nature physics

SUPPLEMENTARY INFORMATION

https://doi.org/10.1038/s41567-020-0988-4

In the format provided by the authors and unedited.

Exact theory for superconductivity in a doped Mott insulator

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SUPPLEMENTARY MATERIALS

Pair binding instability with zero temperature Hatsugai-Kohmoto Gibbs state

Given the Gibbs state $\rho = e^{-\beta H_{\rm HK}}/Z = \sum_n e^{-\beta E_n}|n\rangle\langle n|$, we fix the purification on the doubled Hilbert space with $\mathcal{H}_B \simeq \mathcal{H}_A$

$$|\beta\rangle = \sum_n e^{-\beta E_n/2}\,|n\rangle \otimes |n\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \tag{35}$$

satisfying $\mathrm{tr}_B|\beta\rangle\langle\beta| = \rho$, and restrict

$$A^\dagger = \sum_{k\notin\Omega_2} \alpha_k b_k^\dagger \tag{36}$$

to the singly-occupied and unoccupied regions Ω_1 and Ω_0 of the Brillouin zone. Now $|\psi\rangle = A^\dagger \otimes \mathbb{1}\,|\infty\rangle$ has overlap

$$\begin{aligned}\langle\infty|b_k\otimes 1|\psi\rangle &= \underset{A,B}{\mathrm{tr}}|\infty\rangle\langle\infty|(b_k A^\dagger\otimes\mathbb{1}) &(37)\\ &= \underset{A}{\mathrm{tr}}\,\rho b_k A^\dagger = \langle b_k A^\dagger\rangle &(38)\\ &= \begin{cases}0 & \text{if } k\in\Omega_2,\\ \frac{1}{4}\alpha_k & \text{if } k\in\Omega_1,\\ \alpha_k & \text{if } k\in\Omega_0.\end{cases} &(39)\end{aligned}$$

With $[b_k, H]$ as before, for each $k\in\Omega_1,\Omega_0$ we have

$$\begin{aligned}&C_k i\hbar\partial_t\alpha_k(t=0)\\ &= \langle\infty|e^{i(H\otimes\mathbb{1}+\mathbb{1}\otimes H')t}(b_k\otimes\mathbb{1})e^{-i(H\otimes\mathbb{1}+\mathbb{1}\otimes H')t}|\psi\rangle\Big|_{t=0} &(40)\\ &= \langle\infty|[b_k,H]\otimes\mathbb{1}|\psi\rangle &(41)\\ &= (2\xi_k + U\langle n_{k\downarrow}+n_{-k\uparrow}\rangle)\,C_k\alpha_k\\ &\quad -\frac{g}{L^d}\langle 1-n_{k\uparrow}-n_{-k\downarrow}\rangle\left[\sum_{q\in\Omega_1}\frac{1}{4}\alpha_q + \sum_{q\in\Omega_0}\alpha_q\right] &(42)\end{aligned}$$

where $C_k = 1/4$ if $k\in\Omega_1$ and $C_k = 1$ if $k\in\Omega_0$. Then taking $\alpha_k(t) = e^{-iEt/\hbar}\alpha_k(0)$ recovers the same consistency equation

$$\begin{aligned}1 &= -\frac{g}{L^d}\sum_{k\in\Omega_1,\Omega_0}\frac{\langle 1-n_{k\uparrow}-n_{-k\downarrow}\rangle}{E-2\xi_k-U\langle n_{k\downarrow}+n_{-k\uparrow}\rangle} &(43)\\ &= -g\int_\mu^{W/2} d\epsilon\,\frac{\rho(\epsilon)}{E-2\epsilon+2\mu} &(44)\end{aligned}$$

since the numerator vanishes in the singly-occupied region Ω_1.

Dyson equation for pair susceptibility

In order to relate the pair susceptibility

$$\chi(i\nu_n) \equiv \frac{1}{L^d}\int_0^\beta d\tau\, e^{i\nu_n\tau}\langle T\Delta(\tau)\Delta^\dagger\rangle_g \tag{45}$$

to the bare pair susceptibility, $\chi_0(i\nu_n)$ at $g=0$, we work in the interaction picture where

$$\langle T\Delta(\tau)\Delta^\dagger\rangle_g = \frac{\langle TS(\beta,0)\Delta(\tau)\Delta^\dagger\rangle_0}{\langle TS(\beta,0)\rangle_0} \tag{46}$$

for $\Delta(\tau)$ evolved in the Heisenberg picture (under H) in $\langle\cdots\rangle_g$, and evolved in the interaction picture (under $H_{\rm HK}$) in $\langle\cdots\rangle_0$. Here

$$S(\beta,0) = e^{\beta H_{\rm HK}}e^{-\beta H} = T_{\tau_i}\exp g\int_0^\beta d\tau_1 H_p(\tau_1), \tag{47}$$

for $H_p(\tau_1)$ evolved in the interaction picture. Then the numerator takes the form of a power series in g,

$$\begin{aligned}&\langle TS(\beta,0)\Delta(\tau)\Delta^\dagger\rangle_0\\ &= \sum_{m=0}^\infty\frac{g^m}{m!}\langle T\left(\int_0^\beta d\tau_1 H_p(\tau_1)\right)^m\Delta(\tau)\Delta^\dagger\rangle_0 &(48)\end{aligned}$$

where each term

$$\frac{1}{L^d}\langle T\left(\int_0^\beta d\tau_1 H_p(\tau_1)\right)^m\Delta(\tau)\Delta^\dagger\rangle_0 = \frac{1}{(L^d)^{m+1}}\int\cdots\int_0^\beta d\tau_1\cdots d\tau_m\langle T\Delta^\dagger(\tau_1)\Delta(\tau_1)\cdots\Delta^\dagger(\tau_m)\Delta(\tau_m)\Delta(\tau)\Delta^\dagger\rangle_0 \tag{49}$$

factorizes, as χ_0 does in Eq. (20), because each pair annihilator $b_k(\tau)$ in the sum

$$\Delta(\tau) = \sum_k b_k(\tau) = \sum_k e^{-\tau(2\xi_k+U(n_{k\downarrow}+n_{-k\uparrow}-1))}b_k \tag{50}$$

evolves (in the interaction picture under $H_{\rm HK}$) as a multiple of an unevolved pair annihilator b_k. The denominator then removes all disconnected factorizations with either $\langle T\Delta^\dagger(\tau_i)\Delta(\tau_i)\rangle_0$ for any time τ_i or $\langle T\Delta(\tau)\Delta^\dagger\rangle_0$. This leaves only those factorizations with the form of an m-wise convolution, resulting in

$$\chi(i\nu_n) = \sum_{m=1}^\infty g^{m-1}(\chi_0(i\nu_n))^m = \frac{\chi_0(i\nu_n)}{1-g\chi_0(i\nu_n)}. \tag{51}$$

Example of T_c and Δ calculation

To ease calculations, consider ϵ_k such that $\rho(\omega) = \frac{1}{L^d}\sum_k \delta(\omega-\epsilon_k) = \frac{1}{W}$ for $-\frac{W}{2} < \omega < \frac{W}{2}$. We will focus on the half-filled metal i.e. $U < W$ and $\mu = U/2$.

The single-particle density of states (DOS) can be broken into contributions from different regions of momentum space

$$N(\omega) = N_0(\omega) + N_2(\omega) + \frac{1}{2}N_1(\omega) \tag{52}$$
$$N_0(\omega) = \theta(\omega)\rho(\omega+U/2) \tag{53}$$
$$N_2(\omega) = \theta(-\omega)\rho(\omega-U/2) \tag{54}$$
$$N_1(\omega) = \theta(-\omega)\theta(\omega+U)\rho(\omega+U/2) \tag{55}$$
$$+\,\theta(\omega)\theta(-\omega+U)\rho(\omega-U/2) \tag{56}$$

The effective DOS for calculating T_c and Δ are

$$N'(\omega) = N_0(\omega) + N_2(\omega) + \frac{1}{4}N_1(\omega) \tag{57}$$
$$N''(\omega) = N_0(\omega) + N_2(\omega) + N_1(\omega) \tag{58}$$
$$\tag{59}$$

a. Superconducting temperature T_c. The susceptibility diverges when

$$\frac{1}{g} = \chi_0(0) = \int d\omega\, N'(\omega)\frac{\tanh\frac{\beta\omega}{2}}{2\omega}. \tag{60}$$

Set $x = \beta\omega/2$ and integrate by parts

$$\frac{1}{g} = -\frac{1}{2}\int \mathrm{d}x \ln x\Bigg[N'(\frac{2x}{\beta})\,\mathrm{sech}^2\,x \tag{61}$$
$$+\left(\frac{\mathrm{d}}{\mathrm{d}x}N'\left(\frac{2x}{\beta}\right)\right)\tanh x\Bigg]. \tag{62}$$

For $T \ll U, W$ this becomes

$$\frac{1}{g} = \frac{1}{W}\ln\frac{\beta(W-U)}{4} + \frac{1/4}{W}\ln\frac{\beta U}{4} \tag{63}$$
$$-\,N'(0)\left(-\ln\left(\frac{4}{\pi}\right)-\gamma\right) \tag{64}$$
$$\frac{W}{g} = \ln\frac{\beta(W-U)}{4} + \frac{1}{4}\ln\frac{\beta U}{4} - \frac{5}{4}\left(-\ln\left(\frac{4}{\pi}\right)-\gamma\right) \tag{65}$$

where $\gamma \approx 0.577$ is Euler's constant. The solution gives the transition temperature

$$T_c = (W-U)^{4/5}U^{1/5}\frac{e^\gamma}{\pi}e^{-\frac{4}{5}\frac{W}{g}}. \tag{66}$$

b. Superconducting gap Δ. The gap equation is given by

$$1 = \frac{g}{2}\int \mathrm{d}\omega\,\frac{N''(\omega)}{\sqrt{\omega^2+|\Delta|^2}} \tag{67}$$
$$= \frac{g}{W}\sinh^{-1}\left(\frac{W-U}{2\Delta}\right) + \frac{\alpha g}{W}\sinh^{-1}\left(\frac{U}{2\Delta}\right) \tag{68}$$

For $\Delta \ll U, W$ this becomes

$$\frac{1}{g} = \frac{1}{W}\ln\left(\frac{W-U}{\Delta}\right) + \frac{\alpha}{W}\ln\left(\frac{U}{\Delta}\right) \tag{69}$$

which can be solved to find

$$\Delta = (W-U)^{1/2}U^{1/2}e^{-\frac{W}{2g}} \tag{70}$$

Variational ground state

Consider the variational wave function

$$|\psi\rangle = \prod_{k>0}\left(x_k + y_k b_k^\dagger b_{-k}^\dagger + \frac{z_k}{\sqrt{2}}\left(b_k^\dagger + b_{-k}^\dagger\right)\right)|0\rangle\,. \tag{71}$$

$\langle\psi|\psi\rangle = 1$ is satisfied if $|x_k|^2 + |y_k|^2 + |z_k|^2 = 1$. This generalizes the BCS wavefunction, which corresponds to $x_k = u_k^2$, $y_k = v_k^2$, $z_k = \sqrt{2}u_kv_k$. Furthermore, the state defined by $x_k = 1$ for $k \in \Omega_0$, $z_k = 1$ for $k \in \Omega_1$, and $y_k = 1$ for $k \in \Omega_2$ is a ground state of the HK model. Note that although one signal of pair condensation in the BCS wavefunction is the presence of nonzero $u_kv_k \propto z_k$, this state is not a pair condensate.

In the free fermion case, the ground state in the absence of pairing is the filled Fermi sea, with $u_k = 1$ for $k \in \Omega_0$ and $v_k = 1$ for $k \in \Omega_2$. For a small pairing interaction g, the variational ground state with pairing is very similar but with both u_k and v_k non-zero near the boundary of Ω_0 and Ω_2, namely the Fermi surface. In the HK model with weak pairing ($g \ll U, W$), we similarly expect that both x_k and z_k become nonzero near the boundary of Ω_0 and Ω_1 and both y_k and z_k become nonzero near the boundary of Ω_1 and Ω_2.

Again we try to minimize $\langle\psi|H|\psi\rangle$. For all $k > 0$, $p > 0$, and $k \neq p$,

$$\langle\psi|n_{k\sigma}|\psi\rangle = |y_k|^2 + \frac{|z_k|^2}{2} \tag{72}$$
$$\langle\psi|n_{k\uparrow}n_{k\downarrow}|\psi\rangle = |y_k|^2 \tag{73}$$
$$\langle\psi|b_k^\dagger b_k|\psi\rangle = |y_k|^2 + \frac{|z_k|^2}{2} \tag{74}$$
$$\langle\psi|b_k^\dagger b_{-k}|\psi\rangle = \frac{|z_k|^2}{2} \tag{75}$$
$$\langle\psi|b_k|\psi\rangle = \frac{1}{\sqrt{2}}(x_k^*z_k + z_k^*y_k) \tag{76}$$
$$\langle\psi|b_k^\dagger b_p|\psi\rangle = \frac{1}{2}(z_k^*x_k + y_k^*z_k)\left(x_p^*z_p + z_p^*y_p\right). \tag{77}$$

The same equations apply if we take $k \to -k$, $p \to -p$ on the left hand sides. Combining everything, and ignoring terms like $g'\sum_k \ldots$ that do not scale extensively in the thermodynamic limit,

$$\langle\psi|H|\psi\rangle = \sum_{k>0}\xi_k\Big(4|y_k|^2+2|z_k|^2\Big)+U\Big(2|y_k|^2\Big) \tag{78}$$
$$-\,g'\sum_{k,p>0;k\neq p}2(z_k^*x_k+y_k^*z_k)\big(x_p^*z_p+z_p^*y_p\big) \tag{79}$$
$$=\sum_{k>0}(4\xi_k+2U)|y_k|^2+2\xi_k|z_k|^2 \tag{80}$$
$$-\,2g'\sum_{k,p>0;k\neq p}(z_k^*x_k+y_k^*z_k)\big(x_p^*z_p+z_p^*y_p\big) \tag{81}$$

For each k, introduce a lagrange multiplier λ_k to enforce normalization.

$$0=\frac{\partial}{\partial x_k}\Big[\langle\psi|H|\psi\rangle+\lambda_k\Big(|x_k|^2+|y_k|^2+|z_k|^2-1\Big)\Big] \tag{82}$$
$$=\lambda_k x_k^*-2g'z_k^*\sum_{p>0,p\neq k}\big(x_p^*z_p+z_p^*y_p\big) \tag{83}$$
$$\lambda_k=2\frac{z_k^*}{x_k^*}O, \tag{84}$$

where $O=g'\sum_{p>0}\big(x_p^*z_p+z_p^*y_p\big)$ (now including the contribution $p=k$, which is a $\mathcal{O}\big(1/L^d\big)$ difference).

$$0=\frac{\partial}{\partial y_k^*}[\ldots]=(4\xi_k+2U)y_k-2z_kO+\lambda_k y_k \tag{85}$$
$$=(4\xi_k+2U)y_k-2\Big(z_k-\frac{z_k^*y_k}{x_k^*}\Big)O \tag{86}$$
$$2\xi_k+U=\Big(\frac{z_k}{y_k}-\frac{z_k^*}{x_k^*}\Big)O \tag{87}$$

$$0=\frac{\partial}{\partial z_k^*}[\ldots]=2\xi_k z_k-2(x_kO+y_kO^*)+\lambda_k z_k \tag{88}$$
$$=2\xi_k z_k-2\Bigg(x_kO+y_kO^*-\frac{|z_k|^2}{x_k^*}O\Bigg) \tag{89}$$
$$\xi_k x_k^* z_k=\Big(|x_k|^2-|z_k|^2\Big)O+x_k^*y_kO^* \tag{90}$$
$$\xi_k=\Big(\frac{x_k}{z_k}-\frac{z_k^*}{x_k^*}\Big)O+\frac{y_k}{z_k}O^*. \tag{91}$$

In the last lines, we take the limit $L^d\to\infty$, so we ignore the g' on the LHS of and also replace the sum in O with a sum over all momentum. Subtracting (91) from (87) gives

$$\xi_k+U=\Big(\frac{z_k}{y_k}-\frac{x_k}{z_k}\Big)O-\frac{y_k}{z_k}O^*. \tag{92}$$

Using $\xi_k^l=\xi_k$ and $\xi_k^u=\xi_k+U$, and assuming everything is real,

$$\xi_k^l=\Big(\frac{x_k}{z_k}+\frac{y_k}{z_k}-\frac{z_k}{x_k}\Big)O \tag{93}$$
$$\xi_k^u=-\Big(\frac{x_k}{z_k}+\frac{y_k}{z_k}-\frac{z_k}{y_k}\Big)O. \tag{94}$$

It is straightforward to check for $U=0$, combining these equations produces exactly the BCS result, even though we started with a more general wavefunction. This system of two equations is possible to solve analytically, but requires finding the roots of a quartic equation.

c. Weak coupling $g\ll U$ and $g\ll W$. First, rewrite

$$\xi_k^l x_k z_k=\big(x_k^2-z_k^2+x_ky_k\big)O \tag{95}$$
$$\xi_k^u y_k z_k=\big(z_k^2-y_k^2-x_ky_k\big)O, \tag{96}$$

which now looks very similar to the BCS case, apart from the x_ky_k terms.

If $g\ll U$, we expect that there are still well defined regions $\Omega_0,\Omega_1,\Omega_2$, such that mixing occurs only between x_k and z_k or between y_k and z_k and never x_k and y_k. Is it safe to drop the x_ky_k terms from (95) and (96)? Consider a k point where $\xi_k^l<\xi_k^u<0$. Here we expect $1\approx y_k\gg z_k\gg x_k$. If x_k, y_k, z_k are all positive, (95) can only be satisfied if $z_k^2\gg x_ky_k$. A similar argument can be made for (96). Therefore we drop x_ky_k from both equations and work with

$$\xi_k^l x_k z_k=\big(x_k^2-z_k^2\big)O \tag{97}$$
$$\xi_k^u y_k z_k=\big(z_k^2-y_k^2\big)O. \tag{98}$$

Change variables to

$$x_k^2-z_k^2=\frac{\xi_k^l}{E_k^l}\big(1-y_k^2\big),\quad 2x_kz_k=\frac{\Delta_k^l}{E_k^l}\big(1-y_k^2\big) \tag{99}$$
$$E_k^l=\sqrt{{\xi_k^l}^2+{\Delta_k^l}^2} \tag{100}$$
$$z_k^2-y_k^2=\frac{\xi_k^u}{E_k^u}\big(1-x_k^2\big),\quad 2y_kz_k=\frac{\Delta_k^u}{F_k^u}\big(1-x_k^2\big) \tag{101}$$
$$E_k^u=\sqrt{{\xi_k^u}^2+{\Delta_k^u}^2} \tag{102}$$

to get

$$\Delta_k^l=g'\sum_{p>0}\frac{\Delta_p^l}{E_p^l}\big(1-y_k^2\big)+\frac{\Delta_p^u}{E_p^u}\big(1-x_k^2\big) \tag{103}$$
$$\Delta_k^u=g'\sum_{p>0}\frac{\Delta_p^l}{E_p^l}\big(1-y_k^2\big)+\frac{\Delta_p^u}{E_p^u}\big(1-x_k^2\big) \tag{104}$$

from which we see that there is only a single momentum-independent parameter Δ defined by

$$1=g'\sum_{k>0}\frac{1-y_k^2}{\sqrt{{\xi_k^l}^2+\Delta^2}}+\frac{1-x_k^2}{\sqrt{{\xi_k^u}^2+\Delta^2}} \tag{105}$$
$$1=\frac{g}{2}\int\mathrm{d}\omega\,\frac{N''(\omega)}{\sqrt{\omega^2+\Delta^2}}. \tag{106}$$

This is the same as the BCS gap equation, but with an effective density of states

$$N''(\omega) = \frac{1}{L^d}\sum_k \delta(\omega - \xi_k^l)(1-y_k^2) + \delta(\omega - \xi_k^u)(1-x_k^2) \tag{107}$$

Because we are considering $g \ll U$, to a very good approximation $1 - y_k^2 = \theta(\xi_k^u)$ and $1 - x_k^2 = \theta(-\xi_k^l)$.

$$N''(\omega) = \frac{1}{L^d}\sum_k \delta(\omega - \xi_k^l)\theta(\xi_k^u) + \delta(\omega - \xi_k^u)\theta(-\xi_k^l) \tag{108}$$

Note that $N''(\omega)$ is *not* the single-particle density of states of the HK model. In fact it is larger than or equal to it for all ω, and $\int \mathrm{d}\omega\, N''(\omega) \geq 1$. (108) may be rewritten as

$$N''(\omega) = \sum_{k\in\Omega_0} \delta(\omega - \xi_k^l) + \sum_{k\in\Omega_2}\delta(\omega - \xi_k^u) \tag{109}$$

$$+ \sum_{k\in\Omega_1} \delta(\omega - \xi_k^l) + \delta(\omega - \xi_k^u). \tag{110}$$

Mean-field calculation of spectral function

To calculate the spectral function in Fig. 1, we treat the pairing interaction at the mean-field level and the Mott (U) interaction exactly.

$$H_{pair} = -g'\sum_{kk'} b_k^\dagger b_{k'} \tag{111}$$

$$= -g'\sum_{kk'} \left\langle b_k^\dagger \right\rangle b_{k'} + b_k^\dagger \left\langle b_{k'}\right\rangle \tag{112}$$

$$+ \left(b_k^\dagger - \left\langle b_k^\dagger\right\rangle\right)(b_{k'} - \langle b_{k'}\rangle) - \left\langle b_k^\dagger \right\rangle \langle b_{k'}\rangle\,. \tag{113}$$

In the mean-field approximation, the third term (quadratic in fluctuations) is dropped. The last term is an inconsequential constant. Define

$$\Delta = -g'\sum_k \langle b_k\rangle \tag{114}$$

$$H_{pair}^{MF} = \sum_k \Delta^* b_k + \Delta b_k^\dagger \tag{115}$$

The full Hamiltonian becomes

$$H^{MF} = \sum_k \xi_k(n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow}n_{k\downarrow} + \Delta^* b_k + \Delta b_k^\dagger. \tag{116}$$

Here, the pairing is treated at the mean-field level but the Mott interaction is treated exactly. While the Hamiltonian no longer separates completely in k-space, each k couples only to $-k$.

$$H^{MF} = \sum_{k>0} H_k^{MF} \tag{117}$$

$$H_k^{MF} = \xi_k(n_{k\uparrow} + n_{k\downarrow} + n_{-k\uparrow} + n_{-k\downarrow}) \tag{118}$$

$$+ U(n_{k\uparrow}n_{k\downarrow} + n_{-k\uparrow}n_{-k\downarrow}) \tag{119}$$

$$+ \Delta^*(b_k + b_{-k}) + \Delta\left(b_k^\dagger + b_{-k}^\dagger\right). \tag{120}$$

H^{MF} may be solved by exact diagonalization of each H_k^{MF}, yielding 16 eigenstates and energies for each k. Δ is adjusted for self-consistency such that Eq. 114 is satisfied. The single-particle spectral function is calculated directly from its spectral representation

$$A(k,\omega) = \sum_{nm} |\langle n|c_k|m\rangle|^2(\rho_n + \rho_m)\delta(\omega + E_n - E_m), \tag{121}$$

where $|n\rangle$ and $|m\rangle$ are eigenstates of H_k^{MF}, $\rho_n = e^{-\beta E_n}/Z$, and the partition function $Z = \sum_n e^{-\beta E_n}$.

Calculation of superfluid stiffness

The superfluid stiffness can be calculated as

$$\frac{D_s}{\pi} = \frac{1}{L^d}\left(\langle K_{xx}\rangle - \int_0^\beta d\tau\, \langle J_x(\tau)J_x\rangle\right), \tag{122}$$

where

$$K_{xx} = \sum_{k\sigma}\frac{\partial^2 \epsilon_k}{\partial k_x^2} c_{k\sigma}^\dagger c_{k\sigma} \tag{123}$$

$$J_x = \sum_{k\sigma}\frac{\partial \epsilon_k}{\partial k_x} c_{k\sigma}^\dagger c_{k\sigma}. \tag{124}$$

As for the spectral function, all expectation values are calculated by exact diagonalization of H_k^{MF}. In Fig. 4, a 64×64 grid of k-points is used.

For a nearest neighbor tight-binding band structure as considered throughout this work, $\frac{\partial^2\epsilon_k}{\partial k_x^2} = 2t\cos k_x$, in units where the lattice constant $a = 1$. Therefore $\langle K_{xx}\rangle$ is simply the (negative) kinetic energy along the x-direction bonds. In the spectral representation, we see that

$$\int_0^\beta d\tau\, \langle J_x(\tau)J_x\rangle = \sum_{nm} |\langle n|J_x|m\rangle|^2 \frac{\rho_m - \rho_n}{E_n - E_m} \tag{125}$$

is nonnegative, so $\frac{\pi}{L^d}\langle K_{xx}\rangle$ is an upper bound to the superfluid stiffness[30].

Superconducting energy scales in $d = 1, 2, 3$

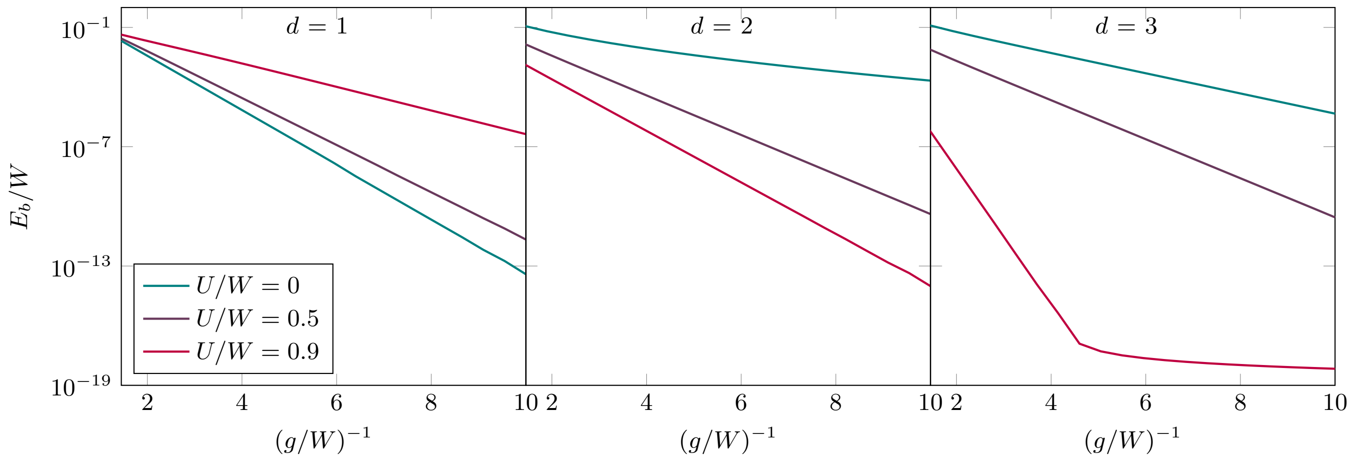

FIG. S1. Cooper pair binding energy E_b at half-filling.

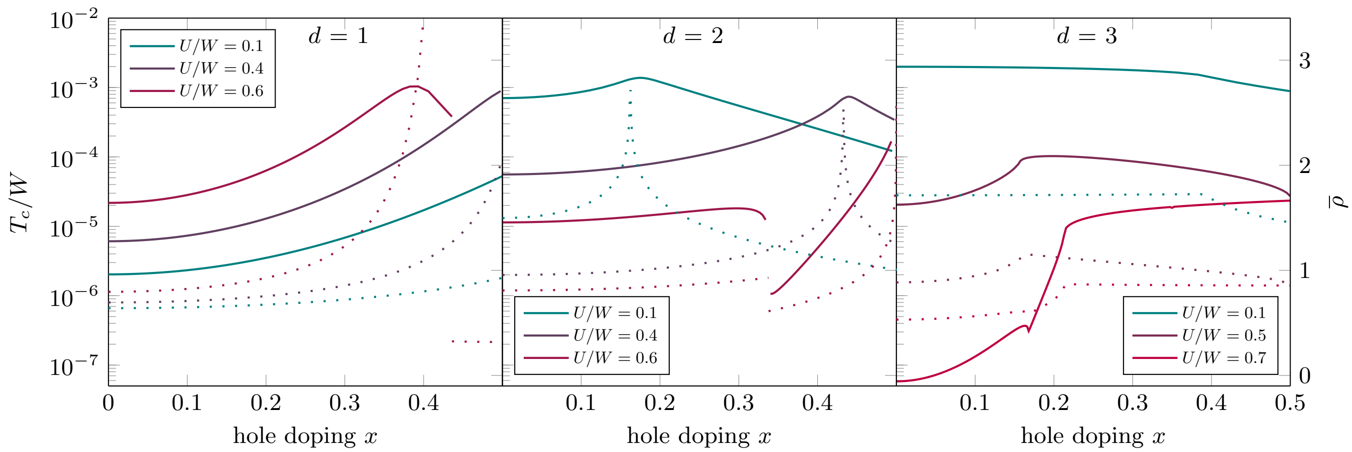

FIG. S2. Superconducting temperature T_c (solid) and mean density of states $\overline{\rho} = \frac{1}{2}(\rho(\mu) + \rho(\mu - U))$ (dotted) at pair coupling $g/W = 0.1$.