Hydrodynamic equations for rapid flows of smooth inelastic spheres, to Burnett order

N. Sela and I. Goldhirsch

Department of Fluid Mechanics and Heat Transfer

Faculty of Engineering, Tel-Aviv University

Ramat-Aviv, Tel-Aviv 69978, Israel

Please find Appendices A, D and E below

A Derivation of some scalar functions

This Appendix presents the method used in this study for the determination of the scalar functions $\hat{\Phi}_v$, $\hat{\Phi}_c$, $\hat{\Phi}_e$ and $\overline{\eta}$. Consider first $\hat{\Phi}_v$, $\hat{\Phi}_c$ and $\hat{\Phi}_e$, which satisfy eqs. (23,30). The isotropy of the operator $\tilde{\mathbf{L}}$ implies that the solutions of these equations assume the forms given in eq. (23) and the text following eq. (30). The remaining task is to determine the scalar functions $\hat{\Phi}_v$, $\hat{\Phi}_c$ and $\hat{\Phi}_e$. This is performed by considering the three parts of the solution (corresponding to the contribution of to the viscosity, heat flow and inelasticity respectively) separately. The 'viscous' contribution satisfies:

$$\tilde{\mathbf{L}}(\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}) = \overline{\tilde{u}_i\tilde{u}_j}. \tag{A.1}$$

It is convenient to assume specific directions e.g. $i = 1, j = 2$. Employing the Fredholm form of $\tilde{\mathbf{L}}$ (cf. Pekeris (1955) and Cercignani (1975)), one obtains:

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}(\tilde{u}_1)\tilde{u}_{1x}\tilde{u}_{1y} + \frac{1}{\pi}\int d\tilde{\mathbf{u}}_\mathbf{2}\hat{\Phi}(\tilde{u}_2)\tilde{u}_{2x}\tilde{u}_{2y}e^{-\tilde{u}_2^2}\left(R - \frac{2}{R}e^{w^2}\right)\right] = \tilde{u}_{1x}\tilde{u}_{1y}, \tag{A.2}$$

where $R \equiv |\tilde{\mathbf{u}}_\mathbf{1} - \tilde{\mathbf{u}}_\mathbf{2}|$, $w \equiv \frac{\tilde{\mathbf{u}}_\mathbf{1}\times\tilde{\mathbf{u}}_\mathbf{2}}{R}$ and $Q(\tilde{u}) \equiv e^{-\tilde{u}^2} + \frac{\sqrt{\pi}}{2}\left(2\tilde{u} + \frac{1}{\tilde{u}}\right)\mathrm{erf}(\tilde{u})$. Since both R and w depend only on $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{\mathbf{u}}_\mathbf{1}\cdot\tilde{\mathbf{u}}_\mathbf{2}$, it is convenient to perform the integration over $\tilde{\mathbf{u}}_\mathbf{2}$ in a (rotated) spherical coordinate system where $\hat{\tilde{\mathbf{u}}}_\mathbf{1}$ defines the z direction. The vector components in this coordinate system are denoted below by primed symbols. It follows that $\tilde{u}_{2x}$ and $\tilde{u}_{2y}$ are given by:

$$\tilde{u}_{2x} = \tilde{u}'_{2x}\cos\theta_1\cos\phi_1 - \tilde{u}'_{2y}\sin\phi_1 + \tilde{u}'_{2z}\sin\theta_1\cos\phi_1. \tag{A.3}$$

$$\tilde{u}_{2y} = \tilde{u}'_{2x}\cos\theta_1\sin\phi_1 + \tilde{u}'_{2y}\cos\phi_1 + \tilde{u}'_{2z}\sin\theta_1\sin\phi_1. \tag{A.4}$$

where θ_1 and ϕ_1 are the spherical angles of $\hat{\tilde{\mathbf{u}}}_\mathbf{1}$ in the original coordinate system. We perform the integration over ϕ'_2 (the azimuthal angle of $\tilde{\mathbf{u}}_\mathbf{2}$ in the rotated frame) first. Since both R and w do not depend on ϕ'_2 one needs to integrate only the term $\tilde{u}_{2x}\tilde{u}_{2y}$. One obtains:

$$\int_0^{2\pi} d\phi'_2\tilde{u}_{2x}\tilde{u}_{2y} = 2\pi\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)^2\tilde{u}_{1x}\tilde{u}_{1y}P_2(\cos\theta'_2), \tag{A.5}$$

where $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second order Legendre polynomial. Next we integrate over the angle between $\tilde{\mathbf{u}}_\mathbf{2}$ and the z direction in the rotated frame, θ'_2, i.e. the angle between $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$. Let $A_n(\tilde{u}_1, \tilde{u}_2)$ be defined by:

$$A_n(\tilde{u}_1, \tilde{u}_2) \equiv \int_0^\pi d\theta_2' \sin\theta_2' \left(R - \frac{2}{R} e^{w^2} \right) P_n(\cos\theta_2'). \tag{A.6}$$

The following equation follows from eq. (A.2):

$$-\frac{1}{\sqrt{\pi}} \left(Q(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1) + \frac{2}{\tilde{u}_1^2} \int_0^\infty d\tilde{u}_2 \tilde{u}_2^4 \hat{\Phi}_v(\tilde{u}_2) A_2(\tilde{u}_1, \tilde{u}_2) e^{-\tilde{u}_2^2} \right) = 1. \tag{A.7}$$

In particular A_2 is given (for $\tilde{u}_2 < \tilde{u}_1$) by Pekeris (1955):

$$A_2 = \frac{1}{\tilde{u}_1^3 \tilde{u}_2^3} \left(\frac{2}{35}\tilde{u}_2^7 - \frac{2}{15}\tilde{u}_2^5 \tilde{u}_1^2 + 3\tilde{u}_1^2 \tilde{u}_2 - 3\tilde{u}_2^3 + 18\tilde{u}_2 + \right.$$
$$\left. + \frac{\sqrt{\pi}}{2}(-6\tilde{u}_2^4 + 2\tilde{u}_1^2\tilde{u}_2^2 - 3\tilde{u}_1^2 + 15\tilde{u}_2^2 - 18) e^{\tilde{u}_2^2} \mathrm{erf}(\tilde{u}_2) \right). \tag{A.8}$$

The value of A_2 for $\tilde{u}_1 < \tilde{u}_2$ is obtained from eq. (A.8) by exchanging $\tilde{u}_1$ and $\tilde{u}_2$. In the literature, e.g. Kogan (1969); Chapman & Cowling (1970); Harris (1971) and Cercignani (1975), the function $\hat{\Phi}_v$ is usually approximated by a truncated series of Sonine polynomials. We prefer to investigate first the symmetry and asymptotic properties of $\hat{\Phi}_v$, and then expand it in a series of functions that obey these symmetry and asymptotic properties. This procedure leads to a more accurate determination of $\hat{\Phi}_v$ for all values of $\tilde{u}$ (since a truncated series of polynomials diverges for large values of $\tilde{u}$ whereas $\hat{\Phi}_v$ decays at large values of $\tilde{u}$ as shown below). Consider eq. (A.7) and let $\tilde{u}_1 \to -\tilde{u}_1$. Clearly Q is formally an even function of u, hence the term of the LHS of eq. (A.7), which is proportional to Q, preserves any parity symmetry of $\hat{\Phi}_v$. The second term on the LHS of eq. (A.7) is even as well. This is seen by considering the definition of A_n (cf. eq. (A.6)), with $n = 2$: changing the integration variable to $dx \equiv \sin\theta_2' d\theta_2'$ one obtains that A_2 is proportional to

$\int_{-1}^{1} dx \left(R - \frac{2}{R}e^{-w^2}\right) P_2(x)$; recalling that $R = \sqrt{\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2}$ and $w^2 = \frac{\tilde{u}_1^2\tilde{u}_2^2(1-x^2)}{R^2}$, it is clear that the transformation $\tilde{u}_1 \to -\tilde{u}_1$ followed by a change of the integration variable $x \to -x$ leaves the integral unchanged. Hence $A_2(-\tilde{u}_1, \tilde{u}_2) = A_2(\tilde{u}_1, \tilde{u}_2)$, i.e. the second term on the LHS of eq. (A.7) is invariant to $\tilde{u}_1 \to -\tilde{u}_1$. Now, since the first term on the LHS preserves the symmetry of $\hat{\Phi}_v$ and the second term is formally even, it follows that $\hat{\Phi}_v(\tilde{u}_1)$ is formally even in $\tilde{u}_1$. Next we investigate the asymptotic properties of $\hat{\Phi}_v$ as $\tilde{u}_1 \gg 1$. Assume $\hat{\Phi}_v(\tilde{u})$ is asymptotically proportional to $\tilde{u}^p$ where p is a constant (this assumption is shown to be justified a-posteriori). Eq. (A.7) can be expressed as $g_1^{(v)} + g_2^{(v)} + g_3^{(v)} = 1$, where:

$$g_1^{(v)} \equiv -\frac{1}{\sqrt{\pi}} Q(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1), \tag{A.9}$$

$$\begin{aligned} g_2^{(v)} \equiv -\frac{2}{\sqrt{\pi}\tilde{u}_1^5}\int_0^{\tilde{u}_1} d\tilde{u}_2 \Bigg[\left(\frac{2}{35}\tilde{u}_2^8 - \frac{2}{15}\tilde{u}_1^2\tilde{u}_2^6 + 3\tilde{u}_1^2\tilde{u}_2^2 - 3\tilde{u}_2^4 + 18\tilde{u}_2^2\right) e^{-\tilde{u}_2^2} + \\ + \frac{\sqrt{\pi}}{2}(-6\tilde{u}_2^5 + 2\tilde{u}_1^2\tilde{u}_2^3 - 3\tilde{u}_1^2\tilde{u}_2 + 15\tilde{u}_2^3 - 18\tilde{u}_2)\mathrm{erf}(\tilde{u}_2)\Bigg] \hat{\Phi}_v(\tilde{u}_2), \end{aligned} \tag{A.10}$$

and,

$$\begin{aligned} g_3^{(v)} \equiv -\frac{2}{\sqrt{\pi}\tilde{u}_1^5}\int_{\tilde{u}_1}^{\infty} d\tilde{u}_2 \Bigg[\left(\frac{2}{35}\tilde{u}_1^7\tilde{u}_2 - \frac{2}{15}\tilde{u}_1^5\tilde{u}_2^3 + 3\tilde{u}_1\tilde{u}_2^3 - 3\tilde{u}_1^3\tilde{u}_2 + 18\tilde{u}_1\tilde{u}_2\right) e^{-\tilde{u}_2^2} + \\ + \frac{\sqrt{\pi}}{2}(-6\tilde{u}_1^4\tilde{u}_2 + 2\tilde{u}_1^2\tilde{u}_2^3 - 3\tilde{u}_2^3 + 15\tilde{u}_1^2\tilde{u}_2 - 18\tilde{u}_2)e^{\tilde{u}_1^2 - \tilde{u}_2^2}\mathrm{erf}(\tilde{u}_2)\Bigg] \hat{\Phi}_v(\tilde{u}_2). \end{aligned} \tag{A.11}$$

In eqs. (A.9-A.11) use has been made of the explicit form of A_2 (cf. eq. (A.8)). Next we investigate the asymptotic leading behavior of the g's for $\tilde{u}_1 \gg 1$. It is clear from the definition of $Q(\tilde{u}_1)$ that its asymptotic leading behavior is proportional to $\tilde{u}_1$, hence the asymptotic leading behavior of $g_1^{(v)}$ is proportional to $\tilde{u}_1^{1+p}$. Next consider $g_2^{(v)}$. In the first part of the integrand (which is proportional to $e^{-\tilde{u}_2^2}$), it is possible, to leading order in $\tilde{u}$, to replace the upper limit of the integral by infinity. It is then clear that, to leading order in $\tilde{u}$, the integral of this term is proportional to $\tilde{u}_1^2$. It is easy to show that the leading asymptotic behavior of the integral of the term which is proportional to $\text{erf}(\tilde{u}_2)$ is proportional to $\tilde{u}_1^{6+p}$. Upon dividing this integral by $\tilde{u}_1^5$, it follows that the asymptotic leading behavior of $g_2^{(v)}$ is proportional to $\tilde{u}_1^{max(1+p,-3)}$. Finally consider $g_3^{(v)}$ and recall that the asymptotic leading behavior of integrals of the form $\int_{\tilde{u}}^{\infty} dx x^{\alpha} e^{-x^2}$ for $\tilde{u} \gg 1$ is proportional to $\tilde{u}^{\alpha-1}e^{-\tilde{u}^2}$. It follows that the asymptotic behavior of the first contribution to the integral (resulting from the term in the integrand which is proportional to $e^{-\tilde{u}_2^2}$) is proportional to $\tilde{u}_1^{7+p}e^{-\tilde{u}_1^2}$. The asymptotic behavior of the second contribution to the integral is proportional to $\tilde{u}_1^{4+p}$. Hence, the leading behavior of $g_3^{(v)}$ is proportional to $\tilde{u}_1^{-1+p}$. Now, since the sum of the g's is unity, the only possible value for p is $p = -1$. Having determined the symmetry and asymptotic properties of $\hat{\Phi}_v(\tilde{u})$, we expand it in a set of functions having the same properties. For this purpose we use the following set of functions: $\phi_n^{(v)}(\tilde{u}) \equiv e^{-\tilde{u}^2} I_{n-1}(\tilde{u}^2)$, where I_n is a modified Bessel function, i.e. we consider the expansion: $\hat{\Phi}_v(\tilde{u}) = \sum_{n=1}^{\infty} a_n^{(v)} \phi_n^{(v)}$. Next, define the kernel $K^{(v)}(\tilde{u}, x) \equiv -\frac{1}{\sqrt{\pi}} \left(Q(\tilde{u})\delta(\tilde{u}-x) + \frac{2x^4}{\tilde{u}^2} A_2(\tilde{u}, x) e^{-x^2} \right)$; eq. (A.7) can be rewritten as follows:

$$\sum_{n=1}^{\infty} a_n^{(v)} \int_0^{\infty} dx K^{(v)}(\tilde{u}, x) \phi_n^{(v)}(x) = 1. \tag{A.12}$$

The solution of eq. (A.12) can be obtained, to any desired degree of accuracy, by truncating the series at a large enough value of $n = N$ and choosing a (large enough) set of points $\{\tilde{u}_m; 1 < m < M\}$ where $N < M$. Next, define the matrix elements

$K^{(v)}_{nm} \equiv \int_0^\infty dx K^{(v)}(\tilde{u}_m, x)\phi_n(x)$ (the integrals over x are carried out numerically). The values of the coefficients are found by minimizing $\frac{1}{M}\sum_{m=1}^{M}\left(\sum_{n=1}^{N} a_n^{(v)} K_{nm}^{(v)} - 1\right)^2$. The squared deviation assumes a value less than 10^{-6} for $N = 10$ and $M = 46$, when the points $\tilde{u}_m$ are evenly distributed between $\tilde{u}_1 = 0$ and $\tilde{u}_{46} = 15$. The components $a_n^{(v)}$ which correspond to the above fit are given in Table 1.

The equation corresponding to the 'heat-flux' term is given by:

$$\tilde{\mathbf{L}}\left[\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\right] = \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i. \tag{A.13}$$

Let $i = 3$ (the z direction). Using the Fredholm form of $\tilde{\mathbf{L}}$ and following consideration which are similar to those employed when calculating $\hat{\Phi}_v$ one obtains:

$$-\frac{1}{\sqrt{\pi}}\left(Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1) + \frac{2}{\tilde{u}_1\left(\tilde{u}_1^2 - \frac{5}{2}\right)}\int_0^\infty d\tilde{u}_2\tilde{u}_2^3\left(\tilde{u}_2^2 - \frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)A_1(\tilde{u}_1, \tilde{u}_2)e^{-\tilde{u}_2^2}\right) = 1, \tag{A.14}$$

where, in particular, A_1 is given (for $\tilde{u}_2 < \tilde{u}_1$) by Pekeris (1955):

$$A_1(\tilde{u}_1, \tilde{u}_2) = \frac{1}{\tilde{u}_1^2\tilde{u}_2^2}\left[\frac{2}{15}\tilde{u}_2^5 - \frac{2}{3}\tilde{u}_2^3\tilde{u}_1^2 - 4\left(\tilde{u}_2 + \frac{\sqrt{\pi}}{2}(\tilde{u}_2^2 - 1)e^{\tilde{u}_2^2}\mathrm{erf}(\tilde{u}_2)\right)\right]. \tag{A.15}$$

The value of A_1 for $\tilde{u}_2 > \tilde{u}_1$ is obtained from eq. (A.15) by exchanging $\tilde{u}_1$ and $\tilde{u}_2$. Consider eq. (A.14) and let $\tilde{u}_1 \to -\tilde{u}_1$. The term on the LHS of eq. (A.14), which is proportional to Q, preserves the parity of $\hat{\Phi}_c$ since Q is even in $\tilde{u}_1$. The second term on the LHS is also invariant under a parity transformation; note that A_1 is proportional to $\int_{-1}^{1} dx\left(R - \frac{2}{R}e^{w^2}\right)P_1(x)$. Thus letting $\tilde{u}_1 \to -\tilde{u}_1$, followed by the change of the integration variable $x \to -x$, yields: $A_1(-\tilde{u}_1, \tilde{u}_2) = -A_1(\tilde{u}_1, \tilde{u}_2)$. This change of sign is canceled by the change of sign of $\tilde{u}_1$ in the denominator of the term multiplying the integral in eq.

(A.14). Hence, the second term on the LHS of eq. (A.14) is invariant to $\tilde{u}_1 \to -\tilde{u}_1$. It follows that $\hat{\Phi}_c(\tilde{u}_1)$ is formally even in $\tilde{u}_1$. Assume that the asymptotic leading behavior of $\hat{\Phi}_c(\tilde{u})$ is proportional to $\tilde{u}^p$ and consider the limit $\tilde{u} \gg 1$. Eq. (A.14) can be written as $g_1^{(c)} + g_2^{(c)} + g_3^{(c)} = 1$, where:

$$g_1^{(c)} \equiv -\frac{1}{\sqrt{\pi}} Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1), \tag{A.16}$$

$$g_2^{(c)} \equiv -\frac{2}{\sqrt{\pi}\tilde{u}_1^3\left(\tilde{u}_1^2 - \frac{5}{2}\right)} \int_0^{\tilde{u}_1} d\tilde{u}_2 \left(\tilde{u}_2^2 - \frac{5}{2}\right)\left[\left(\frac{2}{15}\tilde{u}_2^6 - \frac{2}{3}\tilde{u}_1^2\tilde{u}_2^4 - 4\tilde{u}_2^2\right)e^{-\tilde{u}_2^2} - \right.$$
$$\left. -2\sqrt{\pi}\tilde{u}_2(\tilde{u}_2^2 - 1)\mathrm{erf}(\tilde{u}_2)\right]\hat{\Phi}_c(\tilde{u}_2), \tag{A.17}$$

and,

$$g_3^{(c)} \equiv -\frac{2}{\sqrt{\pi}\tilde{u}_1^3\left(\tilde{u}_1^2 - \frac{5}{2}\right)} \int_{\tilde{u}_1}^{\infty} d\tilde{u}_2 \left(\tilde{u}_2^2 - \frac{5}{2}\right)\left[\left(\frac{2}{15}\tilde{u}_1^5\tilde{u}_2 - \frac{2}{3}\tilde{u}_1^3\tilde{u}_2^3 - 4\tilde{u}_1\tilde{u}_2\right)e^{-\tilde{u}_2^2} - \right.$$
$$\left. -2\sqrt{\pi}\tilde{u}_2(\tilde{u}_1^2 - 1)e^{\tilde{u}_1^2 - \tilde{u}_2^2}\mathrm{erf}(\tilde{u}_1)\right]\hat{\Phi}_c(\tilde{u}_2), \tag{A.18}$$

Proceeding as in the calculation of $\hat{\Phi}_v$, one finds that the asymptotic leading behavior of $g_1^{(c)}$, $g_2^{(c)}$ and $g_3^{(c)}$ is given by $\tilde{u}_1^{1+p}$, $\tilde{u}_1^{max(-3,1+p)}$ and $\tilde{u}_1^{-1+p}$, respectively. This implies that $p = -1$. Hence, the same set of functions $\phi_n^{(c)} = \phi_n^{(v)}$ can be employed for expanding $\hat{\Phi}_c$: $\hat{\Phi}_c(\tilde{u}) = \sum_{n=1}^{\infty} a_n^{(c)}\phi_n^{(c)}(\tilde{u})$. A good approximation (squared deviation less than 10^{-6}) is obtained by taking $N = 10$, $M = 46$, with the points $\tilde{u}_m$ evenly distributed between $\tilde{u}_1 = 0$ and $\tilde{u}_{46} = 15$. The coefficients $a_n^{(c)}$ for the above truncation are given in Table 1. Eq. (A.14) possesses a homogeneous solution of the form: $\frac{b}{\tilde{u}^2 - \frac{5}{2}}$ (where b is a constant). Thus the general solution of this equation is the sum of $\hat{\Phi}_c$, as obtained above, and the homogeneous solution. The value of the coefficient, b, is determined by requiring $\hat{\Phi}_c$ to be orthogonal to the invariants. In this case it is sufficient to consider the orthogonality to the invariants $\tilde{u}_i$: $\int d\tilde{\mathbf{u}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_z^2 e^{-\tilde{u}^2} = 0$. Denoting by $\hat{\Phi}_c^{(0)}$ the solution obtained above (and whose coefficients are tabulated in Table 1), the value of the constant b is given by:

$$b = -\frac{\int_0^\infty dx x^4\left(x^2 - \frac{5}{2}\right)\hat{\Phi}_c^{(0)}(x)e^{-x^2}}{\int_0^\infty dx x^4 e^{-x^2}} \approx -0.1927, \tag{A.19}$$

The integral in the numerator of eq. (A.19) has been evaluated numerically. Thus: $\hat{\Phi}_c$ is given by $\hat{\Phi}_c(\tilde{u}) = \hat{\Phi}_c^{(0)}(\tilde{u}) + \frac{b}{\tilde{u}^2 - \frac{5}{2}}$.

The equation for the first order correction due to the inelasticity is (cf. eq. (30)):

$$\tilde{\mathbf{L}}(\hat{\Phi}_e) = h(\tilde{u}). \tag{A.20}$$

where $h(\tilde{u})$ is defined as the RHS of eq. (30) divided by ϵ. Employing the Fredholm form of the operator $\tilde{\mathbf{L}}$ one obtains:

$$-\frac{1}{\sqrt{\pi}}\left(Q(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1) + 2\int_0^\infty d\tilde{u}_2\tilde{u}_2^2\hat{\Phi}_e(\tilde{u}_2)A_0(\tilde{u}_1,\tilde{u}_2)e^{-\tilde{u}_2^2}\right) = h(\tilde{u}_1), \tag{A.21}$$

where A_0 is given (for $\tilde{u}_2 < \tilde{u}_1$) by:

$$A_0(\tilde{u}_1,\tilde{u}_2) = \frac{1}{\tilde{u}_1\tilde{u}_2}\left(\frac{2}{3}\tilde{u}_2^3 + 2\tilde{u}_1^2\tilde{u}_2 - 2\sqrt{\pi}\mathrm{erf}(\tilde{u}_2)e^{\tilde{u}_2^2}\right). \tag{A.22}$$

The value of $A_0(\tilde{u}_1,\tilde{u}_2)$ for $\tilde{u}_1 < \tilde{u}_2$ is obtained from eq. (A.22) by exchanging the arguments of A_0. The function $A_0(\tilde{u}_1,\tilde{u}_2)$ is even with respect to $\tilde{u}_1$. This fact follows from the definition of A_n, eq. (A.6), with $n = 0$. The function $A_0(\tilde{u}_1,\tilde{u}_2)$ is proportional to $\int_{-1}^1 dx\left(R - \frac{2}{R}e^{w^2}\right)$ where R and w are given in the above. Clearly, letting $\tilde{u}_1 \to -\tilde{u}_1$, followed by a change of the integration variable $x \to -x$, implies that $A_0(-\tilde{u}_1,\tilde{u}_2) =$

$A_0(\tilde{u}_1, \tilde{u}_2)$. Hence, since the RHS of eq. (A.21) is even as well, it is clear that $\hat{\Phi}_e(\tilde{u}_1)$ is an even function of $\tilde{u}_1$. Next, we investigate the asymptotic properties of $\hat{\Phi}_e$. Assume that its asymptotic leading behavior is proportional to $\tilde{u}_1^p$. Eq. (A.21) can be written as: $g_1^{(e)} + g_2^{(e)} + g_3^{(e)} = h(\tilde{u}_1)$, where:

$$g_1^{(e)} \equiv -\frac{1}{\sqrt{\pi}} Q(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1). \tag{A.23}$$

$$g_2^{(e)} \equiv -\frac{2}{\sqrt{\pi}\tilde{u}_1}\int_0^{\tilde{u}_1} d\tilde{u}_2 \left[\left(\frac{2}{3}\tilde{u}_2^4 + 2\tilde{u}_1^2\tilde{u}_2^2\right)e^{-\tilde{u}_2^2} - \frac{8}{\sqrt{\pi}}\tilde{u}_2 \mathrm{erf}(\tilde{u}_2)\right], \hat{\Phi}_e(\tilde{u}_2) \tag{A.24}$$

and,

$$g_3^{(e)} \equiv -\frac{2}{\sqrt{\pi}\tilde{u}_1}\int_{\tilde{u}_1}^{\infty} d\tilde{u}_2 \left[\left(\frac{2}{3}\tilde{u}_1^3\tilde{u}_2 + 2\tilde{u}_2^3\tilde{u}_1\right)e^{-\tilde{u}_2^2} - \frac{8}{\sqrt{\pi}}\tilde{u}_2 \mathrm{erf}(\tilde{u}_1)e^{\tilde{u}_1^2-\tilde{u}_2^2}\right].\hat{\Phi}_e(\tilde{u}_2) \tag{A.25}$$

Arguments, similar to those presented above lead to the conclusion that the for $\tilde{u}_1 \gg 1$, $g_1^{(e)}$, $g_2^{(e)}$ and $g_3^{(e)}$ are proportional to $\tilde{u}_1^{1+p}$, $\tilde{u}_1^{max(1,1+p)}$ and $\tilde{u}_1^{-1+p}$ respectively. Hence, since the asymptotic leading behavior of $h(\tilde{u}_1)$ is proportional to $\tilde{u}_1^3$ one could have concluded that $p = 2$. Note, however, that $\tilde{u}_1^2$ is a summational invariant of $\tilde{\mathbf{L}}$; one can easily check that in this case the coefficient of $\tilde{u}_1^2$ diverges. The same anomaly occurs in the two-dimensional case (Sela & Goldhirsch (1996)); a somewhat tedious but straightforward calculation reveals that the correct asymptotic leading term of $\hat{\Phi}_e$ at large $\tilde{u}$ is proportional to $\tilde{u}^2\log(\tilde{u})$. A particular solution of eq. (A.21), which is denoted by $\hat{\Phi}_e^{(0)}$ can be obtained by expanding this function in a set of functions obeying the above determined symmetry and asymptotic properties of $\hat{\Phi}_e$. We choose to expand it in the following set of functions $\phi_n^{(e)} = (1+\log(1+\tilde{u}^2))(1+\tilde{u}^2)^{\frac{3}{2}} I_{n-1}(\tilde{u}^2)e^{-\tilde{u}^2}$, i.e. $\hat{\Phi}_e^{(0)}(\tilde{u}) = \sum_{n=1}^{\infty} a_n^{(e)}\phi_n^{(e)}(\tilde{u})$. A truncation of the series with a minimization of the error is then performed as in the above. A good approximation (error less then 10^{-6}) is obtained by taking $N = 10$, $M = 46$, the points $\tilde{u}_m$ being evenly distributed between $\tilde{u}_1 = 0$ and $\tilde{u}_{46} = 15$. The resulting coefficients, $a_n^{(e)}$, are presented in Table 1. The general form of $\hat{\Phi}_e$ is a sum of $\hat{\Phi}_e^0$ and a combination of the invariants, 1 and $\tilde{u}^2$ which we denote by: $b_0^* + b_2^*\tilde{u}^2$. The orthogonality conditions (of $\hat{\Phi}_e$ to the invariants) determine the coefficients b_0^* and b_2^* (cf. the text following eq. (30)): $b_0^* \approx 0.0698$ and $b_2^* \approx 1.4769$. All in all, $\hat{\Phi}_e(\tilde{u}) = \hat{\Phi}_e^{(0)}(\tilde{u}) + b_0^* + b_2^*\tilde{u}^2$.

Finally, we consider $\overline{\eta}$ which satisfies:

$$\tilde{\mathbf{L}}(\overline{\eta}) = \overline{\chi}, \tag{A.26}$$

where $\overline{\chi}$ is given in subsection (3.4). Since $\overline{\chi}$ is symmetric in $\tilde{u}$ and its asymptotic behavior is proportional to $\tilde{u}^3$ we can evaluate $\overline{\eta}$ in a similar manner to $\hat{\Phi}_e$. In particular the following representation of $\overline{\eta}$ is employed:

$$\overline{\eta}(\tilde{u}) \approx (1 + \log(1 + \tilde{u}^2))(1 + \tilde{u}^2)^{\frac{3}{2}} e^{-\tilde{u}^2} \sum_{n=1}^{10} a_n^{(\eta)} I_{n-1}(\tilde{u}^2) + c_0^* + c_2^*\tilde{u}^2, \tag{A.27}$$

A calculation similar to those described above yields the values of $a_n^{(\eta)}$ (tabulated in Table 1) and $c_0^* \approx 7.9908$ and $c_2^* \approx 36.729$.

B Proof of Solubility

Not included here

C Constitutive relations at $\mathcal{O}(K\epsilon)$

Not included here

D The integral I_δ

In this Appendix we calculate the integral I_δ defined as:

$$I_\delta \equiv \int_{\hat{\mathbf{k}}\cdot\mathbf{u_{12}}>0} d\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{u_{12}})\delta(\mathbf{u}-\mathbf{u_1}+q(\hat{\mathbf{k}}\cdot\mathbf{u_{12}})\hat{\mathbf{k}}), \tag{D.1}$$

where $q \equiv \frac{1+e}{2}$. Upon expressing the delta function as $\delta(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d\mathbf{w}e^{i\mathbf{w}\cdot\mathbf{x}}$, defining $\mathbf{s} \equiv \mathbf{u}-\mathbf{u_1}$ and changing the order of integration, one obtains:

$$I_\delta = \frac{1}{(2\pi)^3}\int d\mathbf{w}e^{i\mathbf{w}\cdot\mathbf{s}}\int_{\hat{\mathbf{k}}\cdot\mathbf{u_{12}}>0}(\hat{\mathbf{k}}\cdot\mathbf{u_{12}})e^{iq(\hat{\mathbf{k}}\cdot\mathbf{u_{12}})(\hat{\mathbf{k}}\cdot\mathbf{w})}. \tag{D.2}$$

Next, one performs the integration over $\hat{\mathbf{k}}$ in a frame whose z axis coincides with $\mathbf{u_{12}}$. One obtains:

$$I_\delta = \frac{u_{12}}{(2\pi)^3}\int d\mathbf{w}e^{i\mathbf{w}\cdot\mathbf{s}}\int_0^{\frac{\pi}{2}} d\theta\sin\theta\cos\theta\int_0^{2\pi} d\phi e^{iqu_{12}\cos\theta(\cos\theta w_z+\sin\theta w_\perp\cos(\phi-\chi))}, \tag{D.3}$$

where θ and ϕ are the coordinates of $\hat{\mathbf{k}}$ in the rotated frame, w_z and $w_\perp$ are the components of $\mathbf{w}$ parallel and perpendicular to $\mathbf{u_{12}}$ respectively, and χ is the azimuthal angle of the projection of $\mathbf{w}$ on the plane normal to $\mathbf{u_{12}}$. The integration over ϕ yields:

$$I_\delta = \frac{u_{12}}{(2\pi)^2}\int d\mathbf{w}e^{i\mathbf{w}\cdot\mathbf{s}}\int_0^{\frac{\pi}{2}} d\theta\sin\theta\cos\theta e^{iqu_{12}w_z\cos^2\theta}J_0(qw_\perp u_{12}\sin\theta\cos\theta), \tag{D.4}$$

where J_0 is the Bessel function of zeroth order. Next, one performs the integration over $\mathbf{w}$ in a cylindrical coordinate system whose z axis coincides with $\mathbf{u_{12}}$. By decomposing $\mathbf{s}$ in the above system of coordinates, one obtains:

$$I_\delta = \frac{u_{12}}{(2\pi)^2}\int_{-\infty}^{\infty} dw_z\int_0^{\infty} dw_\perp w_\perp\int_0^{2\pi} d\mu e^{i(w_z s_z+w_\perp s_\perp\cos\mu)}\int_0^{\frac{\pi}{2}} d\theta\sin\theta\cos\theta\times$$

$$e^{iqu_{12}w_z\cos^2\theta}J_0(qw_\perp u_{12}\sin\theta\cos\theta), \tag{D.5}$$

where μ is the angle between the projections of $\mathbf{w}$ and $\mathbf{s}$ on the plane perpendicular to $\mathbf{u_{12}}$. The integration over μ is straightforward and one obtains:

$$I_\delta = \frac{u_{12}}{2\pi}\int_{-\infty}^{\infty} dw_z \int_0^{\infty} dw_\perp w_\perp \int_0^{\frac{\pi}{2}} d\theta \sin\theta\cos\theta \times$$

$$e^{iw_z(s_z+qu_{12}\cos^2\theta)} J_0(w_\perp s_\perp) J_0(qw_\perp u_{12}\sin\theta\cos\theta). \quad \text{(D.6)}$$

The integration over w_z yields a delta function and thus eq. (D.6) is transformed to:

$$I_\delta = u_{12}\int_0^{\infty} dw_\perp w_\perp \int_0^{\frac{\pi}{2}} d\theta \sin\theta\cos\theta\delta(s_z + qu_{12}\cos^2\theta)\times$$

$$J_0(w_\perp s_\perp) J_0(qw_\perp u_{12}\sin\theta\cos\theta). \quad \text{(D.7)}$$

The delta function in eq. (D.7) imposes the condition $\cos\theta = \sqrt{\frac{-s_z}{qu_{12}}}$, which implies that: $\sin\theta = \sqrt{\frac{qu_{12}+s_z}{qu_{12}}}$ (recall that s_z is the projection of $\mathbf{s}$ on $\mathbf{u_{12}}$ and it must satisfy the relation: $0 < \frac{-s_z}{qu_{12}}$). The integration over θ amounts to a substitution of the above conditions for $\sin\theta$ and $\cos\theta$ followed by a division by the absolute value of the derivative of the argument of the delta function with respect to θ. One obtains:

$$I_\delta = \frac{1}{2q} H\left(\frac{-s_z}{qu_{12}}\right) H\left(1 + \frac{s_z}{qu_{12}}\right)\int_0^{\infty} dw_\perp w_\perp J_0(w_\perp s_\perp) J_0(w_z\sqrt{-s_z(qu_{12}+s_z)}), \quad \text{(D.8)}$$

where H is the Heaviside function. Next, using the orthogonality property of the Bessel functions: $\int_0^\infty dx x J_0(ax) J_0(bx) = \frac{1}{a}\delta(a-b)$, one obtains:

$$I_\delta = \frac{1}{2qs_\perp} H\left(\frac{-s_z}{qu_{12}}\right) H\left(1 + \frac{-s_z}{qu_{12}}\right)\delta(s_\perp - \sqrt{-s_z(qu_{12}+s_z)}), \quad \text{(D.9)}$$

The delta function in eq. (D.9) implies the following condition: $s^2 = s_\perp^2 + s_z^2 = -qs_z u_{12} = -q\mathbf{s}\cdot\mathbf{u_{12}}$ (the latter equality follows from the definition of s_z). It thus follows from the above conditions that: $\frac{-s_z}{qu_{12}} = \frac{s_z^2}{s^2}$, hence the condition imposed by the Heaviside functions is satisfied once the condition imposed by the delta function is. Thus eq. (D.9) reduces to:

$$I_\delta = \frac{1}{2qs_\perp}\delta(s_\perp - \sqrt{-s_z(qu_{12}+s_z)}) = \frac{1}{q}\delta(s^2 + qs_z u_{12})$$
$$= \frac{1}{q}\delta(s^2 + q\mathbf{s}\cdot\mathbf{u_{12}}). \tag{D.10}$$

The second equality follows from the relation $\delta(f(x)-f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$. Expressing $\mathbf{u_{12}} = \mathbf{u} - \mathbf{s} - \mathbf{u_2}$ it follows from the above condition that:

$$\cos\theta_2' = \hat{\mathbf{s}}\cdot\hat{\mathbf{u}}_\mathbf{2} = \frac{\hat{\mathbf{s}}\cdot\mathbf{u}}{u_2} + \frac{1-q}{q}\frac{s}{u_2}, \tag{D.11}$$

where θ_2' is the angle between $\hat{\mathbf{s}}$ and $\hat{\mathbf{u}}_\mathbf{2}$. The condition, eq. (D.11), restricts the values of u_2 to $u_2 > |\frac{1-q}{q}s + \hat{\mathbf{s}}\cdot\mathbf{u}|$.

Consider next the following integral over the angle θ_2', of the form:

$$\int_0^\pi d\theta_2' \sin\theta_2' F(\cos\theta_2') I_\delta = \frac{1}{q}\int_{-1}^{1} d(\cos\theta_2')F(\cos\theta_2')\delta(s^2 + qs_z u_{12}), \tag{D.12}$$

where F is any smooth function. This integral can be performed by substituting the RHS of eq. (D.11) for $\cos\theta_2'$ and dividing by the absolute value of the derivative of the argument of the delta function with respect to $\cos\theta_2'$. Clearly the argument can be written as $s^2 + q\mathbf{s}\cdot\mathbf{u_{12}} = (1-q)s^2 + q\mathbf{s}\cdot\mathbf{u} - q\mathbf{s}\cdot\mathbf{u_2}$. Thus, the absolute value of the derivative of the argument with respect to $\cos\theta_2'$ is qsu_2. Hence, the result of the integration is:

$$\frac{1}{q^2 s u_2} F\left(\frac{\hat{\mathbf{s}}\cdot\mathbf{u}}{u_2} + \frac{1-q}{q}\frac{s}{u_2}\right) H\left(u_2 - |\frac{1-q}{q}s + \hat{\mathbf{s}}\cdot\mathbf{u}|\right). \tag{D.13}$$

E Derivation of I_1 and I_2

In this Appendix the derivation of the integrals I_1 and I_2 is presented. Recalling eq. (43), I_1 is given by:

$$I_1 = \int d\tilde{\mathbf{u}}_1 \overline{\eta}(\tilde{u}_1) e^{-\tilde{u}_1^2} \tilde{\mathbf{L}}(\Phi_{KK}), \tag{E.1}$$

The term $\tilde{\mathbf{L}}(\Phi_{KK})$ can be extracted from eq. (12) by retaining terms of $\mathcal{O}(K^2)$. One obtains:

$$\tilde{\mathbf{L}}(\Phi_{KK}) = \tilde{\mathbf{D}}_{KK}\log n + 2\sqrt{\frac{3}{2\Theta}}\tilde{u}_i\tilde{\mathbf{D}}_{KK}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathbf{D}}_{KK}\log\Theta +$$
$$\Phi_K\left(\tilde{\mathbf{D}}_K\log n + 2\sqrt{\frac{3}{2\Theta}}\tilde{u}_i\tilde{\mathbf{D}}_K V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathbf{D}}_K\log\Theta\right) + \tilde{\mathbf{D}}_K\Phi_K - \frac{1}{2}\tilde{\mathbf{\Omega}}(\Phi_K, \Phi_K), \tag{E.2}$$

The operation of $\tilde{\mathbf{D}}$ on the hydrodynamic fields is given in eqs. (13-15). At $\mathcal{O}(K^2)$ it reads:

$$\tilde{\mathbf{D}}_{KK}\log n = 0, \tag{E.3}$$

$$\tilde{\mathbf{D}}_{KK}V_i = -K\frac{1}{n}\sqrt{\frac{3}{2\Theta}}\frac{\partial P^K_{ij}}{\partial\tilde{r}_j} = K^2\sqrt{\frac{3}{2}}\tilde{\mu}_0\left(\frac{\partial\log\Theta}{\partial\tilde{r}_j}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j} + 2\frac{\partial}{\partial\tilde{r}_j}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\right), \tag{E.4}$$

and,

$$\tilde{\mathbf{D}}_{KK}\log\Theta = -K^2\frac{2}{n\Theta}\sqrt{\frac{3}{2\Theta}}\left(P^K_{ij}\frac{\partial V_i}{\partial\tilde{r}_j} + \frac{\partial Q^K_j}{\partial\tilde{r}_j}\right) =$$
$$= K^2\left(\frac{8}{3}\sqrt{\frac{3}{2}}\tilde{\mu}_0\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j} + 2\sqrt{\frac{3}{2}}\tilde{\kappa}_0\frac{\partial^2\log\Theta}{\partial\tilde{r}_j\partial\tilde{r}_j} + 3\sqrt{\frac{3}{2}}\tilde{\kappa}_0\frac{\partial\log\Theta}{\partial\tilde{r}_j}\frac{\partial\log\Theta}{\partial\tilde{r}_j}\right). \tag{E.5}$$

Next, using the result of the operation of $\tilde{\mathbf{D}}_K$ on the hydrodynamic fields (cf. eqs. (19-21)), and the explicit form of Φ_K (cf. eq. (23)) one obtains;

$$\Phi_K\left(\tilde{\mathbf{D}}_K\log n + 2\sqrt{\frac{3}{2\Theta}}\tilde{u}_i\tilde{\mathbf{D}}_K V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathbf{D}}_K\log\Theta\right) =$$

$$K^2\left[4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_\ell\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial V_k}{\partial\tilde{r}_\ell}-\frac{4}{3}\hat{\Phi}_v(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial V_k}{\partial\tilde{r}_k}+\right.$$
$$2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial\log n}{\partial\tilde{r}_k}+2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial\log\Theta}{\partial\tilde{r}_k}-$$
$$2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial\log(n\Theta)}{\partial\tilde{r}_k}+2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\partial V_k}{\partial\tilde{r}_j}\frac{\partial\log\Theta}{\partial\tilde{r}_i}-$$
$$\frac{2}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\frac{\partial V_j}{\partial\tilde{r}_j}\frac{\partial\log\Theta}{\partial\tilde{r}_i}+\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\log\Theta}{\partial\tilde{r}_i}\frac{\partial\log n}{\partial\tilde{r}_j}+$$
$$\left.\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\log\Theta}{\partial\tilde{r}_i}\frac{\partial\log\Theta}{\partial\tilde{r}_j}-\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\log\Theta}{\partial\tilde{r}_i}\frac{\partial\log(n\Theta)}{\partial\tilde{r}_j}\right]. \tag{E.6}$$

Next, we consider $\tilde{\mathbf{D}}_K\Phi_K$. Firstly we write Φ_K in dimensional form:

$$\Phi_K(\tilde{\mathbf{u}})=2\ell\hat{\Phi}_v(\tilde{u})\overline{u_iu_j}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_j}+\ell\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)u_i\sqrt{\frac{3}{2\Theta}}\frac{\partial\log\Theta}{\partial r_i}, \tag{E.7}$$

Next, we calculate $\tilde{\mathbf{D}}_K\Phi_K$ separately for each of the terms on the RHS of eq. (E.7). First:

$$\frac{\partial\Phi_K}{\partial\ell}\tilde{\mathbf{D}}_K\ell=-2K^2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial\log n}{\partial\tilde{r}_k}\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}-\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\frac{\partial V_k}{\partial\tilde{r}_k}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\right)$$
$$-K^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}_i\tilde{u}_j\frac{\partial\log n}{\partial\tilde{r}_i}\frac{\partial\log\Theta}{\partial\tilde{r}_j}-\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\frac{\partial V_j}{\partial\tilde{r}_j}\frac{\partial\log\Theta}{\partial\tilde{r}_i}\right). \tag{E.8}$$

The term $\tilde{\mathbf{D}}_K\ell$ has been calculated using eq. (19) and the relation $\ell=\frac{1}{\pi nd^2}$. Next, using the relation $\frac{\partial\tilde{u}^2}{\partial V_i}=-\frac{3}{\Theta}u_i$, one obtains:

$$\frac{\partial\Phi_K}{\partial V_i}\tilde{\mathbf{D}}_KV_i=-2K^2\left[2\hat{\Phi}'_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_\ell\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\overline{\partial V_k}}{\partial\tilde{r}_\ell}-\hat{\Phi}'_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial\log(n\Theta)}{\partial\tilde{r}_k}+\right.$$
$$\left.+2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_k}\frac{\overline{\partial V_k}}{\partial\tilde{r}_j}-\hat{\Phi}_v(\tilde{u})\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_k}\frac{\partial\log(n\Theta)}{\partial\tilde{r}_k}\right]$$
$$-K^2\left[\left(\hat{\Phi}_c(\tilde{u})+\hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right)\left(2\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial\log\Theta}{\partial\tilde{r}_k}-\tilde{u}_i\tilde{u}_j\frac{\partial\log\Theta}{\partial\tilde{r}_i}\frac{\partial\log(n\Theta)}{\partial\tilde{r}_j}\right)+\right.$$

$$\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\overline{\frac{\partial V_j}{\partial \tilde{r}_i}}\frac{\partial \log\Theta}{\partial \tilde{r}_j}-\frac{1}{2}\frac{\partial \log\Theta}{\partial \tilde{r}_i}\frac{\partial \log(n\Theta)}{\partial \tilde{r}_i}\right)\Bigg]. \tag{E.9}$$

Next, the relation $\frac{\partial \tilde{u}^2}{\partial \Theta}=-\frac{\tilde{u}^2}{\Theta}$ is employed to obtain:

$$\frac{\partial \Phi_K}{\partial \log\Theta}\tilde{\mathbf{D}}_K\log\Theta = K^2\left[2\left(\hat{\Phi}_v(\tilde{u})+\frac{2}{3}\hat{\Phi}'_v(\tilde{u})\tilde{u}^2\right)\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\frac{\partial V_k}{\partial \tilde{r}_k}-\right.$$

$$\left.\left(3\hat{\Phi}_v(\tilde{u})+2\hat{\Phi}'_v(\tilde{u})\tilde{u}^2\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\frac{\partial \log\Theta}{\partial \tilde{r}_k}\right]$$

$$-K^2\left[\left(\hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2+\hat{\Phi}_c(\tilde{u})\left(\frac{3}{2}\tilde{u}^2-\frac{5}{4}\right)\right)\tilde{u}_i\tilde{u}_j\frac{\partial \log\Theta}{\partial \tilde{r}_i}\frac{\partial \log\Theta}{\partial \tilde{r}_j}-\right.$$

$$\left.\frac{2}{3}\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\frac{\partial V_j}{\partial \tilde{r}_j}\frac{\partial \log\Theta}{\partial \tilde{r}_i}\right]. \tag{E.10}$$

In addition to the above dependence of Φ_K on the hydrodynamic fields, it also depends on the spatial derivatives of $\mathbf{V}$ and $\log\Theta$. Thus, one has to consider the following contributions to $\tilde{\mathbf{D}}_K\Phi_K$ as well:

$$\frac{\partial \Phi_K}{\partial\left(\frac{\partial V_i}{\partial r_j}\right)}\tilde{\mathbf{D}}_K\frac{\partial V_i}{\partial r_j} = -2K^2\left[\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial \tilde{r}_k}\frac{\partial V_k}{\partial \tilde{r}_j}}-\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\sqrt{\frac{3}{2\Theta}}\frac{\partial}{\partial \tilde{r}_k}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}+\right.$$

$$\left.\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial \log\Theta}{\partial \tilde{r}_i}\frac{\partial \log(n\Theta)}{\partial \tilde{r}_j}}+\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial^2 \log(n\Theta)}{\partial \tilde{r}_i\partial \tilde{r}_j}}\right], \tag{E.11}$$

where $\tilde{\mathbf{D}}_K\frac{\partial V_i}{\partial r_j}$ is obtained by employing (the definition) $\tilde{\mathbf{D}}=\ell\sqrt{\frac{3}{2\Theta}}\mathbf{D}$ and using the fact that $\mathbf{D}$ and $\frac{\partial}{\partial r_i}$ commute (cf. Appendix C). Similarly, one obtains:

$$\frac{\partial \Phi_K}{\partial\left(\frac{\partial \log\Theta}{\partial r_i}\right)}\tilde{\mathbf{D}}_K\frac{\partial \log\Theta}{\partial r_i} = K^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left[\tilde{u}_i\tilde{u}_j\frac{\partial^2 \log\Theta}{\partial \tilde{r}_j\partial \tilde{r}_j}-\right.$$

$$\left.\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\frac{\partial V_j}{\partial \tilde{r}_i}\frac{\partial \log\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\sqrt{\frac{3}{2\Theta}}\frac{\partial^2 V_j}{\partial \tilde{r}_i\partial \tilde{r}_j}\right]. \tag{E.12}$$

The sum of all of the above contributions is a cumbersome expression for $\tilde{\mathbf{L}}(\Phi_{KK})$. Notice, however, that this expression has to be multiplied by $\overline{\eta}(\tilde{u})e^{-\tilde{u}^2}$ and integrated over $\tilde{\mathbf{u}}$ (cf.

eq. (E.1)). The isotropy of the latter function and the orthogonality relations which it satisfies imply that all terms in $\tilde{\mathbf{L}}(\Phi_{KK})$ which are proportional to the summational invariants or whose tensorial structure is: $\tilde{u}_i\tilde{u}_j\tilde{u}_k$ or $\overline{\tilde{u}_i\tilde{u}_j}$, do not contribute to the above integral (with $\overline{\eta}(\tilde{u})e^{-\tilde{u}^2}$). After omitting the non contributing terms, the integral in (E.1) assumes the form:

$$\int d\tilde{\mathbf{u}}\overline{\eta}(\tilde{u})e^{-\tilde{u}^2}\tilde{\mathbf{L}}(\Phi_{KK}) =$$

$$K^2\int d\tilde{\mathbf{u}}\overline{\eta}(\tilde{u})e^{-\tilde{u}^2}\left[4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_\ell\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\partial V_k}{\partial\tilde{r}_\ell} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\text{log}\Theta}{\partial\tilde{r}_i}\frac{\partial\text{log}(n\Theta)}{\partial\tilde{r}_j}+\right.$$
$$\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\text{log}\Theta}{\partial\tilde{r}_i}\frac{\partial\text{log}\Theta}{\partial\tilde{r}_j} - 4\hat{\Phi}'_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_\ell\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_j}\frac{\overline{\partial V_k}}{\partial\tilde{r}_\ell}-$$
$$4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial\tilde{r}_k}\frac{\overline{\partial V_k}}{\partial\tilde{r}_j} + \left(\hat{\Phi}_c(\tilde{u})+\hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right)\tilde{u}_i\tilde{u}_j\frac{\partial\text{log}\Theta}{\partial\tilde{r}_i}\frac{\partial\text{log}(n\Theta)}{\partial\tilde{r}_j}+$$
$$\frac{1}{2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\text{log}\Theta}{\partial\tilde{r}_i}\frac{\partial\text{log}(n\Theta)}{\partial\tilde{r}_i} + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial^2\text{log}\Theta}{\partial\tilde{r}_i\partial\tilde{r}_j}-$$
$$\left.\left(\hat{\Phi}'_c(\tilde{u})\tilde{u}^2\left(\tilde{u}^2-\frac{5}{2}\right)+\hat{\Phi}_c(\tilde{u})\left(\frac{3}{2}\tilde{u}^2-\frac{5}{4}\right)\right)\tilde{u}_i\tilde{u}_j\frac{\partial\text{log}\Theta}{\partial\tilde{r}_i}\frac{\partial\text{log}\Theta}{\partial\tilde{r}_j}\right] - \frac{1}{2}\int d\tilde{\mathbf{u}}\overline{\eta}(\tilde{u})e^{-\tilde{u}^2}\tilde{\mathbf{\Omega}}(\Phi_K,\Phi_K). \tag{E.13}$$

The first integral in eq. (E.13) is carried out rather simply; first an integration over the angular values of $\tilde{\mathbf{u}}$ is performed. The following results are used:

$$\int d\hat{\tilde{\mathbf{u}}}\tilde{u}_i\tilde{u}_j = \frac{4\pi}{3}\delta_{ij}. \tag{E.14}$$

$$\int d\hat{\tilde{\mathbf{u}}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_\ell = \frac{4\pi}{15}(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}). \tag{E.15}$$

The remaining integration over $\tilde{u}$ is then performed numerically. The second integral in eq. (E.13) is carried out using a more complicated procedure. Using the explicit form of Φ_K it reads:

$$\frac{1}{2}\int d\tilde{\mathbf{u}}\overline{\eta}(\tilde{u})e^{-\tilde{u}^2}\tilde{\mathbf{\Omega}}(\Phi_K,\Phi_K) = \frac{K^2}{\pi^{\frac{5}{2}}}\int_{\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}}>0} d\hat{\mathbf{k}}d\tilde{\mathbf{u}}_{\mathbf{1}}d\tilde{\mathbf{u}}_{\mathbf{2}}(\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}})e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\overline{\eta}(\tilde{u}_1)\times$$

$$\left[4\hat{\Phi}_v(\tilde{u}_1')\hat{\Phi}_v(\tilde{u}_2')\tilde{u}_{1i}'\tilde{u}_{1j}'\tilde{u}_{2k}'\tilde{u}_{2\ell}'\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial \tilde{r}_j}\frac{\overline{\partial V_k}}{\partial \tilde{r}_\ell}+\hat{\Phi}_c(\tilde{u}_1')\hat{\Phi}_c(\tilde{u}_2')\left(\tilde{u}_1'^2-\frac{5}{2}\right)\left(\tilde{u}_2'^2-\frac{5}{2}\right)\tilde{u}_{1i}'\tilde{u}_{2j}'\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_j}\right]$$
$$-\frac{K^2}{\pi^{\frac{5}{2}}}\int_{\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}}>0} d\hat{\mathbf{k}}d\tilde{\mathbf{u}}_{\mathbf{1}}d\tilde{\mathbf{u}}_{\mathbf{2}}(\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}})e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\overline{\eta}(\tilde{u}_1)\times$$
$$\left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2\ell}\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial \tilde{r}_j}\frac{\overline{\partial V_k}}{\partial \tilde{r}_\ell}+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_j}\right]. \tag{E.16}$$

Notice that mixed terms, i.e. those which involve products of velocity gradients and temperature gradients are omitted since their contributions vanish by symmetry. Denote the first integral in eq. (E.16) by J_1; this integral can be evaluated by methods similar to those used in Appendix C. Hence, transforming to the primed integration variables, using the elastic relations $d\tilde{\mathbf{u}}_{\mathbf{1}}'d\tilde{\mathbf{u}}_{\mathbf{2}}' = d\tilde{\mathbf{u}}_{\mathbf{1}}d\tilde{\mathbf{u}}_{\mathbf{2}}$, $\tilde{u}_1'^2+\tilde{u}_2'^2=\tilde{u}_1^2+\tilde{u}_2^2$ and $\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}}' = -\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}}$ and exchanging between primed and unprimed quantities, one obtains:

$$J_1 = \frac{K^2}{\pi^{\frac{5}{2}}}\int_{\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}}>0} d\hat{\mathbf{k}}d\tilde{\mathbf{u}}_{\mathbf{1}}d\tilde{\mathbf{u}}_{\mathbf{2}}(\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{\mathbf{12}})e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\overline{\eta}(\tilde{u}_1')\times$$
$$\left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2\ell}\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial \tilde{r}_j}\frac{\overline{\partial V_k}}{\partial \tilde{r}_\ell}+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_j}\right]. \tag{E.17}$$

The integral in eq. (E.17) can be written in the following form:

$$J_1 = \frac{K^2}{\pi^{\frac{5}{2}}}\int d\tilde{\mathbf{u}}d\tilde{\mathbf{u}}_{\mathbf{1}}d\tilde{\mathbf{u}}_{\mathbf{2}}e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\overline{\eta}(\tilde{u})I_\delta^{(0)}\times$$
$$\left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2\ell}\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial \tilde{r}_j}\frac{\overline{\partial V_k}}{\partial \tilde{r}_\ell}+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_j}\right], \tag{E.18}$$

where $I_\delta^{(0)}$ is given in eq. (D.1) with $q=1$. Using manipulations similar to those leading to the evaluation of the integrals in eqs. (C.16) and (C.43) one obtains:

$$J_1 = \frac{K^2}{\pi^{\frac{5}{2}}}\frac{64\pi^3}{15}\frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial \tilde{r}_j}\frac{\overline{\partial V_i}}{\partial \tilde{r}_j}\int_{-1}^{1}dy\int_0^\infty d\tilde{u}\tilde{u}^2\overline{\eta}(\tilde{u})\int_0^\infty d\tilde{s}\tilde{s}\int_0^\infty d\tilde{u}_2\tilde{u}_2(2\tilde{u}^2y^2-\tilde{u}_2^2)\times$$

$$\left(\tilde{s}^2 - 2\tilde{u}\tilde{s}y + \frac{1}{2}\tilde{u}^2(3y^2-1)\right)\hat{\Phi}_v(\sqrt{\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2})\hat{\Phi}_v(\sqrt{\tilde{u}_2^2 + \tilde{u}^2y^2})e^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}e^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}$$

$$+\frac{K^2}{\pi^{\frac{5}{2}}}\frac{16\pi^3}{3}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_j}\int_{-1}^{1} dyy \int_0^\infty d\tilde{u}\tilde{u}^3\overline{\eta}(\tilde{u})\int_0^\infty d\tilde{s}\tilde{s}\int_0^\infty d\tilde{u}_2\tilde{u}_2(\tilde{u}y - \tilde{s})\times$$

$$\hat{\Phi}_c(\sqrt{\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2})\hat{\Phi}_c(\sqrt{\tilde{u}_2^2 + \tilde{u}^2y^2})\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right)\times$$

$$e^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}e^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \tag{E.19}$$

The second integral in eq. (E.16) reads, after integrating over $\hat{\mathbf{k}}$:

$$J_2 = \frac{K^2}{\pi^{\frac{3}{2}}}\frac{6}{\Theta}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\,\overline{\frac{\partial V_k}{\partial \tilde{r}_\ell}}\int d\tilde{\mathbf{u}}_1 d\tilde{\mathbf{u}}_2 \tilde{u}_{12}e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\overline{\eta}(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2\ell}$$

$$+\frac{K^2}{\pi^{\frac{3}{2}}}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_j}\int d\tilde{\mathbf{u}}_1 d\tilde{\mathbf{u}}_2 \tilde{u}_{12}e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\overline{\eta}(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}. \tag{E.20}$$

These integrals are carried out by similar methods to those used in Appendix C - The tensorial structure of the integrands implies that the result of the first integral is proportional to the general isotropic fourth order tensor $a\delta_{ij}\delta_{k\ell} + b(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$ and the second integral is proportional to the general isotropic second order tensor δ_{ij}. Thus, without loss of generality, one may choose in the first integral $i = k = 1$ (x direction), $j = \ell = 2$ (y direction) and multiply the integral by 2. In the second integral we set $i = j = 1$. Next, one integrates over all orientations of $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$. The result is:

$$J_2 = 2\sqrt{\pi}K^2\left[\frac{32}{15}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\,\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\int_0^\infty d\tilde{u}_1\int_0^\infty d\tilde{u}_2\tilde{u}_1^4\tilde{u}_2^4 R_2(\tilde{u}_1,\tilde{u}_2)\overline{\eta}(\tilde{u}_1)e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\right.$$

$$\left.+\frac{4}{3}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\frac{\partial \mathrm{log}\Theta}{\partial \tilde{r}_i}\int_0^\infty d\tilde{u}_1\int_0^\infty d\tilde{u}_2\tilde{u}_1^3\tilde{u}_2^3 R_1(\tilde{u}_1,\tilde{u}_2)\overline{\eta}(\tilde{u}_1)e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right)\right], \tag{E.21}$$

where

$$R_n(\tilde{u}_1, \tilde{u}_2) \equiv \int_{-1}^{1} dx \sqrt{\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 + \tilde{u}_2^2} P_n(x), \tag{E.22}$$

with P_n being the n'th order Legendre polynomial. A numerical evaluation of the integrals in eqs. (E.13,E.19,E.21) yields:

$$I_1 = K^2 \left(\tilde{\alpha}_1 \frac{3}{2\Theta} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} + \tilde{\alpha}_2 \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} + \tilde{\alpha}_3 \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \frac{\partial \text{log}(n\Theta)}{\partial \tilde{r}_i} + \tilde{\alpha}_4 \frac{\partial^2 \text{log}\Theta}{\partial \tilde{r}_i \partial \tilde{r}_i} \right), \tag{E.23}$$

where, $\tilde{\alpha}_1 \approx 12.6469$, $\tilde{\alpha}_2 \approx 73.1575$, $\tilde{\alpha}_3 \approx -19.0089$ and $\tilde{\alpha}_4 \approx 15.7588$

Next, we consider the integral I_2, defined in eq. (40). Substituting the explicit form of Φ_K and omitting mixed terms, whose contributions vanish by symmetry, one obtains:

$$I_2 = K^2 \int d\tilde{\mathbf{u}}_1 d\tilde{\mathbf{u}}_2 \tilde{u}_{12}^3 e^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2\ell} \frac{3}{2\Theta} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} \frac{\overline{\partial V_k}}{\partial \tilde{r}_\ell} \right.$$
$$\left. \hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1i}\tilde{u}_{2j} \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \frac{\partial \text{log}\Theta}{\partial \tilde{r}_j} \right]. \tag{E.24}$$

This integral has been evaluated in a similar way to that used for the calculation of J_2. The result is:

$$J_2 = K^2 \left[\frac{64\pi^2}{15} \frac{3}{2\Theta} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} \int_0^\infty d\tilde{u}_1 \int_0^\infty d\tilde{u}_2 \tilde{u}_1^4 \tilde{u}_2^4 S_2(\tilde{u}_1, \tilde{u}_2) e^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2) \right.$$
$$\left. + \frac{8\pi^2}{3} \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \int_0^\infty d\tilde{u}_1 \int_0^\infty d\tilde{u}_2 \tilde{u}_1^3 \tilde{u}_2^3 S_1(\tilde{u}_1, \tilde{u}_2) e^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \right], \tag{E.25}$$

where

$$S_n(\tilde{u}_1, \tilde{u}_2) \equiv \int_{-1}^{1} dx (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 + \tilde{u}_2^2)^{\frac{3}{2}} P_n(x). \tag{E.26}$$

A numerical evaluation of the integrals in eq. (E.26) yields:

$$I_2 = K^2 \left(\tilde{\beta}_1 \frac{3}{2\Theta} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} \frac{\overline{\partial V_i}}{\partial \tilde{r}_j} + \tilde{\beta}_2 \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \frac{\partial \text{log}\Theta}{\partial \tilde{r}_i} \right), \tag{E.27}$$

where, $\tilde{\beta}_1 \approx 15.3412$ and $\tilde{\beta}_2 \approx -3.4190$.

Please Notice: the table below is referred to in Appendix A

Table 1

n	$a_n^{(v)}$	$a_n^{(c)}$	$a_n^{(e)}$	$a_n^{(\eta)}$
1	-1.1000	-1.6166	-0.4199	-13.5770
2	-1.8487	-2.7330	-0.5091	-13.4559
3	-1.2733	-2.6952	-0.2378	-1.5773
4	-0.2646	1.6729	0.0044	0.1282
5	-1.7862	-4.5166	-0.0441	2.5801
6	3.6306	-13.8131	0.0042	-0.6866
7	-7.3334	60.8587	0.0400	-2.2654
8	9.2746	-107.7763	-0.0419	6.5680
9	-7.1014	91.9285	0.0311	-5.9880
10	2.3928	-32.8111	-0.0043	2.4564