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Correction

Correction to: Spectral Theory for Interacting Particle Systems Solvable by Coordinate Bethe Ansatz

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This is a correction to Theorems 7.3 and 8.12 in [1]. These statements claimed to deduce the spatial Plancherel formula (spatial biorthogonality) of the ASEP and XXZ eigenfunctions from the corresponding statements for the eigenfunctions of the q-Hahn system. Such a reduction is wrong. We are grateful to Yier Lin for pointing this out to us.

We have updated the arXiv version of the paper with the necessary corrections [2]. Below is the summary of the issue and the steps we made to correct the presentation of the ASEP and XXZ applications of our results about the q-Hahn eigenfunctions.

q-Hahn Spatial Biorthogonality

Recall that the q-Hahn left and right eigenfunctions are given by

$$\Psi^{\ell}_{\vec{z}}(\vec{n}) := \sum_{\sigma\in S(k)} \prod_{1\le B<A\le k} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1-z_{\sigma(j)}}{1-\nu z_{\sigma(j)}}\right)^{-n_j},$$

$$\Psi^{r}_{\vec{z}}(\vec{n}) := (-1)^k (1-q)^k q^{\frac{k(k-1)}{2}} \mathfrak{m}_{q,\nu}(\vec{n}) \sum_{\sigma\in S(k)} \prod_{1\le B<A\le k} \frac{z_{\sigma(A)} - q^{-1} z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1-z_{\sigma(j)}}{1-\nu z_{\sigma(j)}}\right)^{n_j}$$

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where $\vec{n} = (n_1 \geq \cdots \geq n_k)$. (Here and below we bring only the essential notation from the original paper [1].) Their spatial biorthogonality written in the small contour form reads [1, Corollary 3.13]

$$\sum_{\lambda \vdash k} \oint_{\gamma_k} \cdots \oint_{\gamma_k} d\mathsf{m}_\lambda^{(q)}(\vec{w}) \prod_{j=1}^{\ell(\lambda)} \frac{1}{(w_j;q)_{\lambda_j}(\nu w_j;q)_{\lambda_j}} \Psi^\ell_{\vec{w}\circ\lambda}(\vec{n})\Psi^r_{\vec{w}\circ\lambda}(\vec{m}) = \mathbf{1}_{\vec{m}=\vec{n}}, \quad (1)$$

with all integration contours being small positively oriented circles around 1, and where

$$d\mathsf{m}_\lambda^{(q)}(\vec{w}) := \frac{(1-q)^k(-1)^k q^{-\frac{k^2}{2}}}{m_1!m_2!\ldots} \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} w_j^{\lambda_j} q^{\frac{\lambda_j^2}{2}} \frac{dw_j}{2\pi \mathbf{i}}.$$

Here, $\vec{w} = (w_1, \ldots, w_{\ell(\lambda)}) \in \mathbb{C}^{\ell(\lambda)}$, and m_j is the number of components of λ equal to j (so that $\lambda = 1^{m_1}2^{m_2}\ldots$), and

$$\vec{w} \circ \lambda := (w_1, qw_1, \ldots, q^{\lambda_1-1}w_1, w_2, qw_2, \ldots, q^{\lambda_2-1}w_2, \ldots, w_{\lambda_{\ell(\lambda)}}, qw_{\lambda_{\ell(\lambda)}}, \ldots, q^{\lambda_{\ell(\lambda)}-1}w_{\lambda_{\ell(\lambda)}}) \in \mathbb{C}^k.$$

ASEP Spatial Biorthogonality

To obtain the ASEP eigenfunctions from the q-Hahn ones, we set $\nu = 1/q = 1/\tau$, where $\tau \in (0,1)$ is the ASEP asymmetry parameter:

$$\Psi_{\vec{z}}^{\mathrm{ASEP}}(x_1, \ldots, x_k) = \Psi^\ell_{-\vec{z}}(x_k, \ldots, x_1)\big|_{q=\nu^{-1}=\tau},$$
$$(\mathcal{R}\Psi_{\vec{z}}^{\mathrm{ASEP}})(x_1, \ldots, x_k) \cdot \mathbf{1}_{x_1<\ldots<x_k} = (\tau^{-1}-1)^{-k}\Psi^r_{-\vec{z}}(x_k, \ldots, x_1)\big|_{q=\nu^{-1}=\tau}.$$

Here, $x_1 < \cdots < x_k$ are the ASEP spatial coordinates. The spatial biorthogonality of the ASEP eigenfunctions reads

$$\oint_{\widetilde{\gamma}_{-1}} \cdots \oint_{\widetilde{\gamma}_{-1}} d\mathsf{m}_{(1^k)}^{(\tau)}(\vec{z}) \prod_{j=1}^{k} \frac{1-1/\tau}{(1+z_j)(1+z_j/\tau)} \Psi_{\vec{z}}^{\mathrm{ASEP}}(\vec{x})(\mathcal{R}\Psi_{\vec{z}}^{\mathrm{ASEP}})(\vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}, \quad (2)$$

where the integration is performed over sufficiently small positively oriented circles around -1. This biorthogonality of the ASEP eigenfunctions follows from the paper by Tracy and Widom [4], as we explain in detail in [2, Proof of Theorem 7.3]. Next we discuss the gap in our original argument.

Why (2) Does Not Follow from (1) as Claimed

The "proof" of ASEP spatial biorthogonality given in [1] claimed to deduce (2) by plugging $\nu = 1/q$ into (1) before performing the integration. Indeed, identity (2) looks as if one takes the q-Hahn small contour formula (1), removes all terms corresponding to partitions $\lambda \neq (1^k)$, and then plugs in $\nu = 1/q$, $q = \tau$. Formula (2) (following from [4]) a posteriori implies that under this specialization, the contribution of all additional terms with $\lambda \neq (1^k)$ vanishes.

First, observe that the substitution $\nu = 1/q$ before the integration might change the value of the integral because of the factors of the form $\frac{1}{1-q\nu w_i}$ in the integrand for $\lambda \neq (1^k)$. Before the substitution $\nu = 1/q$, the residue at $w_i = (q\nu)^{-1}$ was not picked while after the substitution we have $1 - q\nu w_i = 1 - w_i$, so this factor adds an extra pole inside the integration contour.

With the agreement that the substitution $\nu = 1/q$ occurs after the integration, the "proof" of (2) presented in [1] asserted a stronger statement: For each individual $\lambda \neq (1^k)$ and any two permutations $\sigma, \omega \in S(k)$ (coming from $\Psi^{\ell}_{\vec{z}}$ and $\Psi^{r}_{\vec{z}}$, respectively), the corresponding term vanishes after setting $\nu = 1/q$. This assertion is wrong.

For example, take $\vec{x} = (10, 9, 8, 7, 6, 5)$ and $\vec{y} = (5, 4, 3, 2, 1, 0)$. The summand in the integrand in (1) corresponding to $\lambda = (3, 2, 1)$, and permutations $\sigma = 321{,}546$ and $\omega = 645{,}123$ has the form (before setting $q = 1/\nu = \tau$):

$$\text{const} \cdot \frac{(1-\nu q w_1)^7 (1-\nu q w_2)^3}{(1-w_1)^7(1-w_2)^3(1-w_3)}$$
$$\times \frac{(qw_1 - w_2)\left(q^2 w_1 - w_2\right)^2 \left(q^3 w_1 - w_2\right)\left(q^2 w_1 - w_3\right)\left(q^3 w_1 - w_3\right)(q w_2 - w_3)\left(q^2 w_2 - w_3\right)}{(w_1 - w_2)(w_1 - w_3)(w_2 - w_3)(q w_2 - w_1)^2 \left(q^2 w_2 - w_1\right)(q w_3 - w_1)(q w_3 - w_2)}$$
$$\times f_1(w_1) f_2(w_2) f_3(w_3).$$

Here, $f_1(w_1)$ is independent of w_2, w_3 and has no zeroes or poles at $w_1 = 1$ and $w_1 = 1/(q\nu)$, and similarly for $f_2(w_2)$ and $f_3(w_3)$. One can check that the residue of this term at $w_3 = 1$, $w_2 = 1$, and $w_1 = 1$ does not vanish when setting $q = 1/\nu$. (Note that the result of the integration depends on the order of taking the residues for individual summands due to the presence of the factors of the form $w_i - w_j$ in the denominators. These factors cancel out after summing over all permutations σ, ω, and each summand indexed by λ is independent of the order of integration because the result of the summation is a function symmetric in the w_i's.)

Let us mention another (possibly related) subtlety in the spatial biorthogonality of the ASEP eigenfunctions as compared to the general q-Hahn case. Namely, in the q-Hahn situation the contribution of individual permutations coming from the eigenfunctions vanishes, while in the ASEP case this is not the case (see [2, Remark 7.6] for details). The proof of the ASEP statement in [4] employs nontrivial combinatorics to determine cancellations of specific combinations of permutations.

Corrections We Made in the New Version [2] Compared to the Published Version [1]

We have replaced the incorrect "proof" of Theorem 7.3 (spatial biorthogonality of the ASEP eigenfunctions) by its derivation from the earlier result of Tracy and Widom [4]. We have also removed Theorem 8.12 which claimed a spatial biorthogonality statement of the XXZ eigenfunctions based on a similar incorrect direct substitution $\nu = \boldsymbol{\theta}$.

The Same Gap in [3]

The claim similar to (1) but with more general $\nu = q^{-I}$, where I is an arbitrary positive integer, is made in [3, Appendix A] (by a subset of the current authors). When $I = 1$, this identity is correct, but does not follow from the general $\nu \in (0, 1)$ formulas (as explained above). Moreover, for $I \geq 2$ the claimed orthogonality does not seem to hold as stated. A separate erratum will be prepared to address the issues in the work [3].

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