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# Topological phase transition between Jain states and daughter states of the $\nu = 1/2$ fractional quantum Hall state

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**Contents**

## Supplementary Section 1: Evolution of the various phases at $\nu = 1/2$ in wide GaAs quantum wells

For any given well-width, at low densities, 2D electrons at $\nu = 1/2$ show a compressible state of composite fermion (CF) Fermi sea. The fractional quantum Hall states (FQHSs) flanking the CF Fermi sea have been studied extensively and are understood to be the integer quantum Hall states of CFs, or Jain states found at $\nu = p/(2p \pm 1)$; see Extended Data Figs. 1a and 2a for examples. As the density is increased to an intermediate range, the CF Fermi sea at $\nu = 1/2$ makes a transition to a FQHS. With further raise in density, an insulating phase appears for $\nu < 1/2$, eventually engulfing even the 1/2 FQHS; see Ref. [1, 2], Fig. 1e and Extended Data Fig. 2a.

The phase boundaries between the three phase at $\nu = 1/2$ depend on the QW width ($w$) [1, 3, 4]. These boundaries are effectively captured in the phase diagrams presented in Ref. [4] where the stability of the compressible, FQHS, and insulating phases was plotted in different parameter spaces ($w$-$n$, $\Delta_{SAS}$-$n$, $d/l_B$-$\Delta_{SAS}$). In Extended Data Fig. 1b we reproduce the well-width vs density phase diagram reported in Ref. [4]. The 1/2 FQHS is stable in an intermediate density range sandwiched between the compressible and insulating phases. By making the QW narrower, the two phase boundaries (dashed and dotted respectively) are shifted to higher densities and simultaneously the density range for the stability of the 1/2 FQHS becomes wider.

Lastly, within the intermediate density range where the 1/2 FQHS is stable, the strength of the 1/2 FQHS increases with increasing density, shows a maximum and then starts to go down as the insulating phases on its flanks get stronger and eventually engulf it. Quantitatively, this behavior was first captured by the measurements of the 1/2 FQHS energy gap as a function of density, as reported in Ref. [3]. The behavior was also qualitatively displayed in the phase diagrams of Ref. [4] where the size of the data points was used to denote the relative strength of the 1/2 FQHS. (As an example for the strengthening of the 1/2 FQHS with increasing density in Ref. [4], see Extended Data Fig. 1.) The evolution of the 1/2 FQHS in our samples qualitatively shows a similar behavior. For example, as shown in Fig. 2, the energy gap for the 1/2 FQHS increases from 1.48 K to 3.73 K as the density is increased from 1.17 to $1.35 \times 10^{11}$ cm$^{-2}$. This strengthening can also be seen qualitatively in Figs. 1e,f and Extended Data Fig. 2, both of which show a broadening of the 1/2 FQHS $R_{xx}$ minimum and $R_{xy}$ Hall plateau with increasing density.

It is worth emphasizing that the above experimental observations are consistent with the findings of numerous theoretical works which study the 1/2 FQHS in wide QWs for appropriate sample parameters [5–7]. The calculations, too, predict an enhancement of the 1/2 FQHS energy gap as the parameters are tuned so that the sample is deeper in the FQHS regime, and is away from the compressible-FQHS boundary [5, 6].

Now the very important feature of our new data is that we observe a rather abrupt emergence of the daughter states of the 1/2 FQHS (at $\nu = 8/17$ and 7/13) only when the 1/2 FQHS is relatively strong, i.e., when the density is sufficiently large so that the sample parameters (well width and density) are such that we are reasonably far from the compressible-FQHS boundary and well into the FQHS regime (see, e.g., Figs. 1c,d). This makes sense, as it is well known historically that the FQHSs associated with daughter states start to appear only as their "parent" states become sufficiently strong, for example, as a function of increasing density which enhances the Coulomb energy, or as the "quality" is improved by making purer and less disordered samples [8].

To summarize the above discussion, we observe the daughter states of the 1/2 FQHS (at $\nu =$ 8/17 and 7/13) when the parent 1/2 FQHS is sufficiently strong. Both the strengthening of the 1/2 FQHS as we tune the sample deeper into the FQHS regime, and the observation of the daughter states at fillings factors that match those theoretically predicted for a Pfaffian 1/2 FQHS [9], strongly suggest that the 1/2 FQHS we are observing is a Pfaffian.

## Supplementary Section 2: Daughter states of the $\nu = 1/2$ fractional quantum Hall states

The Halperin $\Psi_{331}$ and the $K = 8$ strong-pairing states are Abelian fractional quantum Hall states (FQHSs) at Landau level filling factor $\nu = 1/2$ [10]. Here we derive their daughter states by condensing the minimal-charge quasiparticles and quasiholes of the respective states. We derive these daughter states using Equations 2.29, 2.30 and 2.31 of Ref. [11]; the derivation for the daughter states of the $\Psi_{331}$ state was provided to us by Xiao-Gang Wen whom we thank, and are grateful to for granting us permission to use his unpublished work. We also include a table summarizing the daughter states of different $\nu = 1/2$ FQHSs.

### (a) Halperin $\Psi_{331}$ state

The $\Psi_{331}$ state is an Abelian FQHS at $\nu = 1/2$ [10]. It can be effectively described using the $K$-matrix and charge vector, $t$:

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The filling factor $\nu$ at which the FQHS state occurs is related to its $K$-matrix and charge vector $t$, and can be computed by using Equation 2.31 in Ref. [11]. The above $K$-matrix and $t$ yield:

$$\nu = t^T K^{-1} t = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}.$$

The lowest-lying charge excitations of the $\Psi_{331}$ state are (using Equation 2.30 in Ref. [11]) :

$$Q_{331} = eK^{-1}t = e \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}.$$

The daughter states are also Abelian. They are formed by condensing $e/4$ *quasiparticles* and are effectively described by the $K_m^{qp}$-matrix (Equation 2.29 in Ref. [11]) and its corresponding charge vector, $t_1^{qp}$ (Equation 2.26 in Ref. [11]; the index '1' in $t_1^{qp}$ indicates the first level of hierarchy):

$$K_m^{qp} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & m \end{pmatrix}, \qquad t_1^{qp} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Here, $m$ is a positive, even integer. The daughter states of the $\Psi_{331}$ state occur at:

$$\nu_{331}^{qp} = (t_1^{qp})^T (K_m^{qp})^{-1} t_1^{qp} = \frac{4m-1}{8m-3} = \frac{7}{13}, \frac{15}{29}, \ldots.$$

Similarly, the daughter states formed by condensing $e/4$ *quasiholes* are described by the $K_m^{qh}$-matrix and its corresponding charge vector, $t_1^{qh}$:

$$K_m^{qh} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & -m \end{pmatrix}, \qquad t_1^{qh} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

and occur at:

$$\nu_{331}^{qh} = \frac{4m+1}{8m+3} = \frac{9}{19}, \frac{17}{35}, \ldots.$$

### (b) $K = 8$ strong-pairing state

The $K = 8$ strong-pairing state was proposed by Halperin [10] as a possible description for even-denominator FQHSs. This state describes a Laughlin $\nu = 1/8$ state for charge $2e^-$ bosons which are formed out of two electrons tightly bound to each other. For the $K = 8$ strong-pairing state, the $K$-matrix and charge vector are:

$$K = 8, \qquad t = 2.$$

This state occurs at $\nu = t^T K^{-1} t = 2 \times \frac{1}{8} \times 2 = \frac{1}{2}$, and its lowest-lying quasi-particle/hole excitations carry charge $Q_{K=8} = eK^{-1}t = e/4$.

The daughter states formed by condensing $e/4$ *quasiparticles* are described by the $K_m^{qp}$-matrix and its corresponding charge vector, $t_1^{qp}$:

$$K_m^{qp} = \begin{pmatrix} 8 & -1 \\ -1 & m \end{pmatrix}, \qquad t_1^{qp} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

and occur at:

$$\nu_{K=8}^{qp} = \frac{4m}{8m-1} = \frac{8}{15}, \frac{16}{31}, \ldots .$$

Similarly, the daughter states formed by condensing $e/4$ *quasiholes* are described by the $K_m^{qh}$-matrix and its corresponding charge vector, $t_1^{qh}$:

$$K_m^{qh} = \begin{pmatrix} 8 & 1 \\ 1 & -m \end{pmatrix}, \qquad t_1^{qh} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

and occur at:

$$\nu_{K=8}^{qh} = \frac{4m}{8m+1} = \frac{8}{17}, \frac{16}{33}, \ldots .$$

### (c) Moore-Read state

The Moore-Read Pfaffian (*Pf*) state [12] and its particle-hole conjugate, the anti-Pfaffian (*a-Pf*) state, are also candidates for even-denominator FQHSs [13, 14]. The *Pf* and *a-Pf* states are in different universality classes and have different topological order. The lowest-lying excitations of these states also carry charge $e/4$ and are posited to show non-Abelian statistics. The problem of constructing daughter states of the *Pf* state was tackled by Levin and Halperin [9] who showed that these daughter states occur at:

$$\nu_{Pf}^{qp} = \frac{7}{13}, \frac{15}{29}, \ldots .$$

$$\nu_{Pf}^{qh} = \frac{8}{17}, \frac{16}{33}, \ldots .$$

The daughter states of the *a-Pf* state can be obtained by particle-hole conjugation of the *Pf* daughter states.

$$\nu^{qp}_{a\text{-}Pf} = \frac{9}{17}, \frac{17}{33}, \ldots.$$

$$\nu^{qh}_{a\text{-}Pf} = \frac{6}{13}, \frac{14}{29}, \ldots.$$

**(d) Summary**

In Table S1 we summarize the simplest daughter states of the different 1/2 FQHS candidates. Note that the *Pf* and *A-Pf* states are non-Abelian, while the $\Psi_{331}$ and the strong-pairing states are Abelian. All the daughter states are Abelian. Based on the Levin-Halperin construction, the daughter states obtained by condensing the *Pf*-quasiparticles and the $\Psi_{331}$ quasiparticles are in the same universality class; similarly, condensing quasiholes of either the *Pf* state or the strong-pairing state lead to topologically equivalent states.

| Parent 1/2 FQHS | Halperin $\Psi_{331}$ | Strong-pairing | Pfaffian | anti-Pfaffian |
|---|---|---|---|---|
| quasiparticle | 7/13 ✓ | 8/15 ✗ | 7/13 ✓ | 9/17 ✗ |
| quasihole | 9/19 ✗ | 8/17 ✓ | 8/17 ✓ | 6/13 ✗ |

**Table S1|Simplest daughter states of the different 1/2 FQHS candidate states**. Note that the strong daughter states we observe at 8/17 and 7/13 are consistent with a Pfaffian FQHS at $\nu = 1/2$.


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