M\Big) = \mathrm{Tr}\left[\Big(\int dU\, U\rho U^\dagger\Big) M^\dagger M\right]$$

, where the integration is performed over the uniform Haar measure in the space of unitaries. Using the identity

$$\int dU\, UXU^\dagger = \frac{\mathrm{Tr}(X) I}{d}$$

valid for any linear operator X, for the special case of ρ pure one obtains

$$\int d\rho\; \mathrm{Tr}(\rho M^\dagger M) = \frac{1}{d}\mathrm{Tr}(M^\dagger M). \tag{11}$$

Now, rewriting $\int \mathrm{d}\rho\, \mathrm{Tr}\big(\rho M^\dagger M\big)$ yields

$$\begin{aligned} &\int\; dU\, \mathrm{Tr}\Big(U\rho U^\dagger M\Big)^2 = \\ \sum_{\substack{i,j,k,l\\ m,n,p,q}} &\int\; dU\, U_{ij}U_{k\ell}\bar{U}_{mn}\bar{U}_{pq}\rho_{jn}\rho_{\ell q}M_{mi}M_{pk}. \end{aligned}$$

Collins, Matsumoto, and Novak[1] provide formulae to integrate polynomials of unitary matrices

$$\begin{aligned} &\int_{U_d} dU U_{ij}U_{k\ell}\bar{U}_{mn}\bar{U}_{pq} = \\ \frac{1}{d^2-1}&\Big[(\delta_{im}\delta_{jn}\delta_{kp}\delta_{\ell q} + \delta_{ip}\delta_{jq}\delta_{km}\delta_{\ell n}) \end{aligned}$$

$$-\frac{1}{d}\left(\delta_{im}\delta_{jq}\delta_{kp}\delta_{\ell n}+\delta_{ip}\delta_{jn}\delta_{km}\delta_{\ell q}\right)\Big] \,, \tag{12}$$

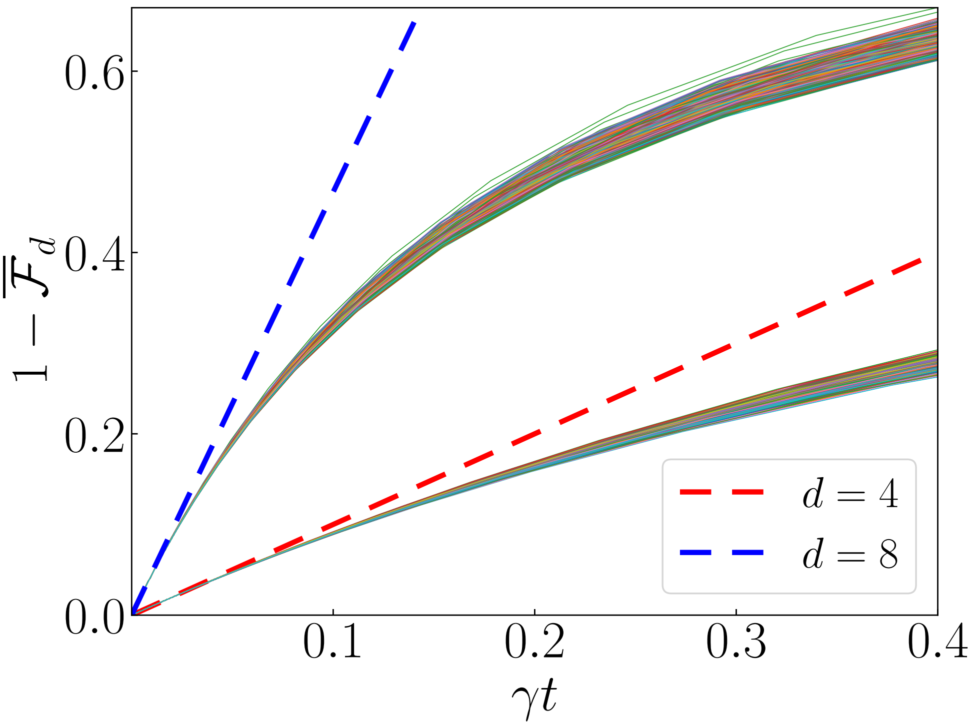
which contracting the indices gives

$$\int d\rho \ |\mathrm{Tr}(\rho M)|^2 = \frac{1}{d(d+1)}\left(\mathrm{Tr}(M^\dagger M)+|\mathrm{Tr}(M)|^2\right).$$

Finally subtracting (12) from (11) leads to (28) from the article.

Supplementary Notes 2 : Higher-order effects of the collapse operators

Supplementary Notes 2a : Complementary figure : deviation from linearity and gate dependence



Supplementary Figure 1 Simulated AGIs of $N_g = 4400$ gates for $d = 4, 8$ in solid lines. The dashed lines correspond to the expected linear behaviour at small γt.

Fig.1 allows for observation of (i) the deviation from linear behaviour as γt increases, (ii) for higher d, this deviation becomes noticeable for smaller values of γt and, (iii) the infidelity becomes increasingly gate-dependent as γt increases.

Supplementary Notes 2b : Full expansion of the density matrix

The density matrix $\rho(t)$ can be decomposed as

$$\rho(t) = \rho^* + \sum_{l=1} \sum_{k=1} \rho_{lk} \gamma^l t^k \ . \tag{13}$$

Substituting (13) in (1) from the article yields the following results:

- $\rho_{11} = \mathscr{D}[\rho^*] := \sum_k L_k \rho^* L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho^*\}$.
- for $l \geq 2$, $k = 1$, $\rho_{l,1} = 0$.
- for $l = 1$, $k \geq 2$,
$$k\rho_{1k} = -\mathrm{i}[H, \rho_{1(k-1)}] - \dot{\rho}_{1(k-1)} \ . \tag{14}$$
- $\forall l, k \geq 2$,
$$k\rho_{lk} = -\mathrm{i}[H, \rho_{l(k-1)}] - \dot{\rho}_{l(k-1)} + \mathscr{D}[\rho_{(l-1)(k-1)}] \ . \tag{15}$$

It can be linked to (3) from the article by noticing that $M = -\rho_{11}$.

Moreover, for $k = l = 2$ we obtain

$$\rho_{22} = \mathscr{D}[\rho_{11}] = \mathscr{D}[\mathscr{D}[\rho^*]] \ . \tag{16}$$

Finally we have

$$\rho_{12} = \frac{\mathrm{i}}{2}\left(\mathscr{D}[[H, \rho^*]] - [H, \rho^*]\right), \tag{17}$$

and

$$\rho_{13} = -\frac{\mathrm{i}}{3}[H, \rho_{12}] - \frac{\dot{\rho}_{12}}{3}. \tag{18}$$

This gives us the following expansion

$$\rho(t) = \rho^* + \gamma t \rho_{11} + \gamma t^2 \rho_{12} + \gamma t^3 \rho_{13} + (\gamma t)^2 \rho_{22} + \varepsilon, \tag{19}$$

with $\varepsilon = O\left(\gamma^l t^k\right)_{l+k \geq 5}$.

Interestingly, it can be proven by induction that if $H = \mathbb{0}_d$ then,

$$\rho(t) = \rho^* + \sum_k (\gamma t)^k \rho_{kk}, \tag{20}$$

with $\rho_{kk} = \frac{1}{k!}\mathscr{D}^{(k)}[\rho^*]$.

References

[1] Collins, B., Matsumoto, S. & Novak, J. The Weingarten Calculus. Preprint at https://arxiv.org/abs/2109.14890 (2021).