



Supplementary materials for

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Proof S1 Proof of Theorem 1

The following proof is composed of two steps. First, we shall certify the boundedness of the estimate of the pseudo-partial-derivative (PPD) matrix $\hat{\mathcal{U}}_i(h)$.

Step 1: boundedness of $\hat{\mathcal{U}}_i(h)$

Define

$$\hat{\mathcal{U}}_i(h) \triangleq \begin{bmatrix} \hat{\omega}_{i,11}(h) & \hat{\omega}_{i,12}(h) & \cdots & \hat{\omega}_{i,1n}(h) \\ \hat{\omega}_{i,21}(h) & \hat{\omega}_{i,22}(h) & \cdots & \hat{\omega}_{i,2n}(h) \\ \vdots & \vdots & & \vdots \\ \hat{\omega}_{i,n1}(h) & \hat{\omega}_{i,n2}(h) & \cdots & \hat{\omega}_{i,nn}(h) \end{bmatrix}, \quad (\text{S1})$$

which can also be rewritten as

$$\hat{\mathcal{U}}_i(h) \triangleq \begin{bmatrix} \hat{\mathcal{U}}_{i,1}(h) \\ \hat{\mathcal{U}}_{i,2}(h) \\ \vdots \\ \hat{\mathcal{U}}_{i,n}(h) \end{bmatrix}, \quad (\text{S2})$$

where $\hat{\mathcal{U}}_{i,j}(h) \triangleq [\hat{\omega}_{i,j1}(h) \ \hat{\omega}_{i,j2}(h) \ \cdots \ \hat{\omega}_{i,jn}(h)]$. Then, at the triggering instant $h = h_s^i$, one has

$$\hat{\mathcal{U}}_i(h_s^i - 1) = \hat{\mathcal{U}}_i(h - 1), u_i(h_s^i - 1) = u_i(h - 1). \quad (\text{S3})$$

The updating algorithm (16) is reformulated as

$$\hat{\mathcal{U}}_{i,j}(h) = \hat{\mathcal{U}}_{i,j}(h - 1) + \frac{\gamma(\Delta y_{i,j}(h) - \hat{\mathcal{U}}_{i,j}(h - 1)\Delta u_i(h - 1))\Delta u_i^T(h - 1)}{\nu + \|\Delta u_i(h - 1)\|^2}, \quad (\text{S4})$$

where $\Delta y_{i,j}(h) = \mathcal{U}_{i,j}(h) - \mathcal{U}_{i,j}(h - 1)$. Defining the estimation error $\tilde{\mathcal{U}}_{i,j} = \mathcal{U}_{i,j}(h) - \hat{\mathcal{U}}_{i,j}(h)$ and combining with Lemma 1 and algorithm (16), we arrive at

$$\tilde{\mathcal{U}}_{i,j}(h) = \tilde{\mathcal{U}}_{i,j}(h - 1) + \mathcal{U}_{i,j}(h) - \mathcal{U}_{i,j}(h - 1) - \frac{\gamma\tilde{\mathcal{U}}_{i,j}(h - 1)\Delta u_i(h - 1)\Delta u_i^T(h - 1)}{\nu + \|\Delta u_i(h - 1)\|^2}. \quad (\text{S5})$$

It is inferred from the fact that $\|\mathcal{U}_i(h)\| \leq m$ in Lemma 1, and we can obtain $\|\mathcal{U}_i(h) - \mathcal{U}_i(h - 1)\| \leq 2m$. Applying the basic inequality yields

$$\begin{aligned} \|\tilde{\mathcal{U}}_{i,j}(h)\| &\leq \|\mathcal{U}_{i,j}(h) - \mathcal{U}_{i,j}(h - 1)\| + \left\| \tilde{\mathcal{U}}_{i,j}(h - 1) - \frac{\gamma\tilde{\mathcal{U}}_{i,j}(h - 1)\Delta u_i(h - 1)\Delta u_i^T(h - 1)}{\nu + \|\Delta u_i(h - 1)\|^2} \right\| \\ &\leq \left\| \tilde{\mathcal{U}}_{i,j}(h - 1) - \frac{\gamma\tilde{\mathcal{U}}_{i,j}(h - 1)\Delta u_i(h - 1)\Delta u_i^T(h - 1)}{\nu + \|\Delta u_i(h - 1)\|^2} \right\| + 2m, \end{aligned} \quad (\text{S6})$$

which is further organized as follows:

$$\begin{aligned} & \left\| \tilde{\mathcal{U}}_{i,j}(h-1) - \frac{\gamma \tilde{\mathcal{U}}_{i,j}(h-1) \Delta u_i(h-1) \Delta u_i^T(h-1)}{\nu + \|\Delta u_i(h-1)\|^2} \right\|^2 \\ &= \left(-2 + \frac{\gamma \|\Delta u_i(h-1)\|}{\nu + \|\Delta u_i(h-1)\|^2} \right) \frac{\gamma \|\tilde{\mathcal{U}}_{i,j}(h-1) \Delta u_i(h-1)\|}{\nu + \|\Delta u_i(h-1)\|^2} + \|\tilde{\mathcal{U}}_{i,j}(h-1)\|^2. \end{aligned} \quad (\text{S7})$$

In addition, it is not tricky to confirm that there exist $\gamma \in (0, 1)$ and $\nu > 0$ such that

$$-2 + \frac{\gamma \|\Delta u_i(h-1)\|}{\nu + \|\Delta u_i(h-1)\|^2} < 0, \quad (\text{S8})$$

which is further concluded that there exists a scalar $\rho \in (0, 1)$ satisfying the following condition:

$$\left\| \tilde{\mathcal{U}}_{i,j}(h-1) - \frac{\gamma \tilde{\mathcal{U}}_{i,j}(h-1) \Delta u_i(h-1) \Delta u_i^T(h-1)}{\nu + \|\Delta u_i(h-1)\|^2} \right\| \leq \rho \|\tilde{\mathcal{U}}_{i,j}(h-1)\|. \quad (\text{S9})$$

Substituting inequality (S9) into inequality (S6) yields

$$\|\tilde{\mathcal{U}}_{i,j}(h)\| \leq \rho \|\tilde{\mathcal{U}}_{i,j}(h-1)\| + 2m \leq \dots \leq \rho^{h-1} \|\tilde{\mathcal{U}}_{i,j}(1)\| + \frac{2m(1-\rho^{h-1})}{1-\rho}, \quad (\text{S10})$$

which implies that $\tilde{\mathcal{U}}_{i,j}(h)$ is bounded. Since $\|\mathcal{U}_i(h)\| \leq m$, it is readily seen from inequality (S10) that both $\tilde{\mathcal{U}}_i(h)$ and $\hat{\mathcal{U}}_i(h)$ are bounded. In addition, it is obvious that $\hat{\mathcal{U}}_i(h)$ remains unchanged over the interval $h \in (h_s^i, h_{s+1}^i)$. Thus, it can be calculated that $\hat{\mathcal{U}}_i(h)$ is bounded at all instants. Because both $\mathcal{U}_i(h)$ and $\hat{\mathcal{U}}_i(h)$ are bounded, one has that $Q(h)$, $A(h)$, and $M(h)$ are bounded matrices. Thus, there exist Nn -dimensional matrices \bar{Q} , \bar{A} , and \bar{M} satisfying $Q^T(h)Q(h) \leq \bar{Q}$, $A^T(h)A(h) \leq \bar{A}$, and $M^T(h)M(h) \leq \bar{M}$.

Step 2: consensus analysis

Construct the following Lyapunov function:

$$V_1(h) = \tilde{y}^T(h) \tilde{y}(h). \quad (\text{S11})$$

Along the trajectory of system (25), the difference of $V_1(h)$ can be evaluated as follows:

$$\begin{aligned} & \Delta V_1(h+1) \\ &= V_1(h+1) - V_1(h) \\ &= [M(h) \tilde{y}(h) + Q(h) \tilde{d}(h-1) + \eta(h+1) + A(h) \beta(h) e(h)]^T \\ & \quad \cdot [M(h) \tilde{y}(h) + Q(h) \tilde{d}(h-1) + \eta(h+1) + A(h) \beta(h) e(h)] - \tilde{y}^T(h) \tilde{y}(h) \\ &= \tilde{y}^T(h) (M^T(h)M(h) - I + \tau) \tilde{y}(h) + \tilde{d}^T(h-1) Q^T(h) Q(h) \tilde{d}(h-1) + \eta^T(h+1) \eta(h+1) \\ & \quad + \beta(h) e^T(h) A^T(h) A(h) \beta(h) e(h) + 2\tilde{d}^T(h-1) Q^T(h) M(h) \tilde{y}(h) + 2\eta^T(h+1) M(h) \tilde{y}(h) \\ & \quad + 2\beta(h) e^T(h) A^T(h) M(h) \tilde{y}(h) + 2\eta^T(h+1) Q(h) \tilde{d}(h-1) + 2\beta(h) e^T(h) A^T(h) \eta(h+1) \\ & \quad + 2\beta(h) e^T(h) A^T(h) Q(h) \tilde{d}(h-1) - \tau \tilde{y}^T(h) \tilde{y}(h). \end{aligned} \quad (\text{S12})$$

By means of Assumption 3, one has $\|\tilde{d}_i(h)\| \leq \alpha(h)d$. With Lemma 4, Eq. (S12) is further manipulated

as follows:

$$\begin{aligned}
& \Delta V_1(h+1) \\
& \leq \tilde{y}^T(h)(\check{\ell}_1 M^T(h)M(h) - (1-\tau)I)\tilde{y}(h) + \check{\ell}_2 \tilde{d}^T(h-1)Q^T(h)Q(h)\tilde{d}(h-1) \\
& \quad + \check{\ell}_3 \eta^T(h+1)\eta(h+1) + \check{\ell}_4 \beta(h)e^T(h)A^T(h)A(h)\beta(h)e(h) \\
& \leq \tilde{y}^T(h)(\check{\ell}_1 \bar{M} - (1-\tau)I)\tilde{y}(h) + \check{\ell}_2 \tilde{d}^T(h-1)\tilde{Q}\tilde{d}(h-1) \\
& \quad + \check{\ell}_3 \eta^T(h+1)\eta(h+1) + \check{\ell}_4 \beta(h)e^T(h)\bar{A}\beta(h)e(h) \\
& \quad - \varepsilon_1 \tilde{d}^T(h-1)\tilde{d}(h-1) + \varepsilon_1 \alpha^2 (h-1)d^2 \\
& \quad - \varepsilon_2 \eta^T(h+1)\eta(h+1) + \varepsilon_2 \eta^T(h+1)\eta(h+1) \\
& \quad + \varepsilon_3 \beta^2(h) \left(\sum_{i=1}^N \theta_i - e^T(h)e(h) \right) - \tau \tilde{y}^T(h)\tilde{y}(h) \\
& = \Omega_1^T(h)\Pi_1\Omega_1(h) - \tau \tilde{y}^T(h)\tilde{y}(h) + \Upsilon_1(h),
\end{aligned} \tag{S13}$$

where $\Omega_1(h) \triangleq [\tilde{y}^T(h) \ \tilde{d}^T(h-1) \ \eta^T(h+1) \ \beta(h)e^T(h)]^T$ and $\Upsilon_1(h) \triangleq \varepsilon_1 \alpha^2 (h-1)d^2 + \varepsilon_2 \eta^T(h+1)\eta(h+1) + \varepsilon_3 \beta^2(h) \sum_{i=1}^N \theta_i$ with ε_1 - ε_3 and $\check{\ell}_1$ - $\check{\ell}_6$ being positive constants.

It follows from inequality (S13) that

$$V_1(h+1) \leq (1-\tau)V_1(h) + \Upsilon_1(h). \tag{S14}$$

Noting that $0 < \tau < 1$, $\sum_{h=0}^{\infty} \tau = \infty$, and $\lim_{h \rightarrow \infty} \frac{\Upsilon_1(h)}{\tau} = 0$, it is simple to deduce from Lemma 2 that $\lim_{h \rightarrow \infty} V_1(h) = 0$. Thus, we can draw the conclusion that $\lim_{h \rightarrow \infty} \|\tilde{y}(h) - y_i(h)\| = 0$. The proof is complete.

Proof S2 Proof of Theorem 2

Construct a Lyapunov function as follows:

$$V_2(h) = \hat{y}^T(h)\hat{y}(h). \tag{S15}$$

Then, calculating the difference of $V_2(h)$ results in

$$\begin{aligned}
& \Delta V_2(h+1) \\
& = V_2(h+1) - V_2(h) \\
& = [M(h)\hat{y}(h) + Q(h)\tilde{d}(h-1) + A(h)\beta(h)\delta(h) + A(h)\beta(h)e(h)]^T \\
& \quad \cdot [M(h)\hat{y}(h) + Q(h)\tilde{d}(h-1) + A(h)\beta(h)\delta(h) + A(h)\beta(h)e(h)] - \hat{y}^T(h)\hat{y}(h) \\
& = \hat{y}^T(h)(M^T(h)M(h) - I)\hat{y}(h) + \tilde{d}^T(h-1)Q^T(h)Q(h)\tilde{d}(h-1) \\
& \quad + \beta(h)\delta^T(h)A^T(h)A(h)\beta(h)\delta(h) + \beta(h)e^T(h)A^T(h)A(h)\beta(h)e(h) \\
& \quad + 2\tilde{d}^T(h-1)Q^T(h)M(h)\hat{y}(h) + 2\beta(h)\delta^T(h)A^T(h)M(h)\hat{y}(h) \\
& \quad + 2\beta(h)e^T(h)A^T(h)M(h)\hat{y}(h) + 2\beta(h)\delta^T(h)A^T(h)Q(h)\tilde{d}(h-1) \\
& \quad + 2\beta(h)e^T(h)A^T(h)A(h)\beta(h)\delta(h) + 2\beta(h)e^T(h)A^T(h)Q(h)\tilde{d}(h-1) - \hat{y}^T(h)\hat{y}(h),
\end{aligned} \tag{S16}$$

which further implies that

$$\begin{aligned}
& \Delta V_2(h+1) \\
& \leq \hat{y}^T(h) (\check{\ell}_5 M^T(h) M(h) - I) \hat{y}(h) + \check{\ell}_6 \tilde{d}^T(h-1) Q^T(h) Q(h) \tilde{d}(h-1) \\
& \quad + \check{\ell}_7 \beta(h) \delta^T(h) A^T(h) A(h) \beta(h) \delta(h) + \check{\ell}_8 \beta(h) e^T(h) A^T(h) A(h) \beta(h) e(h) \\
& \leq \hat{y}^T(h) (\check{\ell}_5 \bar{M} - I) \hat{y}(h) + \check{\ell}_6 \tilde{d}^T(h-1) \bar{Q} \tilde{d}(h-1) + \check{\ell}_7 \beta(h) \delta^T(h) \bar{A} \beta(h) \delta(h) \\
& \quad + \check{\ell}_8 \beta(h) e^T(h) \bar{A} \beta(h) e(h) - \omega_1 \tilde{d}^T(h-1) \tilde{d}(h-1) + \omega_1 \alpha^2 (h-1) d^2 \\
& \quad - \omega_2 \beta(h) \delta^T(h) \beta(h) \delta(h) + \omega_2 \beta^2(h) l^2 + \omega_3 \beta^2(h) \left(\sum_{i=1}^N \theta_i - e^T(h) e(h) \right) \\
& = \Omega_2^T(h) \Pi_2 \Omega_2(h) + \Upsilon_2(h),
\end{aligned} \tag{S17}$$

where $\Omega_2(h) \triangleq [\hat{y}^T(h) \quad \tilde{d}^T(h-1) \quad \beta(h) \delta^T(h) \quad \beta(h) e^T(h)]^T$ and $\Upsilon_2(h) \triangleq \omega_1 \alpha^2 (h-1) d^2 + \omega_2 \beta^2(h) l^2 + \omega_3 \beta^2(h) \sum_{i=1}^N \theta_i$ with $\omega_1 - \omega_3$ and $\ell_7 - \ell_{12}$ being positive constants. Then, we can reasonably calculate that

$$\Omega_2^T(h) \Pi_2 \Omega_2(h) \leq -\hbar \Omega_2^T(h) \Omega_2(h), \tag{S18}$$

where $\hbar \triangleq \rho_{\min}\{-\Pi_2\} > 0$. Taking inequality (S17) into account, we can obtain

$$V_2(h+1) \leq V_2(h) - \hbar \Omega_2^T(h) \Omega_2(h) + \Upsilon_2(h). \tag{S19}$$

Note that $\lim_{h \rightarrow \infty} \Upsilon_2(h) = 0$ and that $\Upsilon_2(h)$ is bounded, which indicates that $\sum_{h=0}^{\infty} \Upsilon_2(h) < \infty$. It can be lightly derived from Lemma 3 that $V(h)$ converges to 0. Hence, we arrive at $\lim_{h \rightarrow \infty} V_2(h) = 0$ and $\lim_{h \rightarrow \infty} \|y^* - y_i(h)\| = 0$. According to the definition of limit, it can be shown that the limits of $y_i(h)$ and $\bar{y}(h)$ exist. Since the limit point of the sequence $\bar{y}(h)$ is unique, based on Theorem 1, one can deduce that $\lim_{h \rightarrow \infty} y_i(h) = \bar{y}(h)$. Thus, we have $\lim_{h \rightarrow \infty} \bar{y}(h) = y^*$. The proof of Theorem 2 is complete.