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Quantized fractional Thouless pumping of solitons

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Supplemental Material: Quantized Fractional Thouless Pumping of Solitons

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S1. ADDITIONAL INFORMATION ON THE NONLINEAR TRANSITION BETWEEN INTEGER AND FRACTIONALLY QUANTIZED PUMPING

In this section we provide more information on the transition from an integer pumped soliton to a fractionally pumped soliton for increasing nonlinearity. Figs. S1a-e show the center of mass positions of the single-band Wannier functions of the first (black, solid) and the maximally localized multi-band Wannier functions of the first and second band (black, dashed), together with the position of the relevant stable instantaneous solitons (red). Figs. S1a-e show one half of a period but behavior in the remaining part of the pump cycle is identical due the symmetric hopping modulation and translation symmetry. For small nonlinearity ($gP/J^{\max}$=0.55), i.e., in Fig. S1a, the soliton follows the path of the Wannier function of the first band, which is the band from which it bifurcates at low power. With increasing power the solitons undergo nonlinear bifurcations (see Fig. S1f). The pitchfork bifurcations first split the originally contiguous path of the soliton (Figs. S1c,d) such that no quantized pumping occurs. For even stronger nonlinearity (Fig. S1e) a new contiguous propagation path forms along the position of the maximally localized multi-band Wannier function of the two lowest bands, which displaces the soliton by one half of a unit cell per cycle.

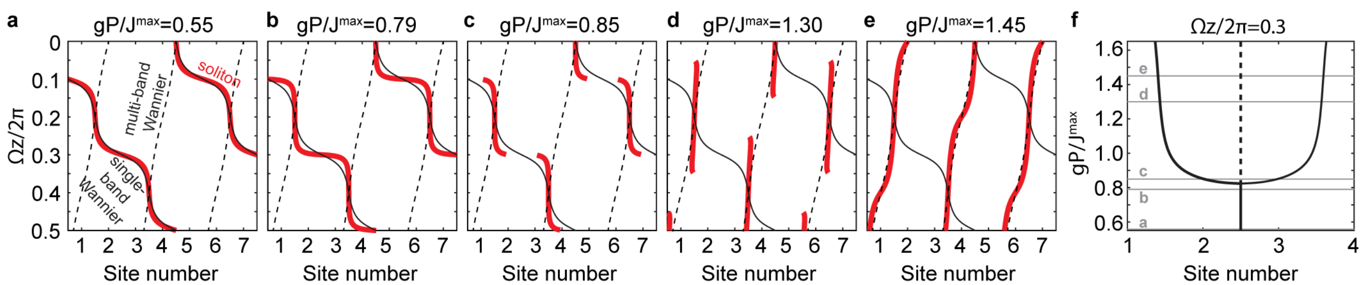

FIG. S1. Nonlinear transition between integer and fractionally quantized pumping. **a-e.** Center of mass position of the soliton (red) with respect to the center of mass position of the single-band Wannier function of the lowest band (black, solid) and the maximally localized multi-band Wannier function of the two lowest bands (black, dashed) for increasing nonlinearity $gP/J^{\max}$ and for 1/2 of a pumping cycle. **f.** Nonlinear pitchfork bifurcation diagram for $\Omega z/2\pi$=0.3 showing the stable solitons from **a-e** with solid lines, and one unstable soliton (not shown in **a-e**) with dashed lines. Horizontal gray lines indicate the strength of nonlinearity for which the z-evolution is shown in **a-e**.

S2. ADDITIONAL INFORMATION ON NUMERICAL SIMULATION OF MULTIPLE FRACTIONAL PLATEAUS WITHIN ONE MODEL

In this section we provide more details on the off-diagonal AAH-model with 13 sites per unit cell, used for Fig. 4 in the main text. The on-site detuning is set to zero and the nearest-neighbor hoppings between site n and n+1 are given by $J_n = K + \kappa\cos(\Omega z + \frac{10\pi}{13}n + \frac{2\pi}{13})$ with $K = 1/b$ and $\kappa = 0.95/b$, where b is an arbitrary length. This model has 13 bands (see Extended Data Fig. 8a) that are distributed symmetrically around zero energy and which can be clustered into five bands at low energy (see Extended Data Fig. 8c), three bands around zero energy (see Extended Data Fig. 8b) and five high energy bands. The group of the lowest five bands has another subgroup of energetically close bands, consisting of the two lowest bands.

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To evaluate the Chern number of these bands, we calculate the position of the instantaneous single-band Wannier functions over one period projected into a single unit cell for each band individually (see Extended Data Fig. 9a). The number of windings is equal to the Chern number of the respective band, resulting in Chern numbers C={-8,5,5,-8,5,5,-8,5,5,-8,5,5,-8}, ordered from bottom to top. Similar to Fig. 2 in the main text, we also evaluate the maximally localized multi-band Wannier functions (Extended Data Fig. 9c-g) together with the position of the instantaneous solitons for $gP/J^{\max}$=0.04, 0.10, 0.78 and 3.08 which closely follow the position of the Wannier functions.

We observe that fractional pumping occurs for groups of energy bands that (1) are energetically close together but (2) are well separated from all other bands. In our case the groups are bands 1 and 2, bands 1 to 5 and bands 1 to 13, which all show quantized soliton pumping. While we discuss here only pumping of solitons that bifurcate from the lowest band at low power, we have numerically also observed fractionally pumped solitons that bifurcate from higher bands.

For the propagation simulation in Fig. 4 in the main text, we take into account that the soliton radiation is larger for low power for fixed Ω due to weaker confinement. We therefore calculate the data shown in Fig. 4 differently for $gP/J^{\max}$ <0.06 and $gP/J^{\max}$ >0.06. For $gP/J^{\max}$ >0.06 the soliton is sufficiently well confined to simulate adiabatic propagation over ten periods in a lattice with 20 unit cells and a periodic length L=$5\cdot 10^3 b$. Ten periods are a suitable choice, as they are the least common multiple of 2 and 5, such that we observe integer quantized displacement for the -3/2 and -1/5 fractional plateaus. Due to the weak soliton confinement at low power ($gP/J^{\max}$ <0.06), and hence the long propagation distances necessary for sufficient adiabaticity, we take advantage of the symmetries of the system and propagate those solitons only for 1/13 of a period (L=$5\cdot 10^6 b$) in a lattice with ten unit cells. At this point, the Hamiltonian and the soliton are both identical to those at $z=0$, apart from a spatial translation, and the motion of the soliton repeats itself. Therefore, the propagation of 1/13 of the period suffices to calculate the displacement after one full period. Furthermore, we numerically check via calculations of the instantaneous soliton that the observed propagation of the first 1/13 of the period repeats for the remaining parts of the period.