Supplementary Information for

Fermi-edge superfluorescence from a quantum-degenerate electron-hole gas

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Theoretical Details

We used the semiconductor Bloch equations (SBEs) to study SF from a high-density electron-hole (e-h) plasma in the presence of many-body Coulomb interactions. The usual form of the SBEs[1] is for a bulk semiconductor or a 2D electron gas, when the states can be labeled by a 3D or 2D wave vector $\vec{k}$. Here we rederive SBEs following the same basic approximations but in a more general form, which accommodates the effects of a finite well width and the quantization of motion in a strong magnetic field.

We begin with a general Hamiltonian in the two-band approximation and e-h representation,

$$\begin{aligned}\mathcal{H} &= \sum_{\alpha}\left[\left(E_g^0 + E_\alpha^e\right)a_\alpha^\dagger a_\alpha + E_\alpha^h b_{\bar{\alpha}}^\dagger b_{\bar{\alpha}}\right] \\ &+ \frac{1}{2}\sum_{\alpha\beta\gamma\delta}\left(V_{\alpha\beta\gamma\delta}^{ee}a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma + V_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}^{hh}b_{\bar{\alpha}}^\dagger b_{\bar{\beta}}^\dagger b_{\bar{\delta}} b_{\bar{\gamma}} + 2V_{\alpha\bar{\beta}\gamma\bar{\delta}}^{eh}a_\alpha^\dagger b_{\bar{\beta}}^\dagger b_{\bar{\delta}} a_\gamma\right) \\ &- \mathcal{E}(t)\sum_{\alpha}\left(\mu_\alpha a_\alpha^\dagger b_{\bar{\alpha}}^\dagger + \mu_\alpha^* b_{\bar{\alpha}} a_\alpha\right)\ ,\end{aligned} \tag{1}$$

where E_g^0 is the unperturbed bandgap, $a_\alpha^\dagger$ and $b_{\bar{\alpha}}^\dagger$ are the creation operators for the electron state α and hole state $\bar{\alpha}$, respectively, $\mathcal{E}(t)$ is the optical field, μ_α is the dipole matrix element, and $V_{\alpha\beta\gamma\delta}$ are Coulomb matrix elements, for example, $V_{\alpha\beta\gamma\delta}^{ee} = \int d\vec{r}_1 \int d\vec{r}_2 \Psi_\alpha^{e*}(\vec{r}_1)\Psi_\beta^{e*}(\vec{r}_2)\frac{e^2}{\epsilon|\vec{r}_1-\vec{r}_2|}\Psi_\gamma^e(\vec{r}_1)\Psi_\delta^e(\vec{r}_2)$. Here we denote the hole state which can be recombined with a given electron state α optically by $\bar{\alpha}$, and assume that there is a one-to-one correspondence between them. For the interband Coulomb interaction, $V_{\alpha\bar{\beta}\gamma\bar{\delta}}^{eh}a_\alpha^\dagger b_{\bar{\beta}}^\dagger b_{\bar{\delta}} a_\gamma$ is the only non-zero matrix element due to the orthogonality between the Bloch functions of the conduction and valence bands[2]. The electron and hole wave functions can be written as $\Psi_\alpha^e(\vec{r}) = \psi_\alpha^e(\vec{r})u_{c0}(\vec{r})$ and $\Psi_{\bar{\alpha}}^h(\vec{r}) = \psi_{\bar{\alpha}}^h(\vec{r})u_{v0}^*(\vec{r})$, respectively. In the problems we study, the conduction band and valence band states connected by an optical transition always have the same envelope wave function, so we take $\psi^h_{\bar{\alpha}}(\vec{r}) = \psi^{e*}_\alpha(\vec{r})$. Then the Coulomb matrix elements are related with each other through $V^{hh}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = V^{ee}_{\gamma\delta\alpha\beta}$ and $V^{eh}_{\alpha\bar{\beta}\gamma\bar{\delta}} = -V^{ee}_{\alpha\delta\gamma\beta}$, and we can drop the superscript by defining $V_{\alpha\beta\gamma\delta} \equiv V^{ee}_{\alpha\beta\gamma\delta}$.

Using the above Hamiltonian, we can obtain the equations of motion for the distribution functions $n^e_\alpha = \langle a^\dagger_\alpha a_\alpha \rangle$ and $n^h_\alpha = \langle b^\dagger_{\bar{\alpha}} b_{\bar{\alpha}} \rangle$, and the polarization $P_\alpha = \langle b_{\bar{\alpha}} a_\alpha \rangle$. Using the Hartree-Fock approximation (HFA) and the random phase approximation (RPA), we arrive at the SBEs:

$$i\hbar \frac{d}{dt} P_\alpha = \left(E^0_g + E^{eR}_\alpha + E^{hR}_\alpha\right) P_\alpha + \left(n^e_\alpha + n^h_\alpha - 1\right) \left[\mu_\alpha \mathcal{E}(t) + \sum_\beta V_{\alpha\beta\beta\alpha} P_\beta\right] + i\hbar \frac{d}{dt} P_\alpha \bigg|_{\text{scatt}} \quad (2)$$

$$\hbar \frac{d}{dt} n^e_\alpha = -2\,\text{Im}\left[\left(\mu_\alpha \mathcal{E}(t) + \sum_\beta V_{\alpha\beta\beta\alpha} P_\beta\right) P^*_\alpha\right] + \hbar \frac{d}{dt} n^e_\alpha \bigg|_{\text{scatt}} , \quad (3)$$

$$\hbar \frac{d}{dt} n^h_\alpha = -2\,\text{Im}\left[\left(\mu_\alpha \mathcal{E}(t) + \sum_\beta V_{\alpha\beta\beta\alpha} P_\beta\right) P^*_\alpha\right] + \hbar \frac{d}{dt} n^h_\alpha \bigg|_{\text{scatt}} , \quad (4)$$

where $E^{eR}_\alpha = \left(E^e_\alpha - \sum_\beta V_{\alpha\beta\beta\alpha} n^e_\beta\right)$ and $E^{hR}_\alpha = \left(E^h_\alpha - \sum_\beta V_{\alpha\beta\beta\alpha} n^h_\beta\right)$ are the renormalized energies, and the scattering terms account for higher-order contributions beyond the HFA and other scattering processes such as scattering with LO-phonons.

These equations, together with Maxwell's equations for the electromagnetic field, can be applied to study the full nonlinear dynamics of interaction between the e-h plasma and radiation. Here we derive the gain for given carrier distributions n^e_α and n^h_α, which was used to plot Fig. 5. Assuming a monochromatic and sinusoidal time dependence for the field $\mathcal{E}(t) = \mathcal{E}_0 e^{-i\omega t}$ and the polarization $P_\alpha(t) = P_{0\alpha} e^{-i\omega t}$, we can find P_α from Eq. (2) and define the quantity $\chi_\alpha(\omega) = P_{0\alpha}/\mathcal{E}_0$, which satisfies the equation below:

$$\chi_\alpha(\omega) = \chi^0_\alpha(\omega) \left[1 + \frac{1}{\mu_\alpha} \sum_\beta V_{\alpha\beta\beta\alpha} \chi_\beta(\omega)\right] , \quad (5)$$

where

$$\chi_\alpha^0(\omega) = \frac{\mu_\alpha\left(n_\alpha^e + n_\alpha^h - 1\right)}{\hbar\omega - \left(E_g^0 + E_\alpha^{eR} + E_\alpha^{hR}\right) + i\hbar\gamma_\alpha} \; . \tag{6}$$

Here we have written the dephasing term phenomenologically as $dP_\alpha/dt|_{\text{scatt}} = -\gamma_\alpha P_\alpha$. The optical susceptibility is then

$$\chi(\omega) = \frac{1}{V}\sum_\alpha \mu_\alpha^* \chi_\alpha(\omega) \; , \tag{7}$$

where V is the normalization volume. The gain spectrum is given by[1]

$$g(\omega) = \frac{4\pi\omega}{n_b c}\text{Im}[\chi(\omega)] \; , \tag{8}$$

where n_b is the background refractive index, and c is the speed of light. We use the above general results to analyze optical properties under different conditions.

In a quantum well of thickness L_{w}, the envelope functions for electrons and holes are $\psi_{n,\vec{k}}^{e,h}(\vec{r})$ = $\varphi_n(z)\exp\left(i\vec{k}\cdot\vec{\rho}\right)/\sqrt{A}$, where $\vec{\rho}$ = (x, y), $\varphi_n(z)$ is the envelope wave function in the growth direction for the n-th subband, and A is the normalization area. To calculate the Coulomb matrix element $V_{\alpha\beta\beta\alpha}$, we define $\tilde{V}_{\alpha\beta} \equiv V_{\alpha\beta\beta\alpha}$ and put $\alpha = \left\{n, \vec{k}, s\right\}$, $\beta = \left\{n', \vec{k}', s'\right\}$, where s denotes the spin quantum index. Then one gets

$$\tilde{V}_{n,\vec{k},s;n',\vec{k}',s'} = V^{2D}(q)F_{nn'n'n}(q)\delta_{ss'} \; , \tag{9}$$

where $q = |\vec{q}| = |\vec{k} - \vec{k}'|$, $V^{2D}(q) = 2\pi e^2/\epsilon A q$, ϵ is the dielectric function, and the form factor $F_{nn'n'n}(q)$ is defined as

$$F_{n1,n2,n3,n4}(q) = \int dz_1 \int dz_2 \varphi_{n1}^*(z_1)\varphi_{n2}^*(z_2)\exp\left(-q\left|z_1 - z_2\right|\right)\varphi_{n3}(z_1)\varphi_{n4}(z_2) \; . \tag{10}$$

Throughout the paper, we will assume that only the lowest subband for electrons and holes is occupied. In this case, we can define $\tilde{V}(q) = V^{2D}(q)F_{1111}(q)$. The dielectric function $\epsilon(\vec{q},\omega)$, which describes the screening of the Coulomb potential, is given by the Lindhard formula for a pure 2D case[1]; it can be generalized to the quasi-2D case as

$$\epsilon(\vec{q},\omega) = 1 + \tilde{V}(q)\left(\Pi_e(\vec{q},\omega) + \Pi_h(\vec{q},\omega)\right) \ , \tag{11}$$

where $\Pi_{e(h)}(\vec{q},\omega)$ is the polarization function of an electron or hole, which is given by

$$\Pi(\vec{q},\omega) = 2\sum_{\vec{k}} \frac{n_{\vec{k}+\vec{q}} - n_{\vec{k}}}{\omega + i0^+ - E_{\vec{k}+\vec{q}} + E_{\vec{k}}} \ . \tag{12}$$

Here, we dropped the subscripts e or h, $n_{\vec{k}}$ is the distribution function, the factor of 2 accounts for the summation over spin, and the spin index is suppressed. For simplicity, we will choose the static limit, namely, $\omega = 0$.

Given the dielectric function $\epsilon(q,0)$, the screened Coulomb matrix element is $\tilde{V}_s(q) = \tilde{V}(q)/\epsilon(q,0)$. For simplicity, we will still write it as $\tilde{V}(q)$. Applying Eq. (5) to the case above, we get the equation for $\chi_{\vec{k}}(\omega)$:

$$\chi_{\vec{k}}(\omega) = \chi^0_{\vec{k}}(\omega)\left[1 + \frac{1}{\mu_{\vec{k}}}\sum_{\vec{k}'}\tilde{V}\left(\left|\vec{k}-\vec{k}'\right|\right)\chi_{\vec{k}'}(\omega)\right] \ , \tag{13}$$

where $\chi^0_{\vec{k}}(\omega)$ becomes

$$\chi^0_{\vec{k}}(\omega) = \frac{\mu_{\vec{k}}\left(n^e_{\vec{k}} + n^h_{\vec{k}} - 1\right)}{\hbar\omega - \left(E^0_g + E^{eR}_{\vec{k}} + E^{hR}_{\vec{k}}\right) + i\hbar\gamma_{\vec{k}}} \ . \tag{14}$$

To solve Eq. (13), we notice that $\chi^0_{\vec{k}}(\omega)$ does not depend on the direction of $\vec{k}$, so $\chi_{\vec{k}}(\omega)$ will not depend on it, either. Then, after converting the summation in Eq. (13) into the integral, the integration over the azimuthal angle is acting on $\tilde{V}\left(\left|\vec{k}-\vec{k}'\right|\right)$ only. If we define

$$\tilde{V}\left(k,k'\right)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\tilde{V}\left(\sqrt{k^2+k'^2-2kk'\cos\phi}\right)\ , \tag{15}$$

then Eq. (13) can be written as

$$\chi_k(\omega)=\chi_k^0(\omega)\left[1+\frac{A}{2\pi\mu_k}\int_0^\infty k'dk'\tilde{V}\left(k,k'\right)\chi_k'(\omega)\right]\ . \tag{16}$$

After discretizing the integral, we have a system of linear equations for $\chi_k(\omega)$, which can be solved by using LAPACK[3]. The band structure for our sample consisting of undoped 8-nm $In_{0.2}Ga_{0.8}As$ wells and 15-nm GaAs barriers on a GaAs substrate is calculated using the parameters given by Vurgaftman *et al.*[4]. The strain effect is included using the results of Sugawara *et al.*[5]. Examples of calculated gain spectra are shown in Figs. 5a and 5b.

For a quantum well structure in a strong perpendicular magnetic field, the electronic states are fully quantized. Considering only the lowest subband in the quantum well, the equation for the susceptibility is written as

$$\chi_{\nu,s}=\chi_{\nu,s}^0\left[1+\frac{1}{\mu_{\nu,s}}\sum_{\nu'}V_{\nu,\nu'}\chi_{\nu',s}\right]\ , \tag{17}$$

where ν is the Landau level index, s is the spin index, and $V_{\nu,\nu'}$ is the Coulomb matrix element given by

$$V_{\nu,\nu'}=\frac{e^2}{2\pi\epsilon}\int_0^{2\pi}d\theta\int_0^\infty dq\left|\int dx e^{iqx\cos\theta}\phi_\nu(x)\phi_{\nu'}^*(x+qa_H^2\sin\theta)\right|^2\ , \tag{18}$$

where $\phi_\nu(x)$ is the x-dependent part of the wavefunction of the ν-th Landau level and $a_H^2=\hbar c/eB$. The renormalized electronic energies in the expression for $\chi_{\nu,s}^0$ are

$$E_{\nu,s}^{eR}=E_{\nu,s}^e-\sum_{\nu'}V_{\nu,\nu'}n_{\nu'}^e\ , \tag{19}$$

and a similar equation holds for holes. The gain is calculated as

$$g(\omega) = \frac{4\pi\omega}{n_b c}\frac{1}{\pi a_H^2}\text{Im}\left[\sum_{\nu}\mu^*_{\nu,s}\chi_{\nu,s}\right] . \tag{20}$$

An example of the calculated gain for $B = 17$ T is shown in Figs. 5c and 5d.

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