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Parameter Symmetry in Perturbed GUE Corners Process and Reflected Drifted Brownian Motions

Leonid Petrov[1,2] · **Mikhail Tikhonov**[2,3]

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Abstract
The perturbed GUE corners ensemble is the joint distribution of eigenvalues of all principal submatrices of a matrix $G+\mathrm{diag}(\mathbf{a})$, where G is the random matrix from the Gaussian Unitary Ensemble (GUE), and $\mathrm{diag}(\mathbf{a})$ is a fixed diagonal matrix. We introduce Markov transitions based on exponential jumps of eigenvalues, and show that their successive application is equivalent in distribution to a deterministic shift of the matrix. This result also leads to a new distributional symmetry for a family of reflected Brownian motions with drifts coming from an arithmetic progression. The construction we present may be viewed as a random matrix analogue of the recent results of the first author and Axel Saenz [17].

Keywords Random matrices · Perturbed GUE corner process · Reflected Brownian motions

1 Introduction

1.1 Couplings for Perturbed GUE Corners Process

The Gaussian Unitary Ensemble (GUE) is the most well-known random matrix model [2,6, 14]. This paper presents a new symmetry of the distribution of the *perturbed GUE ensemble*. By this we mean the random matrix ensemble of the form $H = G+\mathrm{diag}(a_1, \dots, a_N)$, where G is an $N \times N$ GUE random matrix, to which we add a fixed diagonal matrix. This model is often also called *GUE with external source*. We refer to [3–5] and references therein for the history of the perturbed ensemble and various asymptotic results. (In fact, below we consider

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✉ Mikhail Tikhonov
me@mtikhonov.com ; tikhonov.ms15@physics.msu.ru

Leonid Petrov
lenia.petrov@gmail.com

[1] Department of Mathematics, University of Virginia, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA

[2] Institute for Information Transmission Problems, Bolshoy Karetny per. 19, Moscow, Russia 127994

[3] Faculty of Physics, Lomonosov Moscow State University, Leninskie Gory, 1-2, Moscow, Russia 119991

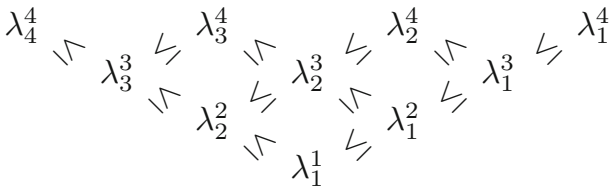

Fig. 1 Interlacing array of eigenvalues of all principal corners of a 4×4 matrix

a slightly more general version of the matrix model involving a time-dependent rescaling; this version is suitable for the application to reflected Brownian motions).

The unperturbed GUE random matrix, corresponding to $a_i \equiv 0$, is unitary invariant in the sense that there is equality in distribution $G \stackrel{d}{=} UGU^*$ for any fixed $N \times N$ unitary matrix U. This implies that the distribution of the eigenvalues of H is *symmetric* in the perturbation parameters $a_1, \ldots, a_N$. The overall goal of the paper is to **explore probabilistic consequences of this symmetry property**.

Together with the eigenvalues $\lambda^N = (\lambda_N^N \leq \ldots \leq \lambda_1^N)$, $\lambda_i^N \in \mathbb{R}$, of the full matrix $H = [h_{ij}]_{i,j=1}^N$, one can also consider its *corners process*,[1] that is, the interlacing collection of eigenvalues of the principal corners $[h_{ij}]_{i,j=1}^k$ of H for all $k = 1, 2, \ldots, N$. (See Fig. 1 for an illustration.) The distribution of the corners process of H is *not symmetric* in the parameters a_i. Moreover, assuming that the a_i's are all distinct, there are $N!$ *different probability distributions* on interlacing collections of eigenvalues at N levels.

In this paper we present explicit couplings between these $N!$ distributions, by showing that each nearest neighbour transposition $a_k \leftrightarrow a_{k+1}$, $k = 1, \ldots, N-1$, of the parameters is equivalent in distribution to a rather simple *Markov swap operator* $\mathcal{S}_k^{a_k - a_{k+1}}$. This swap operator randomly changes the entries λ_i^k on the k-th level of the array given the two adjacent levels λ^{k-1}, λ^{k+1}, while leaving all other entries intact. If $a_k > a_{k+1}$, $\mathcal{S}_k^{a_k - a_{k+1}}$ is realized as an independent collection of instantaneous exponential type jumps of each λ_i^k to the left:[2]

$$\lambda_i^k \mapsto \nu_i^k := \lambda_{i+1}^{k+1} \vee \lambda_i^{k-1} + \mathcal{E}_{a_k - a_{k+1}}^i \wedge \left(\lambda_i^k - \lambda_{i+1}^{k+1} \vee \lambda_i^{k-1}\right), \qquad i = 1, \ldots, k,$$

where $\mathcal{E}_{a_k - a_{k+1}}^i$'s are independent exponential random variables with parameter $a_k - a_{k+1}$ (and mean $1/(a_k - a_{k+1})$). Here by agreement, $\lambda_k^{k-1} = -\infty$. In particular, these left jumps are constrained by the interlacing. For $a_k < a_{k+1}$, the same jumps are performed to the right in a symmetric way:

$$\lambda_i^k \mapsto \mu_i^k := \lambda_i^{k+1} \wedge \lambda_{i-1}^{k-1} - \mathcal{E}_{a_{k+1} - a_k}^i \wedge \left(\lambda_i^{k+1} \wedge \lambda_{i-1}^{k-1} - \lambda_i^k\right), \qquad i = 1, \ldots, k,$$

where, by agreement, $\lambda_0^{k-1} = +\infty$. Finally, if $a_k = a_{k+1}$, then $\mathcal{S}_k^{a_k - a_{k+1}}$ is the identity operation.

Theorem 1.1 *(Follows from Theorem 4.4 below) Assume that $a_k \neq a_{k+1}$. Then the action of the Markov operator $\mathcal{S}_k^{a_k - a_{k+1}}$ (with left jumps for $a_k > a_{k+1}$, and right jumps otherwise) turns the corners distribution of $G + \mathrm{diag}(a_1, \ldots, a_k, a_{k+1}, \ldots, a_N)$ into the one of $G + \mathrm{diag}(a_1, \ldots, a_{k+1}, a_k, \ldots, a_N)$, where G is the $N \times N$ GUE random matrix.*

[1] Also called *minors process* in the literature, cf. [11].

[2] Here and below we use the standard notation $A \vee B = \max(A, B)$, $A \wedge B = \min(A, B)$ for $A, B \in \mathbb{R}$.

We establish this theorem by relying on a perturbed Gibbs structure of the corners distribution of the matrix H. Namely, it is well-known that in the unperturbed case, the conditional distribution of the eigenvalues λ_i^k, $1 \le i \le k \le N-1$, given λ^N, is uniform on the polytope defined by all the interlacing inequalities (known as the Gelfand-Tsetlin polytope). In the perturbed case, the Gibbs structure should be deformed in a certain way by means of the parameters a_i (see Sect. 3.1). The coupling follows by considering the conditional distribution of λ^k given two adjacent levels $\lambda^{k\pm1}$, which reduces to a collection of independent exponential random variables confined to the corresponding intervals. Producing a suitable Markov swap operator for a single such variable (see Proposition 4.1 below), we arrive at the result of Theorem 1.1.

Remark 1.2 Applied twice to $G + \mathrm{diag}(a_1, \dots, a_N)$, the Markov swap operator from Theorem 1.1 returns to the same distribution. That it, the composition of $\mathcal{S}_k^{a_k - a_{k+1}}$ and $\mathcal{S}_k^{a_{k+1}-a_k}$ does not change the distribution of $G + \mathrm{diag}(a_1, \dots, a_N)$. However, this composition is *not* an identity transformation: two random jumps return a particle to the original location with probability 0.

1.2 Perturbation by an Arithmetic Progression

The perturbed GUE corners distributions are compatible for various N, and so one can define the corresponding perturbed GUE corners distribution on infinite interlacing arrays. It depends on an infinite parameter sequence $\mathbf{a} = \{a_i\}_{i \in \mathbb{Z}_{\ge 1}}$. One particular interesting case is when the perturbation parameters form an arithmetic progression $a_i = -(i-1)\alpha$, where $\alpha > 0$. Swapping a_1 with a_2, then a_1 with a_3, and so on all the way to infinity leads to an additive shift in the perturbation parameters, which is equivalent in distribution to a global shift:

Theorem 1.3 *(Theorem 5.2 below) The action of a sequence of left exponential jumps (where the parameter at level k is taken to be $k\alpha$), from level 1 up to infinity, is equivalent in distribution*[3] *to shifting all the elements of the interlacing array by α to the left.*

1.3 Shifting of Reflected Brownian Motions

Let $\alpha > 0$, and let let $X_k(t)$, $k = 1, 2, \dots$, be reflected Brownian motions constructed as follows. First, $X_1(t)$ is the standard driftless Brownian motion started from 0. Inductively, let $X_k(t)$, $k = 2, 3, \dots$, be a new independent Brownian motion with drift $-(k-1)\alpha$, and reflected down off of $X_{k-1}(t)$ by means of subtracting local time when $X_k = X_{k-1}$. For example,

$$X_2(t) = X_2^\circ(t) - L_{1,2}(t),$$

where $X_2^\circ(t)$ is the standard Brownian motion, and $L_{1,2}(t) = \int_0^t \mathbf{1}_{X_1(s)=X_2(s)}\, dL_{1,2}(s)$ is the continuous non-decreasing process which increases only at times when $X_1(s) = X_2(s)$ (in other words, it is twice the semimartingale local time of $X_1 - X_2$ at zero. We refer to [5,19] for further details on the reflection mechanism, and for an explanation on how to start all these reflected processes from zero (which formally results in infinitely many collisions in finite time). Almost surely we have $X_1(t) \ge X_2(t) \ge X_3(t) \dots$ for all t.

[3] Here and below by saying that two operations are "equivalent in distribution" we mean that the random elements resulting from both these operations, applied to the same initial random element, have the same distribution.

Fix t and define

$$X'_k(t) := X_{k+1}(t) + \mathcal{E}_{k\alpha} \wedge (X_k(t) - X_{k+1}(t)), \quad k = 1, 2, \ldots, \tag{1.1}$$

where $\mathcal{E}_{k\alpha}$, $k = 1, 2, \ldots$, are independent exponential random variables with parameters $k\alpha$ (and mean $1/(k\alpha)$).

Theorem 1.4 *For each fixed t, we have equality of joint distributions*

$$\{X'_k(t)\}_{k\in\mathbb{Z}_{\geq 1}} \stackrel{d}{=} \{X_k(t) - \alpha t\}_{k\in\mathbb{Z}_{\geq 1}}.$$

In particular, $X'_1(t) = X_2(t) + \mathcal{E}_\alpha \wedge (X_1(t) - X_2(t))$ is a normal random variable with mean $(-\alpha t)$ and variance t. To the best of our knowledge, even this result for two processes (one a usual Brownian motion, and one reflected off it) is new.

Theorem 1.4 follows from Theorem 1.3 combined with the connection between the reflected drifted Brownian motions and the perturbed GUE corners process due to [5]. We recall this connection in detail in Sect. 2.3 below, and prove Theorem 1.4 in the end of Sect. 5.

As stated, Theorem 1.4 assumes that the time t is fixed. Indeed, naively taking independent exponential shifts at different times t would make the functions $t \mapsto X'_k(t)$ discontinuous. It is interesting to see whether a stochastic process version of Theorem 1.4 holds:

Open problem 1.5 *Is it possible to construct a Markov operator on whole trajectories $t \mapsto \{X_k(t)\}_{k\in\mathbb{Z}_{\geq 1}}$ which is equivalent in distribution to a shift of reflected Brownian motions as stochastic processes?*

Presumably, if such a Markov operator on processes exists, then its construction could be accomplished using the sticky Brownian motion,[4] as exponential random variables arise in the study of this process, e.g., see Theorem 1 in [18]. It seems plausible that the difference process $t \mapsto X_1(t) - X'_1(t) \geq 0$ itself could be distributed as the sticky Brownian motion, as the single-time distributions coincide thanks to the results of [18] and [10, Proposition 14]. However, it is less clear how to extend this idea to all differences $t \mapsto X_k(t) - X'_k(t) \geq 0$.

1.4 Related Discrete Model

The results of this paper might be viewed as a random matrix limit of the ones from the recent work [17]. There, similar Markov swap operators were considered on discrete interlacing arrays as in Fig. 1. A combination of these swap operators together with a certain Poisson-type limit (cf. Sect. 1.5 below) has lead to a Markov chain on distributions of TASEP (totally asymmetric simple exclusion process) which decreases the time parameter. The shifting result for reflected drifted Brownian motions (Theorem 1.4) may be viewed as a certain analogue of the TASEP reversal property. In the Brownian case, instead of decreasing the time, the exponential jumps lead to a deterministic shift.

It should be pointed out that even though the discrete stochastic systems in [17] converge to the reflected Brownian motions [8] (and [5] in the drifted case), here we do not rely on this convergence or the results of [17]. Instead we obtain the results independently using basic mechanisms related to the (perturbed) Gibbs property.

[4] We are grateful to Jon Warren (personal communication) for suggesting this connection.

1.5 Unperturbed Case

In the arithmetic progression setting $a_i = -(i-1)\alpha$ with $\alpha > 0$, when $\alpha \searrow 0$, the perturbed GUE corners process of $H = G + \operatorname{diag}(0, -\alpha, -2\alpha, \ldots)$ becomes the usual GUE corners process, and the system of reflected Brownian motions $\{X_k(t)\}_{k\in\mathbb{Z}_{\geq 1}}$ becomes driftless. It would be very interesting to see whether the Markov operators considered in the present paper have meaningful limits as $\alpha \searrow 0$. However, this limit presents certain immediate issues which we discuss now.

For simplicity, consider the Brownian motion setup. Fix $t > 0$ and suppress this parameter in the notation. As $\alpha \to 0$, the Markov operator $X_k \mapsto X'_k$ (1.1) turns into the (deterministic) identity operator $X_k \mapsto X_k$. Indeed, this is because $\operatorname{Prob}(\mathcal{E}_{k\alpha} > x) = e^{-k\alpha x} \sim 1 - \alpha k x$ for all k and x, and so the minimum in (1.1) is equal to $X_k - X_{k+1}$ with probability of order $1 - O(\alpha)$. Arguing similarly to the discrete case considered in [17, Sect. 6], one can apply the map (1.1) a large number $\lfloor \tau/\alpha \rfloor$ of times, where $\tau \in \mathbb{R}_{>0}$ is the scaled time.

Taking a Poisson-type limit should lead to a continuous time Markov process (with τ as the new time parameter) under which X_k has an exponential clock of rate $k(X_k - X_{k+1})$, and when the clock rings, X_k instantaneously jumps into X'_k selected uniformly from $[X_{k+1}, X_k]$. This jumping mechanism is known as the *Hammersley process* [1,9]. However, applying this continuous time jumping process to the whole system $\{X_k\}_{k\in\mathbb{Z}_{\geq 1}}$ is problematic, as it leads to *infinitely many jumps in finite time* due to the growing jump rates $k(X_k - X_{k+1})$ as $k \to \infty$. Moreover, under this hypothetical process X_k would depend on all X_j for $j > k$, and so one cannot simply restrict the dynamics to finitely many particles where it would make sense.

On the other hand, by Theorem 1.4, the hypothetical continuous time dynamics should be equivalent in distribution to a deterministic shift of the (driftless) reflected Brownian motions by $-\alpha t \lfloor \tau/\alpha \rfloor \sim -t\tau$. It is reasonable to expect that such a deterministic shift of infinitely many X_k's cannot be achieved only by finitely many jumps in finite time. To summarize,

Open problem 1.6 *Do there exist well-defined $\alpha \searrow 0$ limits of the Markov operators acting on the GUE corners process perturbed by an arithmetic progression $a_i = -(i-1)\alpha$ or on the reflected drifted Brownian motions? These hypothetical limits should act on (much more studied) unperturbed GUE corners process and driftless reflected Brownian motions.*

2 Perturbed GUE Corners Process

This section is preliminary. We recall the perturbed GUE corners process [5] (also called the GUE corners process with external source [3]), and its connection to reflected Brownian motions with drifts. The original, unperturbed GUE corners process is due to [11,12], and it was linked to driftless reflected Brownian motions in [19].

2.1 Matrix Model

Take a time parameter $t > 0$ and an infinite sequence of parameters

$$\mathbf{a} = (a_1, a_2, \ldots), \qquad a_i \in \mathbb{R}.$$

Unless otherwise indicated, we assume that the parameters a_i are pairwise distinct. Consider a random matrix $H = t^{1/2} \cdot G + t \cdot \mathrm{diag}\,(\mathbf{a})$ of infinite size with entries:

$$H_{kl} = \begin{cases} t^{1/2} g_{kk} + t\mu_k, & k = l; \\ t^{1/2} g_{kl}, & k < l; \\ t^{1/2} \overline{g}_{lk}, & k > l. \end{cases}$$

Here g_{kk} are independent real standard normal random variables, and g_{kl} are independent complex standard normal random variables (that is, their real and imaginary parts are independent real normal random variables each with mean 0 and variance $\frac{1}{2}$). The matrix H is Hermitian.

For each $m \in \mathbb{Z}_{\geq 1}$, take the $m \times m$ principal corner $[H_{kl}]_{1 \leq k,l \leq m}$ of the infinite matrix H. Let $\lambda^m = (\lambda_1^m \geq \ldots \geq \lambda_m^m)$, $\lambda_i^m \in \mathbb{R}$, be its eigenvalues. At adjacent levels, the eigenvalues *interlace* (notation $\lambda^m \prec \lambda^{m+1}$):

$$\lambda_{m+1}^{m+1} \leq \lambda_m^m \leq \lambda_m^{m+1} \leq \lambda_{m-1}^m \leq \cdots \leq \lambda_2^{m+1} \leq \lambda_1^m \leq \lambda_1^{m+1}. \tag{2.1}$$

We call the joint distribution of all $\{\lambda_j^k\}_{1 \leq j \leq k < \infty}$ the *perturbed GUE corners process*.

2.2 Joint Eigenvalue Density

A standard application of the Harish–Chandra–Itsykson–Zuber integral shows that the joint eigenvalue density of $\{\lambda_i^N\}_{i=1}^N$ at a fixed level N is given by

$$\mathsf{Density}(\lambda^N) = \mathrm{const} \times \det \left[\exp \left\{ -\frac{(\lambda_i^N - t a_j)^2}{2t} \right\} \right]_{i,j=1}^N \frac{\mathsf{V}(\lambda_1^N, \ldots, \lambda_N^N)}{\mathsf{V}(a_1, \ldots, a_N)}, \tag{2.2}$$

where the normalizing constant does not depend on $a_1, \ldots, a_N$. Here and throughout the paper we use the notation

$$\mathsf{V}(b_1, \ldots, b_N) = \prod_{1 \leq i < j \leq N} (b_i - b_j)$$

for the Vandermonde determinant.

Observe from (2.2) that the distribution of $\{\lambda_j^N\}_{i=1}^N$ depends on the parameters a_i in a symmetric way. This should indeed be the case, since the distribution of the eigenvalues of the $N \times N$ matrix $t^{1/2} G_{N \times N} + t\,\mathrm{diag}(a_1, \ldots, a_N)$ does not depend on the order of the a_i's due to the unitary invariance of $G_{N \times N}$. The main goal of this paper is to explore this distributional symmetry from a Markov operator point of view. For this, we will need the joint distribution of eigenvalues of all corners:

Proposition 2.1 ([5, Proposition 2.3]) *The joint density of the eigenvalues $\{\lambda_j^k\}_{1 \leq j \leq k \leq N}$ at the first N levels, where $N \in \mathbb{Z}_{\geq 1}$ is arbitrary, has the following form:*

$$\mathrm{const} \times V\left(\lambda_1^N, \ldots, \lambda_N^N\right) \prod_{i=1}^N e^{-t a_i^2/2 - (\lambda_i^N)^2/(2t)} \exp \left\{ |\lambda^N|\, a_N + \sum_{k=1}^{N-1} |\lambda^k|\, (a_k - a_{k+1}) \right\} \tag{2.3}$$

where we use the notation $|\lambda^k| := \lambda_1^k + \lambda_2^k + \cdots + \lambda_k^k$, and the normalizing constant does not depend on $a_1, \ldots, a_N$.

2.3 Reflected Brownian Motions

Fix a perturbation sequence $\mathbf{a} = \{a_i\}_{i\in\mathbb{Z}_{\geq 1}}$. Consider a family of interacting Brownian motions B_j^k, $1 \leq j \leq k < \infty$, such that:

- All processes start from zero.
- The processes B_j^k, $j = 1, \ldots, k$ have the same drift a_k.
- The evolution of each B_j^k does not depend on any of the B_i^l's with $l > i$.
- The processes interact only through their local times. That is, when the processes are sufficiently far apart, each B_j^k behaves as an independent Brownian motion with drift a_k.
- Each B_j^k belongs to the segment $[B_j^{k-1}, B_{j-1}^{k-1}]$[5] and reflects off both B_j^{k-1} and B_{j-1}^{k-1}. Therefore, at each time t, the random variables $\{B_j^k(t)\}_{1\leq j\leq k<\infty}$ almost surely form an interlacing array as in Fig. 1.

We refer to [5, Sect. 4] (and [19] in the driftless case) for details on the reflection mechanism.

Proposition 2.2 ([5, Theorem 2]) *At each fixed time moment $t \in \mathbb{R}_{\geq 0}$, we have equality of joint distributions of two infinite interlacing arrays:*

$$\{B_j^k(t)\}_{1\leq j\leq k<\infty} \stackrel{d}{=} \{\lambda_j^k\}_{1\leq j\leq k<\infty},$$

where the right-hand side is the perturbed GUE corners process with the same time parameter t and perturbation sequence $\mathbf{a}$.

3 Gibbs Measures

In this section we place the perturbed GUE corners process into a wider family of Gibbs measures on interlacing arrays.

3.1 Gibbs Property and Harmonic Functions

A measure on infinite interlacing arrays $\{\lambda_j^k\}_{1\leq j\leq k<\infty}$ (satisfying inequalities (2.1) between any two consecutive levels) is called $\mathbf{a}$-*Gibbs* if for each N and any fixed configuration λ^N at level N, the density of the conditional distribution of all the lower entries of the array has the form

$$\begin{aligned}\mathsf{Density}(\lambda^1, \ldots, \lambda^{N-1} \mid \lambda^N) = {} & \frac{\mathsf{V}(a_1, \ldots, a_N)}{\det[\exp\{a_i \lambda_j^N\}]_{i,j=1}^N} \\ & \times \exp\left\{ |\lambda^N| a_N + \sum_{k=1}^{N-1} |\lambda^k| (a_k - a_{k+1}) \right\} \mathbf{1}_{\lambda^1 \prec \lambda^2 \prec \ldots \prec \lambda^{N-1} \prec \lambda^N}\end{aligned} \tag{3.1}$$

(if some of the λ_i^N's are equal, the density would have delta components and formula (3.1) should be understood in a limiting sense). Here and below $\mathbf{1}_B$ stands for the indicator of an event B. Proposition 2.1 implies that the perturbed GUE corners process is an example of an $\mathbf{a}$-Gibbs measure.

[5] If one or both ends of the segment are not defined, they should be replaced with infinity of appropriate sign.

Remark 3.1 The fact that the density (3.1) integrates to 1 in $\lambda^1, \dots, \lambda^{N-1}$ can be checked by induction on N.

Remark 3.2 When $a_i \equiv a$ are all equal to each other, the **a**-Gibbs property becomes the usual Gibbs property, with (3.1) replaced by the uniform conditioning provided that the configurations $\lambda^1, \dots, \lambda^{N-1}, \lambda^N$ interlace. A classification of uniform Gibbs measures on interlacing arrays is due to [16]. In fact, performing a suitable exponential change of variables, one can see that when **a** is an arithmetic progression, the space of **a**-Gibbs measures is essentially the same as in the uniform case. This is somewhat parallel to how the two-sided q-Gelfand-Tsetlin graph degenerates to the "graph of spectra" [7,15].

To each **a**-Gibbs measure we can associate a family of **a**-*harmonic functions* as follows:

$$\varphi_N(\lambda^N) := \frac{\mathsf{V}(a_1, \dots, a_N)}{\det[\exp\{a_i \lambda_j^N\}]_{i,j=1}^N} \mathsf{Density}(\lambda^N), \qquad N = 1, 2, \dots, \tag{3.2}$$

where $\mathsf{Density}(\lambda^N)$ is the marginal density of λ^N. The term "harmonic function" comes from the Vershik–Kerov theory of boundaries of branching graphs, cf. [13]. Harmonicity means that the functions satisfy a version of a mean value theorem associated to a directed graph Laplacian. In the context of random matrices the discrete graph is replaced by a suitable continuous analogue, and the graph Laplacian becomes an integral operator. In other words, since the **a**-harmonic functions φ_N for different N's come from the same Gibbs measure, they must be consistent in the following sense:

Lemma 3.3 *For all $N \geq 2$ we have*

$$\varphi_{N-1}(\lambda^{N-1}) = \int_{\lambda^N \,:\, \lambda^N \succ \lambda^{N-1}} \varphi_N(\lambda^N)\, e^{a_N(|\lambda^N| - |\lambda^{N-1}|)} d\lambda^N. \tag{3.3}$$

Identity (3.3) should be viewed as a version of the mean value theorem, as discussed before Lemma 3.3.

Proof of Lemma 3.3 The claim follows by writing down the joint distribution of $\lambda^1, \dots, \lambda^N$ through φ_N and the conditional distribution (3.1), and then integrating out $\lambda^1, \dots, \lambda^{N-2}$ (this produces the factor $\mathsf{V}(a_1, \dots, a_{N-1}) / \det[\exp\{a_i \lambda_j^{N-1}\}]_{i,j=1}^{N-1}$) and λ^N to get the marginal density of λ^{N-1}. The resulting marginal density is expressed through φ_{N-1} via (3.2), which yields the result. □

Lemma 3.4 *For an* **a**-*Gibbs measure, let each φ_k depend on $a_1, \dots, a_k$ in a symmetric way. Then the distribution of λ^k, where $k \in \mathbb{Z}_{\geq 1}$ is fixed, depends on the parameters $a_1, \dots, a_k$ in a symmetric way, too.*

Proof An immediate consequence of (3.2). □

Proposition 3.5 *Any* **a**-*Gibbs measure is uniquely determined by the corresponding family of* **a**-*harmonic functions $\{\varphi_N\}_{N \in \mathbb{Z}_{\geq 1}}$.*

Proof Follows from the Kolmogorov extension theorem. □

Let us emphasize that the results of this subsection (Lemmas 3.3 and 3.4 and Proposition 3.5) are valid not only for the perturbed GUE corners process (which, as we see next, is an example of an **a**-Gibbs measure), but hold in the full generality for any **a**-Gibbs measure.

3.2 Perturbed GUE Corners as a Gibbs Measure

One readily sees that for the perturbed GUE corners process we have the following harmonic functions:

$$\varphi_N^{\mathrm{pertGUE}(\mathbf{a};t)}(\lambda^N) = \mathrm{const} \times \mathsf{V}(\lambda_1^N, \dots, \lambda_N^N) \prod_{i=1}^{N} e^{-ta_i^2/2-(\lambda_i^N)^2/(2t)}, \qquad N = 1, 2, \dots, \tag{3.4}$$

where the constant is the same as in (2.3) and does not depend on the a_j's. One readily checks that the $\mathbf{a}$-Gibbs property (Lemma 3.3) for the perturbed GUE corners process is equivalent to the well-known integral identity for the Vandermonde determinants:

$$\begin{aligned} &\mathsf{V}(\lambda_1^{N-1}, \dots, \lambda_{N-1}^{N-1}) \prod_{i=1}^{N-1} e^{-(\lambda_i^{N-1}-a_N t)^2/(2t)} \\ &\quad = \mathrm{const} \times \int_{\lambda^N \,:\, \lambda^N \succ \lambda^{N-1}} \mathsf{V}(\lambda_1^N, \dots, \lambda_N^N) \prod_{i=1}^{N} e^{-(\lambda_i^N-a_N t)^2/(2t)} d\lambda^N. \end{aligned} \tag{3.5}$$

where the constant does not depend on the a_j's. The shift by $a_N t$ in the exponents in both sides by changing the variables in the integral and renaming the λ_i^{N-1}'s, can also be removed (or replaced with any other shift bt) since the Vandermonde is translation invariant.

In particular, (2.3) together with Lemma 3.4 implies the symmetry (as in this lemma) of the perturbed GUE corners distribution with respect to the parameters a_i.

4 Swap Operators Via Exponential Jumps

In this section we explore the Gibbs property and prove Theorem 1.1 on Markov swap operators.

4.1 Confined Exponential Distribution

Let $c < d$ and α be real numbers. Let us call a random variable on (c, d) with probability density

$$\frac{\alpha}{e^{d\alpha} - e^{c\alpha}} e^{\alpha x}, \qquad x \in (c, d),$$

an *exponential random variable confined to the segment* (c, d), notation $E_\alpha(c, d)$. Note that this definition makes sense regardless of the sign of α (in contrast with the case when the interval (c, d) is half-infinite). If $\alpha = 0$, then $E_0(c, d)$ is simply the uniform random variable on (c, d).

4.2 Elementary Markov Swap Operator

The next observation plays a key role:

Proposition 4.1 *Take real numbers $c < d$ and $\alpha > 0$. Let X be distributed as $E_\alpha(c, d)$, and $\mathcal{E}_\alpha \in (0, +\infty)$ be an independent usual exponential random variable with parameter α (i.e.,*

with density $\alpha e^{-\alpha y}$, $y > 0$). Then the random variable

$$Y := c + \mathcal{E}_\alpha \wedge (X - c) \tag{4.1}$$

is distributed as $E_{-\alpha}(c, d)$.

Proof We have for the conditional distribution of Y given $X = x$:

$$\text{Prob}\,(Y \in [y, y + \mathsf{d}y] \mid X = x) = \mathbf{1}_{x=y} e^{-\alpha(y-c)} + \alpha e^{-\alpha(y-c)} \mathsf{d}y, \qquad 0 \le y \le x. \tag{4.2}$$

The distribution of Y has an atom at $y = x$ (coming from the event $\mathcal{E}_\alpha > X - c$ in (4.1)) and an absolutely continuous part on $(0, x)$. The overall density of Y in the variable y is obtained from the following integral:

$$\begin{aligned}
&\int_y^d \frac{\alpha}{e^{d\alpha} - e^{c\alpha}} e^{\alpha x} \,\text{Prob}\,(Y \in [y, y + \mathsf{d}y] \mid X = x)\, \mathsf{d}x \\
&= \frac{\alpha}{e^{d\alpha} - e^{c\alpha}} e^{\alpha y} e^{-\alpha(y-c)} + \frac{\alpha}{e^{d\alpha} - e^{c\alpha}} \alpha e^{-\alpha(y-c)} \int_y^d e^{\alpha x} \mathsf{d}x \\
&= \frac{-\alpha}{e^{-d\alpha} - e^{-c\alpha}} e^{-\alpha y},
\end{aligned}$$

which completes the proof. □

We will view the operation of passing from X to Y as in (4.1) as a one-step Markov transition operator. One can think that the "particle" $X \in (c, d)$ jumps left into the new location Y, by means of the new exponential random variable $\mathcal{E}_\alpha$. Note that the new location Y depends only on X and not on the right endpoint d of the interval. We call this Markov transition operator the *elementary swap operator* and denote it by S^α. This operator acts on distributions (in our case, densities) as $\mathsf{Density}_Y = \mathsf{Density}_X\, S^\alpha$.

The swap operator S^α is analogous to the jump operator L_α in the discrete situation considered in [17, Sect. 4]. Let us make a number of remarks.

Remark 4.2 (1) When $\alpha = 0$, the swap operator S^α should be understood as the identity map, which is evident from (4.2).

(2) For $\alpha = -\beta < 0$, algebraic manipulations in the proof of Proposition 4.1 make sense, but the new random variable Y obtained by applying $S^{-\beta}$ to $X \sim E_{-\beta}(c, d)$ does not admit a probabilistic interpretation as in (4.1).

(3) Instead of applying $S^{-\beta}$ to $E_{-\beta}(c, d)$, let us consider the operator which moves X to the right symmetrically to how S^α moves the "particle" X to the left. That is, this new operator acts as $Y' = d - \mathcal{E}_\beta \wedge (d - X)$, where $\mathcal{E}_\beta$ is an independent exponential random variable. One can show similarly to Proposition 4.1 that if $X \sim E_{-\beta}(c, d)$, then $Y' \sim E_\beta(c, d)$. All our results for Markov operators built from the left jumps S^α have straightforward analogues for these right jumping operators, and so we will only focus on the left jumps in the paper.

4.3 Swap Operator for Gibbs Measures

Let us fix a perturbation sequence $\mathbf{a}$, and let $\{\lambda_j^m\}_{1 \le j \le m < \infty}$ be a random interlacing array distributed according to some $\mathbf{a}$-Gibbs measure (for example, the perturbed GUE corners process with an arbitrary time parameter $t \ge 0$).

Next, fix a level $k \in \mathbb{Z}_{\geq 1}$, and consider the conditional distribution of λ^k given the two adjacent levels $\lambda^{k-1}, \lambda^{k+1}$ (if $k = 1$, the conditioning is only on λ^2). From (3.1) one readily sees that this conditional distribution takes the form

$$\mathsf{Density}(\lambda^k \mid \lambda^{k-1}, \lambda^{k+1}) = \mathrm{const} \times \exp\left\{\alpha(\lambda_1^k + \ldots + \lambda_k^k)\right\} \mathbf{1}_{\lambda^{k-1} \prec \lambda^k \prec \lambda^{k+1}}, \tag{4.3}$$

where we have denoted $\alpha := a_k - a_{k+1}$. Equivalently, we can describe distribution (4.3) as follows.

Proposition 4.3 *The conditional distribution of λ^k given λ^{k-1} and λ^{k+1} is such that each λ_i^k, $i = 1, \ldots, k$, is an independent random variable distributed as*

$$E_\alpha\left(\lambda_{i+1}^{k+1} \vee \lambda_i^{k-1}, \lambda_i^{k+1} \wedge \lambda_{i-1}^{k-1}\right), \tag{4.4}$$

where $\alpha = a_k - a_{k+1}$. (For $i = k$ we set $\lambda_k^{k-1} = -\infty$, and for $i = 1$ we set $\lambda_0^{k-1} = +\infty$, but both ends of the interval in (4.4) are always finite.)

Proof Readily follows from (4.3). □

Assume that $\alpha = a_k - a_{k+1} > 0$, and take an array $\{\lambda_j^m\}_{1 \leq j \leq m < \infty}$ as above. Let us define a new random interlacing array $\{\nu_j^m\}_{1 \leq j \leq m < +\infty}$ for which $\nu_j^m = \lambda_j^m$ for all $m \neq k$, $j = 1, \ldots, m$, and such that

$$\nu_i^k := \lambda_{i+1}^{k+1} \vee \lambda_i^{k-1} + \mathcal{E}_\alpha^i \wedge \left(\lambda_i^k - \lambda_{i+1}^{k+1} \vee \lambda_i^{k-1}\right), \qquad i = 1, \ldots, k, \tag{4.5}$$

where $\mathcal{E}_\alpha^1, \ldots, \mathcal{E}_\alpha^k$ are independent usual exponential random variables with parameter α. Note that almost surely we have $\nu_i^k \leq \lambda_i^k$, $i = 1, \ldots, k$.

In other words, in (4.5) we independently apply the elementary swap operator S^α to each λ_i^k which is confined to the corresponding interval as in Proposition 4.3. Denote this combination of the swap operators applied at level k by $\mathcal{S}_k^\alpha$. As in Remark 4.2, the Markov operator $\mathcal{S}_k^\alpha$ makes sense only for $\alpha > 0$.

Let τ_k denote the elementary transposition $(k, k+1)$. For a perturbation sequence $\mathbf{a}$, let $\tau_k \mathbf{a} = (a_1, \ldots, a_{k-1}, a_{k+1}, a_k, \ldots)$ be the permuted sequence.

Theorem 4.4 *(Theorem 1.1 in Introduction) Take an $\mathbf{a}$-Gibbs measure for which each harmonic function φ_N depends on the parameters $a_1, \ldots, a_N$ in a symmetric way. If $a_k > a_{k+1}$, then the action of the Markov operator $\mathcal{S}_k^\alpha$ (with $\alpha = a_k - a_{k+1}$) on this $\mathbf{a}$-Gibbs measure results in a $\tau_k\mathbf{a}$-Gibbs measure which corresponds to harmonic functions modified as follows:*

$$\begin{aligned} \varphi_j'(\lambda^k) &= \varphi_j(\lambda^k), \qquad j \neq k; \\ \varphi_k'(\lambda^k) &= \int_{\lambda^{k+1}:\, \lambda^{k+1} \succ \lambda^k} \varphi_{k+1}(\lambda^{k+1})\, e^{a_k(|\lambda^{k+1}| - |\lambda^k|)} d\lambda^{k+1}. \end{aligned} \tag{4.6}$$

Proof Since the action of $\mathcal{S}_k^\alpha$ does not change levels $j \neq k$ (and hence distributions of these levels), we clearly have $\varphi_j' = \varphi_j$ for $j \neq k$.

Thus, it remains to show that under $\mathcal{S}_k^\alpha$ the $\mathbf{a}$-Gibbs property becomes $\tau_k\mathbf{a}$-Gibbs. This can be seen by representing the conditional distributions as

$$\mathrm{Prob}(\lambda^1, \ldots, \lambda^k \mid \lambda^{k+1}) = \mathrm{Prob}(\lambda^1, \ldots, \lambda^{k-1} \mid \lambda^{k+1}) \cdot \mathrm{Prob}(\lambda^k \mid \lambda^{k-1}, \lambda^{k-1}). \tag{4.7}$$

The left-hand side depends on $a_1, \ldots, a_{k+1}$ in a symmetric way. One can readily check that $\mathrm{Prob}(\lambda^1, \ldots, \lambda^{k-1} \mid \lambda^{k+1})$ depends on the parameters a_k, a_{k+1} in a symmetric way, too. Indeed, this conditional distribution corresponds to integrating (3.1) (with $N = k+1$) over

λ^k. The non-exponential prefactor in (3.1) is already symmetric, and for the exponential part we have

$$e^{a_{k+1}|\lambda^{k+1}|+\sum_{j=1}^{k-1}|\lambda^j|(a_j-a_{j+1})}\int e^{|\lambda^k|(a_k-a_{k+1})}d\lambda^k$$
$$= e^{a_{k+1}|\lambda^{k+1}|+\sum_{j=1}^{k-1}|\lambda^j|(a_j-a_{j+1})}\prod_{i=1}^{k}\frac{e^{\alpha(\lambda_{i+1}^{k+1}\vee\lambda_i^{k-1})}-e^{\alpha(\lambda_i^{k+1}\wedge\lambda_{i-1}^{k-1})}}{\alpha}, \tag{4.8}$$

where we used the normalizing constant for the confined exponential distribution, and $\alpha = a_k - a_{k+1}$. Swapping the parameters as $a_k \leftrightarrow a_{k+1}$ brings $\exp\{-\alpha(|\lambda^{k+1}|+|\lambda^{k-1}|)\}$ from the exponential factor in front of the product in (4.8). This factor compensates the product of the expressions $\exp\left\{\alpha\left(\lambda_{i+1}^{k+1}\vee\lambda_i^{k-1}+\lambda_i^{k+1}\wedge\lambda_{i-1}^{k-1}\right)\right\}$ over $i = 1,\dots,k$, coming out of the product in (4.8) after the same swap. Thus, (4.8) is symmetric under $a_k \leftrightarrow a_{k+1}$.

The action of $\mathcal{S}_k^\alpha$ affects only the part $\mathrm{Prob}(\lambda^k \mid \lambda^{k-1}, \lambda^{k-1})$ in the right-hand side of (4.7). Before the action of $\mathcal{S}_k^\alpha$, each λ_i^k was distributed as E_α on the corresponding interval (see Proposition 4.3). By Proposition 4.1, after the action of $\mathcal{S}_k^\alpha$, these random variables turn into the $E_{-\alpha}$'s, which corresponds to a $\tau_k\mathbf{a}$-Gibbs structure. Combining this with the symmetries in (4.7) described above and using Lemma 3.4 and Proposition 3.5, we arrive at the claim. □

In particular, for $a_k > a_{k+1}$, the perturbed GUE corners process coming from the random matrix $H = t^{1/2}\cdot G + t\cdot\mathrm{diag}(\mathbf{a})$ (cf. Sect. 2.1), after the application of $\mathcal{S}_k^{a_k-a_{k+1}}$, turns into the corners process for the random matrix $\mathcal{T}_k H \mathcal{T}_k = t^{1/2}\cdot G + t\cdot\mathrm{diag}(\tau_k\mathbf{a})$:

$$H \dashrightarrow^{\mathcal{S}_k^{a_k-a_{k+1}}} \mathcal{T}_k H \mathcal{T}_k, \tag{4.9}$$

where $\mathcal{T}_k$ is the permutation matrix of $\tau_k = (k, k+1)$. In other words, applying the exponential jumps $\mathcal{S}_k^{a_k-a_{k+1}}$ on the level of eigenvalues is equivalent in distribution to the change of basis $\mathrm{e}_k \leftrightarrow \mathrm{e}_{k+1}$ in the space corresponding to the random matrix.

5 Global Shift and Reflected Brownian Motions

In this section we consider a special case when the perturbation sequence is an arithmetic progression, and prove Theorems 1.3 and 1.4.

Set

$$a_j = -(j-1)\alpha, \qquad j = 1, 2, \dots, \tag{5.1}$$

where $\alpha > 0$ is fixed. Denote the corresponding random matrix by

$$H^\alpha = t^{1/2}\cdot G + t\cdot\mathrm{diag}(0, -\alpha, -2\alpha, \dots), \tag{5.2}$$

and its corners distribution on infinite interlacing arrays by $\mathbf{M}^\alpha$. To $\mathbf{M}^\alpha$ we will apply the sequence of swap operators $\mathcal{S}_k^{k\alpha}$, first with $k = 1$, then with $k = 2$, and so on. Denote the resulting Markov operator which acts on the infinite interlacing array by $\mathbb{S}^\alpha$.

Lemma 5.1 *The Markov transition operator $\mathbb{S}^\alpha$ is well-defined.*

Proof Let $\{\lambda_j^m\}_{1\le j\le m<\infty}$ be the random interlacing array to which we apply $\mathbb{S}^\alpha$. The resulting random interlacing array $\{\nu_j^m\}_{1\le j\le m<\infty}$ is defined inductively: for each k, the k-th level configuration ν^k is the result of the action of $\mathcal{S}_k^{k\alpha}$ on λ^k given ν^{k-1} and λ^{k+1}. For this action, the configuration ν^{k-1} was defined on the previous step of the induction. This implies that $\mathbb{S}^\alpha$ is well-defined. □

Acting on $\mathbf{a}$-Gibbs measures with $\mathbf{a} = (0, -\alpha, -2\alpha, \ldots)$ (5.1), $\mathcal{S}_1^{\alpha}$ interchanges 0 with $-\alpha$, then $\mathcal{S}_2^{2\alpha}$ interchanges 0 (which is now the new a_2) with -2α, and so on. After infinitely many swaps, the parameter 0 disappears, and one expects that the resulting distribution would be $\mathbf{a}$-Gibbs with $\mathbf{a} = (-\alpha, -2\alpha, -3\alpha, \ldots)$. For the special choice of the perturbed GUE corners process (5.2), the action of $\mathbb{S}^{\alpha}$ is, moreover, equivalent in distribution to a global shift. We establish the following result:

Theorem 5.2 *(Theorem 1.3 in Introduction) The action of $\mathbb{S}^{\alpha}$ on $\mathbf{M}^{\alpha}$ is equivalent in distribution to a deterministic shift of the whole infinite interlacing array to the left by αt. In terms of random matrices, we have*

$$H^{\alpha} \dashrightarrow^{\mathbb{S}^{\alpha}} H^{\alpha} - \alpha t\, \mathbf{I}, \tag{5.3}$$

where $\mathbf{I}$ is the infinite identity matrix.

Proof Informally, one can think that (5.3) follows by a sequential application of the change of basis (4.9) under a single-level action $\mathcal{S}_k^{k\alpha}$. The shift by αt is precisely the difference between $t \cdot \mathrm{diag}(\mathbf{a})$ before and after the modification of $\mathbf{a}$. We will now prove this claim more formally, using Theorem 4.4 on how Gibbs measures change under swap operators.

Take the harmonic functions $\varphi_N = \varphi_N^{\mathrm{pertGUE}(\mathbf{a};t)}$ as in (3.4) with $\mathbf{a} = (0, -\alpha, -2\alpha, \ldots)$. The action of each $\mathcal{S}_k^{k\alpha}$ changes only the k-th function φ_k as in (4.6) and leaves all other functions intact. Therefore, the action of the whole $\mathbb{S}^{\alpha}$ replaces $\{\varphi_k\}$ by the family

$$\varphi_k'(\lambda^k) = \int_{\lambda^{k+1}\colon \lambda^{k+1} \succ \lambda^k} \varphi_{k+1}(\lambda^{k+1})\, d\lambda^{k+1}. \tag{5.4}$$

Here we took $a_k = 0$ because this is precisely the perturbation parameter that is being swapped with $a_{k+1} = -k\alpha$ under the action of $\mathcal{S}_k^{k\alpha}$. The integral in the right-hand side of (5.4) can be computed using (3.5) (with $a_N = 0$ in that formula), and we obtain

$$\varphi_k'(\lambda^k) = \mathrm{const} \times \mathsf{V}(\lambda_1^k, \ldots, \lambda_k^k) \prod_{i=1}^{k} e^{-(\lambda_i^k)^2/(2t)} \prod_{j=1}^{k+1} e^{-t((j-1)\alpha)^2/2} = C_0 \varphi_k(\lambda^k)\, e^{-tk^2\alpha^2/2}.$$

Here both const and C_0 are some constants which are independent of α. Sequentially applying Theorem 4.4, we see that the new the harmonic functions $\{\varphi_k'\}$ satisfy Gibbs property with the sequence $\mathbf{a} = (-\alpha, -2\alpha, -3\alpha, \ldots)$, and hence (by Proposition 3.5) correspond to a Gibbs measure with shifted parameters. Let us now identify this particular Gibbs measure.

The modified density of λ^k after the application of $\mathbb{S}^{\alpha}$ reads, by (3.2),

$$\begin{aligned}
\mathsf{Density}'(\lambda^k) &= \frac{\det[\exp\{(-i\alpha)\lambda_j^k\}]_{i,j=1}^k}{\mathsf{V}(-\alpha, -2\alpha, \ldots, -k\alpha)}\, \varphi_k'(\lambda^k) \\
&= C_0\, \frac{\det[\exp\{-i\alpha\lambda_j^k\}]_{i,j=1}^k}{\mathsf{V}(-\alpha, -2\alpha, \ldots, -k\alpha)}\, \varphi_k(\lambda^k)\, e^{-tk^2\alpha^2/2} \\
&= C_0\, \frac{\det[\exp\{-i\alpha\lambda_j^k\}]_{i,j=1}^k}{\mathsf{V}(-\alpha, -2\alpha, \ldots, -k\alpha)}\, \frac{\mathsf{V}(0, -\alpha, -2\alpha, \ldots, -(k-1)\alpha)}{\det[\exp\{-(i-1)\alpha\lambda_j^k\}]_{i,j=1}^k}\, \mathsf{Density}(\lambda^k)\, e^{-tk^2\alpha^2/2} \\
&= C_0\, e^{-\alpha|\lambda^k| - tk^2\alpha^2/2} \mathsf{Density}(\lambda^k),
\end{aligned}$$

where $\mathsf{Density}(\cdot)$ is the original density before applying $\mathbb{S}^{\alpha}$. In the last step, the two Vandermondes are equal by their translation invariance, and the ratio of the determinants is $e^{-\alpha|\lambda^k|}$

(indeed, factor out $e^{-\lambda_j^k}$ from each j-th column of the determinant in the numerator). Now, using (2.2) we have

$$C_0\, e^{-\alpha|\lambda^k|-tk^2\alpha^2/2}\,\mathsf{Density}(\lambda^k) \quad = C_0\mathrm{const}\times e^{-\alpha|\lambda^k|-tk^2\alpha^2/2}\det$$
$$\left[\exp\left\{-\frac{(\lambda_i^k+t(j-1)\alpha)^2}{2t}\right\}\right]_{i,j=1}^{k}\frac{\mathsf{V}(\lambda_1^k,\ldots,\lambda_k^k)}{\mathsf{V}(0,-\alpha,\ldots,-(k-1)\alpha)}.$$

Here const is the normalizing constant in (2.2) which is independent of α. Observe that in the exponents inside the determinant we have

$$-\frac{1}{2t}\left(\lambda_i^k+t(j-1)\alpha\right)^2=-\frac{1}{2t}\left(\lambda_i^k+\alpha t+t(j-1)\alpha\right)^2+\alpha\lambda_i^k+\frac{t\alpha^2}{2}(2j-1).$$

Factoring out the last two terms from each j-th column, we get a factor which precisely cancels with $e^{-\alpha|\lambda^k|-tk^2\alpha^2/2}$. Therefore, we see that

$$\mathsf{Density}'(\lambda^k)=C_0\mathsf{Density}(\lambda^k+\alpha t).$$

Normalizing, this implies that $C_0=1$. Thus, we see that applying $\mathbb{S}^\alpha$ is indeed equivalent to the global shift by αt to the left, as desired. □

We can now establish the shifting property for the reflected Brownian motions:

Proof of Theorem 1.4 Fix t, and use the identification $\{\lambda_j^k\}\overset{d}{=}\{B_j^k(t)\}$ of the GUE corners distribution with that of the reflected Brownian motions from Proposition 2.2 Denote $X_k(t):=B_k^k(t)$, then these are exactly the reflected Brownian motions from Theorem 1.4. Observe that the action of the operator $\mathcal{S}_k^{k\alpha}$ (4.5) on these $\lambda_k^k\overset{d}{=}X_k$ depends only on λ_k^k and λ_{k+1}^{k+1} and is the same as the Markov operator (1.1) in Theorem 1.4. Combining this observation with the shifting property from Theorem 5.2 we obtain the desired claim. □

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References

1. Adler, M., van Moerbeke, P., Wang, D.: Random matrix minor processes related to percolation theory. Random Matrices **2**, 1350008 (2013)
2. Aldous, D., Diaconis, P.: Hammersley's interacting particle process and longest increasing subsequences. Probab. Theory Relat. Fields **103**(2), 199–213 (1995)
3. Anderson, G.W., Guionnet, A., Zeitouni, O.: An Introduction to Random Matrices. Cambridge University Press, Cambridge (2010)
4. Desrosiers, P., Forrester, P.J.: Asymptotic correlations for Gaussian and Wishart matrices with external source. Intern. Math. Res. Notices **2006**, 27395 (2006)
5. Ferrari, P., Frings, R.: Perturbed GUE minor process and Warren's process with drifts. J. Stat. Phys. **154**(1–2), 356–377 (2014)
6. Gorin, V., Olshanski, G.: A quantization of the harmonic analysis on the infinite-dimensional unitary group. J. Funct. Anal. **270**(1), 375–418 (2016)
7. Gorin, V., Shkolnikov, M.: Limits of multilevel TASEP and similar processes. Annales de Inst. H. Poincaré, Probabilités et Statistiques **51**, 18–27 (2015)
8. Hammersley, J.M.: A few seedlings of research. Proc. Sixth Berkeley Symp. Math. Stat. Probab. **1**, 345–394 (1972)

9. Howitt, C., Warren, J.: Dynamics for the Brownian web and the erosion flow. Stochast. Proc. Appl. **119**(6), 2028–2051 (2009)
10. Johansson, K., Nordenstam, E.: Eigenvalues of GUE minors. Electron. J. Probab **11**(50), 1342–1371 (2006)
11. Johansson, K., Nordenstam, E.: Erratum to eigenvalues of gue minors. Electron. J. Probab. **12**(37), 1048–1051 (2007)
12. Kerov, S., Okounkov, A., Olshanski, G.: The boundary of Young graph with Jack edge multiplicities. Int. Math. Res. Notices **4**, 173–199 (1998)
13. Mehta, M.L.: Random Matrices. Academic Press, London (2004)
14. Olshanski, G., Vershik, A.: *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices*, Contemporary Mathematical Physics. F.A. Berezi's memorial volume. American Mathematical Society Translations, (Advances in the Mathematical Sciences — 31), 2, vol. 175, AMS, arXiv:math/9601215v1 [math.RT], pp. 137–175 (1996)
15. Olshanski, G.: Extended gelfand-tsetlin graph, its q-boundary, and q-b-splines. Funct. Anal. Appl. **50**(2), 107–130 (2016)
16. Peter, J.: Forrester. Log-Gases and Random Matrices. Princeton University Press, Princeton (2010)
17. Petrov, L., Saenz, A.: *Mapping tasep back in time*. (2019), arXiv:1907.09155
18. Warren, J.: Branching processes, the ray-knight theorem, and sticky Brownian motion. Séminaire de Probabilités XXXI, pp. 1–15. Springer, New York (1997)
19. Warren, J.: Dyson's Brownian motions, intertwining and interlacing. Electron. J. Probab. **12**(19), 573–590 (2007)