Online appendix for the paper

Approximation Fixpoint Theory and the Well-Founded Semantics of Higher-Order Logic Programs

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Appendix A: Mathematical Preliminaries and Proofs of Section 4

A partially ordered set (or poset) $(L, \leq)$ is called a *lattice* if for all $x, y \in L$ there exists a least upper bound and a greatest lower bound. A lattice $(L, \leq)$ is called *complete* if for all $S \subseteq L$, there exists a least upper bound and a greatest lower bound, denoted by $\bigvee S$ and $\bigwedge S$ respectively. Every complete lattice has a least element and a greatest element, denoted by $\perp$ and $\top$ respectively. We will use the following two convenient equivalent definitions of complete lattices (Davey and Priestley 2002, Theorem 2.31, page 47):

Theorem 4
A partially ordered set $(L, \leq)$ is a complete lattice if L has a least element and every non-empty subset $S \subseteq L$ has a least upper bound in L. Alternatively, $(L, \leq)$ is a complete lattice if L has a greatest element and every non-empty subset $S \subseteq L$ has a greatest lower bound in L.

Given a partially ordered set $(L, \leq)$, every linearly ordered subset S of L will be called a *chain*. A partially ordered set is *chain-complete* if it has a least element $\perp$ and every chain $S \subseteq L$ has a least upper bound.

Proposition 1
Let D be a nonempty set. For every predicate type π, $([\![\pi]\!]_D, \leq_\pi)$ is a complete lattice and $([\![\pi]\!]_D, \preceq_\pi)$ is a chain complete poset.

Proof
Consider the first statement and let π be an arbitrary predicate type. Recall that $\perp_{\leq_\pi}$ exists; it suffices to show that for every non-empty subset S of $[\![\pi]\!]_D$, the least upper bound of S exists and belongs to $[\![\pi]\!]_D$.

The least upper bound can be defined inductively on the structure of predicate types. If $\pi = o$, then $\bigvee_{\leq_o} S$ is defined in the obvious way. For $\pi = \iota \to \pi_1$, we define for all $d \in D$, $(\bigvee_{\leq_{\iota \to \pi_1}} S)(d) = \bigvee_{\leq_{\pi_1}} \{f(d) \mid f \in S\}$. Finally, if $\pi = \pi_1 \to \pi_2$, we define for all $d \in [\![\pi_1]\!]_D$, $(\bigvee_{\leq_{\pi_1 \to \pi_2}} S)(d) = \bigvee_{\leq_{\pi_2}} \{f(d) \mid f \in S\}$. We need to verify that for type $\pi_1 \to \pi_2$ the least upper bound is a Fitting-monotonic function. This is a consequence of the following auxiliary statement, which we need to establish for every predicate type π:

Auxiliary statement: Let I be a non-empty index-set and let $d_i, d'_i \in [\![\pi]\!]_D$, $i \in I$. If for all $i \in I$, $d_i \preceq_\pi d'_i$, then $\bigvee_{\leq_\pi} \{d_i \mid i \in I\} \preceq_\pi \bigvee_{\leq_\pi} \{d'_i \mid i \in I\}$.

The proof of the auxiliary statement is by a simple induction on the structure of π. For type $\pi = o$ the statement follows by a case analysis on the value of $\bigvee_{\leq_\pi}\{d_i \mid i \in I\}$. For types $\iota \to \pi_1$ and $\pi_1 \to \pi_2$, the statement follows directly by the induction hypothesis. The auxiliary statement implies that $(\bigvee_{\leq_{\pi_1 \to \pi_2}} S)$ is a Fitting-monotonic function. More specifically, for all $d, d' \in [\![\pi_1]\!]_D$ with $d \preceq_{\pi_1} d'$, it holds $f(d) \preceq_{\pi_2} f(d')$ for every $f \in S$ (because the members of S are Fitting-monotonic functions). Then, the auxiliary statement implies that $\bigvee_{\leq_{\pi_2}}\{f(d) \mid f \in S\} \preceq_{\pi_2} \bigvee_{\leq_{\pi_2}}\{f(d) \mid f \in S\}$ which is equivalent to $(\bigvee_{\leq_{\pi_1 \to \pi_2}} S)(d) \preceq_{\pi_2} (\bigvee_{\leq_{\pi_1 \to \pi_2}} S)(d')$, which means that $(\bigvee_{\leq_{\pi_1 \to \pi_2}} S)$ is Fitting-monotonic.

Consider now the second statement. Notice that $([\![\pi]\!]_D, \preceq_\pi)$ is not a complete lattice (for example, the set $\{\mathit{false}, \mathit{true}\}$ does not have a least upper bound with respect to $\preceq_o$). However, it is a chain complete poset. For every type π, $\perp_{\preceq_\pi}$ exists. Moreover, given a chain S of elements of $[\![\pi]\!]_D$, it suffices to verify that $\bigvee_{\preceq_\pi} S$ exists and belongs to $[\![\pi]\!]_D$. The proof is by induction on the structure of π. For type $\pi = o$ it is obvious. For $\pi = \iota \to \pi_1$, define $(\bigvee_{\preceq_{\iota \to \pi_1}} S)(d) = \bigvee_{\preceq_{\pi_1}}\{f(d) \mid f \in S\}$. For $\pi = \pi_1 \to \pi_2$ define $(\bigvee_{\preceq_{\pi_1 \to \pi_2}} S)(d) = \bigvee_{\preceq_{\pi_2}}\{f(d) \mid f \in S\}$. We need to verify that $(\bigvee_{\preceq_{\pi_1 \to \pi_2}} S)$ is a Fitting-monotonic function, i.e., that for all $d, d' \in [\![\pi_1]\!]_D$ with $d \preceq_{\pi_1} d'$, it holds $(\bigvee_{\preceq_{\pi_1 \to \pi_2}} S)(d) \preceq_{\pi_2} (\bigvee_{\preceq_{\pi_1 \to \pi_2}} S)(d')$, or equivalently that $\bigvee_{\preceq_{\pi_2}}\{f(d) \mid f \in S\} \preceq_{\pi_2} \bigvee_{\preceq_{\pi_2}}\{f(d') \mid f \in S\}$, which holds because for every $f \in S$, $f(d) \preceq_{\pi_2} f(d')$. $\square$

The proof of the above lemma has as a direct consequence the following corollary:

Corollary 1
Let D be a nonempty set and π a predicate type. Let I be a non-empty index-set and let $d_i, d'_i \in [\![\pi]\!]_D$, $i \in I$. If for all $i \in I$, $d_i \preceq_\pi d'_i$, then $\bigvee_{\leq_\pi}\{d_i \mid i \in I\} \preceq_\pi \bigvee_{\leq_\pi}\{d'_i \mid i \in I\}$.

Appendix B: Proofs of Section 5

Proposition 2
Let D be a nonempty set. For every predicate type π, $([\![\pi]\!]_D^{\mathsf{ma}}, \leq_\pi)$ and $([\![\pi]\!]_D^{\mathsf{am}}, \leq_\pi)$ are complete lattices.

Proof
We give the proof for the case $([\![\pi]\!]_D^{\mathsf{ma}}, \leq_\pi)$; the case $([\![\pi]\!]_D^{\mathsf{am}}, \leq_\pi)$ is symmetrical and omitted. The proof is by induction on the structure of π. For $\pi = o$ the result is immediate. We show the result for types $\iota \to \pi$ and $\pi_1 \to \pi_2$, assuming it holds for π, π_1 and π_2.

Consider first the set $[\![\iota \to \pi]\!]_D^{\mathsf{ma}} = D \to [\![\pi]\!]^{\mathsf{ma}}$. This set has a least element, namely the function that assigns to each $d \in D$ the bottom element of type π. Let $S \subseteq D \to [\![\pi]\!]^{\mathsf{ma}}$ be a nonempty set. For every $d \in D$ we define $(\bigvee_{\leq_{\iota \to \pi}} S)(d) = \bigvee_{\leq_\pi}\{f(d) \mid f \in S\}$, which by the induction hypothesis exists and belongs to $[\![\pi]\!]_D^{\mathsf{ma}}$.

Consider now the set $[\![\pi_1 \to \pi_2]\!]_D^{\mathsf{ma}} = [([\![\pi_1]\!]_D^{\mathsf{ma}} \otimes [\![\pi_1]\!]_D^{\mathsf{am}}) \overset{\mathsf{ma}}{\to} [\![\pi_2]\!]_D^{\mathsf{ma}}]$. This set has a least element, namely the function that assigns to each pair $(x, y) \in ([\![\pi_1]\!]_D^{\mathsf{ma}} \otimes [\![\pi_1]\!]_D^{\mathsf{am}})$ the bottom element of type $\perp_{\pi_2}$; this function is constant and therefore obviously monotone-antimonotone. Let $S \subseteq [([\![\pi_1]\!]_D^{\mathsf{ma}} \otimes [\![\pi_1]\!]_D^{\mathsf{am}}) \overset{\mathsf{ma}}{\to} [\![\pi_2]\!]_D^{\mathsf{ma}}]$ be a nonempty set. For every $(x, y) \in ([\![\pi_1]\!]_D^{\mathsf{ma}} \otimes [\![\pi_1]\!]_D^{\mathsf{am}})$ we define $(\bigvee_{\leq_{\pi_1 \to \pi_2}} S)(x, y) = \bigvee_{\leq_{\pi_2}}\{f(x, y) \mid f \in S\}$, which by the induction hypothesis exists and belongs to $[\![\pi_2]\!]_D^{\mathsf{ma}}$. It remains to show that $\bigvee S$

is monotone-antimonotone. Consider $(x, y), (x', y') \in (\llbracket \pi_1 \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi_1 \rrbracket_D^{\mathsf{am}})$ and assume that $x \le x'$ and $y \ge y'$. It suffices to show that $(\bigvee_{\le_{\pi_1 \to \pi_2}} S)(x, y) \le_{\pi_2} (\bigvee_{\le_{\pi_1 \to \pi_2}} S)(x', y')$. Since every element of S is monotone-antimonotone, for every $f \in S$ it holds $f(x, y) \le_{\pi_2} f(x', y')$. Therefore, $\bigvee_{\le_{\pi_2}} \{f(x, y) \mid f \in S\} \le_{\pi_2} \bigvee_{\le_{\pi_2}} \{f(x', y') \mid f \in S\}$, and thus $(\bigvee S_{\le_{\pi_1 \to \pi_2}})(x, y) \le_{\pi_2} (\bigvee S_{\le_{\pi_1 \to \pi_2}})(x', y')$. $\square$

The proof of Proposition 3 requires the following lemma which can be established by induction on the structure of π:

Lemma 6
Let D be a nonempty set and let π be a predicate type. Let $S \subseteq \llbracket \pi \rrbracket_D^{\mathsf{ma}}$ and $g \in \llbracket \pi \rrbracket_D^{\mathsf{am}}$.

- If for all $f \in S$, $f \le g$, then $\bigvee S \le g$.
- If for all $f \in S$, $f \ge g$, then $\bigwedge S \ge g$.

Proposition 3
Let D be a nonempty set. For each predicate type π, $\llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}$ is a complete lattice with respect to $\le_\pi$ and a chain-complete poset with respect to $\preceq_\pi$.

Proof
For every π it is straightforward to define the bottom elements of the partially ordered sets $(\llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}, \le_\pi)$ and $(\llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}, \preceq_\pi)$.

Given $S \subseteq \llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}$, we define $\bigvee_{\le_\pi} S = (\bigvee_{\le_\pi} \{f \mid (f, g) \in S\}, \bigvee_{\le_\pi} \{g \mid (f, g) \in S\})$. It can be easily seen that $\bigvee_{\le_\pi} S \in \llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}$ due to Proposition 2, Lemma 6 and the fact that for every pair $(f, g) \in S$, $f \le_\pi g$.

On the other hand, let $S \subseteq \llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}$ be a chain. We define $\bigvee_{\preceq_\pi} S = (\bigvee_{\le_\pi} \{f \mid (f, g) \in S\}, \bigwedge_{\le_\pi} \{g \mid (f, g) \in S\})$. It is straightforward to show that $\bigvee_{\preceq_\pi} S$ is the $\preceq_\pi$-least upper bound of the chain. Moreover, $(\bigvee_{\preceq_\pi} S) \in \llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}$ because $\bigvee_{\le_\pi} \{f \mid (f, g) \in S\} \le_\pi \bigwedge_{\le_\pi} \{g \mid (f, g) \in S\}$ (this can easily be shown using basic properties of lubs and glbs, Lemma 6, and the fact that S is a chain; see also Proposition 2.3 in (Denecker et al. 2004)). $\square$

Proposition 4
Let D be a nonempty set and let π be a predicate type. Then, for every $f, g \in \llbracket \pi \rrbracket_D$ and for every $(f_1, f_2), (g_1, g_2) \in \llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}}$, the following statements hold:

1. $\tau_\pi(f) \in (\llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}})$ and $\tau_\pi^{-1}(f_1, f_2) \in \llbracket \pi \rrbracket_D$.
2. If $f \preceq_\pi g$ then $\tau_\pi(f) \preceq_\pi \tau_\pi(g)$.
3. If $f \le_\pi g$ then $\tau_\pi(f) \le_\pi \tau_\pi(g)$.
4. If $(f_1, f_2) \preceq_\pi (g_1, g_2)$ then $\tau_\pi^{-1}(f_1, f_2) \preceq_\pi \tau_\pi^{-1}(g_1, g_2)$.
5. If $(f_1, f_2) \le_\pi (g_1, g_2)$ then $\tau_\pi^{-1}(f_1, f_2) \le_\pi \tau_\pi^{-1}(g_1, g_2)$.

Proof
The five statements are shown by a simultaneous induction on the structure of π. We give the proofs for Statement 1, Statement 2 (the proof of Statement 3 is analogous and omitted) and Statement 4 (the proof of Statement 5 is similar and omitted).

The basis case is for $\pi = o$ and is straightforward for all statements. We assume the statements hold for π, π_1 and π_2. We demonstrate that they hold for $\iota \to \pi$ and for $\pi_1 \to \pi_2$.

Statement 1: Consider first the case of $\iota \to \pi$. It suffices to show that $\tau_{\iota\to\pi}(f) \in (\llbracket \iota \to \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \iota \to \pi \rrbracket_D^{\mathsf{am}})$. By the induction hypothesis, $\tau_\pi(f(d)) \in (\llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}})$. Therefore, $[\tau_\pi(f(d))]_1 \leq [\tau_\pi(f(d))]_2$, and consequently $(\lambda d.[\tau_\pi(f(d))]_1, \lambda d.[\tau_\pi(f(d))]_2) \in (\llbracket \iota \to \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \iota \to \pi \rrbracket_D^{\mathsf{am}})$. We next show that $\tau_\pi^{-1}(f_1, f_2) \in \llbracket \iota \to \pi \rrbracket_D$. Since $(f_1, f_2) \in (\llbracket \iota \to \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \iota \to \pi \rrbracket_D^{\mathsf{am}})$, $f_1 \leq f_2$ and $(f_1(d), f_2(d)) \in (\llbracket \pi \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi \rrbracket_D^{\mathsf{am}})$. By the induction hypothesis, $\tau_\pi^{-1}(f_1(d), f_2(d)) \in \llbracket \pi \rrbracket_D$ and $\lambda d.\tau_\pi^{-1}(f_1(d), f_2(d)) \in \llbracket \iota \to \pi \rrbracket_D$.

Consider the case $\pi_1 \to \pi_2$. We show that $\tau_{\pi_1\to\pi_2}(f) \in (\llbracket \pi_1 \to \pi_2 \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi_1 \to \pi_2 \rrbracket_D^{\mathsf{am}})$. Let $(d_1, d_2) \in (\llbracket \pi_1 \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi_1 \rrbracket_D^{\mathsf{am}})$. By the induction hypothesis $\tau_{\pi_1}^{-1}(d_1, d_2) \in \llbracket \pi_1 \rrbracket_D$, $f(\tau_{\pi_1}^{-1}(d_1, d_2)) \in \llbracket \pi_2 \rrbracket_D$, and $\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2))) \in (\llbracket \pi_1 \rrbracket_D^{\mathsf{ma}} \otimes \llbracket \pi_1 \rrbracket_D^{\mathsf{am}})$, which has as a direct consequence that $[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1 \leq [\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2$. Therefore, $\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1 \leq \lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2$. It remains to show that the function $\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1$ is monotone-antimonotone and the function $\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2$ is antimonotone-monotone. This follows by using the induction hypothesis for Statement 4, the Fitting-monotonicity of f, and the induction hypothesis of Statement 2. The fact that $\tau_{\pi_1\to\pi_2}^{-1}(f_1, f_2) \in \llbracket \pi_1 \to \pi_2 \rrbracket_D$ follows using similar arguments as above.

Statement 2: Consider first the case of $\iota \to \pi$. It suffices to show that:

$$(\lambda d.[\tau_\pi(f(d))]_1, \lambda d.[\tau_\pi(f(d))]_2) \preceq (\lambda d.[\tau_\pi(g(d))]_1, \lambda d.[\tau_\pi(g(d))]_2)$$

or equivalently that $\lambda d.[\tau_\pi(f(d))]_1 \leq \lambda d.[\tau_\pi(g(d))]_1$ and $\lambda d.[\tau_\pi(f(d))]_2 \geq \lambda d.[\tau_\pi(g(d))]_2$, or equivalently that for every d, $[\tau_\pi(f(d))]_1 \leq [\tau_\pi(g(d))]_1$ and $[\tau_\pi(f(d))]_2 \geq [\tau_\pi(g(d))]_2$. This holds because, since $f \preceq g$, it holds $f(d) \preceq g(d)$ and by the induction hypothesis, $\tau_\pi(f(d)) \preceq \tau_\pi(g(d))$. Consider now the case of $\pi_1 \to \pi_2$. It suffices to show that:

$$\begin{gathered}(\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1, \lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2) \preceq \\ (\lambda(d_1, d_2).[\tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1, \lambda(d_1, d_2).[\tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2)\end{gathered}$$

or equivalently that $\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1 \leq \lambda(d_1, d_2).[\tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1$ and $\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2 \geq \lambda(d_1, d_2).[\tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2$, or equivalently that for all d_1, d_2, $[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1 \leq [\tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1$ and $[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2 \geq [\tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2$. Since $f \preceq g$, it holds that $f(\tau_{\pi_1}^{-1}(d_1, d_2))) \preceq g(\tau_{\pi_1}^{-1}(d_1, d_2)))$, and by the induction hypothesis $\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2))) \preceq \tau_{\pi_2}(g(\tau_{\pi_1}^{-1}(d_1, d_2)))$, which is the desired result.

Statement 4: Consider first the case of $\iota \to \pi$. It suffices to show that:

$$\lambda d.\tau_\pi^{-1}(f_1(d), f_2(d)) \preceq \lambda d.\tau_\pi^{-1}(g_1(d), g_2(d))$$

or equivalently that for every d, $\tau_\pi^{-1}(f_1(d), f_2(d)) \preceq \tau_\pi^{-1}(g_1(d), g_2(d))$. Since $(f_1, f_2) \preceq (g_1, g_2)$, it holds $(f_1(d), f_2(d)) \preceq (g_1(d), g_2(d))$, and the result follows from the induction hypothesis. Consider now the case of $\pi_1 \to \pi_2$. It suffices to show that:

$$\lambda d.\tau_{\pi_2}^{-1}(f_1(\tau_{\pi_1}(d)), f_2(\tau_{\pi_1}(d))) \preceq \lambda d.\tau_{\pi_2}^{-1}(g_1(\tau_{\pi_1}(d)), g_2(\tau_{\pi_1}(d)))$$

or equivalently that for every d, $\tau_{\pi_2}^{-1}(f_1(\tau_{\pi_1}(d)), f_2(\tau_{\pi_1}(d))) \preceq \tau_{\pi_2}^{-1}(g_1(\tau_{\pi_1}(d)), g_2(\tau_{\pi_1}(d)))$. Since $(f_1, f_2) \preceq (g_1, g_2)$, it holds $(f_1(\tau_{\pi_1}(d)), f_2(\tau_{\pi_1}(d))) \preceq (g_1(\tau_{\pi_1}(d)), g_2(\tau_{\pi_1}(d)))$, and the result follows from the induction hypothesis. $\square$

Proposition 5

Let D be a nonempty set and let π be a predicate type. Then, for every $f \in [\![\pi]\!]_D$, $\tau_\pi^{-1}(\tau_\pi(f)) = f$, and for every $(f_1, f_2) \in [\![\pi]\!]_D^{\mathsf{ma}} \otimes [\![\pi]\!]_D^{\mathsf{am}}$, $\tau_\pi(\tau_\pi^{-1}(f_1, f_2)) = (f_1, f_2)$.

Proof

The proof of the two statements is by a simultaneous induction on the structure of π. The case $\pi = o$ is immediate. Assume the two statements hold for π, π_1 and π_2. We demonstrate that they hold for $\iota \to \pi$ and for $\pi_1 \to \pi_2$.

We have:

$$\begin{array}{rl}
& \tau_{\iota\to\pi}^{-1}(\tau_{\iota\to\pi}(f)) \;= \\
= & \tau_{\iota\to\pi}^{-1}(\lambda d.[\tau_\pi(f(d))]_1, \lambda d.[\tau_\pi(f(d))]_2) \\
& \text{(Definition of } \tau_{\iota\to\pi}) \\
= & \lambda d.\tau_\pi^{-1}([\tau_\pi(f(d))]_1, [\tau_\pi(f(d))]_2) \\
& \text{(Definition of } \tau_{\iota\to\pi}^{-1}) \\
= & \lambda d.\tau_\pi^{-1}(\tau_\pi(f(d))) \\
& \text{(Definition of } [\cdot]_1 \text{ and } [\cdot]_2) \\
= & \lambda d.f(d) \\
& \text{(Induction Hypothesis)} \\
= & f
\end{array}$$

Also:

$$\begin{array}{rl}
& \tau_{\iota\to\pi}(\tau_{\iota\to\pi}^{-1}(f_1, f_2)) \;= \\
= & \tau_{\iota\to\pi}(\lambda d.\tau_\pi^{-1}(f_1(d), f_2(d))) \\
& \text{(Definition of } \tau_{\iota\to\pi}^{-1}) \\
= & (\lambda d.[\tau_\pi(\tau_\pi^{-1}(f_1(d), f_2(d)))]_1, \lambda d.[\tau_\pi(\tau_\pi^{-1}(f_1(d), f_2(d)))]_2) \\
& \text{(Definition of } \tau_{\iota\to\pi}) \\
= & (\lambda d.[(f_1(d), f_2(d))]_1, \lambda d.[(f_1(d), f_2(d))]_2) \\
& \text{(Induction Hypothesis)} \\
= & (\lambda d.f_1(d), \lambda d.f_2(d)) \\
& \text{(Definition of } [\cdot]_1 \text{ and } [\cdot]_2) \\
= & (f_1, f_2)
\end{array}$$

Consider now the case of $\pi_1 \to \pi_2$. We have:

$$\begin{array}{rl}
& \tau_{\pi_1\to\pi_2}^{-1}(\tau_{\pi_1\to\pi_2}(f)) \;= \\
= & \tau_{\pi_1\to\pi_2}^{-1}(\lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_1, \lambda(d_1, d_2).[\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(d_1, d_2)))]_2) \\
& \text{(Definition of } \tau_{\pi_1\to\pi_2}) \\
= & \lambda d.\tau_{\pi_2}^{-1}([\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(\tau_{\pi_1}(d))))]_1, [\tau_{\pi_2}(f(\tau_{\pi_1}^{-1}(\tau_{\pi_1}(d))))]_2) \\
& \text{(Definition of } \tau_{\pi_1\to\pi_2}^{-1}) \\
= & \lambda d.\tau_{\pi_2}^{-1}([\tau_{\pi_2}(f(d))]_1, [\tau_{\pi_2}(f(d))]_2) \\
& \text{(Induction Hypothesis)} \\
= & \lambda d.\tau_{\pi_2}^{-1}(\tau_{\pi_2}(f(d))) \\
& \text{(Definition of } [\cdot]_1 \text{ and } [\cdot]_2) \\
= & \lambda d.f(d) \\
& \text{(Induction Hypothesis)} \\
= & f
\end{array}$$

Also:

$$
\begin{array}{rl}
& \tau_{\pi_1\to\pi_2}(\tau^{-1}_{\pi_1\to\pi_2}(f_1,f_2)) \;= \\
= & \tau_{\pi_1\to\pi_2}(\lambda d.\tau^{-1}_{\pi_2}(f_1(\tau_{\pi_1}(d)), f_2(\tau_{\pi_1}(d))) \\
& \text{(Definition of } \tau^{-1}_{\pi_1\to\pi_2}) \\
= & (\lambda(d_1,d_2).[\tau_{\pi_2}(\tau^{-1}_{\pi_2}(f_1(\tau_{\pi_1}(\tau^{-1}_{\pi_1}(d_1,d_2))), f_2(\tau_{\pi_1}(\tau^{-1}_{\pi_1}(d_1,d_2)))))]_1, \\
& \;\lambda(d_1,d_2).[\tau_{\pi_2}(\tau^{-1}_{\pi_2}(f_1(\tau_{\pi_1}(\tau^{-1}_{\pi_1}(d_1,d_2))), f_2(\tau_{\pi_1}(\tau^{-1}_{\pi_1}(d_1,d_2)))))]_2) \\
& \text{(Definition of } \tau_{\pi_1\to\pi_2}) \\
= & (\lambda(d_1,d_2).[f_1(d_1,d_2), f_2(d_1,d_2)]_1, \lambda(d_1,d_2).[f_1(d_1,d_2), f_2(d_1,d_2)]_2) \\
& \text{(Induction Hypothesis)} \\
= & (\lambda(d_1,d_2).f_1(d_1,d_2), \lambda(d_1,d_2).f_2(d_1,d_2)) \\
& \text{(Definition of } [\cdot]_1 \text{ and } [\cdot]_2) \\
= & (f_1,f_2)
\end{array}
$$

The above completes the proof of the proposition. □

Appendix C: An Extension of Consistent Approximation Fixpoint Theory

In this appendix we propose a mild extension of the theory of consistent approximating operators developed in (Denecker et al. 2004). We briefly highlight the main idea behind the work in (Denecker et al. 2004) and then justify the necessity for our extension.

Let $(L, \leq)$ be a complete lattice. The authors in (Denecker et al. 2004) consider the set $L^c = \{(x, y) \in L \times L \mid x \leq y\}$. Intuitively speaking, a pair $(x, y) \in L^c$ can be viewed as an approximation to all elements $z \in L$ such that $x \leq z \leq y$. An operator $A : L^c \to L^c$ is called in (Denecker et al. 2004) a *consistent approximating operator* if it is $\preceq$-monotone (see below) and for every $x \in L$, $A(x, x)_1 = A(x, x)_2$ (the subscripts 1 and 2 denote projection to the first and second elements respectively of the pair returned by A). In Section 3 of (Denecker et al. 2004), an elegant theory is developed whose purpose is to demonstrate how, under specific conditions, one can characterize the *well-founded fixpoint* of a given consistent approximating operator A. Since approximating operators emerge in many non-monotonic formalisms, the theory developed in (Denecker et al. 2004) provides a useful tool for the study of the semantics of such formalisms.

In our work, the immediate consequence operator T_{P} is not an approximating operator in the sense of (Denecker et al. 2004). More specifically, T_{P} is a function in $(\mathcal{H}^{\mathsf{ma}}_{\mathsf{P}} \otimes \mathcal{H}^{\mathsf{am}}_{\mathsf{P}}) \to (\mathcal{H}^{\mathsf{ma}}_{\mathsf{P}} \otimes \mathcal{H}^{\mathsf{am}}_{\mathsf{P}})$. In other words, there is not just a single lattice L involved in the definition of T_{P}, but instead two lattices, namely $\mathcal{H}^{\mathsf{ma}}_{\mathsf{P}}$ and $\mathcal{H}^{\mathsf{am}}_{\mathsf{P}}$. Moreover, the condition "for every $x \in L$, $A(x, x)_1 = A(x, x)_2$" required in (Denecker et al. 2004), does not hold in our case, because the two arguments of T_{P} range over two different sets (namely $\mathcal{H}^{\mathsf{ma}}_{\mathsf{P}}$ and $\mathcal{H}^{\mathsf{am}}_{\mathsf{P}}$). We therefore need to define an extension of the material in Section 3 of (Denecker et al. 2004), that suits our purposes.

In the following, we develop the above mentioned extension following closely the statements and proofs of (Denecker et al. 2004). The material is presented in an abstract form (as in (Denecker et al. 2004)), with the purpose of having a wider applicability than the present paper. In order to retrieve the connections with the present paper, one can take $A = T_{\mathsf{P}}$, $L_1 = \mathcal{H}^{\mathsf{ma}}_{\mathsf{P}}$ and $L_2 = \mathcal{H}^{\mathsf{am}}_{\mathsf{P}}$.

Let $(L, \leq)$ be a partially ordered set and assume that L contains a least element $\bot$ and a greatest element $\top$ with respect to $\leq$. Let $L_1, L_2 \subseteq L$ be non-empty sets such that $L_1 \cup L_2 = L$ and $(L_1, \leq)$ and $(L_2, \leq)$ are complete lattices that both contain the elements $\bot$ and $\top$. We will denote the least upper bound operations in the two lattices by lub_{L_1} and lub_{L_2} respectively (we will also use $\bigvee_{L_1}$ and $\bigvee_{L_2}$). We denote the greatest lower bound operations by glb_{L_1} and glb_{L_2} (also denoted by $\bigwedge_{L_1}$ and $\bigwedge_{L_2}$). We assume that our lattices satisfy the following two properties:

1. **Interlattice Lub Property:** Let $b \in L_2$ and $S \subseteq L_1$ such that for every $x \in S$, $x \leq b$. Then, $\bigvee_{L_1} S \leq b$.
2. **Interlattice Glb Property:** Let $a \in L_1$ and $S \subseteq L_2$ such that for every $x \in S$, $x \geq a$. Then, $\bigwedge_{L_2} S \geq a$.

Remark: It can be easily verified (see Lemma 6 in Appendix B) that both the Interlattice Lub Property and the Interlattice Glb Property hold when we take $L_1 = \mathcal{H}_{\mathsf{P}}^{\mathsf{ma}}$ and $L_2 = \mathcal{H}_{\mathsf{P}}^{\mathsf{am}}$.

Given $(x, y), (x', y') \in L_1 \times L_2$, we will write $(x, y) \preceq (x', y')$ if $x \leq x'$ and $y' \leq y$. We will write:

$$L_1 \otimes L_2 = \{(x, y) \mid x \in L_1, y \in L_2, x \leq y\}$$

The above set is non-empty since $(\bot, \top) \in L_1 \otimes L_2$.

Definition 22
A function $A : L_1 \otimes L_2 \to L_1 \otimes L_2$ is called a *consistent approximating operator* if it is $\preceq$-monotonic.

We will write $Appx(L_1 \otimes L_2)$ for the set of all consistent approximating operators over $L_1 \otimes L_2$. In the following results we assume we work with a given consistent approximating operator A (and therefore the symbol A will appear free in most definitions and results).

Definition 23
The pair $(a, b) \in L_1 \otimes L_2$ will be called A-reliable if $(a, b) \preceq A(a, b)$.

Given $a \in L_1$ and $b \in L_2$, we write $[a, b]_{L_1} = \{x \in L_1 \mid a \leq x \leq b\}$. Symmetrically, $[a, b]_{L_2} = \{x \in L_2 \mid a \leq x \leq b\}$.

Proposition 8
For all $a \in L_1$ and $b \in L_2$, the sets $[\bot, b]_{L_1}$ and $[a, \top]_{L_2}$ are complete lattices.

Proof
We use Theorem 4 of Appendix A. Consider first the set $[\bot, b]_{L_1}$ which obviously has a least element (since $\bot$ is the least element of both L_1 and L_2 and therefore $\bot \in [\bot, b]_{L_1}$). Let S be a non-empty subset of $[\bot, b]_{L_1}$. Since L_1 is a complete lattice, $\bigvee_{L_1} S \in L_1$. It suffices to show that $\bigvee_{L_1} S \in [\bot, b]_{L_1}$. Since $S \subseteq [\bot, b]_{L_1}$, for every $x \in S$ it holds $x \leq b$. By the Interlattice Lub Property, $\bigvee_{L_1} S \leq b$, and therefore $\bigvee_{L_1} S \in [\bot, b]_{L_1}$.

The proof for the case of $[a, \top]_{L_2}$ is symmetrical and uses the Interlattice Glb Property instead. $\square$

The following proposition corresponds to Proposition 3.3 in (Denecker et al. 2004):

Proposition 9
Let $(a, b) \in L_1 \otimes L_2$ and assume that (a, b) is A-reliable. Then, for every $x \in [\bot, b]_{L_1}$, it holds $\bot \leq A(x, b)_1 \leq b$. Moreover, for every $x \in [a, \top]_{L_2}$, it holds $a \leq A(a, x)_2 \leq \top$.

Proof
Define $a^* = lub_{L_1}\{y \in L_1 \mid y \leq b\}$. By the fact that $a \leq b$ and the definition of a^*, we get that $a \leq a^*$. By the Interlattice Lub Property we get that $a^* \leq b$ and therefore $(a^*, b) \in L_1 \otimes L_2$. Moreover, $(x, b) \preceq (a^*, b)$. Due to the $\preceq$-monotonicity of A we have $A(x, b) \preceq A(a^*, b)$, and therefore $A(x, b)_1 \leq A(a^*, b)_1$. Then:

$$\begin{array}{rcll} A(a^*, b)_1 & \leq & A(a^*, b)_2 & \text{(Consistency of } A\text{)} \\ & \leq & A(a, b)_2 & (a \leq a^* \text{ and } A \text{ is } \preceq\text{-monotone}) \\ & \leq & b & (A\text{-reliability}) \end{array}$$

For the second part of the proof, define $b^* = glb_{L_2}\{y \in L_2 \mid y \geq a\}$. By the fact that $b \geq a$ and the definition of b^*, we get that $b^* \leq b$. By the Interlattice Glb Property we get that $b^* \geq a$ and therefore $(a, b^*) \in L_1 \otimes L_2$. Moreover, $(a, x) \preceq (a, b^*)$. Due to the $\preceq$-monotonicity of A we have $A(a, x) \preceq A(a, b^*)$, and therefore $A(a, x)_2 \geq A(a, b^*)_2$. Then:

$$\begin{array}{rcll} A(a, b^*)_2 & \geq & A(a, b^*)_1 & \text{(Consistency of } A\text{)} \\ & \geq & A(a, b)_1 & (b^* \leq b \text{ and } A \text{ is } \preceq\text{-monotone}) \\ & \geq & a & (A\text{-reliability}) \end{array}$$

This completes the proof of the proposition. □

The above proposition implies that for every A-reliable pair (a, b), the restriction of $A(., b)_1$ to $[\bot, b]_{L_1}$ and the restriction of $A(a, .)_2$ to $[a, \top]_{L_2}$ are in fact operators (namely functions $[\bot, b]_{L_1} \to [\bot, b]_{L_1}$ and $[a, \top]_{L_2} \to [a, \top]_{L_2}$) on these intervals. Since by Proposition 8 we know that $([\bot, b]_{L_1}, \leq)$ and $([a, \top]_{L_2}, \leq)$ are complete lattices, the operators $A(\cdot, b)_1$ and $A(a, \cdot)_2$ have least fixpoints in the corresponding lattices. We define:

$$b^{\downarrow} = lfp(A(\cdot, b)_1)$$

and

$$a^{\uparrow} = lfp(A(a, \cdot)_2)$$

In the following, we will call the function mapping the A-reliable pair (a, b) to $(b^{\downarrow}, a^{\uparrow})$, the *stable revision operator* for the approximating operator A. We will denote this mapping by $\mathcal{C}_A$, namely:

$$\mathcal{C}_A(x, y) = (y^{\downarrow}, x^{\uparrow}) = (lfp(A(\cdot, y)_1), lfp(A(x, \cdot)_2))$$

We have the following proposition, which corresponds to Proposition 3.6 of (Denecker et al. 2004):

Proposition 10
Let $A \in Appx(L_1 \otimes L_2)$. For every A-reliable pair (a, b), $b^{\downarrow} \leq b$, $a \leq a^{\uparrow} \leq b$, and $(b^{\downarrow}, a^{\uparrow}) \in L_1 \otimes L_2$.

Proof
The inequalities $b^{\downarrow} \leq b$ and $a \leq a^{\uparrow}$ follow from the definition of the stable revision operator. By the A-reliability of (a, b) we have $A(a, b)_2 \leq b$ and therefore b is a pre-fixpoint of $A(a, \cdot)_2$. Since $a^{\uparrow}$ is the least pre-fixpoint of $A(a, \cdot)_2$, we conclude that $a^{\uparrow} \leq b$.

Let $a^* = lub_{L_1}\{x \in L_1 \mid x \leq a^{\uparrow}\}$. Since $a \in \{x \in L_1 \mid x \leq a^{\uparrow}\}$ and since a^* is the *lub* of this set, it holds $a \leq a^*$. Moreover, notice that a^* is in the domain of $A(\cdot, b)_1$ because (by the Interlattice Lub Property) $a^* \leq a^\uparrow$, and since $a^\uparrow \leq b$ we get $a^* \leq b$. We have:

$$\begin{array}{rcll} A(a^*,b)_1 & \leq & A(a^*,a^\uparrow)_1 & (A \text{ is } \preceq\text{-monotonic}) \\ & \leq & A(a^*,a^\uparrow)_2 & (A \text{ is consistent}) \\ & \leq & A(a,a^\uparrow)_2 & (A \text{ is } \preceq\text{-monotonic}) \\ & = & a^\uparrow & (a^\uparrow \text{ fixpoint of } A(a,\cdot)_2) \end{array}$$

Consequently, $A(a^*,b)_1 \leq a^\uparrow$ and therefore $A(a^*,b)_1 \in \{x \in L_1 \mid x \leq a^\uparrow\}$. But $a^* = lub_{L_1}\{x \in L_1 \mid x \leq a^\uparrow\}$ and therefore $A(a^*,b)_1 \leq a^*$. It follows that a^* is a pre-fixpoint of the operator $A(\cdot,b)_1$. Thus, $b^\downarrow = lfp(A(\cdot,b)_1) \leq a^* \leq a^\uparrow$. □

Definition 24
An A-reliable approximation (a,b) is A-prudent if $a \leq b^\downarrow$.

Proposition 11
Let $A \in Appx(L_1 \otimes L_2)$ and let $(a,b) \in L_1 \otimes L_2$ be A-prudent. Then, $(a,b) \preceq (b^\downarrow, a^\uparrow)$ and $(b^\downarrow, a^\uparrow)$ is A-prudent.

Proof
By Proposition 10, it holds $b^\downarrow \leq b$, $a \leq a^\uparrow$ and $a^\uparrow \leq b$. Since (a,b) is A-prudent, we get $(a,b) \preceq (b^\downarrow, a^\uparrow)$.

Notice now that by the $\preceq$ monotonicity of A we get that $b^\downarrow = A(b^\downarrow,b)_1 \leq A(b^\downarrow,a^\uparrow)_1$ and $a^\uparrow = A(a,a^\uparrow)_2 \geq A(b^\downarrow,a^\uparrow)_2$. This implies that $(b^\downarrow,a^\uparrow)$ is A-reliable.

Observe now that since $a^\uparrow \leq b$ and A is $\preceq$-monotonic, it holds that for every $x \in [\bot, a^\uparrow]_{L_1}$, $A(x,b)_1 \leq A(x,a^\uparrow)_1$. Therefore, each pre-fixpoint of $A(\cdot,a^\uparrow)_1$ is a pre-fixpoint of $A(\cdot,b)_1$. By the proof of Proposition 10 we have that $A(a^*,a^\uparrow)_1 \leq a^\uparrow$, and by the definition of a^* in that same proof, it follows that $A(a^*,a^\uparrow)_1 \leq a^*$. Therefore the set of pre-fixpoints of $A(\cdot,a^\uparrow)_1$ is non-empty. Consequently, $b^\downarrow = lfp(A(\cdot,b)_1) \leq lfp(A(\cdot,a^\uparrow)_1) = (a^\uparrow)^\downarrow$, and therefore $(b^\downarrow,a^\uparrow)$ is A-prudent. □

The following proposition (corresponding to Proposition 2.3 in (Denecker et al. 2004)) now requires in its proof the Interlattice Lub Property.

Proposition 12
Let $\{(a_\kappa,b_\kappa)\}_{\kappa<\lambda}$, where λ is an ordinal, be a chain in $L_1 \otimes L_2$ ordered by the relation $\preceq$. Then:

1. $\bigvee_{L_1}\{a_\kappa \mid \kappa < \lambda\} \leq \bigwedge_{L_2}\{b_\kappa \mid \kappa < \lambda\}$.
2. The least upper bound of the chain with respect to $\preceq$ exists, and is equal to $(\bigvee_{L_1}\{a_\kappa \mid \kappa < \lambda\}, \bigwedge_{L_2}\{b_\kappa \mid \kappa < \lambda\})$.

Proof
We demonstrate the first statement; the proof of the second part is easy and omitted. For the proof of the first part, notice that since the chain is ordered by $\preceq$, $\bigwedge_{L_2}\{b_\kappa \mid \kappa < \lambda\} = b_0$. Moreover, for every $\kappa < \lambda$ it holds $a_\kappa \leq b_\kappa$ because $(a_\kappa,b_\kappa) \in L_1 \otimes L_2$; since $b_\kappa \leq b_0$, it is $a_\kappa \leq b_0$ for all $\kappa < \lambda$. By the Interlattice Lub Property, we get $\bigvee_{L_1}\{a_\kappa \mid \kappa < \lambda\} \leq b_0 = \bigwedge_{L_2}\{b_\kappa \mid \kappa < \lambda\}$. □

The following proposition (corresponding to Proposition 3.10 in (Denecker et al. 2004)) and the subsequent theorem (corresponding to Theorem 3.11 in (Denecker et al. 2004)) have identical proofs to the ones given in (Denecker et al. 2004) (the only difference being that our underlying domain is $L_1 \otimes L_2$):

Proposition 13
Let $A \in Appx(L_1 \otimes L_2)$ and let $\{(a_\kappa, b_\kappa)\}_{\kappa<\lambda}$, where λ is an ordinal, be a chain of A-prudent pairs from $L_1 \otimes L_2$. Then, $\bigvee_{\preceq}\{(a_\kappa, b_\kappa)\}_{\kappa<\lambda}$, is A-prudent.

Theorem 5
Let $A \in Appx(L_1 \otimes L_2)$. The set of A-prudent elements of $L_1 \otimes L_2$ is a chain-complete poset under $\preceq$ with least element $(\bot, \top)$. The stable revision operator is a well-defined, increasing and monotone operator in this poset, and therefore it has a least fixpoint which is A-prudent and can be obtained as the limit of the following sequence:

$$\begin{array}{lcll}(a_0, b_0) & = & (\bot, \top) & \\ (a_{\lambda+1}, b_{\lambda+1}) & = & \mathcal{C}_A(a_\lambda, b_\lambda) & \\ (a_\lambda, b_\lambda) & = & \bigvee_{\preceq}\{(a_\kappa, b_\kappa) : \kappa < \lambda\} & \text{for limit ordinals } \lambda\end{array}$$

The proof of the following theorem is also a straightforward generalization of the proof of Theorem 19 in (Denecker et al. 2000):

Theorem 6
Every fixpoint of the stable revision operator $\mathcal{C}_A$ is a $\leq$-minimal pre-fixpoint of A.

Appendix D: Proofs of Section 6

Before providing the proofs of the results of Section 6, we notice that Proposition 4 extends to the case of Herbrand interpretations as follows:

Proposition 14
Let P be a program. Then, for every $\mathcal{I}, \mathcal{J} \in \mathcal{H}_\mathsf{P}$ and for every $(I_1, J_1), (I_2, J_2) \in (\mathcal{H}_\mathsf{P}^{\mathsf{ma}} \otimes \mathcal{H}_\mathsf{P}^{\mathsf{am}})$, the following statements hold:

1. $\tau(\mathcal{I}) \in (\mathcal{H}_\mathsf{P}^{\mathsf{ma}} \otimes \mathcal{H}_\mathsf{P}^{\mathsf{am}})$ and $\tau^{-1}(I_1, J_1) \in \mathcal{H}_\mathsf{P}$.
2. If $\mathcal{I} \preceq \mathcal{J}$ then $\tau(\mathcal{I}) \preceq \tau(\mathcal{J})$.
3. If $\mathcal{I} \leq \mathcal{J}$ then $\tau(\mathcal{I}) \leq \tau(\mathcal{J})$.
4. If $(I_1, J_1) \preceq (I_2, J_2)$ then $\tau^{-1}(I_1, J_1) \preceq \tau^{-1}(I_2, J_2)$.
5. If $(I_1, J_1) \leq (I_2, J_2)$ then $\tau^{-1}(I_1, J_1) \leq \tau^{-1}(I_2, J_2)$.

Lemma 3
Let P be a program and let $(I_1, J_1), (I_2, J_2) \in \mathcal{H}_\mathsf{P}^{\mathsf{ma}} \otimes \mathcal{H}_\mathsf{P}^{\mathsf{am}}$. If $(I_1, J_1) \preceq (I_2, J_2)$ then $T_\mathsf{P}(I_1, J_1) \preceq T_\mathsf{P}(I_2, J_2)$.

Proof
It follows directly from the definition of Ψ_P together with Lemma 2 and Corollary 1 in Appendix A that Ψ_P is $\preceq$-monotonic. It follows from Proposition 14 that $\tau^{-1}(I_1, J_1) \preceq \tau^{-1}(I_2, J_2)$. Since Ψ_P is $\preceq$-monotonic we get $\Psi_\mathsf{P}(\tau^{-1}(I_1, J_1)) \preceq \Psi_\mathsf{P}(\tau^{-1}(I_2, J_2))$. By applying again Proposition 4 we have that $T_\mathsf{P}(I_1, J_1) \preceq T_\mathsf{P}(I_2, J_2)$ that concludes the proof. □

Lemma 4
Let P be a program. If $(I, J) \in \mathcal{H}_\mathsf{P}^{\mathsf{ma}} \otimes \mathcal{H}_\mathsf{P}^{\mathsf{am}}$ is a pre-fixpoint of T_P then $\tau^{-1}(I, J)$ is a model of P.

Proof
From the definition of T_{P} and using the fact that (I, J) is a pre-fixpoint of T_{P}, it follows that $\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I, J))) = T_{\mathsf{P}}(I, J) \leq (I, J)$. By applying τ^{-1} to both sides of the statement and using Proposition 14 we get that $\tau^{-1}(\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I, J)))) \leq \tau^{-1}(I, J)$ which gives $\Psi_{\mathsf{P}}(\tau^{-1}(I, J)) \leq \tau^{-1}(I, J)$. From the definition of Ψ_{P} and the definition of model, it follows that $\tau^{-1}(I, J)$ is model of P. $\square$

Lemma 5
Let $\mathcal{M} \in \mathcal{H}_{\mathsf{P}}$ be a model of P. Then, $\tau(\mathcal{M})$ is a pre-fixpoint of T_{P}.

Proof
By the definition of Ψ_{P} we have that for every predicate constant p in P, $\Psi_{\mathsf{P}}(\mathcal{M})(\mathsf{p}) = \bigvee_{\leq}\{[\![\mathsf{E}]\!](\mathcal{M}) \mid (\mathsf{p} \leftarrow \mathsf{E}) \in \mathsf{P}\}$. Since $\mathcal{M}$ is a model of P it follows that $[\![\mathsf{E}]\!](\mathcal{M}) \leq \mathcal{M}(\mathsf{p})$ for every clause $\mathsf{p} \leftarrow \mathsf{E}$ in P, i.e., $\mathcal{M}(\mathsf{p})$ is an upper bound of the set $\{[\![\mathsf{E}]\!](\mathcal{M}) \mid (\mathsf{p} \leftarrow \mathsf{E}) \in \mathsf{P}\}$. Therefore, $\bigvee_{\leq}\{[\![\mathsf{E}]\!](\mathcal{M}) \mid (\mathsf{p} \leftarrow \mathsf{E}) \in \mathsf{P}\} \leq \mathcal{M}(\mathsf{p})$, which implies that $\Psi_{\mathsf{P}}(\mathcal{M}) \leq \mathcal{M}$. By Proposition 14 it follows that $\tau(\Psi_{\mathsf{P}}(\mathcal{M})) \leq \tau(\mathcal{M})$. Moreover, by the definition of T_{P} and Proposition 14 we have that $T_{\mathsf{P}}(\tau(\mathcal{M})) = \tau(\Psi_{\mathsf{P}}(\tau^{-1}(\tau(\mathcal{M})))) = \tau(\Psi_{\mathsf{P}}(M)) \leq \tau(\mathcal{M})$, and therefore $\tau(\mathcal{M})$ is a pre-fixpoint of T_{P}. $\square$

In order to establish Theorem 2 that follows, we need the following lemma:

Lemma 7
Let P be a program. If $(I, J) \in \mathcal{H}_{\mathsf{P}}^{\mathsf{ma}} \otimes \mathcal{H}_{\mathsf{P}}^{\mathsf{am}}$ is a minimal pre-fixpoint of T_{P} then $\tau^{-1}(I, J)$ is a minimal model of P.

Proof
Let $\mathcal{M} = \tau^{-1}(I, J)$. By Lemma 4, $\mathcal{M}$ is a model of P. Assume there exists a model $\mathcal{N} \in \mathcal{H}_{\mathsf{P}}$ of P such that $\mathcal{N} \leq \mathcal{M}$. Applying τ to both sides and using Proposition 14 we get that $\tau(\mathcal{N}) \leq \tau(\mathcal{M})$. By Lemma 5, $\tau(\mathcal{N})$ is a pre-fixpoint of T_{P} and since $\tau(\mathcal{M}) = (I, J)$ is a minimal pre-fixpoint of T_{P}, we get that $\tau(\mathcal{N}) = \tau(\mathcal{M})$. Applying τ^{-1} to both sides, we get $\mathcal{N} = \mathcal{M}$. $\square$

Theorem 2
Let P be a program. Then, $\mathcal{M}_{\mathsf{P}}$ is a $\leq$-minimal model of P.

Proof
By Theorem 6 (see Appendix C) every fixpoint of $\mathcal{C}_{T_{\mathsf{P}}}$ is a *minimal pre-fixpoint* of T_{P}. Since by Theorem 1 $(I_\delta, J_\delta) = \tau(\mathcal{M}_{\mathsf{P}})$ is a fixpoint of $\mathcal{C}_{T_{\mathsf{P}}}$, $\tau(\mathcal{M}_{\mathsf{P}})$ is a minimal pre-fixpoint of T_{P}. By Lemma 7, $\tau^{-1}(\tau(\mathcal{M}_{\mathsf{P}})) = \mathcal{M}_{\mathsf{P}}$ is a minimal model of P. $\square$

Theorem 3
For every propositional program P, $\mathcal{M}_{\mathsf{P}}$ coincides with the well-founded model of P.

Proof
In (Denecker et al. 2004)[Section 6, pages 107-108], the well-founded semantics of propositional logic programs (allowing arbitrary nesting of conjunction, disjunction and negation in clause bodies) is derived. By a careful inspection of the steps used in the above reference, it can be seen that the construction given therein is a special case of the technique used in the present paper. $\square$

Appendix E: The Model $\mathcal{M}_{\mathsf{P}}$ for an Example Program

Consider the following program P which is a simplified non-recursive version of a program taken from (Rondogiannis and Symeonidou 2017). Initially we use a Prolog-like syntax:

```
s(Q,V) ← Q(V)
p(R) ← R
q(R) ← ∼ w(R)
w(R) ← ∼ R
```

In the above example, the type of `p`, `q` and `w` is $o \to o$, and the type of `s` is $(o \to o) \to o \to o$. In $\mathcal{HOL}$ notation the program can be written as follows:

```
s ← λQ.λV.(Q V)
p ← λR.R
q ← λR.∼ (w R)
w ← λR.(∼ R)
```

Notice now that the bodies of the clauses of `s`, `q` and `w` do not involve other predicate constants, and therefore the calculation of their meaning can be performed in a more direct way. On the other hand, the body of the clause concerning `q` involves the predicate constant `w`, and therefore the calculation of the meaning of `q` is more involved.

The first approximation to the well-founded model of P is the pair $(I_0, J_0) = (\bot, \top)$ (see Theorem 1). Consider now (I_1, J_1). We have:

$$I_1 = \mathit{lfp}([T_{\mathsf{P}}(\cdot, \top)]_1) = \mathit{lfp}([\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\cdot, \top)))]_1)$$

and

$$J_1 = \mathit{lfp}([T_{\mathsf{P}}(\bot, \cdot)]_2) = \mathit{lfp}([\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot, \cdot)))]_2)$$

where, as discussed in Appendix C, the *lfp* in the case of I_1 is the least upper bound of the sequence $I_1^0, I_1^1, \ldots$, defined as follows:

$$\begin{array}{lcl} I_1^0 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot, \top)))]_1 \\ I_1^1 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1^0, \top)))]_1 \\ & \cdots & \\ I_1^{\alpha+1} & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1^\alpha, \top)))]_1 \\ & \cdots & \end{array}$$

and the *lfp* in the case of J_1 is the least upper bound of the sequence $J_1^0, J_1^1, \ldots$, defined as follows:

$$\begin{array}{lcl} J_1^0 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot, \bot)))]_2 \\ J_1^1 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot, J_1^0)))]_2 \\ & \cdots & \\ J_1^{\alpha+1} & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot, J_1^\alpha)))]_2 \\ & \cdots & \end{array}$$

For the predicate constant w we have:

$$
\begin{array}{l}
I_1^0(\mathtt{w}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot,\top)))]_1(\mathtt{w}) = [\tau([\![\lambda\mathtt{R}.\sim \mathtt{R}]\!](\tau^{-1}(\bot,\top)))]_1 = [\tau(\lambda v.v^{-1})]_1\\
I_1^1(\mathtt{w}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1^0,\top)))]_1(\mathtt{w}) = [\tau([\![\lambda\mathtt{R}.\sim \mathtt{R}]\!](\tau^{-1}(I_1^0,\top)))]_1 = [\tau(\lambda v.v^{-1})]_1\\
\qquad\qquad \cdots\\
I_1^{\alpha+1}(\mathtt{w}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1^\alpha,\top)))]_1(\mathtt{w}) = [\tau([\![\lambda\mathtt{R}.\sim \mathtt{R}]\!](\tau^{-1}(I_1^\alpha,\top)))]_1 = [\tau(\lambda v.v^{-1})]_1\\
\qquad\qquad \cdots
\end{array}
$$

Similarly, we can show that for every ordinal α, $J_1^\alpha(\mathtt{w}) = [\tau(\lambda v.v^{-1})]_2$. The above imply that $\mathcal{M}_{\mathsf{P}}(\mathtt{w}) = \lambda v.v^{-1}$. In other words, the denotation of w is the *not* function over our 3-valued truth domain. In a similar way, it follows that $\mathcal{M}_{\mathsf{P}}(\mathtt{p}) = \lambda v.v$. In other words, the denotation of p is the identity function over our 3-valued domain.

Consider now the predicate constant q. We have:

$$
\begin{array}{l}
I_1^0(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot,\top)))]_1(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(\bot,\top)))]_1 = [\tau(\lambda v.\mathit{undef})]_1\\
I_1^1(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1^0,\top)))]_1(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_1^0,\top)))]_1 = [\tau(f)]_1\\
\qquad\qquad \cdots\\
I_1^{\alpha+1}(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1^\alpha,\top)))]_1(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_1^\alpha,\top)))]_1 = [\tau(f)]_1\\
\qquad\qquad \cdots
\end{array}
$$

where f is the function such that $f(\mathit{true}) = f(\mathit{undef}) = \mathit{undef}$ and $f(\mathit{false}) = \mathit{false}$. Similarly, we have:

$$
\begin{array}{l}
J_1^0(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot,\bot)))]_2(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(\bot,\bot)))]_2 = [\tau(\lambda v.\mathit{true})]_2\\
J_1^1(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot,J_1^0)))]_2(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(\bot,J_1^0)))]_2 = [\tau(g)]_2\\
\qquad\qquad \cdots\\
J_1^{\alpha+1}(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot,J_1^\alpha)))]_2(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(\bot,J_1^\alpha)))]_2 = [\tau(g)]_2\\
\qquad\qquad \cdots
\end{array}
$$

where g is the function such that $g(\mathit{false}) = g(\mathit{undef}) = \mathit{undef}$ and $g(\mathit{true}) = \mathit{true}$.

Consider now (I_2, J_2). We have:

$$I_2 = \mathit{lfp}([T_{\mathsf{P}}(\cdot, J_1)]_1) = \mathit{lfp}([\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\cdot, J_1)))]_1)$$

and

$$J_2 = \mathit{lfp}([T_{\mathsf{P}}(I_1, \cdot)]_2) = \mathit{lfp}([\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, \cdot)))]_2)$$

where the *lfp* in the case of I_2 is the least upper bound of the sequence $I_2^0, I_2^1, \ldots$ defined as follows:

$$
\begin{array}{lcl}
I_2^0 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\bot, J_1)))]_1\\
I_2^1 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_2^0, J_1)))]_1\\
 & \cdots & \\
I_2^{\alpha+1} & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_2^\alpha, J_1)))]_1\\
 & \cdots &
\end{array}
$$

and the *lfp* in the case of J_2 is the least upper bound of the sequence $J_2^0, J_2^1, \ldots$ defined as follows:

$$
\begin{array}{lcl}
J_2^0 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, I_1^*)))]_2\\
J_2^1 & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, J_2^0)))]_2\\
 & \cdots & \\
J_2^{\alpha+1} & = & [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, J_2^\alpha)))]_2\\
 & \cdots &
\end{array}
$$

where I_1^* is the least interpretation in $\mathcal{H}_{\mathsf{P}}^{\mathsf{am}}$ such that $I_1 \leq I_1^*$ (namely, the bottom antimonotone-monotone element of the interval $[I_1, \perp]$, see the construction in Appendix C).

Consider again the predicate constant $\mathtt{q}$. We have:

$$\begin{array}{l} I_2^0(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(\perp, J_1)))]_1(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim\!(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(\perp, J_1)))]_1 \\ I_2^1(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_2^0, J_1)))]_1(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim\!(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_2^0, J_1)))]_1 = [\tau(\lambda v.v)]_1 \\ \qquad\quad \cdots \\ I_2^{\alpha+1}(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_2^\alpha, J_1)))]_1(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim\!(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_2^\alpha, J_1)))]_1 = [\tau(\lambda v.v)]_1 \\ \qquad\quad \cdots \end{array}$$

because for all ordinals α, $I_2^\alpha(\mathtt{w}) = [\tau(\lambda v.v^{-1})]_1$ and $J_1(\mathtt{w}) = [\tau(\lambda v.v^{-1})]_2$. Similarly, we have:

$$\begin{array}{l} J_2^0(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, I_1^*)))]_2(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim\!(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_1, I_1^*)))]_2 \\ J_2^1(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, J_2^0)))]_2(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim\!(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_1, J_2^0)))]_2 = [\tau(\lambda v.v)]_2 \\ \qquad\quad \cdots \\ J_2^{\alpha+1}(\mathtt{q}) = [\tau(\Psi_{\mathsf{P}}(\tau^{-1}(I_1, J_2^\alpha)))]_2(\mathtt{q}) = [\tau([\![\lambda\mathtt{R}.\sim\!(\mathtt{w}\ \mathtt{R})]\!](\tau^{-1}(I_1, J_2^\alpha)))]_2 = [\tau(\lambda v.v)]_2 \\ \qquad\quad \cdots \end{array}$$

because $I_1(\mathtt{w}) = [\tau(\lambda v.v^{-1})]_1$ and for all ordinals α, $J_2^\alpha(\mathtt{w}) = [\tau(\lambda v.v^{-1})]_2$. The above imply that $\mathcal{M}_{\mathsf{P}}(\mathtt{q}) = \lambda v.v$. In other words, the denotation of $\mathtt{q}$ is the identity function over our 3-valued truth domain. Notice that despite their different definitions, $\mathtt{p}$ and $\mathtt{q}$ denote the same 3-valued relation (in some sense, the two negations in the definition of $\mathtt{q}$ cancel each other).

Finally, consider the predicate constant $\mathtt{s}$. We have:

$$\begin{array}{l} I_1^0(\mathtt{s}) = [\tau([\![\lambda\mathtt{Q}.\lambda\mathtt{V}.(\mathtt{Q}\ \mathtt{V})]\!](\tau^{-1}(\perp, J_1)))]_1 = [\tau(\lambda q.\lambda v.(q\,v))]_1 \\ I_1^1(\mathtt{s}) = [\tau([\![\lambda\mathtt{Q}.\lambda\mathtt{V}.(\mathtt{Q}\ \mathtt{V})]\!](\tau^{-1}(I_1^0, \top)))]_1 = [\tau(\lambda q.\lambda v.(q\,v))]_1 \\ \qquad\quad \cdots \\ I_1^{\alpha+1}(\mathtt{s}) = [\tau([\![\lambda\mathtt{Q}.\lambda\mathtt{V}.(\mathtt{Q}\ \mathtt{V})]\!](\tau^{-1}(I_1^\alpha, \top)))]_1 = [\tau(\lambda q.\lambda v.(q\,v))]_1 \\ \qquad\quad \cdots \end{array}$$

and also:

$$\begin{array}{l} J_1^0(\mathtt{s}) = [\tau([\![\lambda\mathtt{Q}.\lambda\mathtt{V}.(\mathtt{Q}\ \mathtt{V})]\!](\tau^{-1}(I_1, \perp)))]_2 = [\tau(\lambda q.\lambda v.(q\,v))]_2 \\ J_1^1(\mathtt{s}) = [\tau([\![\lambda\mathtt{Q}.\lambda\mathtt{V}.(\mathtt{Q}\ \mathtt{V})]\!](\tau^{-1}(\perp, J_1^0)))]_2 = [\tau(\lambda q.\lambda v.(q\,v))]_2 \\ \qquad\quad \cdots \\ J_1^{\alpha+1}(\mathtt{s}) = [\tau([\![\lambda\mathtt{Q}.\lambda\mathtt{V}.(\mathtt{Q}\ \mathtt{V})]\!](\tau^{-1}(\perp, J_1^\alpha)))]_2 = [\tau(\lambda q.\lambda v.(q\,v))]_2 \\ \qquad\quad \cdots \end{array}$$

The above imply that $\mathcal{M}_{\mathsf{P}}(\mathtt{s}) = \lambda q.\lambda v.(q\,v)$.