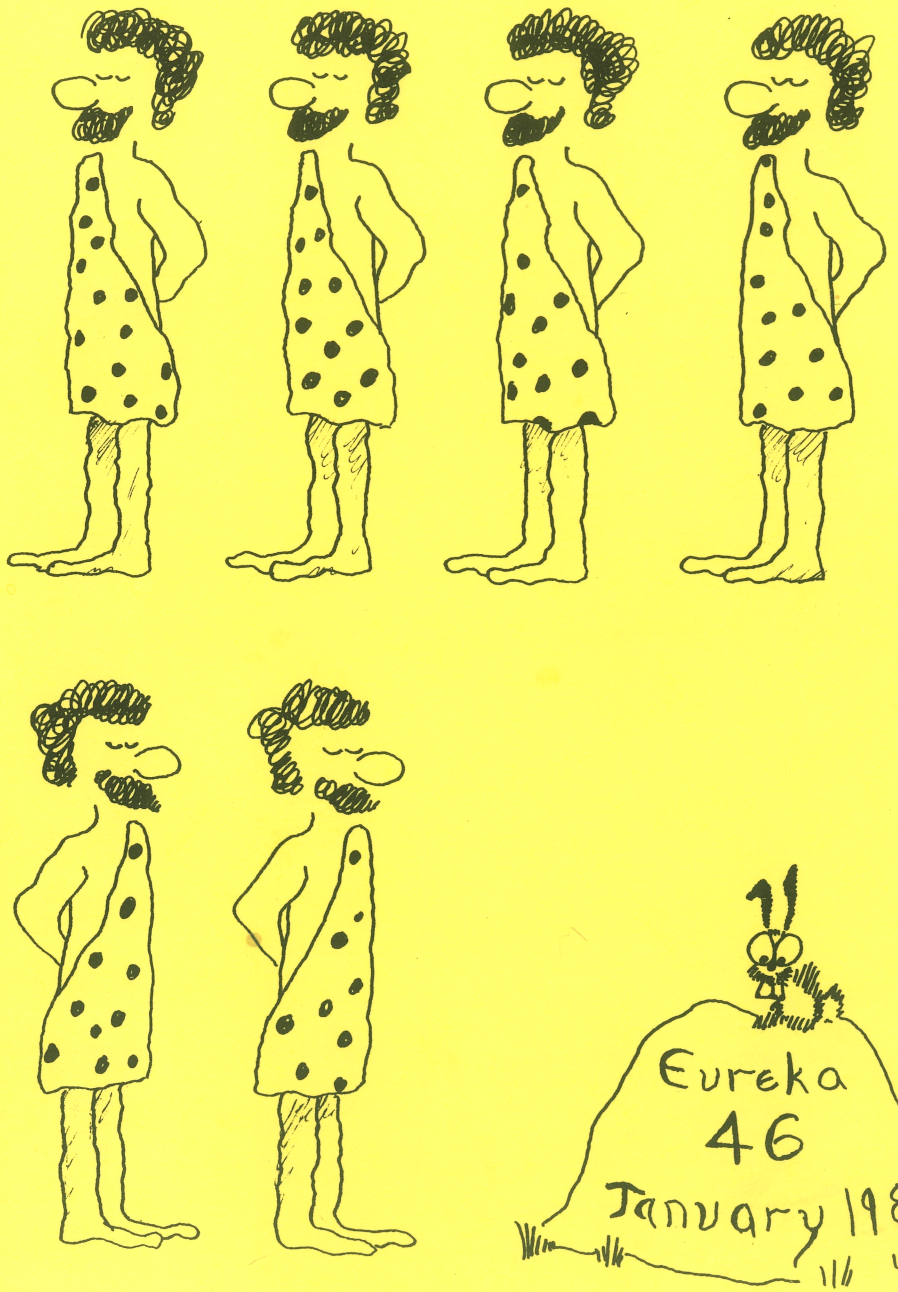
JANUARY 1986



Prelude

Miranda Mowbray, Richard Pennington and Edward Welbourne.

The editor has spent a very great deal of time obtaining Prof. Conway's article on The Elements of Audioactive Chemistry, and feels that some of you may be interested in the somewhat tortuous route whereby the problem came to Prof. Conway's attention.

The earliest appearance of the problem that we know of dates back to 1977, at the International Mathematical Olympiad in Belgrade, Yugoslavia. It is a well-known sociological phenomenon that, if one puts a sufficiently large number of mathematicians into a sufficiently small area, someone is going to start setting puzzles - and, in this case, some very interesting puzzles were indeed set, entirely independently of the official competition. The Dutch contingent perpetrated the following puzzle against the British team:

```
          1
         11
         21
       1211
     111221
     312211
What is the next line?
```

(Try solving it yourself! Answer in Prof. Conway's article.)

The Dutch team may have cooked this up themselves but we suspect that it is older; however, Richard was on the relevant British team, and duly brought the problem to Cambridge, where he inflicted it on Miranda, who relates that...

" When I first showed this puzzle to one of my friends, he thought for some time and then gave an agonised cry, 'I've solved it - but you need a really TWISTED mind to think of that!' I showed it to several arts students, who were all baffled, which is surprising as it requires no mathematical skills beyond counting. From my mathematical friends I got the same response as the initial one; silence and furious thinking for between two and thirty minutes followed by anguished howling. If hideous noises were heard echoing down the corridors of Newnham it was a good bet I'd asked that puzzle again."

Miranda published the puzzle in **2-Manifold**, of which Richard showed a copy to Eddy, who in turn tormented a few people. Prof. Conway lectured the Algebra III course in Michaelmas 1983 with a lecture to spare at the end, so his students invited him to a party in lecture room A at the time set aside for the spare lecture. In the course of this Eddy told the puzzle to the good Professor, who immediately started looking at the general case, rapidly noticing the few properties that had previously been seen - such as that digits other than 1,2 and 3 don't occur naturally - and then went on to guess that the strings would split; further developments came later. The following Spring he met Eddy in Sainsbury's, mentioned the discovery of the elements and promissed an article to **Eureka**. A year and a half later, here it is!

The Weird and Wonderful Chemistry of Audioactive Decay

J.H.Conway

Introduction.

Suppose we start with a string of numbers (i.e. positive integers), say

5 5 5 5 5.

We might describe this in words in the usual way as 'five fives', and write down the derived string

5 5.

This we describe as 'two fives', so it yields the next derived string

2 5

which is 'one two, one five',giving

1 2 1 5

namely 'one one, one two, one one, one five', or

1 1 1 2 1 1 1 5

and so on. What happens when an arbitrary string of positive integers is repeatedly **derived** like this?

I note that more usually one is given a sequence such as

55555 ; 55 ; 25 ; 1215 ; 11121115 ;

and asked to guess the generating rule or the next term. The history of this problem is described elsewhere in this issue of ***Eureka***.

The numbers in our strings are usually single-digit ones, so we'll call them digits and usually cram them together as we have just done. But occasionally we want to indicate the way the numbers in the string were obtained, and we can do this neatly by inserting commas recalling the commas and quotes in our verbal descriptions, thus:

```
5 5 5 5 5
 ,5 5,
 ,2 5,
,1 2,1 5,
,1 1,1 2,1 1,1 5,
 ...  .
```

The insertion of these commas into a string or portion thereof is called parsing.

We'll often denote repetitions by indices in the usual way, so that the derivation rule is

$$a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}\cdots \to \alpha a\beta b\gamma c\delta d\cdots$$

When we do this it is always to be understood that the repetitions are collected maximally, so that we must have

$$a\neq b,\ b\neq c,\ c\neq d,\ \cdots .$$

Since what we write down is often only a chunk of the entire string (that is, a consecutive subsequence of its terms), we often use the square brackets "[" or "]" to indicate that the apparent left or right end really is the end. We also introduce the formal digits

0, as an index, to give an alternative way of indicating the ends (see below)
X for an arbitrary digit, possibly 0, and
≠n for any digit (maybe 0) other than n.

Thus $X^0a^{\alpha}b^{\beta}c^{\gamma}$ means the same as $[a^{\alpha}b^{\beta}c^{\gamma}$

$a^{\alpha}b^{\beta}c^{\gamma}X^0$ means the same as $a^{\alpha}b^{\beta}c^{\gamma}]$

$a^{\alpha}b^{\beta}c^{\gamma}X^{\neq 0}$ means $a^{\alpha}b^{\beta}c^{\gamma}$ followed by another digit, and

$a^{\alpha}b^{\beta}c^{\gamma}(\neq 2)^{\neq 0}$ means that this digit is not a 2.

I'm afraid that this heap of conventions makes it quite hard to check the proofs, since they cover many more cases than one naively expects. To separate these cases would make this article very long and tedious, and the reader who really wants to check all the details is advised first to spend some time practising the derivation process. Note that when we write

$$L \to L' \to L'' \to \cdots$$

we mean just that every string of type L derives to one of type L', every string of type L' derives to one of type L", and so on. So when in our proof of the Ending Theorem we have

$$n^{n]} \xrightarrow{(n\neq 2)} n^{\neq n]} \longrightarrow n^{1]}$$

the fact that the left arrow is asserted only when n≠2 does not excuse us from checking the right arrow for n=2. (But, since n>1 is enforced at that stage in the proof, we needn't check either of them for n=1.)

By applying the derivation process n times to a string L we obtain what we call its n'th descendant, L_n. The string itself is counted among its descendants, as the 0^{th}.

Sometimes a string factors as the product LR of two strings L and R whose descendants never interfere with each other, in the sense that $(LR)_n = L_nR_n$ for all n. In this case, we say that LR splits as L.R (dots in strings will always have this meaning). It is plain that this happens just when (L or R is empty or) the last digit of L_n always differs from the first one of R_n. Can you find a simple criterion for this to happen? (When you give up, you'll find the answer in our Splitting Theorem.)

Obviously, we call a string with no non-trivial splittings an atom, or element. Then every string is the split product, or compound, of a certain number of elements, which we call the elements it involves. There are infinitely many distinct elements, but most of them only arise from specially chosen starting strings. However, there are some very interesting elements that are involved in the descendants of **every** string except the boring ones [] and [22]. Can you guess how many of them there are? (Hint: we have given them the names

Hydrogen, Helium, Lithium, ··· , Uranium.)

It's also true (but **ASTONISHINGLY** hard to prove) that every string eventually decays into a compound of these elements, together with perhaps a few others (namely isotopes of **Plutonium** and **Neptunium**). Moreover, all strings except the two boring ones increase in length exponentially at the same constant rate. (This rate is roughly 1.30357726903: it can be precisely defined as the largest root of a certain algebraic equation of degree 71.) Also, the relative abundances of the elements settle down to fixed numbers (zero for Neptunium and Plutonium). Thus, of every million atoms about 91790 on average will be of Hydrogen, the commonest element, while about 27 will be of Arsenic, the rarest one.

You should get to know the common elements, as enumerated in our Periodic Table (pages 8 and 9). The abundance (in atoms per million) is given first, followed by the atomic number and symbol as in ordinary chemistry. The actual digit-string defining the element is the numerical part of the remainder of the entry, which, when read in full, gives the derivate of the element of next highest atomic number, split into atoms. Thus, for example, the last line of the Periodic Table tells us that Hydrogen (*H*) is our name for the digit-string 22, and that the next higher element, Helium (*He*), derives to the compound

Hf.Pa.H.Ca.Li

which we might call

"Hafnium-Protactinium-Hydrogen-Calcium Lithide"!

Not everything is in the Periodic Table! For instance, the single digit string "1" isn't. But watch:

1
11
21
1211
111221
312211
13112221
11132.13211 = *Hf.Sn*

after a few moves it has become Hafnium Stannide! This is an instance of our "Cosmological Theorem", which asserts that the exotic elements (such as "1"), all disappear soon after the Big Bang.

The Periodic Table. (Uranium to Silver)

abundance:-	n	E_n	E_n inside the derivate of E_{n+1} :
102.56285249	92	*U*	3
9883.5986392	91	*Pa*	13
7581.9047125	90	*Th*	1113
6926.9352045	89	*Ac*	3113
5313.7894999	88	*Ra*	132113
4076.3134078	87	*Fr*	1113122113
3127.0209328	86	*Rn*	3113112222113
2398.7998311	85	*At*	*Ho*.1322113
1840.1669683	84	*Po*	1113222113
1411.6286100	83	*Bi*	3113322113
1082.8883285	82	*Pb*	*Pm*.123222113
830.70513293	81	*Tl*	111213322113
637.25039755	80	*Hg*	311211232222113
488.84742982	79	*Au*	1321122211213322113
375.00456738	78	*Pt*	11131221222211211232222113
287.67344775	77	*Ir*	3113111221132211222112112133222113
220.68001229	76	*Os*	132113212221132221222112112323222113
169.28801808	75	*Re*	11131221131211322113321132211221121133221113
315.56655252	74	*W*	*Ge.Ca*.312211322212221121123222113
242.07736666	73	*Ta*	131122211332113221122112133221113
2669.0970363	72	*Hf*	11132.*Pa.H.Ca.W*
2047.5173200	71	*Lu*	311312
1570.6911808	70	*Yb*	1321131112
1204.9083841	69	*Tm*	11131221133112
1098.5955997	68	*Er*	311311222.*Ca.Co*
47987.529438	67	*Ho*	1321132.*Pm*
36812.186418	66	*Dy*	1113122113112
28239.358949	65	*Tb*	31131122221131112
21662.972821	64	*Gd*	*Ho*.13221133112
20085.668709	63	*Eu*	1113222.*Ca.Co*
15408.115182	62	*Sm*	311332
29820.456167	61	*Pm*	132.*Ca.Zn*
22875.863883	60	*Nd*	111312
17548.529287	59	*Pr*	31131112
13461.825166	58	*Ce*	1321133112
10326.833312	57	*La*	11131.*H.Ca.Co*
7921.9188284	56	*Ba*	311311
6077.0611889	55	*Cs*	13211321
4661.8342720	54	*Xe*	11131221131211
3576.1856107	53	*I*	3113112221131111221
2743.3629718	52	*Te*	*Ho*.1322113312211
2104.4881933	51	*Sb*	*Eu.Ca*.3112221
1614.3946687	50	*Sn*	*Pm*.13211
1238.4341972	49	*In*	11131221
950.02745646	48	*Cd*	3113112211
728.78492056	47	*Ag*	132113212221

The Periodic Table. (Palladium to Hydrogen)

abundance:-	n	E_n	E_n inside the derivate of E_{n+1} :
559.06537946	46	*Pd*	111312211312113211
428.87015041	45	*Rh*	3113112221131112211312211
328.99480576	44	*Ru*	*Ho*.1322113312211311211
386.07704943	43	*Tc*	*Eu.Ca*.311322113212221
296.16736852	42	*Mo*	13211322211312113211
227.19586752	41	*Nb*	11131221133221131111221131221
174.28645997	40	*Zr*	*Er*.12322211331222113112211
133.69860315	39	*Y*	1112133.*H.Ca.Tc*
102.56285249	38	*Sr*	3112112.*U*
78.678000089	37	*Rb*	1321122112
60.355455682	36	*Kr*	11131221222112
46.299868152	35	*Br*	31131122113221122
35.517547944	34	*Se*	131113212221132221122
27.246216076	33	*As*	11131221131211322113322112
1887.4372276	32	*Ge*	3113112221131112211322.*Na*
1447.8905642	31	*Ga*	*Ho*.13221133122211331
23571.391336	30	*Zn*	*Eu.Ca.Ac.H.Ca*.312
18082.082203	29	*Cu*	131112
13871.124200	28	*Ni*	11133112
45645.877256	27	*Co*	*Zn*.32112
35015.858546	26	*Fe*	13122112
26861.360180	25	*Mn*	1113112222112
20605.882611	24	*Cr*	31132.*Si*
15807.181592	23	*V*	13211312
12126.002783	22	*Ti*	11131221131112
9302.0974443	21	*Sc*	31131122211331122
56072.543129	20	*Ca*	*Ho.Pa.H*.12.*Co*
43014.360913	19	*K*	1112
32997.170122	18	*Ar*	3112
25312.784218	17	*Cl*	132112
19417.939250	16	*S*	1113122112
14895.886658	15	*P*	3113112222112
32032.812960	14	*Si*	*Ho*.1322112
24573.006696	13	*Al*	1113222112
18850.441228	12	*Mg*	3113322112
14481.448773	11	*Na*	*Pm*.123222112
11109.006821	10	*Ne*	111213322112
8521.9396539	9	*F*	311211232222112
6537.3490750	8	*O*	1321122112133221122
5014.9302464	7	*N*	11131221222112112322211 2
3847.0525419	6	*C*	3113112211322112211213322112
2951.1503716	5	*B*	132113212221132221222112112322211 2
2263.8860325	4	*Be*	11131221131211322113321132211221121133221122
4220.0665982	3	*Li*	*Ge.Ca*.312211322212221121123222112
3237.2968588	2	*He*	131122211332113221122112133221122
91790.383216	1	*H*	*Hf.Pa*.22.*Ca.Li*

The Theory.

We start with some easy theorems that restrict the possible strings after the first few moves. Any chunk of a string that has lasted at least n moves will be called an n-day-old string.

The One-Day Theorem. Chunks of types

$$,a\ x,b\ x, \qquad x^4 \text{ or more} \qquad \text{and } x^3y^3$$

don't happen in day-old lists. (Note that the first one has a given parsing.)

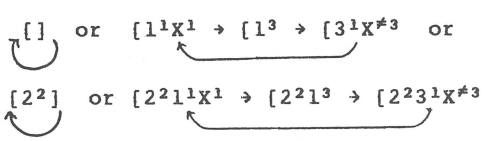
Proof. The first possibility comes from $x^a x^b$, which, however, should have been written x^{a+b}, in the previous day's string. The other two, however parsed, imply cases of the first.

The Two-Day Theorem. No digit 4 or more can be born on or after the second day. Also, a chunk 3x3 (in particular 3^3) can't appear in any 2-day-old list.

Proof. The first possibility comes from a chunk x^4 or more, while the second, which we now know must parse ,3 x,3 y, can only come from a chunk x^3y^3, of the previous day's string.

When tracking particular strings later, we'll use these facts without explicit mention.

The Starting Theorem. Let R be any chunk of a 2-day-old string, considered as a string in its own right. Then the starts of its descendants ultimately cycle in one of the ways



If R is not already in such a cycle, at least three distinct digits appear as initial digits of its descendants.

Proof. If R is non-empty and doesn't start with 2^2, then it

either starts with a 1 and is of one of the types

$[1^1X^0$ or ^{1} or $[1^1(2^2$ or ^{3} or $3^2)$ or $[1^2X^1$ or $\neq 1$ or $[1^3$

or starts with a 2 and is of one of the types

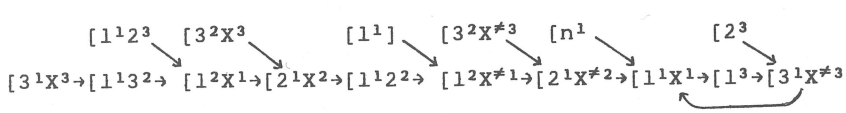
$[2^1X^2$ or $\neq 2$ or $[2^3$

or starts with a 3 and is of one of the types

$[3^1X^3$ or $\neq 3$ or $[3^2X^3$ or $\neq 3$

or starts with some n>3 and has form $[n^1$.

It is therefore visible in



which establishes the desired results for it.

This proves the theorem except for strings of type $[2^2R'$ all of whose descendants start with 2^2. This happens only if no descendant of R' starts with a 2, and so we can complete the proof by applying the results we've just found to R'.

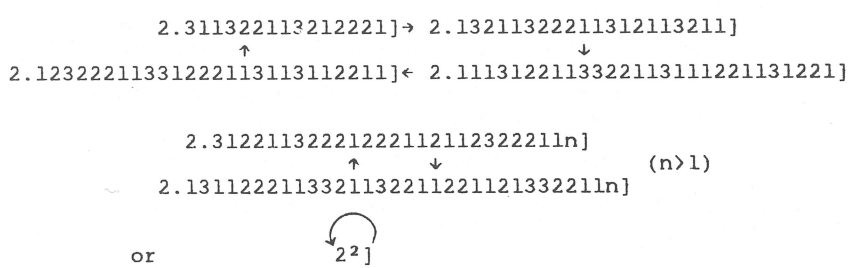
The Splitting Theorem. A 2-day-old string LR splits as L.R just if one of L and R is empty or L and R are of the types shown in one of:

L	R
n]	[m (n≥4,m≤3)
2]	$[1^1X^1$ or $[1^3$ or $[3^1X^{\neq 3}$ or $[n^1$ (n≥4)
≠2]	$[2^21^1X^1$ or $[2^21^3$ or $[2^23^1X^{\neq 3}$ or $[2^2n^{(0 \text{ or } 1)}$ (n≥4)

Proof. This follows immediately from the Starting Theorem applied to R and the obvious fact that the last digit of L is constant.

Now we investigate the evolution of the end of the string!

The Ending Theorem. The end of a string ultimately cycles in one of the ways:



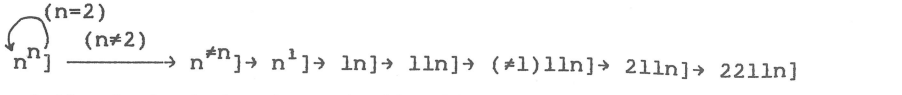
[Note: our splitting theorem shows that these strings actually do split at the dots, although we don't use this.]

Proof. A string with last digit 1 must end in one of the ways visible in

$1^{\geq 3}] \to (\neq 2)^X 1^1] \to (\neq 2)^X 1^2] \to 2^1$ or $\geq 4\,1^1] \to 2^1$ or $\geq 3\,1^2] \to 2^2 1^1] \to 2^2 1^2] \to 2^3 1^1]$

and its subsequent evolution is followed on the right hand side of figure 1.

A string with last digit n>1 must end $n^n]$ or $n^{\neq n}]$ and so evolves via



and the last string here is the first or second on the left of Figure 1.

```
(≠2)2211n]   (n>1)        (≠2)2221]
(≠2)2221ln]                   3211]
    3221ln]                   31221]
    3222lln]                  3112211]
(≠3)3322lln]                  3212221]
    23222lln]                 312113211]
    2133221ln]                3111221131221]
    2112322211n]          (≠3)331222113112211]
    221121332211n]            2.311322113212221]
    2211211232221ln]          2.13211322211312113211] ←
    22112211213322lln]        2.11131221133221131112211131221]
    221222112112322211n]      2.311311222.123222113312221131122111]
    21132211221121332211n]    2.1112133.22.12.311322113212221]
    221132221222112112322211n]
    2211332113221122112133221ln]                  (period 4)
    22.12.312211322212221122112322211n]
    2.131122211332113221122112133221ln] ←         (period 2)
    2.11132.13.22.12.312211322212221122112322211n]
```

Figure 1. The evolution of endings other than $2^2]$.

This figure proves the theorem except for the trivial case $2^2]$. (When any of these strings contains a dot, its subsequent development is only followed from the digit just prior to the rightmost dot.)

We are now ready for our first major result:

The Chemical Theorem.

a) The descendants of any of the 92 elements in our Periodic Table are compounds of those elements.

b) All sufficiently late descendants of any of these elements other than Hydrogen involve all of the 92 elements simultaneously.

c) The descendants of any string other than [] or [22] also ultimately involve all of those 92 elements simultaneously.

d) These 92 elements are precisely the common elements as defined in the introduction.

Proof. a) follows instantly from the form in which we have presented the Periodic Table.

b) It also follows that if the element E_n of atomic number n appears at some time t, then for any m<n, all of the elements on the E_m line of the table will appear at the later time t+n-m. In particular,

E_n at t $\Longrightarrow$ *Hf&Li* at t+n-1 (if n≥2),
Hf&Li at t $\Longrightarrow$ *Hf&Li* at t+2 & t+71,
Hf at t $\Longrightarrow$ *Sr&U* at t+72-38,
U at t $\Longrightarrow$ E_n at t+92-n.

From these we successively deduce that if any of these 92 elements other than Hydrogen is involved at some time t_0, Hafnium and Lithium will simultaneously be involved at some strictly later time $\leq t_0+100$, and then both will exist at all times $\geq t_0+200$, Uranium at all times $\geq t_0+300$, and every other one of these 92 elements at all times $\geq t_0+400$.

In other words, once you can fool some of the elements into appearing some of the time, then soon you'll fool some of them all of the time, and ultimately you'll be fooling all of the elements all of the time!

c) If L is not of form $L'2^2]$, this now follows from the observation that Calcium (digit-string 12) is a descendant of L, since it appears in both the bottom lines of Figure 1. Otherwise we can replace L by L', which does not end in a 2.

d) follows from a),b),c) and the definition of the common elements.

Now we'll call an arbitrary string common just if it's a compound of common atoms.

The Arithmetical Theorem.

a) The lengths of all common strings other than boring old [] and [22] increase exponentially at the same rate $\lambda>1$.

b) The relative abundances of the elements in such strings tend to certain fixed values, all strictly positive.

Notes. Since each common element has at least 1 and at most 42 digits we can afford to measure the lengths by either digits or atoms - we prefer to use atoms. The numerical value of λ is 1.30357726903; the abundances are tabulated in the Periodic Table.

Proof. Let v be the 92-component vector whose (i)-entry is the number of atoms of atomic number i in some such string. Then at each derivation step, $\mathbf{v}$ is multiplied by the matrix $\mathbf{M}$ whose (i,j)-entry is the number of times E_j is involved in the derivate of E_i. Now our Chemical Theorem shows that some power of $\mathbf{M}$ has strictly positive (i,j)-entries for all i≠1 (the (1,j)-entry will be 0 for j≠1, 1 for j=1, since every descendant of a single atom of Hydrogen is another such).

Let λ be an eigenvalue of $\mathbf{M}$ with the largest possible modulus, and $\mathbf{v}_0$ a corresponding eigenvector. Then the non-zero entries of $\mathbf{v}_0\mathbf{M}^n$ are proportional to λ^n, while the entries in the successive images of all other vectors grow at *at most* this rate. Since the 92 coordinate vectors (which we'll call **H,He,⋯ ,U** in the obvious way) span the space, at least one of them must increase at rate λ.

On the other hand, our Chemical Theorem shows that the descendants of each of **He,Li,⋯ ,U** increase as fast as any of them, and that this is at some rate >1, while **H** is a fixed vector (rate 1). These remarks establish our Theorem.

[We have essentially proved the **Frobenius-Perron Theorem**, that the dominant eigenvalue of a matrix with positive entries is positive and occurs just once, but I didn't want to frighten you with those long names.]

The Transuranic Elements.

For each number n≥4, we define two particular atoms:-

an isotope of **Plutonium** (*Pu*) : 3122113222122211211232221ln
an isotope of **Neptunium** (*Np*) : 1311222113321132211221121332211n

For n=2 these would be Lithium (*Li*) and Helium (*He*); for n=3 they would be Tungsten (*W*) and Tantalum (*Ta*), while for n≥4 they are called the transuranic elements. We won't bother to specify the number n in our notation.

We can enlarge our 92-dimensional vector space by adding any number of new pairs of coordinate vectors **Pu**, **Np** corresponding to pairs of transuranic elements.

Our proof of the Ending Theorem shows that every digit 4 or more ultimately lands up as the last digit in one of the appropriate pair of transuranic elements, and (see the bottom left of Figure 1) that we have the decomposition

Pu → *Np* → *Hf.Pa.H.Ca.Pu*.

Now **Pu±Np** is an eigenvector of eigenvalue ±1 *modulo the subspace corresponding to the common elements*, since **Pu⟷Np** modulo that space. Because these eigenvalues are strictly less than λ in modulus, the relative abundances of the transuranic elements tend to 0.

So far, I can proudly say that this magnificent theory is essentially all my own work. However, the next theorem, the finest achievement so far in Audioactive Chemistry, is the result of the combined labours of three brilliant investigators.

The Cosmological Theorem.

Any string decays into a compound of common and transuranic elements after a bounded number of derivation steps. As a consequence, every string other than the two boring ones increases at the magic rate λ, and the relative abundances of the atoms in its descendants approach the values we have already described.

Proof of the Cosmological Theorem would fill the rest of *Eureka!* Richard Parker and I found a proof over a period of about a month of very intensive work (or, rather, play!). We first produced a very subtle and complicated argument which (almost) reduced the problem to tracking a few hundred cases, and then handled these on dozens of sheets of paper (now lost). Mike Guy found a simpler proof that used tracking and backtracking in roughly equal proportions. Guy's proof still filled lots of pages (almost all lost), but had the advantage that it found the longest-lived of the exotic elements, namely the isotopes of **Methuselum** (223332221ln; see Figure 2). Can you find a proof in only a few pages? Please!

223332221ln (n>1)
22333211n
22332211n
22233211n
3223211n
1322133211n
111322112322211n
311322211213322111n
13211332211211232221ln
*La.H.*1232221122112133221ln
1112133221222112112322211n
*Sr.*32211322112211213322111n
132221132221222112112322211ln
11133221133211322112211213322111n
3123222.*Ca.*(*Li* **or** *W* **or** *Pu*)
1311121332
11133112112.*Zn*
*Zn.*321122112
131221222112
11131122113221l2
311321222113222112
13211312113221133221l2
11131221131112211322.*Na*
3113112221133122211332
Ho.Pa.H.Ca.Ac.H.Ca.Zn

Figure 2. The descendants of Methuselum.

The degree of λ.

Plainly λ is an algebraic number of degree at most 92. We first reduce this bound to 71 by exhibiting a 21-dimensional invariant subspace on which the eigenvalues of **M** are 0 or ±1. To do this, we define the vectors

$$\mathbf{v}_1=\mathrm{H},\ \mathbf{v}_2=\mathrm{He-Ta},\ \mathbf{v}_3=\mathrm{Li-W},\ \cdots\ ,\mathbf{v}_{20}=\mathrm{Ca-Pa},$$

or, in atomic number notation,

$$\mathbf{v}_1=\mathbf{E}_1,\ \mathbf{v}_2=\mathbf{E}_2-\mathbf{E}_{73},\ \mathbf{v}_3=\mathbf{E}_3-\mathbf{E}_{74},\ \cdots\ ,\mathbf{v}_{20}=\mathbf{E}_{20}-\mathbf{E}_{91},$$

and also define

$$\mathbf{v}_{21}=\{\mathrm{Sc+Sm-H-Ni-Er-3U}\}/2,$$

then observe that

$$\mathbf{v}_{21}\to\mathbf{v}_{20}\to\mathbf{v}_{19}\to\cdots\to\mathbf{v}_4\to\mathbf{v}_3\leftrightarrow\mathbf{v}_2,\ \mathbf{v}_1\to\mathbf{v}_1.$$

An alternative base for this space consists of the eigenvectors

$$\mathbf{v}_1 \text{ and } \mathbf{v}_3\pm\mathbf{v}_2$$

of **M** with the respective eigenvalues

$$1 \text{ and } \pm1,$$

together with the following Jordan block of size 18 for the eigenvalue 0

$$v_{21}-v_{19} \to v_{20}-v_{18} \to \cdots \to v_5-v_3 \to v_4-v_2 \to 0.$$

[This shows that M is one of those "infinitely rare" matrices that cannot be diagonalised. Don't expect to follow these remarks unless you've understood more of linear algebra than I fear most of your colleagues have!]

Richard Parker and I have recently proved that the residual 71st degree equation for λ is irreducible, even when it is read modulo 5. We use the fact that the numbers in a finite field of order q all satisfy $x^q = x$ (since the non-zero ones form a group of order q-1, and so satisfy $x^{q-1} = 1$).

Working always modulo 5, we used a computer to evaluate the sequence of matrices

$$M_0 = M,\ M_1 = {M_0}^5,\ M_2 = {M_1}^5,\ M_3 = {M_2}^5,\ \ldots,\ M_{73} = {M_{72}}^5,$$

and to verify that the nullity (modulo 5) of $M_{n+2} - M_2$ was 21 for $1 \leqslant n \leqslant 70$, but 92 for $n = 71$. Note that the 21 vectors of the above 'alternative base' are <u>eigenvectors</u> of M_2 whose eigenvalues (modulo 5) lie in the field of order 5.

If the 71st degree equation were reducible modulo 5, then M_2 would have an eigenvector linearly independent of these with eigenvalue lying in some extension field of order $q = 5^n$ $(1 \leqslant n \leqslant 70)$. But then the eigenvalues ϕ of these 22 eigenvectors would all satisfy $\phi^q = \phi$, and the 22 eigenvectors would be null-vectors for

$$(M_2)^q - M_2 = M_{n+2} - M_2,$$

contradicting our computer calculations.

It is rather nice that we were able to do this without being able to write down the polynomial. However, Professor Oliver Atkin of Chicago has since kindly calculated the polynomial explicitly, and has also evaluated its largest root λ as

1.30357726903429639125709911215254 98

approximately. The polynomial is

$$\begin{aligned}
&x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} \\
&- x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} \\
&+ 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} \\
&- 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} \\
&- 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} \\
&+ 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11} \\
&- 2x^{10} + 5x^9 + x^7 - 7x^6 + 7x^5 - 4x^4 + 12x^3 - 6x^2 + 3x - 6
\end{aligned}$$