Supplementary Information

Etienne Granet and Henrik Dreyer
Quantinuum, Leopoldstrasse 180, 80804 Munich, Germany

Optimal number of shots per circuit.

In our algorithm, an expectation value is expressed as an average over random circuits. Each of these random circuits is then implemented on a quantum computer with a certain number of shots. Let us denote S the number of shots per circuit and N/S the number of circuits. At fixed N, we would like to find the value of S that minimizes the variance on the result, taking into account both the shot noise and the statistical error due to the random circuits.

Given a random circuit U drawn from the Poisson process, we denote $-1 \leq m_U \leq 1$ the real part of the complex amplitude that we measure with our algorithm. Given the real part of a complex amplitude $-1 \leq m \leq 1$, we denote x_m the Bernoulli random variable that takes value $+1$ with probability $\frac{1+m}{2}$ and -1 with probability $\frac{1-m}{2}$. We denote $\mathbb{E}$ the expectation value with respect to the random circuit U, and $\langle\rangle$ the expectation value with respect to the shot noise. The exact value of the real part of the complex amplitude that we measure after averaging over random circuits and shots is

$$m = \mathbb{E}[\langle x_{m_U} \rangle] \,. \tag{1}$$

With S shots to evaluate m_U and N/S different circuits U, the estimated value m_{est} of m is

$$m_{\text{est}} = \frac{1}{N/S} \sum_{p=1}^{N/S} \frac{1}{S} \sum_{i=1}^{S} x^{(i)}_{m_{U_p}} \,, \tag{2}$$

where the U_p's are independent circuits drawn from the Poisson process, and where the $x^{(i)}_{m_{U_p}}$'s are independent Bernoulli random variables with parameter $\frac{1+m_{U_p}}{2}$. Let us compute the variance of this quantity. We have

$$\begin{aligned} m_{\text{est}}^2 =& \frac{1}{N^2} \sum_{i,j,p,q} x^{(i)}_{m_{U_p}} x^{(j)}_{m_{U_q}} \\ =& \frac{1}{N} + \frac{1}{N^2} \sum_{\substack{i,j,p,q \\ i\neq j \text{ or } p\neq q}} x^{(i)}_{m_{U_p}} x^{(j)}_{m_{U_q}} \,, \end{aligned} \tag{3}$$

where we used that $(x^{(i)}_{m_{U_p}})^2 = 1$. Then, averaging over the shots

$$\begin{aligned} \langle m_{\text{est}}^2 \rangle &= \frac{1}{N} + \frac{1}{N^2} \sum_{\substack{i,j,p,q \\ i\neq j \text{ or } p\neq q}} m_{U_p} m_{U_q} \\ &= \frac{1}{N} + \frac{S^2}{N^2} \left(\sum_p m_{U_p} \right)^2 - \frac{S}{N^2} \sum_p m_{U_p}^2 \,. \end{aligned} \tag{4}$$

In the case where we have only one circuit U, i.e. $S = N$, we have $\langle m_{\text{est}}^2 \rangle - \langle m_{\text{est}} \rangle^2 = \frac{1-m_U^2}{N}$, which is the usual variance of a Bernoulli random variable taking values ± 1 with mean m_U. In the general case, after averaging over the circuits we have

$$\mathbb{E}[\langle m_{\text{est}}^2 \rangle] = \frac{1}{N} + \frac{S^2}{N^2} \left(\frac{N^2}{S^2} \mathbb{E}[m_U]^2 + \frac{N}{S} \mathbb{E}[m_U^2] - \frac{N}{S} \mathbb{E}[m_U]^2 \right) - \frac{1}{N} \mathbb{E}[m_U^2] \,. \tag{5}$$

Denoting $v = \mathbb{E}[m_U^2] - \mathbb{E}[m_U]^2$ the variance over the circuits, it yields

$$\mathbb{E}[\langle m_{\text{est}}^2 \rangle] - \mathbb{E}[\langle m_{\text{est}} \rangle]^2 = \frac{1-v}{N} + \frac{S}{N} v \,. \tag{6}$$

Since $v \geq 0$, we obtain that at fixed total number of shots N, doing only one shot $S = 1$ per circuit minimizes the total variance.

Optimal gate angles.

When measuring an observable $\mathcal{M}$ with our algorithm with gate angles τ_n, in absence of noise, the total number of rotations to perform to reach a precision ϵ on the result is $R(\{\tau_n\})/\epsilon^2$, where

$$R(\{\tau_n\}) = 2t \sum_{n=1}^{N} \frac{|c_n|}{\sin \tau_n} \exp\left(4t \sum_{n=1}^{N} |c_n| \tan(\tau_n/2)\right) . \tag{7}$$

Writing $\partial_{\tau_n} R = 0$ at the optimal gate angle τ_n^*, we find that it satisfies the equation

$$\frac{2t}{\cos^2(\tau_n^*/2)} \sum_{p=1}^{N} \frac{|c_p|}{\sin \tau_p^*} = \frac{\cos \tau_n^*}{\sin^2 \tau_n^*} . \tag{8}$$

The left-hand side is bounded from below by $2t\mu$. Hence at large t, we necessarily have $\tau_n \to 0$ for all n to make the right-hand side go to ∞. In that limit we have

$$2t \sum_{p=1}^{N} \frac{|c_p|}{\tau_p^*} \sim (\tau_n^*)^{-2} . \tag{9}$$

The left-hand side is independent of n, so the leading behaviour of τ_n^* when $t \to \infty$ is independent of n. Denoting this leading behaviour τ^*, we thus have $2t\mu(\tau^*)^{-1} = (\tau^*)^{-2}$, and so

$$\tau_n^* = \frac{1}{2t\mu} \tag{10}$$

Let us now consider the noisy case where each rotation $e^{i\tau_n O_n}$ comes with an attenuation factor e^{-r_n}. The damping due to noise per circuit is thus $\exp\left(-2t \sum_{n=1}^{N} \frac{|c_n| r_n}{\sin \tau_n}\right)$. The total number of rotations to reach a precision ϵ is thus $R(\{\tau_n\})/\epsilon^2$, with now

$$R(\{\tau_n\}) = 2t \sum_{n=1}^{N} \frac{|c_n|}{\sin \tau_n} \exp\left(4t \sum_{n=1}^{N} |c_n| \tan(\tau_n/2) + 4t \sum_{n=1}^{N} \frac{|c_n| r_n}{\sin \tau_n}\right) . \tag{11}$$

Writing again $\partial_{\tau_n} R = 0$ at the optimal gate angle τ_n^*, we find that it satisfies the equation

$$2t \left(\frac{1}{\cos^2(\tau_n^*/2)} - \frac{2r_n \cos \tau_n^*}{\sin^2 \tau_n^*}\right) \sum_{p=1}^{N} \frac{|c_p|}{\sin \tau_p^*} = \frac{\cos \tau_n^*}{\sin^2 \tau_n^*} . \tag{12}$$

Let us assume that when $t \to \infty$, the quantity $Q \equiv \frac{1}{\cos^2(\tau_n^*/2)} - \frac{2r_n \cos \tau_n^*}{\sin^2 \tau_n^*}$ does not go to 0. Then the same reasoning as above would apply and we would have $\tau_n^* \to 0$. But then we would have $Q \to -\infty$, and so the right-hand side would become negative, which cannot be. Hence we must have $Q \to 0$. This yields an equation for τ_n^* when $t \to \infty$

$$\frac{1}{\cos^2(\tau_n^*/2)} = \frac{2r_n \cos \tau_n^*}{\sin^2 \tau_n^*} . \tag{13}$$

Assuming r_n small, this is

$$\tau_n^* = \sqrt{2r_n} . \tag{14}$$

Numerical data.

We provide in Supplementary Table I the numerical data plotted in the Figures in the main text.

Figure 2a		
time	noisy	noiseless
0.025		3.466
0.02625	2.558	
0.0275		1.935
0.02875	1.5854	
0.03		1.289
0.03125	1.033	
0.0325		0.862
0.03375	0.726	
0.035		0.580
0.03625	0.479	
0.0375		0.371
0.03875	0.263	
0.04		0.164

Figure 2b		
time	exact	this algo
0	1.000	-2.000 $\pm$ 0
0.2	-2.045	-2.045 $\pm$ 0.001
0.4	-2.161	-2.164 $\pm$ 0.003
0.6	-2.306	-2.311 $\pm$ 0.005
0.8	-2.445	-2.438 $\pm$ 0.008
1.	-2.560	-2.562 $\pm$ 0.011
1.2	-2.648	-2.656 $\pm$ 0.015
1.4	-2.709	-2.700 $\pm$ 0.019
1.6	-2.748	-2.746 $\pm$ 0.024
1.8	-2.772	-2.780 $\pm$ 0.030
2.	-2.785	-2.739 $\pm$ 0.037

Figure 3				
time	exact	Trotter	qDRIFT	this algo
0	1.000	1.000	1.000 $\pm$ 0	1.000$\pm$ 0
0.1	-0.619	-0.577	-0.529 $\pm$ 0.006	-0.622$\pm$ 0.008
0.2	0.258	0.194	0.199 $\pm$ 0.005	0.253$\pm$ 0.011
0.3	-0.184	-0.151	-0.129 $\pm$ 0.004	-0.194$\pm$ 0.011
0.4	0.205	0.161	0.122 $\pm$ 0.004	0.201$\pm$ 0.012
0.5	-0.239	-0.192	-0.125$\pm$ 0.003	-0.245$\pm$ 0.014
0.6	0.268	0.207	0.106 $\pm$ 0.004	0.268$\pm$ 0.018
0.7	-0.288	-0.218	-0.098 $\pm$ 0.003	-0.297$\pm$0.024
0.8	0.285	0.202	0.084$\pm$ 0.003	0.279$\pm$0.031
0.9	-0.272	-0.176	-0.063 $\pm$ 0.003	-0.254$\pm$0.037
1	0.238	0.128	0.045 $\pm$ 0.003	0.263$\pm$0.047

TABLE I. Numerical data for Figure 2a,2b and 3 in the main text.