Supplementary Information: Oscillations of Highly Magnetized Non-rotating Neutron Stars

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Supplementary Note 1: Comparison to the Cowling approximation

It is known that the Cowling approximation (i.e. fixed spacetime) overestimates the oscillation mode frequencies of NSs up to a factor of ~ 2. Simulations with dynamical spacetime are essential for more accurate results. In this supplementary information, we show that the Cowling approximation is indeed inadequate under the circumstances described in this work. The corresponding visualizations of eigenfunctions of the excited oscillation modes are shown in Supplementary Figure 1. Supplementary Table 1 summarizes the measured eigenfrequencies of the six modes in the 12 different NSs with the undetermined eigenfrequencies due to unsatisfactory data quality denoted by 'N/A'. We plot in Supplementary Figure 2 the eigenfrequencies f_{eig} of the six modes with and without Cowling approximation as functions of the magnetic to binding energy ratio $\mathscr{H}/\mathscr{W}$ of the NS model. We further summarize the relative differences in f_{eig} with the Cowling approximation with respect to those without the approximation in Supplementary Table 2 to better illustrate the discrepancies.

We found a similar qualitative relation between f_{eig} of the six modes and $\mathscr{H}/\mathscr{W}$ for both simulations with and without the Cowling approximation. However, a clear quantitative difference in the values of f_{eig} is observed. The largest discrepancy is found in the F-mode, where the eigenfrequencies in all NS models are overestimated by a factor of ~ 2 under the Cowling approximation. The discrepancies are also significant for H_1-mode with relative differences of $\gtrsim 20$ per cent in most cases and even up to ~ 65 per cent in the NS model with the strongest magnetic field strength. The discrepancies are less severe for higher-order modes (i.e. $\ell = 2$ and $\ell = 4$ modes). The relative differences are less than 20 per cent for ${}^2f$ and ${}^2p_1$ modes, while they are less than 10 per cent for ${}^4f$ and ${}^4p_1$ modes.

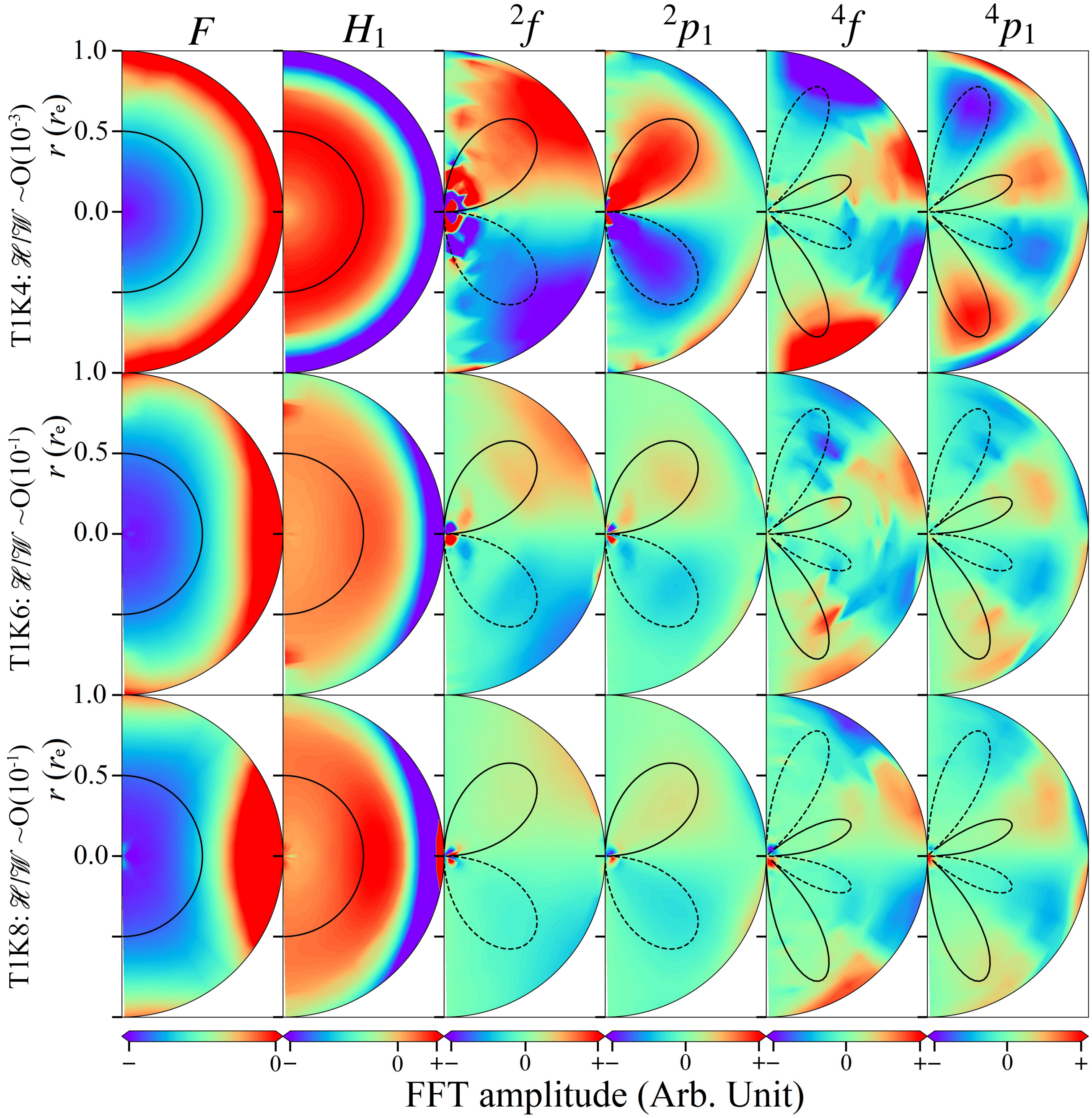
Hence, imposing the Cowling approximation in our simulations also results in overestimating the correct eigenfrequency up to a factor of ~ 2. Dynamical spacetime is still necessary for magnetized NS simulations to obtain more realistic oscillation mode frequencies, especially for the lower-order modes.

Model	F	H_1	${}^2f$	${}^2p_1$	${}^4f$	${}^4p_1$
REF	2.70	4.61	N/A	4.22	2.63	5.26
T1K1	2.70	4.61	N/A	4.22	2.63	5.28
T1K2	2.70	4.61	N/A	4.22	2.64	5.26
T1K3	2.70	4.62	N/A	4.23	2.61	5.26
T1K4	2.70	4.51	1.90	4.10	2.60	5.10
T1K5	2.60	4.42	1.91	4.02	2.51	5.01
T1K6	2.10	3.63	1.60	3.30	2.10	4.14
T1K7	1.80	3.20	1.40	2.83	1.91	3.60
T1K8	1.50	2.70	1.20	2.40	1.60	3.01
T1K9	1.20	2.20	0.99	1.97	1.31	2.40
T1K10	N/A	1.70	0.78	1.51	0.99	1.89
T1K11	0.68	1.27	0.59	N/A	N/A	1.40

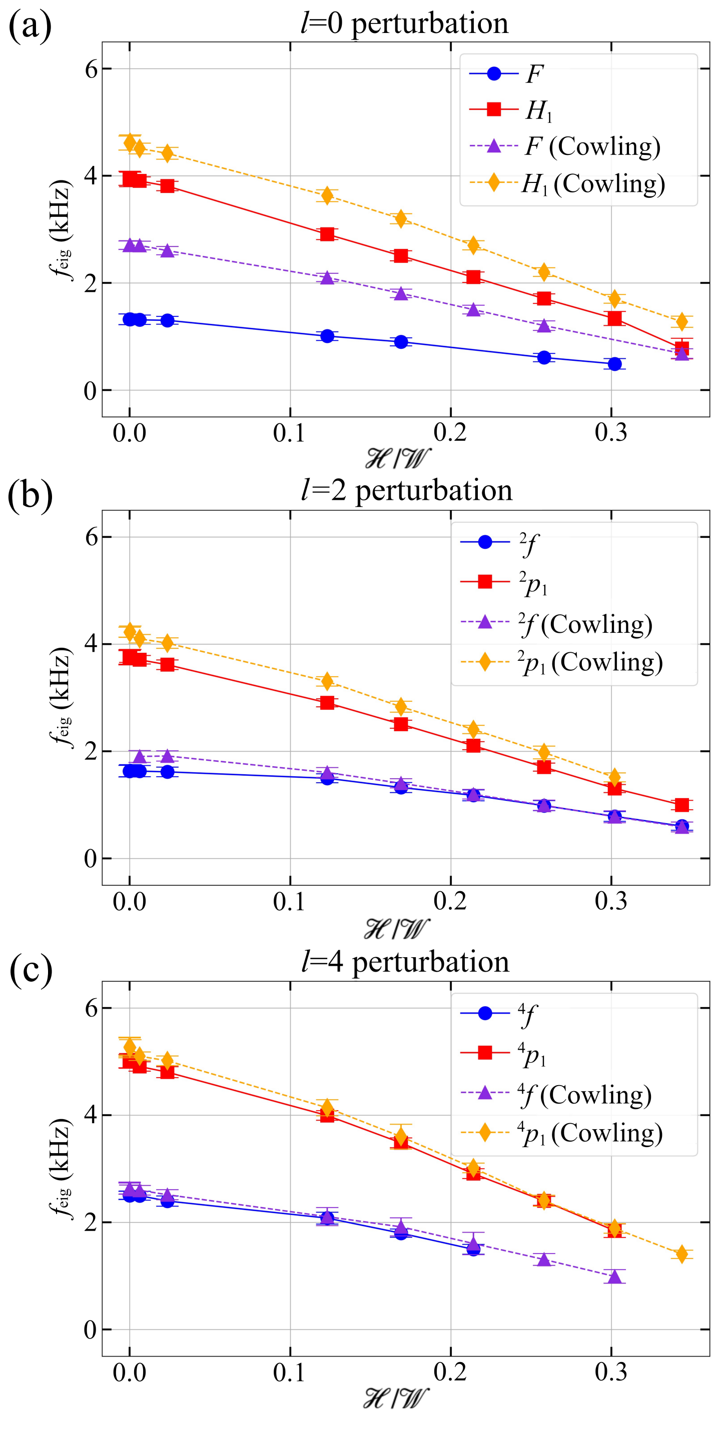
Supplementary Table 1. Measured eigenfrequencies of the six dominant oscillation modes **with the Cowling approximation** in the 12 NS models, including the fundamental quasi-radial ($\ell = 0$) mode F and its first overtone H_1, the fundamental quadrupole ($\ell = 2$) mode ${}^2f$ and its first overtone ${}^2p_1$, as well as the fundamental hexadecapole ($\ell = 4$) mode ${}^4f$ and its first overtone ${}^4p_1$, all predominantly excited under the perturbation with the corresponding l index. All eigenfrequencies are in kHz and rounded off to two decimal places. The undetermined eigenfrequencies in specific models are denoted by 'N/A'. The missing eigenfrequencies are due to unsatisfactory data quality in `Gmunu` simulations under the perturbations.

Model	F	H_1	${}^2f$	${}^2p_1$	${}^4f$	${}^4p_1$
REF	+104.55	+16.71	N/A	+12.53	+5.20	+5.20
T1K1	+104.55	+16.71	N/A	+12.53	+5.20	+5.60
T1K2	+104.55	+16.71	N/A	+12.53	+5.60	+4.99
T1K3	+104.55	+17.56	N/A	+11.90	+3.98	+4.99
T1K4	+106.11	+15.64	+16.56	+10.81	+4.00	+3.87
T1K5	+100.00	+16.01	+18.63	+11.36	+4.58	+4.38
T1K6	+107.92	+24.74	+7.38	+13.79	+0.96	+3.76
T1K7	+100.00	+28.00	+6.06	+13.20	+6.11	+3.45
T1K8	N/A	+28.57	+1.69	+14.29	+7.38	+3.44
T1K9	+100.00	+29.41	+1.02	+15.88	N/A	0.00
T1K10	N/A	+27.82	0.00	+16.15	N/A	+2.72
T1K11	N/A	+64.94	-1.67	N/A	N/A	N/A

Supplementary Table 2. Relative difference in per cent in the frequencies of the six dominant oscillation modes **with the Cowling approximation** with respect to those without the approximation. The models containing undetermined eigenfrequencies are denoted by 'N/A'. All relative differences are rounded off to two decimal places.



Supplementary Figure 1. Visualizations of eigenfunctions of the six dominant oscillation modes **with the Cowling approximation** using the data of 3 equilibrium models (T1K4, T1K6 and T1K8). The fundamental quasi-radial ($\ell = 0$) mode F and its first overtone H_1 are predominantly excited under $\ell = 0$ perturbation; the fundamental quadrupole ($\ell = 2$) mode ${}^2f$ and its first overtone ${}^2p_1$ are predominantly excited under $\ell = 2$ perturbation; the fundamental hexadecapole ($\ell = 4$) mode ${}^4f$ and its first overtone ${}^4p_1$ are predominantly excited under $\ell = 4$ perturbation. Each polar color plot shows the spatial map of FFT amplitude at the eigenfrequency of the mode, where the radial axis is normalized to the equatorial radius r_e of each model. On top of each color plot, there is a polar line plot visualizing the θ-part of the spherical harmonic in the corresponding perturbation function, where the distance from the origin to the line measures the magnitude of the spherical harmonic in that θ-direction, while the solid and dotted portions represent the positive and negative parts of the spherical harmonic respectively. Each line plot is scaled arbitrarily for clearer illustration. It can be seen that the eigenfunctions of the higher-order quadrupole ($\ell = 2$) and hexadecapole ($\ell = 4$) modes have more nodes in the θ-direction compared to the quasi-radial ($\ell = 0$) modes, while the eigenfunction of each first overtone has more nodes in the r-direction compared to its fundamental mode. Furthermore, each eigenfunction qualitatively agrees with the spherical harmonic in the corresponding perturbation function.



Supplementary Figure 2. Comparison of eigenfrequencies $f_{\rm eig}$ of the excited oscillation modes **with and without the Cowling approximation** against the magnetic to binding energy ratio $\mathscr{H}/\mathscr{W}$ of the NS model, if $\ell=0$ (a), $\ell=2$ (b), and $\ell=4$ (c) perturbations are applied respectively.