Supplementary Material: Group interactions modulate critical mass dynamics in social convention

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Supplementary Note 1. MEAN FIELD APPROACH WITH DIFFERENT CONDITION RULES FOR GROUP AGREEMENT

In the main text we have restricted most of our attention to one particular rule for group agreement that we called *unanimity condition*. At the essence of this rule there is the requirement that the spoken name has to be present in all the vocabularies of the hearers. As such, we can mathematically refer to this condition as an "intersection" rule. Here, we complement the mean field description of the proposed model by describing, all together, a "intersection" agreement rule, a "union" agreement rule, and a pairwise case for comparison.

As for the main text, we consider the simplest case in which there are only two names, A and B, so that each vocabulary can have only three states, A, B, and AB. We further assume an homogeneous mixing population that interacts by means of 2-interactions, that is hyperedges of size 3 (triangles). In this case, given a speaker with vocabulary S that communicates a name A to two hearers with vocabularies X and Y, the two agreement rules we have defined reduce to:

Intersection rule:

- if $A \in X \cap Y$:
  - with probability β: $S \Rightarrow \{A\}$, $X \Rightarrow \{A\}$, $Y \Rightarrow \{A\}$
  - with probability $1-\beta$: $X \Rightarrow X \cup \{A\}$, $Y \Rightarrow Y \cup \{A\}$
- if $A \notin X \cap Y$: $X \Rightarrow X \cup \{A\}$, $Y \Rightarrow Y \cup \{A\}$

Union rule:

- if $A \in X \cup Y$:
  - with probability β: $S \Rightarrow \{A\}$, $X \Rightarrow \{A\}$, $Y \Rightarrow \{A\}$
  - with probability $1-\beta$: $X \Rightarrow X \cup \{A\}$, $Y \Rightarrow Y \cup \{A\}$
- if $A \notin X \cup Y$: $X \Rightarrow X \cup \{A\}$, $Y \Rightarrow Y \cup \{A\}$

We denote with n_A, n_B and n_{AB} the fraction of agents in each state (omitting the temporal index for readability purposes), while p denotes the fraction of agents committed to A. Considering both rules, the equations for the evolution of the fraction of agents in each state read:

"Intersection" rule:

$$
\begin{aligned}
d_t n_A &= -2n_A^2 n_B + (\frac{5}{2}\beta - 1)n_A^2 n_{AB} - 2n_A n_B^2 - 3n_A n_B n_{AB} + (4\beta - 1)n_A n_{AB}^2 \\
&\quad + \frac{3}{2}\beta n_{AB}^3 + \frac{5}{2}\beta p^2 n_{AB} + p[-2n_A n_B + (5\beta - 1)n_A n_{AB} + 4\beta n_{AB}^2] \qquad \text{(S1a)} \\
d_t n_B &= -2n_B^2 n_A + (\frac{5}{2}\beta - 1)n_B^2 n_{AB} - 2n_B n_A^2 - 3n_B n_A n_{AB} + (4\beta - 1)n_B n_{AB}^2 \\
&\quad + \frac{3}{2}\beta n_{AB}^3 - 2p^2 n_B - p[4n_A n_B + 2n_B^2 + 3n_b n_{AB}] \qquad \text{(S1b)} \\
n_{AB} &= 1 - n_A - n_B - p \qquad \text{(S1c)}
\end{aligned}
$$

"Union" rule:

$$
\begin{aligned}
d_t n_A &= 2(\beta - 1)n_A^2 n_B + (\frac{5}{2}\beta - 1)n_A^2 n_{AB} - 2n_A n_B^2 + (6\beta - 3)n_A n_B n_{AB} \\
&\quad + (4\beta - 1)n_A n_{AB}^2 + 3\beta n_B n_{AB}^2 + \frac{3}{2}\beta n_{AB}^3 + p^2[2\beta n_B + \frac{5}{2}\beta n_{AB}] \\
&\quad + p[2(2\beta - 1)n_A n_B + (5\beta - 1)n_A n_{AB} + 6\beta n_B n_{AB} + 4\beta n_{AB}^2] \qquad \text{(S2a)} \\
d_t n_B &= 2(\beta - 1)n_B^2 n_A + (\frac{5}{2}\beta - 1)n_B^2 n_{AB} - 2n_B n_A^2 + (6\beta - 3)n_B n_A n_{AB} \\
&\quad + (4\beta - 1)n_B n_{AB}^2 + 3\beta n_A n_{AB}^2 + \frac{3}{2}\beta n_{AB}^3 \\
&\quad - 2p^2 n_B - p[2n_B^2 + (3 - 3\beta)n_B n_{AB} - 2\beta n_{AB}^2 + 4n_a n_B] \qquad \text{(S2b)} \\
n_{AB} &= 1 - n_A - n_B - p \qquad \text{(S2c)}
\end{aligned}
$$

We also consider the pairwise case as a benchmark, whose equations are:

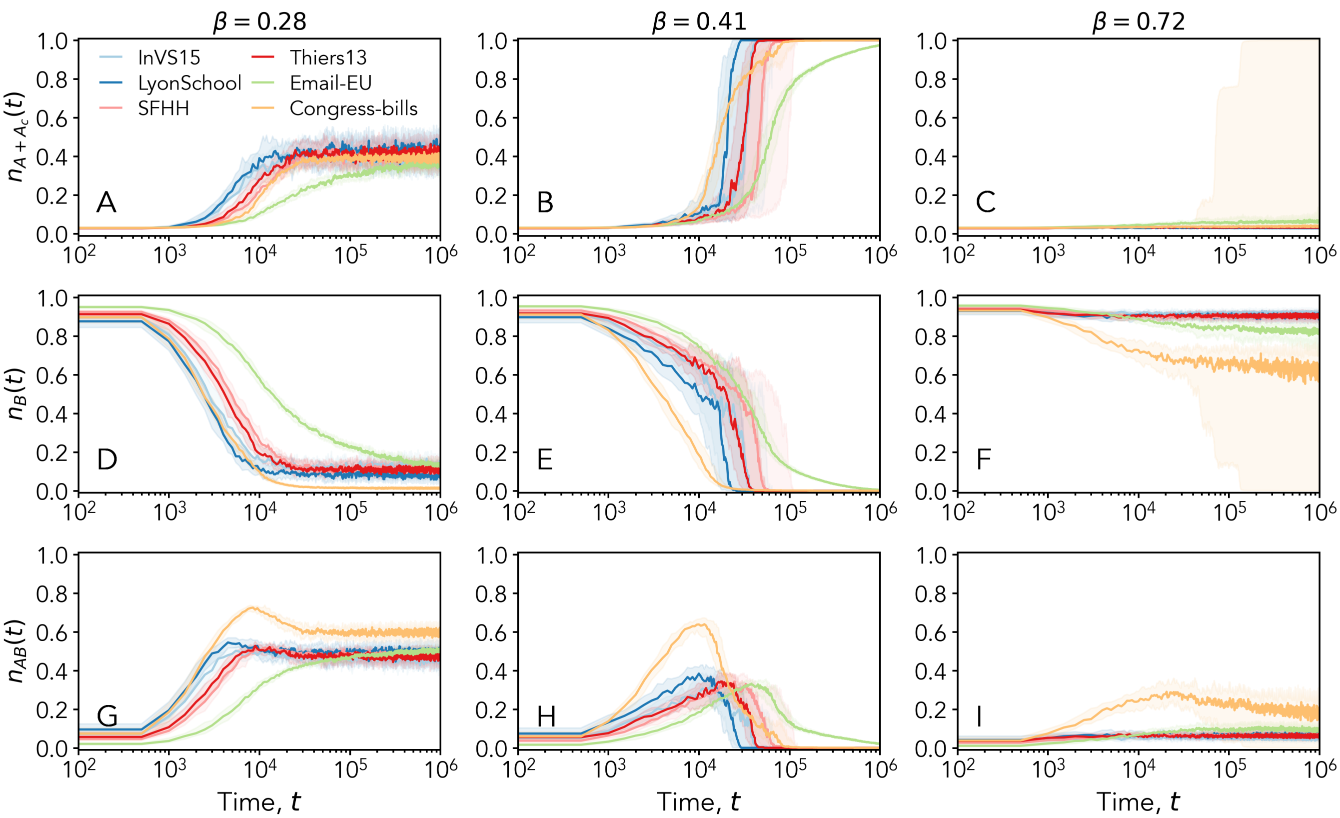
Pairwise case:

$$d_t n_A = -n_A n_B + \frac{1}{2}(3\beta - 1) n_A n_{AB} + \beta n_{AB}^2 + \frac{3}{2}\beta p n_{AB} \tag{S3a}$$

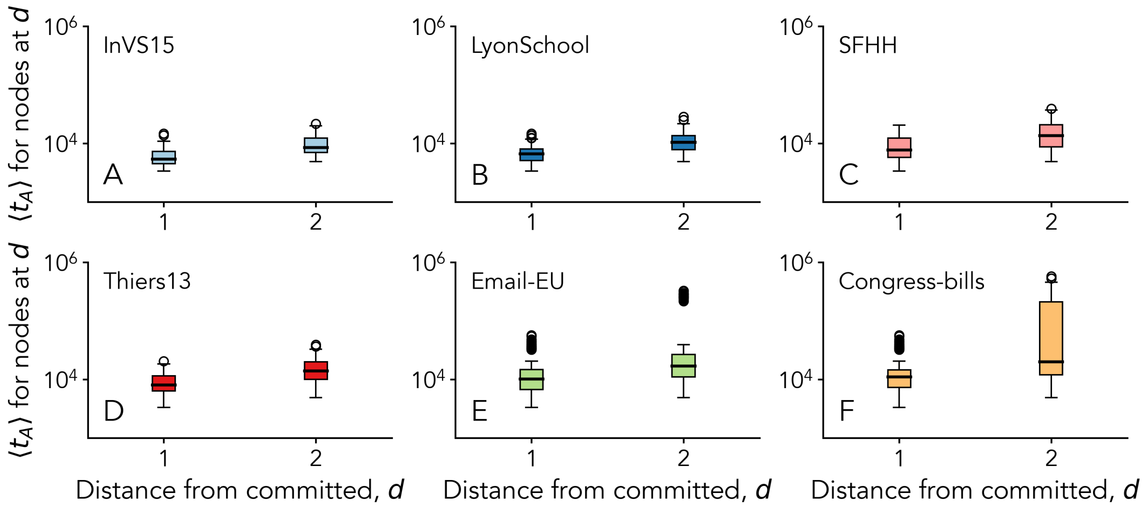
$$d_t n_B = -n_A n_B + \frac{1}{2}(3\beta - 1) n_B n_{AB} + \beta n_{AB}^2 - p n_B \tag{S3b}$$

$$n_{AB} = 1 - n_A - n_B - p \tag{S3c}$$

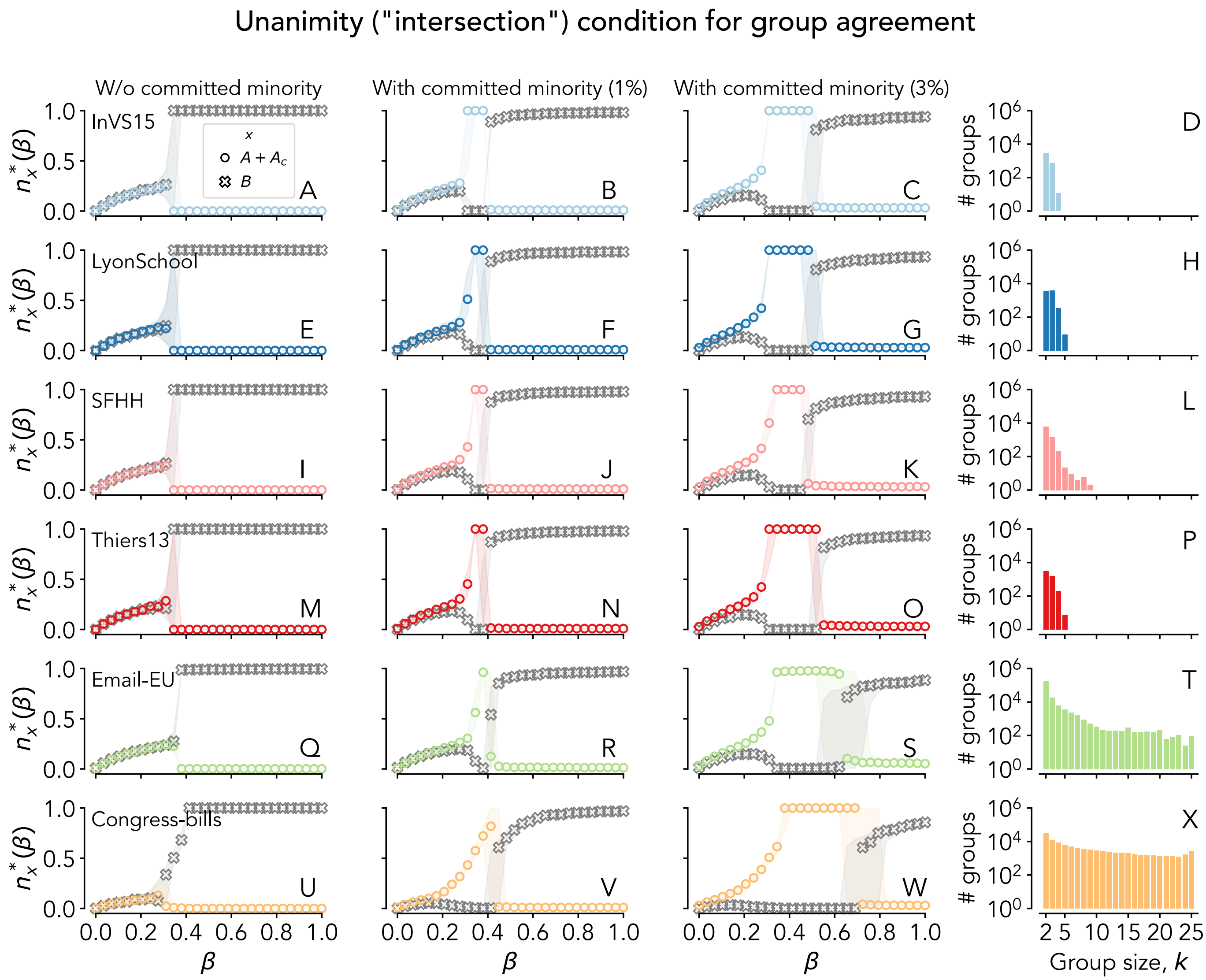
Supplementary Note 2. SUPPLEMENTARY RESULTS



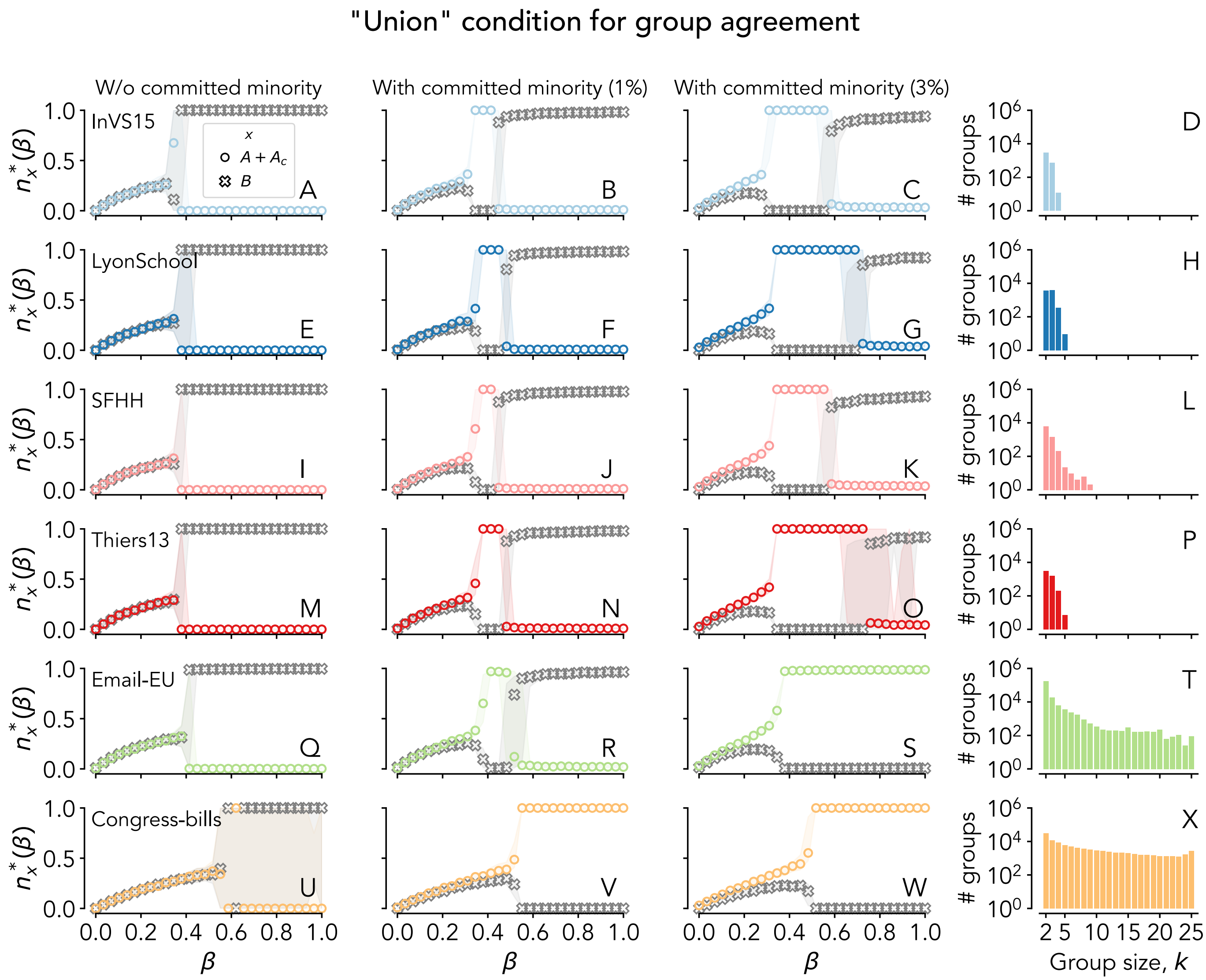
Supplementary Figure 1. Temporal evolution of the dynamics with committed minorities on empirical higher-order structures. As for Fig. 2 of the main text, the considered social structures are constructed from empirical data sets collected in six different context: workplace (InVS15), a primary school (LyonSchool), a conference (SFHH), a high school (Thiers13), email communications (Email-EU) and a political congress (Congress-bills). The temporal evolution of the fraction of nodes supporting name A, B and A, B (with 3% of committed individuals) is reported in panels (*A*-*C*), (*D*-*F*) and (*G*-*I*), respectively, for three different values of the communication efficiency, namely $\beta = 0.28$ (*A*,*D*,*G*), $\beta = 0.41$ (*B*,*E*,*F*) and $\beta = 0.72$ (*C*,*F*,*I*). The results averaged over different runs of stochastic simulations are reported as solid curves and shaded areas, representing median values and values contained within the 25th and 75th percentiles.



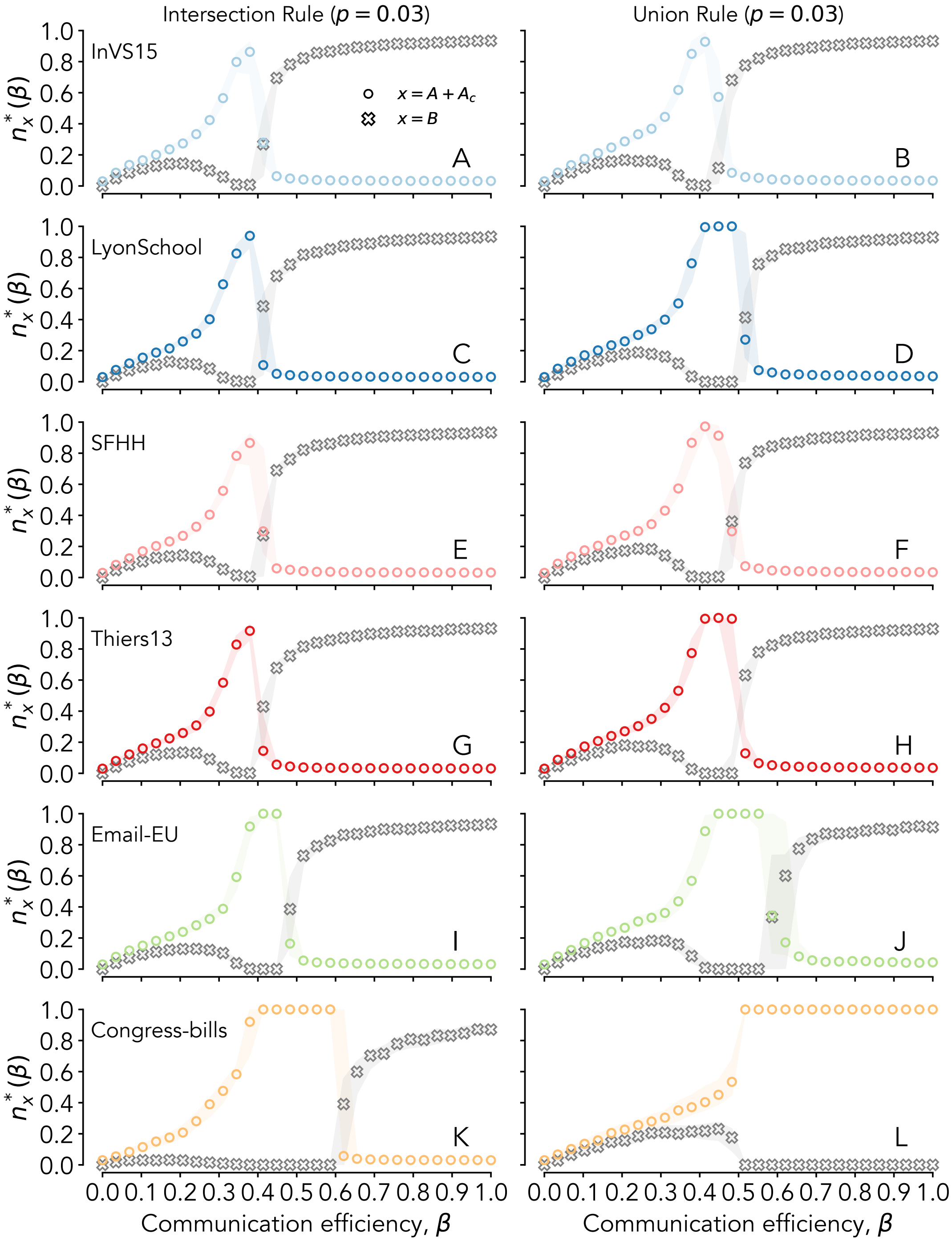
Supplementary Figure 2. Adoption dynamics around committed individuals for simulations of the NG with the unanimity ("intersection") condition for group agreement on empirical higher-order structures. The parameters are $p = 0.03$ and $\beta = 0.4$. The structures correspond to six different contexts: workplace (InVS15), a primary school (LyonSchool), a conference (SFHH), a high school (Thiers13), email communications (Email-EU) and a political congress (Congress-bills). For each data set, the boxplot shows the average time $\langle t_A \rangle$ that it takes for nodes at (graph) distance d from committed individuals to switch to A. Notice that this differs from just having A in the vocabulary, which instead has to be solely composed by A.



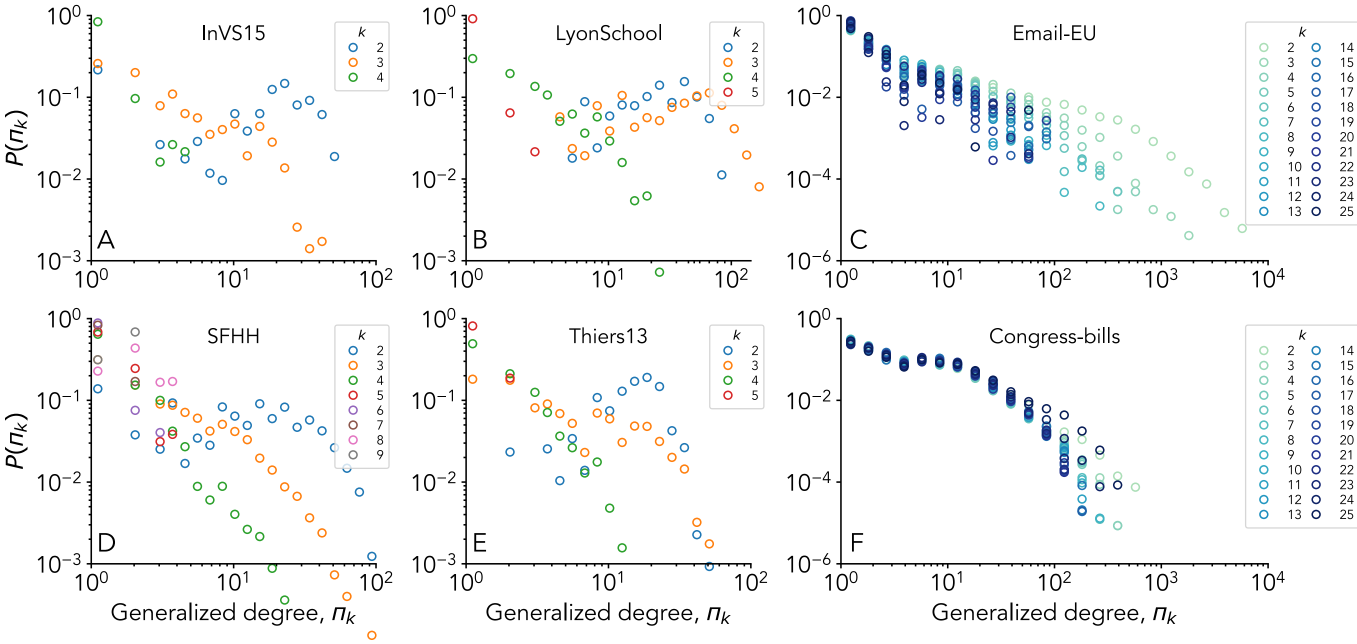
Supplementary Figure 3. Simulations of the stationary state of the dynamics on empirical higher-order structures for a NG with the unanimity ("intersection") condition for group agreement and associated group size frequencies. The considered social structures are the same of Fig. 2 of the main text and correspond to empirical data sets collected in six different context: workplace (InVS15), a primary school (LyonSchool), a conference (SFHH), a high school (Thiers13), email communications (Email-EU) and a political congress (Congress-bills). The phase diagrams in panels (A,E,I,M,Q,U), panels (B,F,J,N,R,V) and panels (C, G, K, O, S, W) correspond to simulation with a different fraction of committed minority, respectively set to $p = 0$, $p = 0.01$ and $p = 0.03$. The fraction of nodes supporting name x in the stationary state $n^*_x(\beta)$, obtained by means of numerical simulations on each data set (row), is plotted as a function of the communication efficiency β. The results averaged over different runs of stochastic simulations are reported as points (circles and crosses, respectively associated to name A and B) and shaded areas, representing median values and values contained within the 25th and 75th percentiles. Panels (D,H,L,P,T,X) show the group size frequencies associated to each data set, where a group of size k represents a higher-order interaction between the k nodes involved.



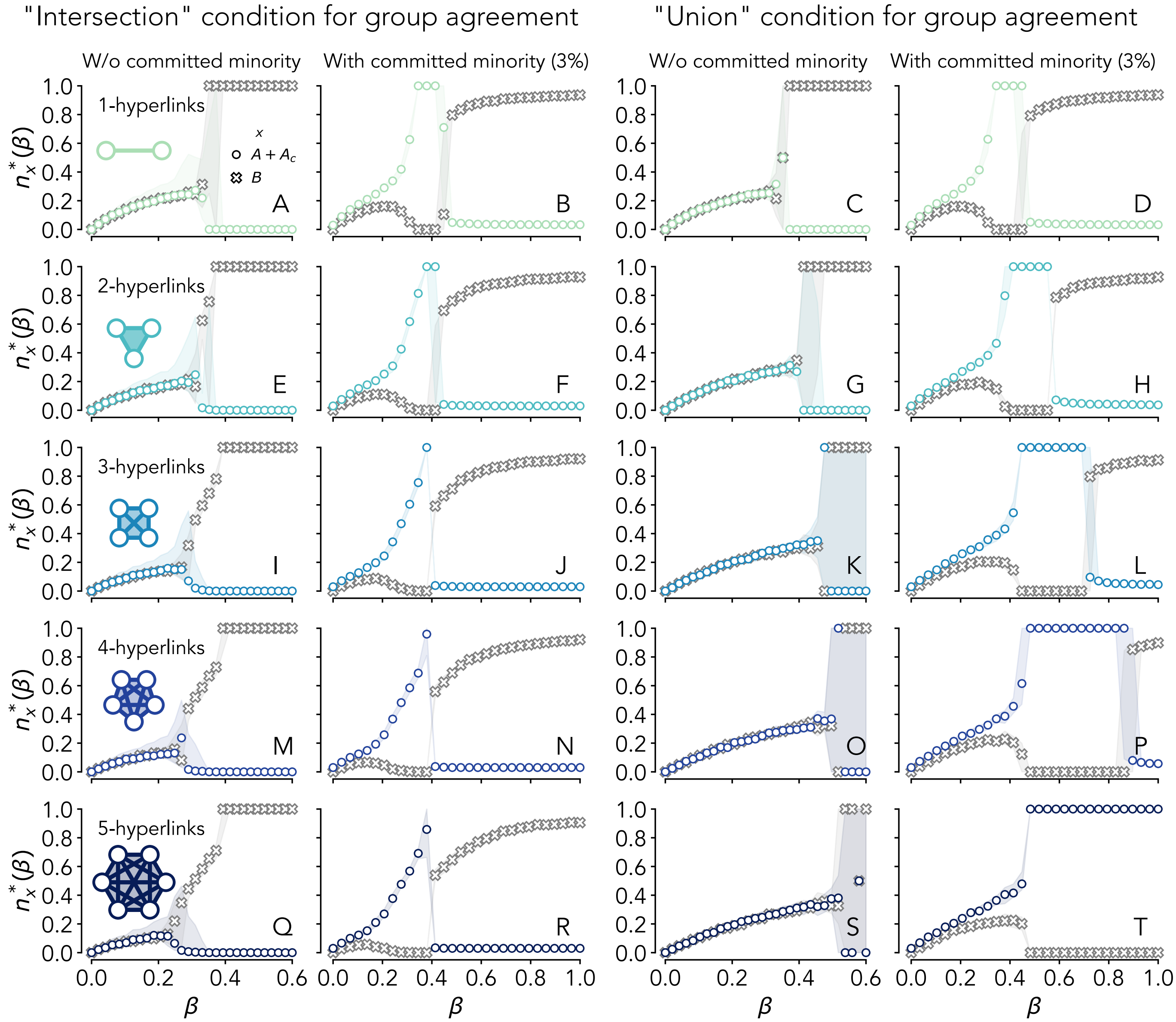
Supplementary Figure 4. Simulations of the stationary state of the dynamics on empirical higher-order structures for a NG with the "union" condition for group agreement and associated group size frequencies. The considered social structures are the same of Fig. 2 of the main text and correspond to empirical data sets collected in six different context: workplace (InVS15), a primary school (LyonSchool), a conference (SFHH), a high school (Thiers13), email communications (Email-EU) and a political congress (Congress-bills). The phase diagrams in panels (A,E,I,M,Q,U), panels (B,F,J,N,R,V) and panels (C, G, K, O, S, W) correspond to simulation with a different fraction of committed minority, respectively set to $p = 0$, $p = 0.01$ and $p = 0.03$. The fraction of nodes supporting name x in the stationary state $n_x^*(\beta)$, obtained by means of numerical simulations on each data set (row), is plotted as a function of the communication efficiency β. The results averaged over different runs of stochastic simulations are reported as points (circles and crosses, respectively associated to name A and B) and shaded areas, representing median values and values contained within the 25th and 75th percentiles. Panels (D,H,L,P,T,X) show the group size frequencies associated to each data set, where a group of size k represents a higher-order interaction between the k nodes involved.



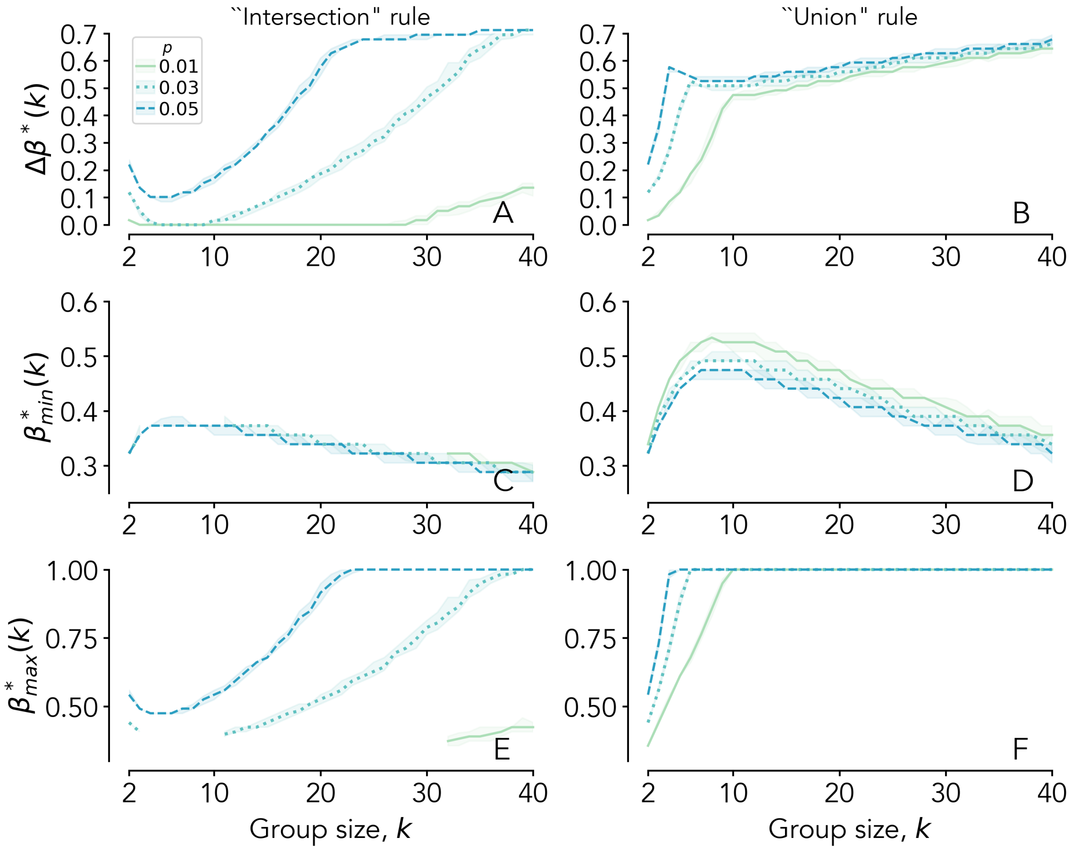
Supplementary Figure 5. Group-size-based mean-field simulations of a NG on empirical higher-order structures constructed from data sets collected in six different context: workplace (InVS15), a primary school (LyonSchool), a conference (SFHH), a high school (Thiers13), email communications (Email-EU) and a political congress (Congress-bills). In all scenarios, simulations are initiated with a random selection of 3% of committed minorities among the population. Left (A,C,E,G,I,K) and right (B,D,F,H,J,L) panels correspond to simulations respectively performed with the unanimity (intersection) and the union condition rule for group agreement. Simulations are performed in the group-size-based mean-field approximation, that means that at each time step a size k is sampled from the group size distribution of the correspondent data set, and then k nodes are selected at random among the entire population. The fraction of nodes supporting name x in the stationary state $n_x^*(\beta)$, obtained by means of 50 numerical simulations on each data set (row), is plotted as a function of the communication efficiency β. Points (circles and crosses) refer to the median values (respectively associated to name $A + A_c$ and B), with shaded areas representing the values contained within the 25th and 75th percentiles. As these simulations discard the correlations present in the original data, the results are only influenced by the group size distribution. In particular, the broad range on which the minority wins for the Congress-bills data set with the union rule can be attributed to the much broader distribution of group sizes with respect to the other data sets.



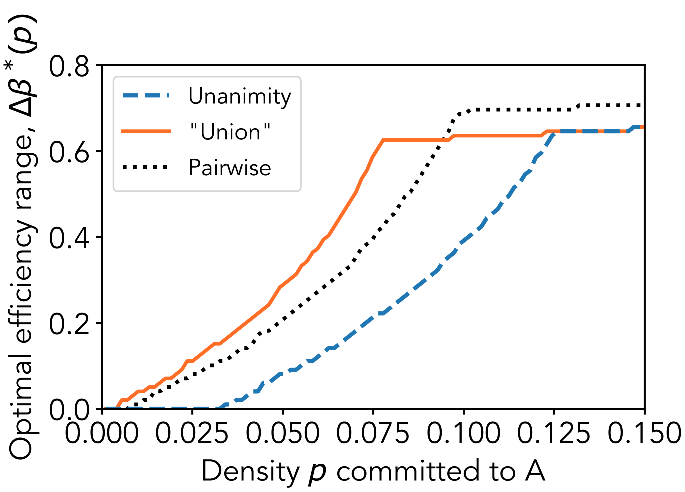
Supplementary Figure 6. Generalised degree distributions of the considered empirical hypergraphs, each constructed from data sets collected in six different context: workplace (InVS15), a primary school (LyonSchool), a conference (SFHH), a high school (Thiers13), email communications (Email-EU) and a political congress (Congress-bills). The generalised degree π_k of a node denotes the number of groups of size k it is part of.



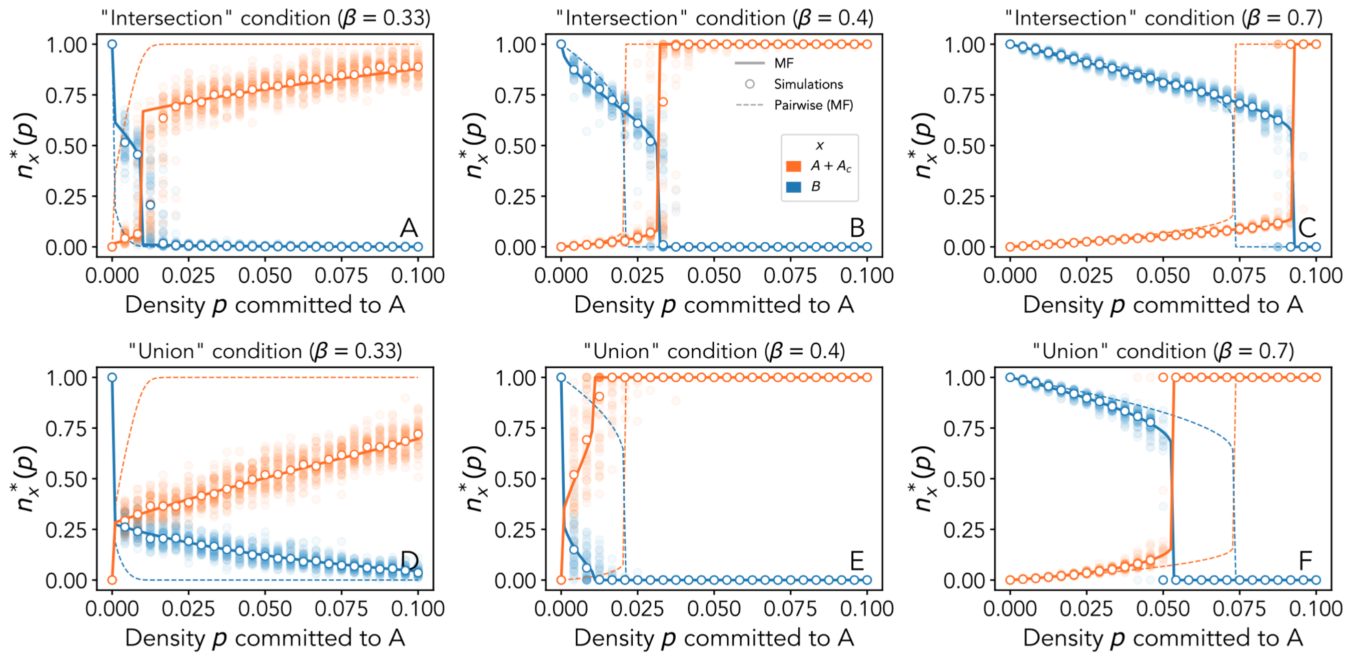
Supplementary Figure 7. Higher-order (group) effects in NG with variable β. The fraction of nodes supporting name x in the stationary state, $n_x^*(\beta)$, obtained by means of numerical simulations, is shown as a function of β. Lines (continuous and dashed, respectively associated to norms $A + A_c$ and B) and shaded areas refer to the median values and values contained within the 25th and 75th percentiles of the 50 numerical simulations. Panels in the first and third column correspond to simulation without committed minorities ($p = 0$), while panels in the second and fourth column correspond to simulation with a non-zero fraction of committed minorities supporting name A ($p = 0.03$). Note that the scales on the x-axes vary for different values of p. Two different conditions for group agreement are considered: unanimity ("intersection") rule (first two columns) and "union" rule (last two columns).



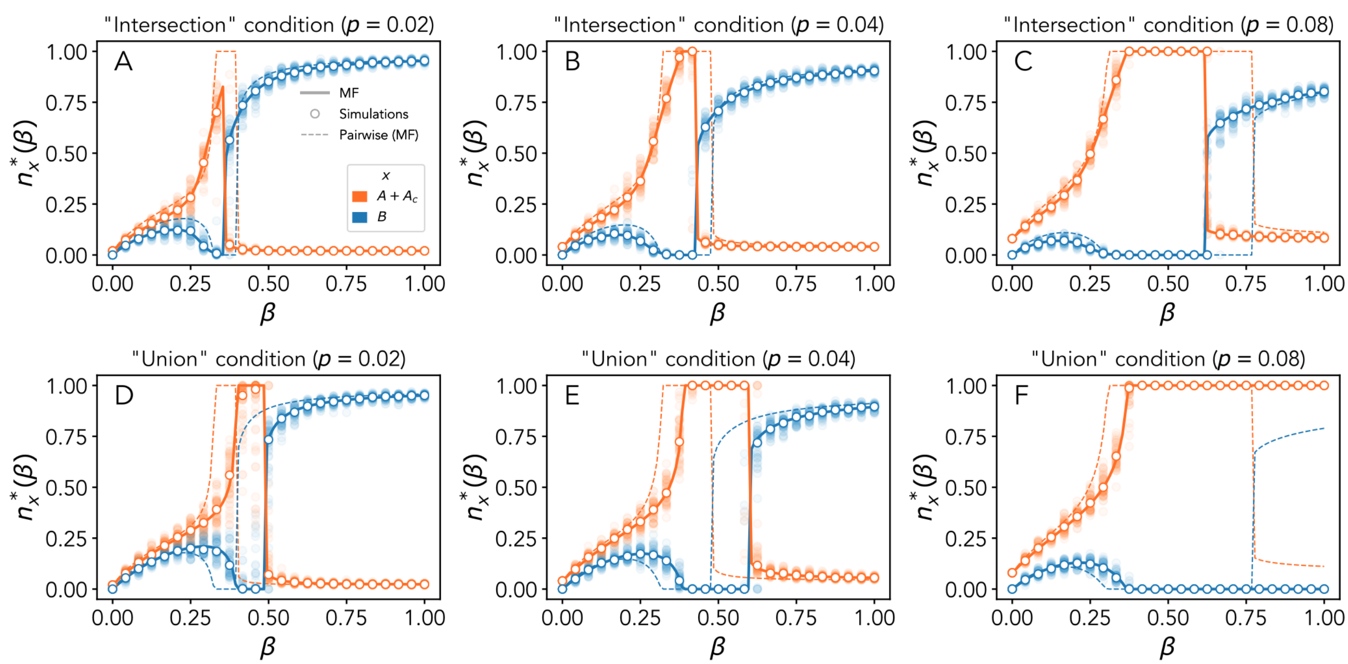
Supplementary Figure 8. Higher-order (group) effects in NG for different values of group size k and group agreement rule. Left (A, C, E) and right (B, D, F) panels correspond to simulations respectively performed with the unanimity (intersection) and the union condition for group agreement. We consider $(k-1)-$uniform hypergraphs, i.e. regular structures in which each interaction involves exactly k agents. (A, B) The range $\Delta\beta^*$ of β values for which $n_A^* = 1$ (i.e., the committed minority manages to convert the whole population), is plotted as a function of the group size k and for different values of p (see legend). The associated minimal value β^*_{min} and maximal value β^*_{max} are respectively shown in panels (C, D) and (E, F).



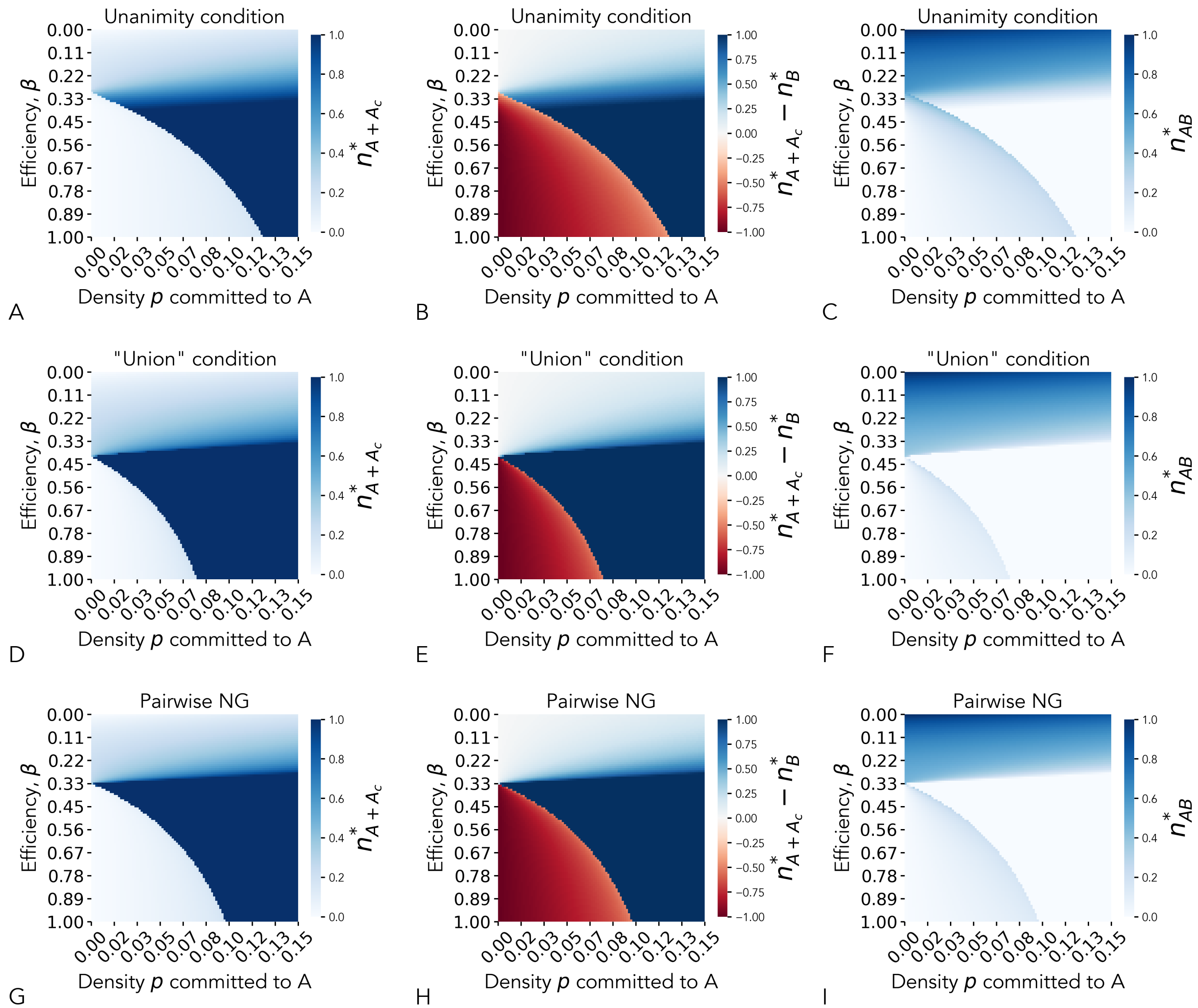
Supplementary Figure 9. Mean field approximation on 2-uniform hypergraphs. The range $\Delta\beta^*$ of β values for which $n_A^* = 1$ (i.e., the committed minority manages to convert the whole population), is plotted as a function of the the fraction p of agents committed to A for different conditions for group agreement, namely unanimity (A) and "union" (B). For comparison, (C) corresponds to the pairwise version of the NG model where no group agreement condition can be defined. Values for are obtained through numerical integration of the correspondent system of equations, Eq. (S1), Eq. (S2) and Eq. (S3), after setting p, $n_A(0) = 0$, $n_B(0) = 1 - p$, and $n_{AB}(0) = 0$.



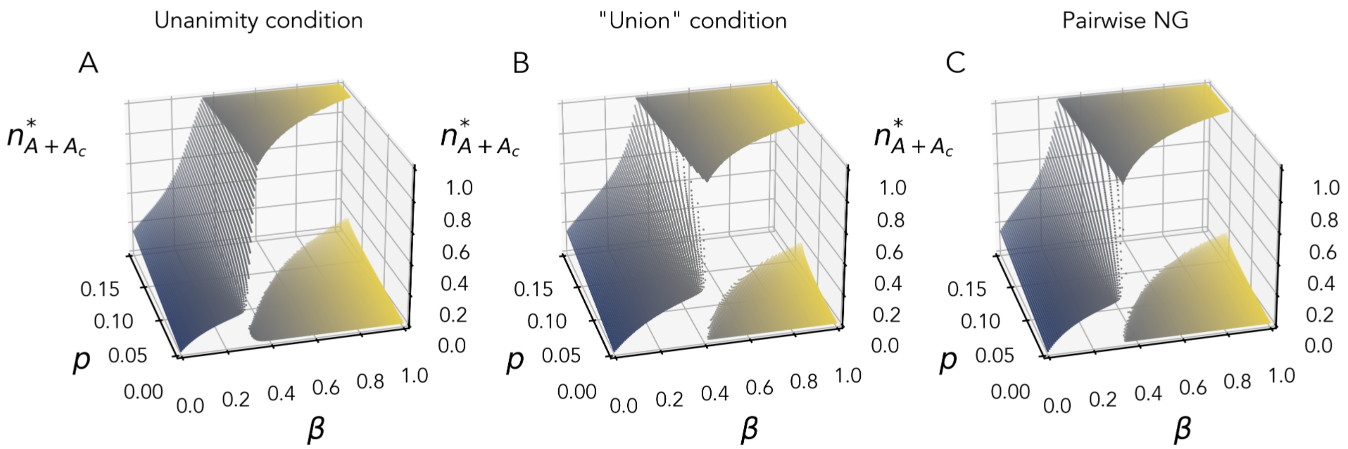
Supplementary Figure 10. Testing the mean field approach against simulations. Stationary fraction $n_x^*(\beta)$ of agents supporting name x as a function of the fraction $p > 0$ of agents committed to A for a NG with "intersection" (A-C) and "union" (D-F) conditions for group agreement. Different columns correspond to different values of β. Continuous lines are obtained through numerical integration of the mean field equations Eq. (S1), Eq. (S2) after setting $p > 0$, $n_A(0) = 0$, $n_B(0) = 1 - p$, and $n_{AB}(0) = 0$. Points are the results of Monte Carlo simulations. The pairwise case is plotted with dashed lines for comparison.



Supplementary Figure 11. Testing the mean field approach against simulations. Stationary fraction $n_x^*(\beta)$ of agents supporting name x as a function of the efficiency of social influence β for a NG with "intersection" (A-C) and "union" (D-F) conditions for group agreement. Different columns correspond to different values of the fraction $p > 0$ of agents committed to A . Continuous lines are obtained through numerical integration of the mean field equations Eq. (S1), Eq. (S2) after setting $p > 0$, $n_A(0) = 0$, $n_B(0) = 1 - p$, and $n_{AB}(0) = 0$. Points are the results of Monte Carlo simulations. The pairwise case is plotted with dashed lines for comparison.



Supplementary Figure 12. Two-dimensional phase diagrams of the NG in the mean field approximation. Heatmaps of the stationary fraction of agents supporting a certain name as a function of the efficiency of social infulence β and the fraction p of agents committed to name A for unanimity (A-C) and "union" (D-F) condition for group agreement on 2-uniform hypergraphs. For comparison, the pairwise version of the NG model is reported in (G-I), where no group agreement condition can be defined. Values for each rule are obtained through numerical integration of the correspondent system of equations, Eq. (S1), Eq. (S2) and Eq. (S3), after setting p, $n_A(0) = 0$, $n_B(0) = 1 - p$, and $n_{AB}(0) = 0$. Panels (A, D, G) show the stationary fraction of agents supporting name A, while its difference with respect to the ones supporting B is displayed in (B,E,H). Finally, panels (C,F,I) show the stationary fraction of agents holding both names A, B.



Supplementary Figure 13. Three-dimensional phase diagram of the NG in the mean-field approximation. The stationary fraction of agents supporting name A, which includes committed agents, is plotted (z-axis) as a function of the efficiency of social influence β and the fraction p of agents committed to A. Different panels correspond to different conditions for group agreement, namely unanimity (A) and "union" (B), on 2-uniform hypergraphs. The pairwise version of the NG model is reported in (C) for comparison, where no group agreement condition can be defined. Values for each rule are obtained through numerical integration of the correspondent system of equations, namely Eq. (S1), Eq. (S2) and Eq. (S3), after setting p, $n_A(0) = 0$, $n_B(0) = 1 - p$, and $n_{AB}(0) = 0$.