

Colored Interacting Particle Systems on the Ring: Stationary Measures from Yang–Baxter Equation

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Abstract

Recently, there has been much progress in understanding stationary measures for colored (also called multi-species or multi-type) interacting particle systems, motivated by asymptotic phenomena and rich underlying algebraic and combinatorial structures (such as nonsymmetric Macdonald polynomials).

In this paper, we present a unified approach to constructing stationary measures for most of the known colored particle systems on the ring and the line, including (1) the Asymmetric Simple Exclusion Process (multispecies ASEP, or mASEP); (2) the q -deformed Totally Asymmetric Zero Range Process (TAZRP) also known as the q -Boson particle system; (3) the q -deformed Pushing Totally Asymmetric Simple Exclusion Process (q -PushTASEP). Our method is based on integrable stochastic vertex models and the Yang–Baxter equation. We express the stationary measures as partition functions of new “queue vertex models” on the cylinder. The stationarity property is a direct consequence of the Yang–Baxter equation.

For the mASEP on the ring, a particular case of our vertex model is equivalent to the multiline queues of Martin [Mar20]. For the colored q -Boson process and the q -PushTASEP on the ring, we recover and generalize known stationary measures constructed using multiline queues or other methods by Ayyer–Mandelshtam–Martin [AMM22], [AMM23], and Bukh–Cox [BC22]. Our proofs of stationarity use the Yang–Baxter equation and bypass the Matrix Product Ansatz (used for the mASEP by Prolhac–Evans–Mallick [PEM09]).

On the line and in a quadrant, we use the Yang–Baxter equation to establish a general colored Burke’s theorem, which implies that suitable specializations of our queue vertex models produce stationary measures for particle systems on the line. We also compute the colored particle currents in stationarity.

Contents

1	Introduction	2
1.1	Background	2
1.2	Main results and methods	4
1.3	Related work in progress	7
1.4	Outline	8
1.5	Acknowledgments	9
2	Colored vertex weights and Yang–Baxter equation	9
2.1	Vertex weights	9
2.2	Yang–Baxter equation	11
2.3	Fused weights	12

2.4	Queue specialization	13
2.5	Queue vertex model on the cylinder	17
3	Stationarity of the queue vertex model	20
3.1	Twisted cylinder Markov operator	20
3.2	Straight cylinder Markov operator	22
4	Multi-species ASEP from twisted cylinder	25
4.1	Multi-species ASEP on the ring	25
4.2	Vertex model for the mASEP stationary distribution	27
4.3	Matching to multiline queues	31
4.4	Connection to Matrix Product Ansatz	35
5	Colored stochastic q-Boson process from straight cylinder	39
5.1	Colored q -Boson process on the ring	39
5.2	Vertex model for the q -Boson stationary distribution	41
5.3	Matching to multiline queues	43
6	Colored q-PushTASEP from straight cylinder	46
7	Stationarity in the quarter plane and on the line	51
7.1	Queue steady state	51
7.2	Colored Burke's theorem via Yang–Baxter equation	53
7.3	Specialization to stochastic six-vertex model and q -PushTASEP	57
7.4	Specialization to q -Boson	61
A	Merging of colors in stationary measures on the line	63

1 Introduction

1.1 Background

This work connects stationary measures for colored (also referred to as multi-species or multi-type) systems of interacting particles hopping on a one-dimensional lattice (the ring or the whole line) to solvable lattice models. One of the particle systems we consider is the multi-species Asymmetric Simple Exclusion Process (*mASEP*). On the ring with N sites, the n -colored mASEP is a continuous time Markov process under which each pair of neighboring particles of colors $0 \leq i \neq j \leq n$ and locations $(k, k + 1) \pmod{N}$ swap with rate 1 if $i < j$, and rate $q \in [0, 1)$ if $i > j$ (color 0 represents holes). See Figure 1 for an illustration.¹ We also consider the multi-species q -TAZRP (q -deformed Totally Asymmetric Zero Range Process, also known as the colored stochastic q -Boson particle system), and the colored q -PushTASEP (q -deformed Pushing Totally Asymmetric Simple Exclusion Process). We refer to Sections 5.1 and 6 for definitions of these models on the ring.

¹Throughout the paper, we say that an event occurs at rate $\alpha > 0$ if the random time ζ till the occurrence is exponentially distributed with parameter α , that is, $\mathbb{P}(\zeta > t) = e^{-\alpha t}$ for $t \geq 0$.

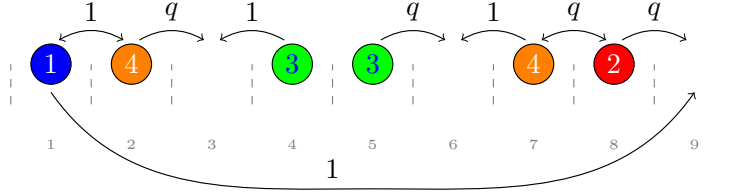


Figure 1: Rates of all possible particle hops in the mASEP on the ring with $N = 9$ sites and $n = 4$ colors. The mASEP preserves the number of particles of each type, and has a unique stationary measure in each “sector” determined by fixing the number N_m of particles of each color $m = 1, \dots, n$.

The stationary measures of these particle systems have been the subject of a systematic investigation recently. On the one hand, their properties and asymptotic behavior highlight interesting physical and probabilistic phenomena (for example, a particle of a different color may follow a “microscopic characteristic”). On the other hand, they have a rich underlying algebraic and combinatorial structure (in particular, deep connections to Macdonald symmetric functions and their nonsymmetric counterparts).

The stationary measure for the single-color mASEP (simply called *ASEP*) on the ring with a given number of particles is uniform among all possible arrangements of these particles. On the line, translation invariant stationary measures are all product Bernoulli (that is, each site is independently a particle or a hole with probability ρ); see Liggett [Lig05, Ch. VIII].

Prolhac–Evans–Mallick [PEM09] constructed stationary measures for mASEP with an arbitrary number of colors using the *Matrix Product Ansatz*, an algebraic formalism utilizing commutation relations of a family of matrices. For previous partial Matrix Product Ansatz results see the references in [PEM09]. We discuss the Matrix Product Ansatz approach in the beginning in Section 4.4 in the text. Methods for sampling from the mASEP stationary measures using combinatorial structures known as *multiline diagrams* or *multiline queues* were developed by Angel [Ang06], Ferrari–Martin [FM07], and Martin [Mar20].

Connections to Macdonald symmetric and nonsymmetric polynomials served as another inspiration for studying stationary measures of interacting particle systems on the ring. Macdonald polynomials are a centerpiece of the symmetric functions theory and have wide applications to representation theory and geometry; see Macdonald [Mac95, Ch. VI], [Mar99]. We do not focus on symmetric and nonsymmetric functions in this paper, but here we mention the necessary background.

A relationship between the mASEP stationary measures and Macdonald polynomials is first observed by Cantini–de Gier–Wheeler [CdGW15]. Nonsymmetric Macdonald polynomials are constructed via multiline queues by Corteel–Mandelshtam–Williams [CMW22]. An integrable vertex model for nonsymmetric Macdonald polynomials is given by Borodin–Wheeler [BW22b]. One can say that the vertex model construction unifies the two points of view.

The stochastic single-color q -Boson process (also referred to as the q -TAZRP) was introduced by Sasamoto–Wadati [SW98]. Its dual process, the q -TASEP, was extensively studied on the line by Borodin–Corwin [BC14], Borodin–Corwin–Sasamoto [BCS14], Borodin–Corwin–Petrov–Sasamoto [BCPS15], Barraquand [Bar15], and others. On the ring, Wang–Waugh [WW16] and

Liu–Saenz–Wang [LSW20] obtained integral formulas for transition probabilities and other observables of the q -Boson process.

The multi-species generalization of the q -Boson process is due to Takeyama [Tak15] and Kuniba–Maruyama–Okado [KMO16]. Stationary measures for this process on the ring were recently connected to modified Macdonald polynomials by Ayer–Mandelstam–Martin [AMM22], [AMM23] (see also the earlier work of Garbali–Wheeler [GW20] connecting modified Macdonald polynomials to vertex models). In contrast with the mASEP, the q -Boson process has spectral parameters (rapidities) attached to the sites on the ring, and these parameters enter the modified Macdonald polynomials. Ayer–Mandelstam–Martin revealed symmetries of the stationary measures in the parameters and utilized them to compute observables in the stationary model.

The colored q -PushTASEP on the ring is less studied. On the line, it was introduced by Borodin–Wheeler [BW22a, Section 12.5] as a degeneration of the colored stochastic higher spin six-vertex model. Like the q -Boson process, it contains spectral parameters, and also the capacity parameter $P \in \mathbb{Z}_{\geq 1}$ which is the maximum number of particles allowed at each site. A discrete time variant of the colored q -PushTASEP (with $q = 0$ and $P = 1$) on the ring was introduced (under the name “frog model”) by Bukh–Cox [BC22] in connection with the problem of the longest common subsequence of a random and a periodic word. In particular, Bukh–Cox [BC22] constructed and investigated stationary measures of the frog model.

1.2 Main results and methods

We unify and generalize existing constructions of stationary measures of the multi-species ASEP, the colored q -Boson process, and the colored q -PushTASEP on the ring and the line. We prove the stationarity property graphically in all cases, using the Yang–Baxter equation (discussed below in this subsection). We obtain the following results for the three particle systems:

- For mASEP, we present a vertex model interpretation of the multiline queues of Martin [Mar20], which is very close to the Matrix Product Ansatz construction of Prolhac–Evans–Mallick [PEM09]. Moreover, we recover the main ingredient of the Matrix Product Ansatz proof of the stationarity (the “hat relation”) directly from the Yang–Baxter equation. On the ring, we add extra parameters to our vertex model that do not affect the stationary measures. We also connect our vertex models with extra parameters to other variants of multiline queues, including the one considered by Martin [Mar20, Section 7].
- For the colored q -Boson process, we present a vertex model construction of stationary measures on the ring and the line. On the ring, the vertex model has extra parameters that do not affect the stationary measures, but they need to be specialized to zero on the line. In a particular case of at most one particle of each color, our vertex model is equivalent to the multiline queues of Ayer–Mandelstam–Martin [AMM22]. On the line, the parameters of the vertex model which survive are in one-to-one correspondence with colored particle densities, and we compute the colored currents in stationarity (as implicit functions of the densities of particles of each color).
- For the colored q -PushTASEP, we also construct vertex models for the stationary measures on the ring and the line. On the ring, the vertex model carries extra parameters that do not affect the stationary measures. On the line, some extra parameters are lost, and again the remaining ones exactly parameterize colored particle densities. A particular $P = 1$ case of the

q -PushTASEP has the same stationary measures as the mASEP and the colored stochastic six-vertex model. On the line, we compute the colored currents for the stochastic six-vertex model and the q -PushTASEP in stationarity.

To obtain all our results, we utilize *integrable stochastic vertex models*, that is, whose vertex weights have a stochastic normalization and satisfy the Yang–Baxter equation. We start from stochastic $U_q(\widehat{\mathfrak{sl}}_{n+1})$ vertex weights whose explicit form was first obtained by Kuniba–Mangazeev–Maruyama–Okado [KMMO16] and which were studied in the stochastic context by Bosnjak–Mangazeev [BM16], Garbali–de Gier–Wheeler [GdGW17], Kuan [Kua18]. The vertex model foundation of our work was developed by Borodin–Wheeler [BW22a] who listed various stochastic degenerations of the general $U_q(\widehat{\mathfrak{sl}}_{n+1})$ weights, leading to the mASEP, the colored q -Boson process, and the colored q -PushTASEP.

In a limit when the number of particles (corresponding to vertical arrows in vertex models) of a given color goes to infinity, the $U_q(\widehat{\mathfrak{sl}}_{n+1})$ stochastic weights degenerate into what we call the *queue vertex weights*. Putting them on the cylinder, we obtain our main object — the *queue vertex model*. Its normalized partition functions serve as stationary measures for our colored stochastic particle systems on the ring. The stationarity of the normalized partition functions is a direct consequence of the Yang–Baxter equation.

For each particle system, we perform the following sequence of steps:

1. We define a colored (higher spin) stochastic transfer matrix \mathfrak{T} on the cylinder. In a Poisson-type limit when the discrete time becomes continuous, large powers of \mathfrak{T} converge to the Markov semigroup (also called the propagator) of the given stochastic process living on configurations of colored particles on the ring. Each of our processes (the mASEP, the colored q -Boson process, and the colored q -PushTASEP) preserves the number of particles of each color, while \mathfrak{T} before the Poisson-type limit may not.
2. We construct a multiparameter family of transfer matrices \mathfrak{Q} on the cylinder using our queue vertex weights (see Figure 2 for an illustration), such that
 - (a) Their matrix elements $\langle \emptyset | \mathfrak{Q} | \mathbf{V} \rangle$ (which are, by definition, partition functions — the sums of weights of all allowed arrow configurations) are nonzero for all terminal states \mathbf{V} . Here $\langle \emptyset |$ is the empty state (no particles present on the ring), and \mathbf{V} encodes a state with prescribed particle locations and colors.
 - (b) The Yang–Baxter equation implies that $\mathfrak{Q}\mathfrak{T} = \mathfrak{T}\mathfrak{Q}$. In detail, we glue the cylinder with the output \mathbf{V} to a sequence of stochastic R matrices, as shown in Figure 3. Their composition is the operator \mathfrak{T} . Together with the queue weights, these R matrices satisfy the Yang–Baxter equation which allows to commute \mathfrak{T} through the queue weights \mathfrak{Q} .

Since $\langle \emptyset | \mathfrak{T} = \langle \emptyset |$, after a degeneration of the parameters of \mathfrak{Q} (corresponding to the Poisson-type limit in \mathfrak{T}), the quantity $\langle \emptyset | \mathfrak{Q}$ viewed as a row vector (or, equivalently, an unnormalized probability distribution on the space of particle configurations on the ring) becomes stationary under the Markov semigroup. By restricting $\langle \emptyset | \mathfrak{Q}$ to particle configurations with fixed numbers of particles of each color (a *sector*), we obtain a stationary measure for the corresponding particle system. Note that part (a) above seems to violate the conservation of particles (the higher spin analog of the ice rule), which holds for models of six-vertex type. It is the limit to the queue vertex weights which allows this violation.

3. For each of the particle systems on the ring, we compute the queue weights explicitly (after the corresponding degeneration of the parameters of \mathfrak{Q}), and provide conditions on the remaining complex parameters guaranteeing the positivity of the unnormalized weights. Note that the normalized weights are automatically positive by the stationarity property and the Perron–Frobenius theorem.

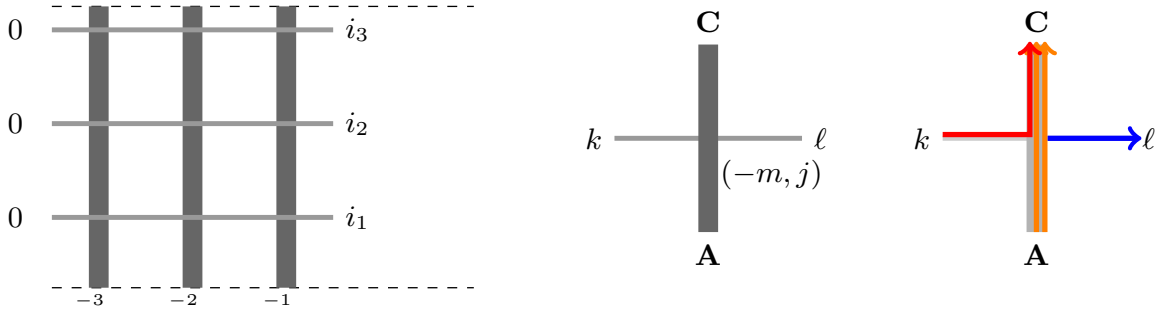


Figure 2: Left: The queue vertex model partition function $\langle \emptyset | \mathfrak{Q} | \mathbf{V} \rangle$ on the cylinder with $n = 3$ colors and $N = 3$ sites (the dotted lines are identified). There are no arrows incoming from the left, and the outgoing terminal state $\mathbf{V} = (i_1, i_2, i_3)$ has $i_1, i_2, i_3 \in \{0, 1, 2, 3\}$. Center: The queue vertex weight $\mathbb{W}_{\text{parameters}(m,j)}^{(-m)}$ (see Definition 2.6 below) is in column $-m$, $1 \leq m \leq N$ at position $j = 1, \dots, N$ in the transfer matrix \mathfrak{Q} . Right: An allowed path configuration at a queue vertex with $m = 1$. Colors 1, 2, 3 are, respectively, blue, orange, and red. We have $\mathbf{A} = (\infty, 0, 2)$, $k = 2$, $\mathbf{C} = (\infty, 1, 2)$, and $\ell = 1$. Infinitely many blue arrows (not depicted) pass through vertically, which allows a blue arrow to exit from the right even though none entered from the left.

We call the weights on the cylinder the *queue vertex weights* because each column $(-m)$ of a cylinder resembles a queueing system for which the “unused service times” are assigned the new color m . Indeed, for the mASEP and (with certain restrictions) the q -Boson process, the output of our vertex model is the same (in distribution) as that of the multiline queues of Martin [Mar20] and Ayyer–Mandelstam–Martin [AMM22], respectively.

For the mASEP, we explain how to set the parameters to exactly match a certain specialization of our stationary measure with the matrix product stationary measure constructed by Prolhac–Evans–Mallick [PEM09]. We also show how the underlying algebraic apparatus for the Matrix Product Ansatz can be derived from the Yang–Baxter equation. Namely, we realize the product ansatz matrices satisfying the so-called “hat relation” (identity (4.15) in the text) as vertex model partition functions. This highlights the Yang–Baxter structure of the Matrix Product Ansatz for the multispecies ASEP which was previously unknown, cf. Arita–Ayyer–Mallick–Prolhac [AAMP12]. Furthermore, we describe how more general solutions A, D, E of the quadratic algebra relations of the Matrix Product Ansatz (formula (4.13) in the text) can be derived from our queue vertex weights.

In Section 7, we use similar vertex model techniques to construct stationary distributions for the mASEP, the colored q -Boson process, and the colored q -PushTASEP on the line \mathbb{Z} . In this setting, we again use the Yang–Baxter equation to prove stationarity in a certain quarter plane setting (a colored analog of Burke’s theorem, see Figure 4), and then we take a bulk limit

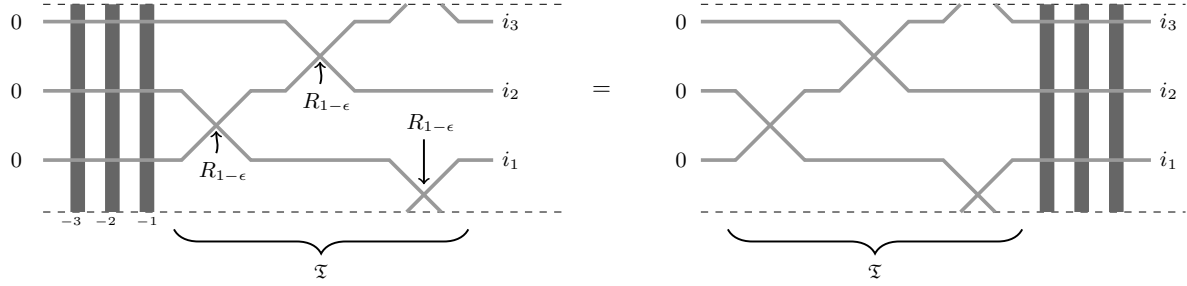


Figure 3: The main commutation relation $\Omega \mathfrak{X} = \mathfrak{X} \Omega$ leading to stationarity. This configuration of R matrices is specific to the mASEP. For small $\epsilon > 0$, the arrows of all colors pass through each $R_{1-\epsilon}$ in a zigzag manner with probability $1 - O(\epsilon)$ (that is $R_{1-\epsilon}(i, j; j, i)$ is close to 1). When $\epsilon \rightarrow 0$, the powers $(\mathfrak{X})^{\lfloor t/\epsilon \rfloor}$ converge to $\mathfrak{F}_{\text{mASEP}}(t)$, the semigroup operators of the mASEP on the ring (here $t \in \mathbb{R}_{\geq 0}$ is the continuous time). For the other systems, the colored q -Boson process and the colored q -PushTASEP, we need a different configuration on the cylinder (given in Figures 12 and 13 in the text) and a slightly different Poisson-type limit transition.

to obtain the stationarity on \mathbb{Z} . The leftover parameters of the queue vertex weights on the line correspond to the densities of particles of each color. Using our quarter plane construction, we also obtain a general description of analogs of the Kardar–Parisi–Zhang (KPZ) pure phases (translation invariant ergodic Gibbs measures) for the colored stochastic six-vertex model and compute the corresponding slope relations. In the single-color case, this was done by Aggarwal [Agg18] using the single-color queue vertex weights at the left boundary of the quadrant.

Our method of producing stationary distributions as partition functions is conceptually reminiscent of the Bethe Ansatz. In Bethe Ansatz, eigenvectors of a transfer matrix of a quantum integrable model are constructed by applying certain other transfer matrices (with specially chosen parameters satisfying algebraic equations) to the vacuum vector $|\emptyset\rangle$. For stationary measures of our Markov processes, we only need the leading (Perron–Frobenius) eigenvector of \mathfrak{X} having the eigenvalue 1. We do not investigate further connections of our constructions to the Bethe Ansatz or the algebraic equations.

1.3 Related work in progress

While preparing the manuscript, we learned about two related works in progress. One by Corteel–Keating [CK23] defines a new particle system with zero-range interactions on the ring. Its stationary distribution is expressed as a queue-like vertex model on the cylinder with fermionic weights related to the algebra $U_q(\widehat{\mathfrak{sl}}(1|n))$. These weights were recently investigated by Corteel–Gitlin–Keating–Meza [CGKM22] and Aggarwal–Borodin–Wheeler [ABW21]. In particular, instead of the Macdonald polynomials, they are related to the Lascoux–Leclerc–Thibon (LLT) polynomials [LLT97].

Another work in progress by Angel–Ayyer–Martin [AAM23] considers stationary measures for the colored q -PushTASEP on the ring (with capacity $P = 1$), and connects them to the multiline queues of Corteel–Mandelshtam–Williams [CMW22]. The approach to this proof is different

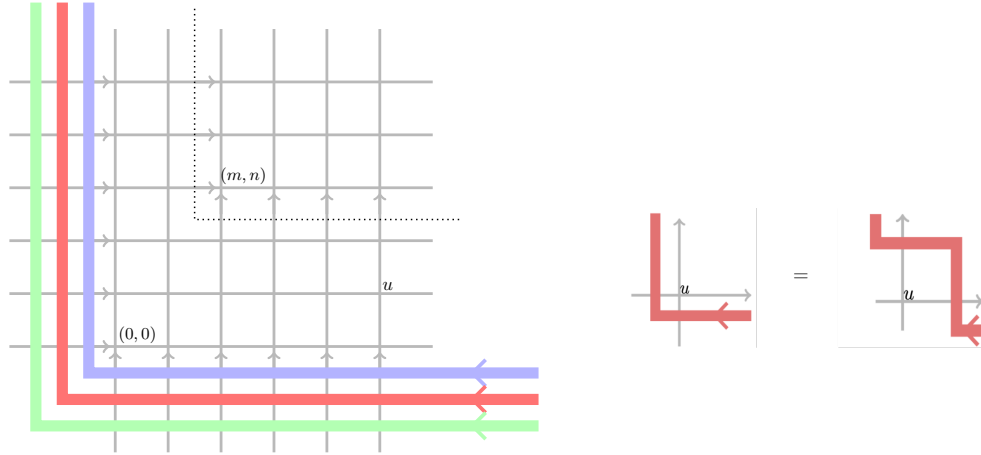


Figure 4: The construction of the stationary quarter plane configuration for the colored stochastic six-vertex model. The arrow configuration entering the quadrant $\mathbb{Z}_{\geq 0}^2$ is generated by the stochastic vertex model with thick colored lines along the boundary (the vertex weights here are the *queue vertex weights*), and has the same distribution as the configuration of arrows crossing the boundary of the (m, n) -shifted quadrant (the dotted line). The configuration of arrows entering along the thick lines is sampled from a particular distribution (namely, the stationary distribution of a Markov chain describing the evolution of “queues in tandem”). Stationarity can be proved by repeated applications of the Yang–Baxter equation, shown on the right. Details of this construction are in Section 7.

from ours, and it relies on properties of symmetric and nonsymmetric Macdonald polynomials and related functions.

1.4 Outline

In Section 2, we recall the colored stochastic vertex weights (together with their fully fused version) and the Yang–Baxter equations for them. We define the queue vertex weights, which arise in the limit of the stochastic vertex weights when the number of vertical arrows of a given color goes to infinity. Putting the queue vertex weights on the cylinder, we obtain our main object — the queue vertex model.

In Section 3, we employ the Yang–Baxter equation to show that the measure on particle configurations on the ring coming from the queue vertex model is stationary under the twisted and the straight cylinder Markov transition operators. In full generality, these Markov operators are formal (may have negative matrix elements).

In Section 4, we take a scaling limit under which the twisted cylinder Markov operator converges to the infinitesimal generator of the mASEP (an actual, not formal, Markov operator). The corresponding limit of the queue vertex model yields a stationary measure for the mASEP. Under a specialization of the parameters, we identify our queue vertex model with the multiline queue system of Martin [Mar20] and connect our constructions to the Matrix Product Ansatz of Prolhac–Evans–Mallick [PEM09]. Moreover, a different parameter specialization presumably relates our queue vertex model to the alternative multiline queue system considered also by Martin

[Mar20, Section 7].

In Sections 5 and 6, we treat two other colored particle systems — the q -Boson process (also known as the q -TAZRP) and the q -PushTASEP. For the q -Boson process, in a particular case of at most one particle of each color, we identify our queue vertex model with the multiline queueing system recently introduced by Ayyer–Mandelshtam–Martin [AMM22].

In Section 7, we consider colored stochastic vertex models in the quadrant and use the Yang–Baxter equation to prove a new colored generalization of Burke’s theorem. It implies that when put on the infinite vertical strip instead of the cylinder, our queue vertex models produce stationary distributions for the interacting particle systems (mASEP, colored q -Boson process, and q -PushTASEP) on the line. In Appendix A, we show that the stationary measures for our colored particle systems on the line respect the operations of color merging (when two or more colors are declared the same). The proof also relies on applying the Yang–Baxter equation to a stochastic vertex model in the quadrant.

1.5 Acknowledgments

We are grateful to Alexei Borodin, Kirone Mallick, James Martin, Ananth Sridhar, and Lauren Williams for helpful discussions. Amol Aggarwal was partially supported by a Packard Fellowship for Science and Engineering, a Clay Research Fellowship, by NSF grant DMS-1926686, and by the IAS School of Mathematics. The work of Matthew Nicoletti and Leonid Petrov was partially supported by the NSF grants DMS-1664617 and DMS-2153869, by the 4-VA Virginia Collaborative Research Initiative, and by the Simons Collaboration Grant for Mathematicians 709055.

2 Colored vertex weights and Yang–Baxter equation

In this section, we collect the necessary formulas for the vertex weights of the colored stochastic vertex model from [BW22a]. Algebraically, this model is powered by the quantum affine Lie algebra $U_q(\widehat{\mathfrak{sl}}_{n+1})$, where n is the number of colors. A stochastic normalization of the $U_q(\widehat{\mathfrak{sl}}_{n+1})$ vertex weights first appeared in [KMMO16]; see also [BM16], [GdGW17], [Kua18]. In certain degenerations, which we recall in Sections 4 and 5 below, the colored stochastic vertex model turns into multi-species interacting particle systems. We start with the vertex weights and the Yang–Baxter equation; then, we proceed to the fused weights and define their new *queue specialization*.

2.1 Vertex weights

Fix the number of colors $n \geq 1$. The higher spin $U_q(\widehat{\mathfrak{sl}}_{n+1})$ stochastic vertex weights $L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell)$ are indexed by the following data:

- The quantum parameter $q \in [0, 1)$, which is fixed throughout this section;
- The spectral parameter x and the spin parameter s , which may depend on the vertex;
- The configurations of incoming and outgoing arrows $(\mathbf{A}, k; \mathbf{B}, \ell)$, where $k, \ell \in \{0, 1, \dots, n\}$, and \mathbf{A}, \mathbf{B} are n -tuples $(A_1, \dots, A_n), (B_1, \dots, B_n)$, where $A_i, B_j \in \mathbb{Z}_{\geq 0}$. Here $k \geq 1$ represents an arrow of color k entering from the left, and $k = 0$ corresponds to no arrows entering from the left; similarly, ℓ encodes the exiting arrows to the right. Each A_i is the number of arrows of

color i entering from the bottom, and B_j is the number of arrows of color j exiting from the top.

The arrow counts $(\mathbf{A}, k; \mathbf{B}, \ell)$ must satisfy the *arrow conservation property* (a higher spin analog of the *ice rule*), with the understanding that all arrows go in the up or right direction. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis in \mathbb{Z}^n , then, the arrow conservation is

$$\mathbf{A} + \mathbf{e}_k \mathbf{1}_{k \geq 1} = \mathbf{B} + \mathbf{e}_\ell \mathbf{1}_{\ell \geq 1}.$$

Here and throughout the paper, $\mathbf{1}_E$ denotes the indicator of the event or condition E . For $1 \leq k, \ell \leq n$, define

$$\mathbf{A}_k^+ := \mathbf{A} + \mathbf{e}_k, \quad \mathbf{A}_k^- := \mathbf{A} - \mathbf{e}_k, \quad \mathbf{A}_{k\ell}^{+-} := \mathbf{A} + \mathbf{e}_k - \mathbf{e}_\ell, \quad |\mathbf{A}| := \sum_{k=1}^n A_k, \quad A_{[k,\ell]} := \sum_{i=k}^{\ell} A_i.$$

The values of all the vertex weights $L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell)$ are listed in the table in Figure 5. In [BW22a, Section 2], they are denoted by $\tilde{L}_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell)$, and they differ from [BW22a, (2.2.2)] by the factor $(-s)^{\mathbf{1}_{\ell > 0}}$. However, in this paper, we work with stochastic weights from the beginning, and remove the tilde from the notation.

$\begin{array}{c} \mathbf{A} \\ \\ 0 \text{ --- } \text{---} 0 \\ \\ \mathbf{A} \end{array}$ $\frac{1 - sxq^{ \mathbf{A} }}{1 - sx}$	$\begin{array}{c} \mathbf{A} \\ \\ k \text{ --- } \text{---} k \\ \\ \mathbf{A} \end{array}$ $\frac{(-sx + s^2q^{A_k})q^{A_{[k+1,n]}}}{1 - sx}$	$\begin{array}{c} \mathbf{A}_k^- \\ \\ 0 \text{ --- } \text{---} k \\ \\ \mathbf{A} \end{array}$ $\frac{-sx(1 - q^{A_k})q^{A_{[k+1,n]}}}{1 - sx}$
$\begin{array}{c} \mathbf{A}_k^+ \\ \\ k \text{ --- } \text{---} 0 \\ \\ \mathbf{A} \end{array}$ $\frac{1 - s^2q^{ \mathbf{A} }}{1 - sx}$	$\begin{array}{c} \mathbf{A}_{k\ell}^{+-} \\ \\ k \text{ --- } \text{---} \ell \\ \\ \mathbf{A} \end{array}$ $\frac{-sx(1 - q^{A_\ell})q^{A_{[\ell+1,n]}}}{1 - sx}$	$\begin{array}{c} \mathbf{A}_{\ell k}^{+-} \\ \\ \ell \text{ --- } \text{---} k \\ \\ \mathbf{A} \end{array}$ $\frac{-s^2(1 - q^{A_k})q^{A_{[k+1,n]}}}{1 - sx}$

Figure 5: Colored stochastic higher spin vertex weights $L_{s,x}$. Here $1 \leq k < \ell \leq n$, and all other values of $L_{s,x}$ are zero.

Proposition 2.1. *The vertex weights $L_{s,x}$ satisfy the sum-to-one property*

$$\sum_{\mathbf{B} \in \mathbb{Z}_{\geq 0}^n} \sum_{\ell=0}^n L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell) = 1 \tag{2.1}$$

for any fixed \mathbf{A}, k . Moreover, if $q \in [0, 1)$,

$$-sx > 0, \quad -s^2 \geq 0, \quad -sx + s^2 \geq 0, \tag{2.2}$$

then all the vertex weights $L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell)$ are nonnegative.

Proof. The sum-to-one property is [BW22a, Proposition 2.5.1]. The nonnegativity under conditions (2.2) is straightforward. \square

Conditions (2.2) mean that s and x must be purely imaginary numbers. Observe that the weights $L_{s,x}$ contain s^2 and $-sx$ as two independent parameters, and they are more convenient to formulate the nonnegativity.

A natural point of view is to interpret the vertex weights $L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell)$ as matrix elements of a linear operator $\mathbb{C}^{n+1} \otimes V \rightarrow \mathbb{C}^{n+1} \otimes V$, where \mathbb{C}^{n+1} has the standard basis $\{|j\rangle\}_{j=0}^n$, and V has the basis $\{|\mathbf{A}\rangle\}_{\mathbf{A} \in \mathbb{Z}_{\geq 0}^n}$. For either space, we denote vectors of the dual basis by $\langle \mathbf{v} |$, and for tensor products of vectors (or dual vectors), we use the notation $|\mathbf{v}, \mathbf{A}\rangle := |\mathbf{v}\rangle \otimes |\mathbf{A}\rangle$. For the tensor product $\mathbb{C}^{n+1} \otimes V$ we use the basis $\{|j, \mathbf{A}\rangle\}_{j \in \{0, \dots, n\}, \mathbf{A} \in \mathbb{Z}_{\geq 0}^n}$, and for its dual we use the basis $\{\langle j, \mathbf{A} | \}_{j \in \{0, \dots, n\}, \mathbf{A} \in \mathbb{Z}_{\geq 0}^n}$. In these bases the operator $\mathcal{L}_{s,x}$ corresponding to $L_{s,x}$ acts as

$$\langle k, \mathbf{A} | \mathcal{L}_{s,x} | \ell, \mathbf{B} \rangle = L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell). \quad (2.3)$$

In this way, pairs of dual and primal basis vectors of $\mathbb{C}^{n+1} \otimes V$ with nonzero $\mathcal{L}_{s,x}$ matrix entries correspond precisely to the allowed local configurations of paths at a vertex, displayed in Figure 5.

Remark 2.2 (Finite-spin reduction). For generic $s \in \mathbb{C}$, the operator $\mathcal{L}_{s,x}$ has infinitely many nonzero matrix entries, so any number of paths can occupy the vertical edges. If, on the other hand, $s = q^{-\frac{M}{2}}$ for some $M \in \mathbb{Z}_{\geq 1}$, then let us set by definition $L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell) = 0$ unless $|\mathbf{A}|, |\mathbf{B}| \leq M$. Note that $L_{s,x}(\mathbf{A}, k; \mathbf{A}_k^+, 0)$ vanishes automatically if $|\mathbf{A}| = M$, so vertices with $|\mathbf{A}| > M$ or $|\mathbf{B}| > M$ cannot be created from the stochastic evolution started from a configuration where all $|\mathbf{A}| \leq M$. We see that for $s = q^{-M/2}$, at most M vertical paths may occupy the vertical edges. In this case, $\mathcal{L}_{s,x}$ acts in the finite-dimensional subspace $\mathbb{C}^{n+1} \otimes V_M$, with $V_M \subset V$ spanned by $\{|\mathbf{A}\rangle\}_{|\mathbf{A}| \leq M}$.

2.2 Yang–Baxter equation

Let us define the following cross vertex weights $R_z(i, j; k, \ell)$, originally due to [Jim86] and [Baz85]:

$$\begin{aligned} R_z(i, i; i, i) &:= 1, \quad i \in \{0, 1, \dots, n\}, \\ \left. \begin{aligned} R_z(j, i; j, i) &:= \frac{q(1-z)}{1-qz}, & R_z(i, j; i, j) &:= \frac{1-z}{1-qz} \\ R_z(j, i; i, j) &:= \frac{1-q}{1-qz}, & R_z(i, j; j, i) &:= \frac{(1-q)z}{1-qz} \end{aligned} \right\} \quad i, j \in \{0, 1, \dots, n\}, \quad i < j. \end{aligned} \quad (2.4)$$

These weights also satisfy the sum-to-one property: $\sum_{k, \ell=0}^n R_z(i, j; k, \ell) = 1$ for any i, j . They are nonnegative if $0 \leq z \leq 1$.

The vertex weights $R_z(i, j; k, \ell)$ can also be regarded as matrix elements of an operator \mathcal{R}_z acting in $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$, namely, $\langle j, i | \mathcal{R}_z | \ell, k \rangle = R_z(i, j; k, \ell)$.

One can check that R_z is the spin- $\frac{1}{2}$ reduction of $L_{s,x}$ (as in Remark 2.2):

$$R_z(i, j; k, \ell) = L_{q^{-1/2}, z^{-1}q^{-1/2}}(\mathbf{e}_i \mathbf{1}_{i \geq 1}, j; \mathbf{e}_k \mathbf{1}_{k \geq 1}, \ell), \quad i, j, k, \ell \in \{0, 1, \dots, n\}. \quad (2.5)$$

In the right-hand side, if $i = 0$, then $\mathbf{e}_i \mathbf{1}_{i \geq 1} = (0, \dots, 0)$ (n zeroes), which corresponds to no arrows at an edge.

The weights $L_{s,x}$ and R_z satisfy the following Yang–Baxter equation:

Proposition 2.3 ([BW22a, (2.3.1) and Corollary B.4.3]). *For any fixed $i_1, i_2, j_1, j_2 \in \{0, 1, \dots, n\}$ and $\mathbf{A}, \mathbf{B} \in \mathbb{Z}_{\geq 0}^n$, we have*

$$\begin{aligned} \sum_{\mathbf{K} \in \mathbb{Z}_{\geq 0}^n} \sum_{k_1, k_2=0}^n L_{s,y}(\mathbf{A}, i_2; \mathbf{K}, k_2) L_{s,x}(\mathbf{K}, i_1; \mathbf{B}, k_1) R_{y/x}(k_2, k_1; j_2, j_1) \\ = \sum_{\mathbf{K} \in \mathbb{Z}_{\geq 0}^n} \sum_{k_1, k_2=0}^n R_{y/x}(i_2, i_1; k_2, k_1) L_{s,y}(\mathbf{K}, k_2; \mathbf{B}, j_2) L_{s,x}(\mathbf{A}, k_1; \mathbf{K}, j_1). \end{aligned} \quad (2.6)$$

See Figure 6 for an illustration. Note that the summations in both sides of (2.6) are actually finite.

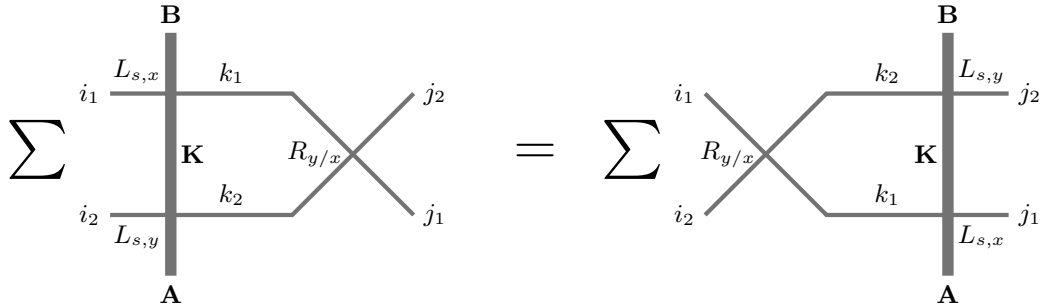


Figure 6: An illustration of the Yang–Baxter equation (2.6) in Proposition 2.3, where the sums in both sides are taken over all $k_1, k_2 \in \{0, 1, \dots, n\}$ and $\mathbf{K} \in \mathbb{Z}_{\geq 0}^n$.

2.3 Fused weights

The vertex weights $L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell)$ are higher spin (that is, they allow multiple arrows per edge) in the vertical direction. For $s = q^{-1/2}$, they reduce to the fundamental R -matrix $R_z(a, k; b, \ell)$, where $a, k, b, \ell \in \{0, 1, \dots, n\}$; see (2.5). The inverse procedure for constructing $L_{s,x}$ from R_z is called *fusion* and dates back to [KRS81]. In the uncolored case, it was put into probabilistic context in [CP16], [BP18a]. The colored fusion is described in, e.g., [BW22a, Appendix B]. The formula for the fully fused stochastic colored vertex weights $W_{x,L,M}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ coming from $U_q(\widehat{\mathfrak{sl}}_{n+1})$ is obtained in [BM16]; see also [BW22a, Appendix C]. Here we recall the stochastic vertex weights fused in both the horizontal and vertical directions.

We need the standard q -Pochhammer symbols notation:

$$(a; q)_k := (1 - a)(1 - aq) \dots (1 - aq^{k-1}), \quad k \in \mathbb{Z}_{\geq 0},$$

and $(z; q)_\infty := \prod_{i=0}^{\infty} (1 - zq^i)$ is a convergent infinite product because $q \in [0, 1)$.

For $\mathbf{A}, \mathbf{B} \in \mathbb{Z}_{\geq 0}^n$ such that $A_i \leq B_i$ for all i , define

$$\Phi(\mathbf{A}, \mathbf{B}; x, y) := \frac{(x; q)_{|\mathbf{A}|} (y/x; q)_{|\mathbf{B}-\mathbf{A}|}}{(y; q)_{|\mathbf{B}|}} (y/x)^{|\mathbf{A}|} q^{\sum_{1 \leq i < j \leq n} (B_i - A_i) A_j} \prod_{i=1}^n \frac{(q; q)_{B_i}}{(q; q)_{A_i} (q; q)_{B_i - A_i}}. \quad (2.7)$$

For any fixed $\mathbf{B} \in \mathbb{Z}_{\geq 0}^n$, we have

$$\sum_{\mathbf{A} \in \mathbb{Z}_{\geq 0}^n : A_i \leq B_i \text{ for all } i} \Phi(\mathbf{A}, \mathbf{B}; x, y) = 1. \quad (2.8)$$

With this notation, if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geq 0}^n$, we define the vertex weights

$$\begin{aligned} W_{x,L,M}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &:= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \cdot x^{|\mathbf{D}-\mathbf{B}|} (q^L)^{|\mathbf{A}|} (q^M)^{-|\mathbf{D}|} \\ &\times \sum_{\mathbf{P}} \Phi(\mathbf{C}-\mathbf{P}, \mathbf{C}+\mathbf{D}-\mathbf{P}; q^{L-M}x, q^{-M}x) \Phi(\mathbf{P}, \mathbf{B}; q^{-L}/x, q^{-L}). \end{aligned} \quad (2.9)$$

The sum in (2.9) is finite and is taken over all $\mathbf{P} \in \mathbb{Z}_{\geq 0}^n$ such that $0 \leq P_i \leq \min(B_i, C_i)$ for all i .

Remark 2.4. The parameters L, M enter the vertex weights (2.9) only through the powers q^L, q^M . Moreover, the weights depend on q^L and q^M in a rational manner. Specializing L, M , or both to positive integers leads to finite-spin reduction as in Remark 2.2. The integers L and M correspond to the horizontal and the vertical edge capacities, respectively. Outside of the finite-spin specializations, we may view q^L, q^M as independent complex parameters of the weights.

The weights $W_{x,L,M}$ (2.9) satisfy a version of the Yang–Baxter equation, see formulas (C.1.2)–(C.1.3) in [BW22a]. Graphically, this equation is similar to the one illustrated in Figure 6, but the weights $L_{s,x}, L_{s,y}$, and $R_{y/x}$ must be replaced with, respectively, $W_{\frac{x}{z}, L, N}$, $W_{\frac{y}{z}, M, N}$, and $W_{\frac{x}{y}, L, M}$. The summation in the Yang–Baxter equation for the W 's goes over triples of elements from $\mathbb{Z}_{\geq 0}^n$.

The weights (2.9) sum to one [BW22a, (C.1.5)],

$$\sum_{\mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geq 0}^n} W_{x,L,M}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 1, \quad \mathbf{A}, \mathbf{B} \in \mathbb{Z}_{\geq 0}^n, \quad (2.10)$$

and reduce to the weights $L_{s,x}$ (spin- $\frac{1}{2}$ in the horizontal direction and higher spin in the vertical direction) and R_z (spin- $\frac{1}{2}$ in both directions) from Sections 2.1 and 2.2 as follows [BW22a, Proposition C.1.4 and formula (C.2.2)]:

$$\begin{aligned} R_z(i, j; k, \ell) &= W_{z^{-1}, 1, 1}(\mathbf{e}_i \mathbf{1}_{i \geq 1}, \mathbf{e}_j \mathbf{1}_{j \geq 1}; \mathbf{e}_k \mathbf{1}_{k \geq 1}, \mathbf{e}_\ell \mathbf{1}_{\ell \geq 1}), \\ L_{s,x}(\mathbf{A}, b; \mathbf{C}, d) &= W_{x/s, 1, N}(\mathbf{A}, \mathbf{e}_b \mathbf{1}_{b \geq 1}; \mathbf{C}, \mathbf{e}_d \mathbf{1}_{d \geq 1}) \Big|_{q^N = s^{-2}}, \end{aligned} \quad (2.11)$$

where $i, j, k, \ell, b, d \in \{0, 1, \dots, n\}$ and $\mathbf{A}, \mathbf{C} \in \mathbb{Z}_{\geq 0}^n$. In the second line in (2.11), the parameter $s = q^{-N/2}$ is real for $q \in [0, 1)$ and $N \in \mathbb{Z}_{\geq 1}$. However, for the parameters (s, x) to satisfy (2.2), we let N not to be an integer, and treating q^N as a generic complex parameter, see Remark 2.4.

2.4 Queue specialization

Here we define a procedure which we call the *queue specialization* of the fully fused stochastic colored vertex weights (2.9). This specialization depends on an integer $1 \leq m \leq n$ and on three parameters u, s_1, s_2 , and proceeds in the following manner:

- First, encode the parameters \mathbf{L} and \mathbf{M} through $s_1, s_2 \in \mathbb{C}$ as $q^{-\mathbf{L}} = s_1^2, q^{-\mathbf{M}} = s_2^2$, and let the spectral parameter be $x = s_1 s_2^{-1} u$. This is just a change of variables which will be useful in subsequent computations.
- After that, take the limit of $W_{s_1 s_2^{-1} u, \mathbf{L}, \mathbf{M}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ as $A_m, C_m \rightarrow +\infty$ such that all other coordinates of \mathbf{A}, \mathbf{C} , as well as the whole tuples \mathbf{B}, \mathbf{D} are fixed, and $A_m - C_m = D_m - B_m$ is also fixed.² The latter condition follows from the arrow conservation.

Lemma 2.5. *The limit of the vertex weights $W_{s_1 s_2^{-1} u, \mathbf{L}, \mathbf{M}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ described above exists. It is given by*

$$\begin{aligned}
\lim_{A_m, C_m \rightarrow +\infty} W_{s_1 s_2^{-1} u, \mathbf{L}, \mathbf{M}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \cdot \mathbf{1}_{D_1=\dots=D_{m-1}=0} \cdot \frac{(s_1^{-1} s_2 u; q)_\infty}{(s_1 s_2 u; q)_\infty} \\
&\times \sum_{\mathbf{P}} \frac{(s_1 s_2 / u; q)_{|\mathbf{P}|} (s_1 u / s_2; q)_{|\mathbf{B}-\mathbf{P}|}}{(s_1^2; q)_{|\mathbf{B}|}} q^{\sum_{1 \leq i < j \leq n} (B_i - P_i) P_j} \prod_{i=1}^n \frac{(q; q)_{B_i}}{(q; q)_{P_i} (q; q)_{B_i - P_i}} \\
&\times \left(\frac{s_1 s_2}{u} \right)^{|\mathbf{B}| - |\mathbf{P}|} \left(\frac{u s_2}{s_1} \right)^{|\mathbf{D}|} q^{\sum_{m \leq i < j \leq n} D_i (C_j - P_j)} \frac{(s_2^2; q)_{|\mathbf{D}|}}{(q; q)_{D_m}} \prod_{i=m+1}^n \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}},
\end{aligned} \tag{2.12}$$

where the sum is over $\mathbf{P} \in \mathbb{Z}_{\geq 0}^n$ with $0 \leq P_i \leq \min(B_i, C_i)$ for all i .

Proof. Set $x = s_1 s_2^{-1} u$. Since \mathbf{B} stays fixed, the second factor

$$\Phi(\mathbf{P}, \mathbf{B}; q^{-\mathbf{L}}/x, q^{-\mathbf{L}}) = \Phi(\mathbf{P}, \mathbf{B}; s_1 s_2 / u, s_1^2)$$

in the sum in (2.9) is also fixed. Moreover, the summation multi-index \mathbf{P} belongs to a fixed finite set where $P_i \leq B_i$ for all i . Thus, it remains to consider the limit of

$$\begin{aligned}
&(q^{\mathbf{L}})^{|\mathbf{A}|} \Phi(\mathbf{C} - \mathbf{P}, \mathbf{C} + \mathbf{D} - \mathbf{P}; q^{\mathbf{L}-\mathbf{M}} x, q^{-\mathbf{M}} x) \\
&= (q^{\mathbf{L}})^{|\mathbf{A}|} \frac{(q^{\mathbf{L}-\mathbf{M}} x; q)_{|\mathbf{C}-\mathbf{P}|} (q^{-\mathbf{L}}; q)_{|\mathbf{D}|}}{(q^{-\mathbf{M}} x; q)_{|\mathbf{C}-\mathbf{P}+\mathbf{D}|}} (q^{-\mathbf{L}})^{|\mathbf{C}-\mathbf{P}|} q^{\sum_{1 \leq i < j \leq n} D_i (C_j - P_j)} \prod_{i=1}^n \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}}. \\
&= s_1^{-2|\mathbf{D}-\mathbf{B}+\mathbf{P}|} \frac{(s_1^{-1} s_2 u; q)_{|\mathbf{C}-\mathbf{P}|} (s_2^2; q)_{|\mathbf{D}|}}{(s_1 s_2 u; q)_{|\mathbf{C}-\mathbf{P}+\mathbf{D}|}} q^{\sum_{1 \leq i < j \leq n} D_i (C_j - P_j)} \prod_{i=1}^n \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}}.
\end{aligned} \tag{2.13}$$

We have $C_m - P_m \rightarrow +\infty$, so (2.13) converges to zero unless $D_i = 0$ for all $i < m$ (as $q \in [0, 1)$). This leads to the indicator $\mathbf{1}_{D_1=\dots=D_{m-1}=0}$ in (2.12). Next, if $D_i = 0$ for all $i < m$, then all other factors in (2.13) behave well, and the desired limit of the whole vertex weight exists. Taking the limit as $C_m \rightarrow +\infty$, we immediately obtain (2.12). This completes the proof. \square

Definition 2.6 (Queue specialization of the vertex weights). Let $1 \leq m \leq n$ and $u, s_1, s_2 \in \mathbb{C}$. We denote the limiting vertex weights (2.12) in Lemma 2.5 by $\mathbb{W}_{s_1, s_2, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$, and call them the *queue specialization* of the fully fused stochastic colored vertex weights.

²Throughout this limit, we assume that $0 < q < 1$, and later we will specialize the limit to $q = 0$ where needed. In other words, q^0 should be treated as 1 before and after the limit.

The term “queue specialization” comes from connections with multiline queues described (in two different degenerations) in Sections 4.3 and 5.3 below. The label $(-m)$ will be useful when we later place the vertices $\mathbb{W}_{s_1, s_2, u}^{(-m)}$ on a lattice.

Remark 2.7. In the queue vertex weights $\mathbb{W}_{s_1, s_2, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$, we abuse the notation of the tuples $\mathbf{A}, \mathbf{C} \in \mathbb{Z}_{\geq 0}^n$ by setting $A_m, C_m = +\infty$. That is, the tuples with infinitely many arrows of color m are not elements of $\mathbb{Z}_{\geq 0}^n$. However, for the uniformity of notation, we will still sometimes treat \mathbf{A}, \mathbf{C} as elements of $\mathbb{Z}_{\geq 0}^n$, while explicitly stating that $A_m, C_m = +\infty$.

Remark 2.8. From (2.12) we see that for fixed $B_1, \dots, B_{m-1} \geq 0$, the weights $\mathbb{W}_{s_1, s_2, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ are *independent* of $A_1, \dots, A_{m-1}, C_1, \dots, C_{m-1}$ provided that $C_i = A_i + B_i$ for all $i < m$. For example, we can set $A_i = 0$ and $C_i = B_i$ for all $i < m$. Note also that since $A_m, C_m = +\infty$, we may have $D_m > 0$ even if $B_m = 0$. The latter property is essential for our constructions.

The next lemma states the independence of the queue vertex weights under B_m , too, provided that no lower colors are present:

Lemma 2.9. *Let $B_1 = \dots = B_{m-1} = 0$. Then the queue vertex weight $\mathbb{W}_{s_1, s_2, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ is independent of B_m .*

Proof. By Remark 2.8, we may set $A_i = C_i = D_i = 0$ for all $i < m$. Set, for simplicity, $B_m = b$, $B_{m+1} + \dots + B_n = b'$, $P_m = p$, $P_{m+1} + \dots + P_n = p'$, and $D_m = d$. Then the part of (2.12) which depends only on b and p has the form

$$\begin{aligned} & \sum_{p=0}^b \frac{(s_1 s_2 / u; q)_{p+p'} (s_1 u / s_2; q)_{b+b'-p-p'}}{(s_1^2; q)_{b+b'}} \frac{(q; q)_b}{(q; q)_p (q; q)_{b-p}} (s_1 s_2 / u)^{b-p} q^{(b-p)p'} \\ &= \frac{(s_1 s_2 / u; q)_{p'} (s_1 u / s_2; q)_{b'-p'}}{(s_1^2; q)_{b'}} \sum_{p=0}^b \frac{(q^{p'} s_1 s_2 / u; q)_p (q^{b'-p'} s_1 u / s_2; q)_{b-p}}{(q^{b'} s_1^2; q)_b} \frac{(q; q)_b}{(q; q)_p (q; q)_{b-p}} (q^{p'} s_1 s_2 / u)^{b-p} \\ &= \frac{(s_1 s_2 / u; q)_{p'} (s_1 u / s_2; q)_{b'-p'}}{(s_1^2; q)_{b'}}. \end{aligned}$$

In the last equality we used the sum-to-one property (2.8) for the single-color case $n = 1$ (with $x = q^{b'-p'} s_1 u / s_2$, $y = q^{b'} s_1^2$, and $\mathbf{B} = b$). We see that the resulting expression does not depend on b , as desired. \square

Proposition 2.10. *For any $m \in \{1, \dots, n\}$, the queue vertex weights $\mathbb{W}_{s_1, s_0, \frac{u_1}{u_0}}^{(-m)}$, $\mathbb{W}_{s_2, s_0, \frac{u_2}{u_0}}^{(-m)}$ (2.12), and the fused cross vertex weight $W_{\frac{s_1 u_1}{s_2 u_2}, \mathbf{L}, \mathbf{M}}$ (2.9), where $q^{-\mathbf{L}} = s_1^2$ and $q^{-\mathbf{M}} = s_2^2$, satisfy the Yang–Baxter equation given in Figure 7. In symbols, for all fixed $\mathbf{A}, \mathbf{I}_1, \mathbf{I}_2, \mathbf{B}, \mathbf{J}_1, \mathbf{J}_2$ with $A_m, B_m = +\infty$, we have*

$$\begin{aligned} & \sum_{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3} \mathbb{W}_{s_2, s_0, \frac{u_2}{u_0}}^{(-m)}(\mathbf{A}, \mathbf{I}_2; \mathbf{K}_3, \mathbf{K}_2) \mathbb{W}_{s_1, s_0, \frac{u_1}{u_0}}^{(-m)}(\mathbf{K}_3, \mathbf{I}_1; \mathbf{B}, \mathbf{K}_1) W_{\frac{s_1 u_1}{s_2 u_2}, \mathbf{L}, \mathbf{M}}(\mathbf{K}_2, \mathbf{K}_1; \mathbf{J}_2, \mathbf{J}_1) \\ &= \sum_{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3} W_{\frac{s_1 u_1}{s_2 u_2}, \mathbf{L}, \mathbf{M}}(\mathbf{I}_2, \mathbf{I}_1; \mathbf{K}_2, \mathbf{K}_1) \mathbb{W}_{s_1, s_0, \frac{u_1}{u_0}}^{(-m)}(\mathbf{A}, \mathbf{K}_1; \mathbf{K}_3, \mathbf{J}_1) \mathbb{W}_{s_2, s_0, \frac{u_2}{u_0}}^{(-m)}(\mathbf{K}_3, \mathbf{K}_2; \mathbf{B}, \mathbf{J}_2). \end{aligned} \quad (2.14)$$

Proof. This is the queue specialization of the Yang–Baxter equation [BW22a, (C.1.2)] for the fully fused stochastic weights. The queue specialization is taken along the vertical line carrying the parameters $z = s_0 u_0$ and N with $q^{-N} = s_0^2$. \square

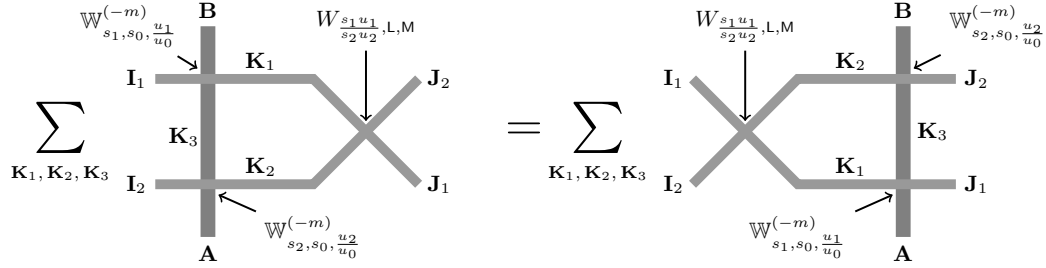


Figure 7: Yang–Baxter equation for the queue specialization. The sums in both sides are taken over all $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3 \in \mathbb{Z}_{\geq 0}^n$, and the inputs and outputs $\mathbf{I}_1, \mathbf{I}_2, \mathbf{J}_1, \mathbf{J}_2, \mathbf{A}, \mathbf{B} \in \mathbb{Z}_{\geq 0}^n$ are fixed.

Recall that $0 \leq q < 1$. Let us define the following two subsets of the parameters (s_1, s_2, u) in the queue vertex weights:

- (higher horizontal spin) $s_1, s_2 \in [-1, 1]$ such that $0 \leq s_1 s_2 < u < \min(\frac{s_1}{s_2}, \frac{s_2}{s_1}, \frac{1}{s_1 s_2})$;
 - (finite horizontal spin) $s_1 = q^{-\frac{1}{2}}$ for some $L \in \mathbb{Z}_{\geq 1}$ and $u = q^{\frac{1}{2}} u'$, with purely imaginary u', s_2 satisfying $s_2 u' \leq s_2^2 \leq 0$ and $s_2 / u' \geq q^L$.
- (2.15)

These subsets present convenient sufficient nonnegativity conditions:

Proposition 2.11. *Under (2.15), the queue vertex weights $\mathbb{W}_{s_1, s_2, u}^{(-m)}$ (2.12) are nonnegative, and*

$$\sum_{C_{m+1}, C_{m+2}, \dots, C_n, D_m, D_{m+1}, \dots, D_n=0}^{\infty} \mathbb{W}_{s_1, s_2, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 1, \quad (2.16)$$

where \mathbf{A}, \mathbf{B} are fixed, $A_m, C_m = +\infty$, and $C_i = A_i + B_i$, $D_i = 0$ for all $i < m$.

Proof. Let us first consider the higher horizontal spin case of (2.15). One can check that all arguments of the q -Pochhammer symbols in (2.12) are in $[0, 1)$. Moreover, the total sign coming from the powers $s_1^{|\mathbf{B}|-|\mathbf{D}|-|\mathbf{P}|} s_2^{|\mathbf{B}+|\mathbf{D}|-|\mathbf{P}|} u^{|\mathbf{D}|-|\mathbf{B}+|\mathbf{P}|}$ is always nonnegative. Thus, we get the nonnegativity of the queue vertex weights.

To see that they sum to one, we need to take the limit of the sum-to-one identity (2.10) for the fully fused weights $W_{x, L, M}$ as $A_m, C_m \rightarrow +\infty$. One can check that under our conditions (2.15), the weight $W_{x, L, M}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ (with $x = s_1 u / s_2$, $q^{-L} = s_1^2$, $q^{-M} = s_2^2$) decays exponentially fast when $|\mathbf{D}| \rightarrow +\infty$ due to the presence of the power $(u s_2 / s_1)^{|\mathbf{D}|}$. Thus, since our queue specialization requires $|\mathbf{D}|$ to stay fixed, this decay ensures that identity (2.10) yields (2.16).

Let us now consider the finite horizontal spin case of (2.15). In this case, $|\mathbf{D}|$ must stay finite (cf. Remarks 2.2 and 2.4), so the sum-to-one property (2.16) is automatic from (2.10). To see

the nonnegativity, observe that since $s_1 = q^{-\frac{1}{2}} \geq 1$, the term $(s_1^2; q)_{|\mathbf{D}|}$ produces the sign $(-1)^{|\mathbf{D}|}$, which is compensated by $(us_2/s_1)^{|\mathbf{D}|}$. The other combined powers $(s_1s_2/u)^{|\mathbf{B}|-|\mathbf{P}|}$ are always nonnegative. Next, we have

$$\frac{(s_1^{-1}s_2u; q)_\infty}{(s_1s_2u; q)_\infty} = \frac{1}{(s_2u'; q)_L} \geq 0.$$

In the remaining q -Pochhammer symbols, we have

$$s_1s_2/u = q^{-L}s_2/u' \geq 1, \quad s_1u/s_2 = u'/s_2 \geq 1, \quad s_1^2 = q^{-L} \geq 1.$$

Thus, the quantities $(s_1s_2/u; q)_{|\mathbf{P}|}(s_1u/s_2; q)_{|\mathbf{B}-\mathbf{P}|}(s_1^2; q)_{|\mathbf{B}|}^{-1}$ are nonnegative for all \mathbf{P} provided that $|\mathbf{B}| \leq L$. This completes the proof. \square

2.5 Queue vertex model on the cylinder

Let us fix the number n of colors, and another integer $N \geq 1$ which is the size of the cylinder. In this subsection we define a linear operator \mathfrak{Q} whose matrix elements $\langle \emptyset | \mathfrak{Q} | \mathbf{V} \rangle$ are partition functions coming from the queue vertex weights on the cylinder $\{-n, \dots, -1\} \times \mathbb{Z}/N\mathbb{Z}$. The vertices on the cylinder are indexed by $(-m, j)$, $m = 1, \dots, n$, $j = 1, \dots, N$, see Figure 8 for an illustration. No paths enter from the left. The paths exiting horizontally from the right are encoded by the integer tuples $\mathbf{V}(1), \dots, \mathbf{V}(N) \in \mathbb{Z}_{\geq 0}^n$. Define the vector space for the states on the cylinder,

$$V^{\otimes N} := \text{Span}(\{ |\mathbf{V}\rangle = |(\mathbf{V}(1), \dots, \mathbf{V}(N))\rangle : \mathbf{V}(j) \in \mathbb{Z}_{\geq 0}^n \forall j = 1, \dots, N \}), \quad (2.17)$$

and similarly let $\langle \mathbf{V} |$ denote the dual basis in $V^{\otimes N}$. We also need the subspace $V_{\text{full}}^{\otimes N}$ of $V^{\otimes N}$ spanned by the vectors $|\mathbf{V}\rangle$ satisfying

$$\sum_{j=1}^N \mathbf{V}(j)_k > 0 \quad \text{for all } k = 1, \dots, n$$

(above, $\mathbf{V}(j)_k$ denotes the k -th coordinate of $\mathbf{V}(j)$). In words, each state $|\mathbf{V}\rangle \in V_{\text{full}}^{\otimes N}$ must contain at least one arrow of each of the n colors.

Definition 2.12. Fix complex parameters

$$\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{C}^N, \quad \mathbf{s}^{(h)} = (s_1^{(h)}, \dots, s_N^{(h)}), \quad \mathbf{v} = (v_1, \dots, v_n), \quad \mathbf{s}^{(v)} = (s_1^{(v)}, \dots, s_n^{(v)}), \quad (2.18)$$

The linear operator $\mathfrak{Q} = \mathfrak{Q}(\mathbf{u}; \mathbf{s}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)})$ on $V^{\otimes N}$, called the *queue transfer matrix*, is defined via its matrix elements $\langle \mathbf{V}' | \mathfrak{Q} | \mathbf{V} \rangle$ as follows. First, if $|\mathbf{V}\rangle \notin V_{\text{full}}^{\otimes N}$, we set $\langle \mathbf{V}' | \mathfrak{Q} | \mathbf{V} \rangle = 0$. Otherwise, $\langle \mathbf{V}' | \mathfrak{Q} | \mathbf{V} \rangle$ is the partition function of the queue vertex weights on the cylinder $\{-n, \dots, -1\} \times \mathbb{Z}/N\mathbb{Z}$ with the following data:

- The entering arrow configurations $\mathbf{V}'(j)$ along the left horizontal edges $(-n-1, j) \rightarrow (-n, j)$, $j = 1, \dots, N$.
- The terminal arrow configurations $\mathbf{V}(j)$ along the right horizontal edges $(-1, j) \rightarrow (0, j)$, $j = 1, \dots, N$.

- Queue vertex weights $\mathbb{W}_{s_j^{(h)}, s_m^{(v)}, u_j/v_m}^{(-m)}$ at each vertex $(-m, j)$ in the cylinder, where $j = 1, \dots, N$, $m = 1, \dots, n$, and $u_j, s_j^{(h)}$ and $v_m, s_m^{(v)}$ are the horizontal and the vertical parameters, respectively.

In detail, the partition function $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle$ is equal to the sum

$$\sum_{\mathbf{M}} \sum_{\mathcal{C} \in \mathcal{P}_{\mathbf{M}, \mathbf{V}', \mathbf{M}, \mathbf{V}}} \prod_{m=1}^n \prod_{j=1}^N \mathbb{W}_{s_j^{(h)}, s_m^{(v)}, u_j/v_m}^{(-m)} (\mathbf{A}(m, j), \mathbf{B}(m, j); \mathbf{C}(m, j), \mathbf{D}(m, j)),$$

where $\mathbf{M} = (\mathbf{M}(-n), \dots, \mathbf{M}(-1))$ encodes the paths winding around the cylinder. The sum over \mathbf{M} , by definition, has

$$\mathbf{M}(-m)_m = +\infty \quad \text{and} \quad \mathbf{M}(-m)_i = 0, \quad i < m, \quad \text{for all } m = 1, \dots, n. \quad (2.19)$$

For each \mathbf{M} , the sum over \mathcal{C} runs over all path configurations in the rectangle with the boundary conditions $\mathbf{M}, \mathbf{V}', \mathbf{M}, \mathbf{V}$ at the bottom, left, top, and right, respectively. The tuples $\mathbf{A}(m, j)$, $\mathbf{B}(m, j)$, $\mathbf{C}(m, j)$, and $\mathbf{D}(m, j)$ encode the arrow configurations at each vertex $(-m, j)$ of the rectangle.

See Figure 8 for an example when $\mathbf{V}' = (\mathbf{0}, \dots, \mathbf{0})$.

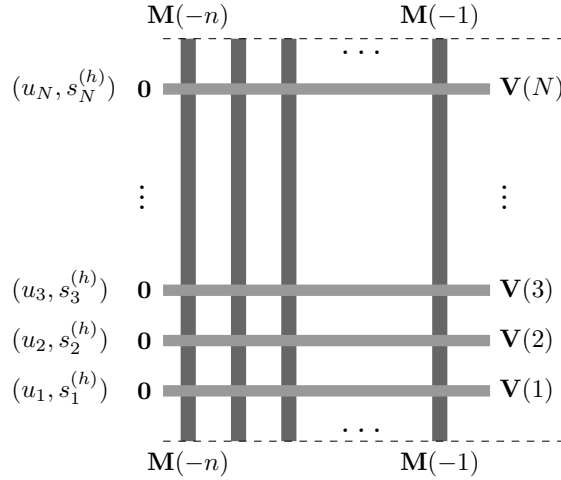


Figure 8: A vertex model on the cylinder whose partition function (indexed by the tuples $\mathbf{V}(1), \dots, \mathbf{V}(N)$) is equal to $\langle \emptyset | \mathcal{Q} | \mathbf{V} \rangle$. We identify the top and the bottom boundaries (dashed lines), and sum over all possible tuples $\mathbf{M}(-n), \dots, \mathbf{M}(-1) \in \mathbb{Z}_{\geq 0}^n$ encoding the paths which wind around the cylinder an arbitrary number of times. The left vector $\langle \emptyset |$ is empty, which corresponds to no paths entering from the left.

Observe that the partition function $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle$ in Definition 2.12 involving the sum over $\mathbf{M}(-n), \dots, \mathbf{M}(-1)$ cannot be interpreted as a probability in a stochastic vertex model because we are summing over input path configurations at vertices, as well as over the output ones. Moreover, this sum may even be divergent. Therefore, we need to make sure that the quantities $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle$ are well-defined:

Lemma 2.13. For any $|\mathbf{V}\rangle \in V_{\text{full}}^{\otimes N}$, the sum over $\mathbf{M}(-n), \dots, \mathbf{M}(-1)$ in Definition 2.12 is convergent for any $|q| < 1$.

Proof. By Lemma 2.5, paths of colors $i < m$ cannot leave the column $(-m)$. The fact that the configuration $|\mathbf{V}\rangle \in V_{\text{full}}^{\otimes N}$ contains at least one arrow of each color implies that each column $(-m)$ must horizontally emit at least one path of color m . Recall again that after we fix the entering horizontal arrows \mathbf{B} at a lattice site, the summation multi-index \mathbf{P} in each vertex weight belongs to a fixed finite set where $P_i \leq B_i$ for all i . Furthermore, with \mathbf{V} fixed, the size of tuples \mathbf{B}, \mathbf{D} at each vertex is bounded from above. Thus, the factor $q^{-D_i P_j}$ in each vertex weight is bounded from above. Therefore, the presence of the factors $q^{\sum_{m \leq i < j \leq n} D_i C_j}$ in the vertex weights (2.12) implies that the weight of the whole column $(-m)$ with fixed winding path counts $\mathbf{M}(-m)_i$, $i > m$, contains the factor $q^{d \sum_{i=m+1}^n \mathbf{M}(-m)_i}$ for some $d \geq 1$. This implies that for any fixed m , the sum over the quantities $\mathbf{M}(-m)_i$, $i > m$, is finite. This completes the proof. \square

Remark 2.14. The condition that $|\mathbf{V}\rangle \in V_{\text{full}}^{\otimes N}$ is essential for Lemma 2.13. Indeed, for $n = 3$, we have

$$\mathbb{W}_{s_1, s_2, u}^{(-2)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{(s_2 u / s_1; q)_{\infty}}{(s_1 s_2 u; q)_{\infty}},$$

where $\mathbf{A} = \mathbf{C} = (0, \infty, k)$ and $\mathbf{B} = \mathbf{D} = (0, 0, 0)$. This expression is independent of k . For $N = 1$, we need to sum it over all k , which leads to divergence. However, if $\mathbf{D} = (0, d, 0)$, the weight $\mathbb{W}_{s_1, s_2, u}^{(-2)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ contains the power q^{kd} , eliminating this problem of divergence.

The partition functions $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle$ are essentially independent of the entrance state $\langle \mathbf{V}' |$:

Proposition 2.15. Let $|\mathbf{V}\rangle \in V_{\text{full}}^{\otimes N}$. If the entering configuration \mathbf{V}' contains at least one path of color strictly less than n , then $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle = 0$. Otherwise, if \mathbf{V}' contains only paths of color n , then we have $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle = \langle \emptyset | \mathcal{Q} | \mathbf{V} \rangle$.

Proof. Lemma 2.5 guarantees that no paths of color strictly less than n leave column $(-n)$. Since we cannot have paths of color not equal to n infinitely wind in the column $(-n)$, the partition function $\langle \mathbf{V}' | \mathcal{Q} | \mathbf{V} \rangle$ must vanish if \mathbf{V}' contains at least one path of color strictly less than n . This establishes the first claim. The second claim immediately follows from Lemma 2.9. \square

Remark 2.16 (Trace formula for queue partition functions). The queue vertex model partition function can be interpreted through a product of the following N operators, where N is the size of the ring in the cross-section of the cylinder:

$$\langle \emptyset | \mathcal{Q} | \mathbf{V} \rangle = \text{Trace}^{\bullet} \left(\mathcal{X}_{\mathbf{V}(1)}(u_1, s_1^{(h)}) \cdots \mathcal{X}_{\mathbf{V}(N)}(u_N, s_N^{(h)}) \right). \quad (2.20)$$

Here each $\mathcal{X}_{\mathbf{V}}(u, s)$, $\mathbf{V} \in \mathbb{Z}_{\geq 0}^n$, acts in the n -fold tensor product $V_{-n} \otimes \dots \otimes V_{-1}$, where V_{-m} has basis $|\mathbf{M}(-m)\rangle$, $\mathbf{M}(-m) \in \mathbb{Z}_{\geq 0}^n$, $m = 1, \dots, n$. The matrix elements of $\mathcal{X}_{\mathbf{V}}(u, s)$ are partition functions of the queue weights on the lattice $\{1\} \times \{-n, \dots, -1\}$. See Figure 9 for an illustration. The operation Trace^{\bullet} in (2.20) means that we restrict the summation to the tuples \mathbf{M} satisfying (2.19). One can turn Trace^{\bullet} into a genuine trace by suitably modifying the definition of the spaces V_{-m} .

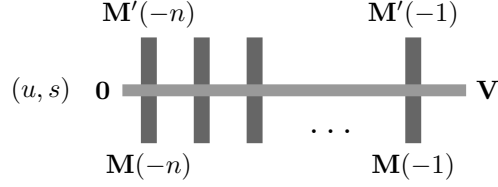


Figure 9: The matrix element $\langle \mathbf{M}(-n), \dots, \mathbf{M}(-1) | \mathcal{X}_{\mathbf{V}}(u, s) | \mathbf{M}'(-n), \dots, \mathbf{M}'(-1) \rangle$ of one of the operators $\mathcal{X}_{\mathbf{V}}(u, s)$ entering the trace formula (2.20). At the vertex indexed by $-m$, $m = 1, \dots, n$, we have the queue vertex weight $\mathbb{W}_{s, s_m^{(v)}, u/v_m}^{(-m)}$.

3 Stationarity of the queue vertex model

In this section, we establish two general stationarity properties of the queue vertex model on the cylinder. Specifically, we construct two Markov operators which, when applied to the empty state $\langle \emptyset |$ on the cylinder, commute with the queue transfer matrix \mathfrak{Q} from Definition 2.12. Both of these commutation relations follow directly from the Yang–Baxter equation.

In this section we work with *formal* Markov operators, that is, we do not assume that their matrix elements are nonnegative. The matrix elements only need to satisfy the corresponding sum-to-one properties. In the future sections we specify the ranges of parameters making the matrix elements nonnegative.

3.1 Twisted cylinder Markov operator

Recall that n is the number of colors, and N is the size of the ring in the vertical cross-section of the cylinder $\{-n, \dots, -1\} \times \mathbb{Z}/N\mathbb{Z}$ carrying the queue transfer matrix $\mathfrak{Q} = \mathfrak{Q}(\mathbf{u}; \mathbf{s}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)})$ of Definition 2.12. For our first relation, we take the spin- $\frac{1}{2}$ specialization in the horizontal direction. That is, the horizontal spin parameters in (2.18) are all equal to $q^{-1/2}$:

$$\mathbf{s}^{(h)} = (s_1^{(h)}, \dots, s_N^{(h)}) = \mathbf{s}_{\frac{1}{2}}^{(h)} := (q^{-1/2}, \dots, q^{-1/2}).$$

We can take each tensor component in the space $V^{\otimes N}$ (2.17) to be $V = \mathbb{C}^{n+1}$.

We need some extra notation. Let R_z be the R -matrix (2.4), and denote by $\check{\mathfrak{R}}_z$ the operator in $V \otimes V$ with matrix elements (see the end of Section 2.1 for basis vector notations)

$$\langle i, j | \check{\mathfrak{R}}_z | \ell, k \rangle := R_z(i, j; k, \ell). \quad (3.1)$$

When $\check{\mathfrak{R}}_z$ acts on the k -th and the ℓ -th tensor components of $V^{\otimes N}$, we denote it by $\check{\mathfrak{R}}_z^{(k\ell)}$.

Definition 3.1. Fix two spectral parameters u, u_1 . Let the *twisted cylinder Markov operator* $\mathfrak{T}(u, u_1)$ be the linear operator on $V^{\otimes N}$ defined as

$$\mathfrak{T}(u, u_1) := \check{\mathfrak{R}}_{u_1 u^{-1}}^{(12)} \check{\mathfrak{R}}_{u_1 u^{-1}}^{(23)} \dots \check{\mathfrak{R}}_{u_1 u^{-1}}^{(N,1)}, \quad (3.2)$$

Pictorially, $\mathfrak{T}(u, u_1)$ is given in Figure 10. The product in (3.2) is interpreted as a product of Markov operators, that is, we first apply $\check{\mathfrak{R}}_{u_1 u^{-1}}^{(12)}$ to a fixed configuration of arrows on the cylinder, get a random configuration, then apply $\check{\mathfrak{R}}_{u_1 u^{-1}}^{(23)}$ to the new configuration, and so on.

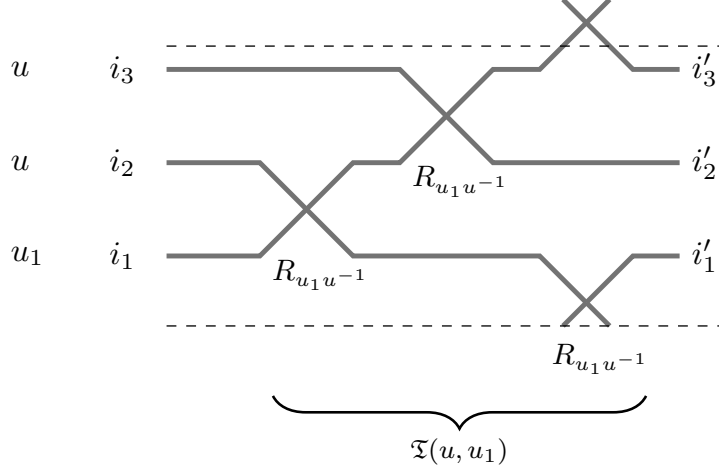


Figure 10: The configuration of vertices whose partition function is the matrix element of the twisted cylinder Markov operator $\langle \mathbf{V}' | \mathfrak{T}(u, u_1) | \mathbf{V} \rangle$, where $\mathbf{V} = (i_1, i_2, i_3)$, $\mathbf{V}' = (i'_1, i'_2, i'_3)$, and $i_k, i'_k \in \{0, \dots, n\}$. The size of the ring is $N = 3$. After all the N crossings, the spectral parameters attached to the strands on the right are the same as on the left. Note that at each crossing, we use the vertex weights $R_{u_1 u^{-1}}$ in the same way as in the Yang–Baxter equation in Figure 6; see also Remark 3.2 for a discussion of the notation.

Remark 3.2. The swapping of the indices in the operator $\check{\mathfrak{R}}_z$ compared to the R -matrix R_z (see (3.1)) is *purely notational*. We employ it for the following convenience. When Figure 10 is read from left to right, the space $V^{\otimes N}$ encoding configurations on the ring stays the same after every single crossing of the strands. In particular, passing to $\check{\mathfrak{R}}_z$ does not affect the random mechanism: the crosses act in Figure 10 in the same way as in the diagram of the Yang–Baxter equation in Figure 7.

Iterating the sum-to-one property of the R -matrix (2.4), we see that for any initial configuration $\mathbf{V}' = (i'_1, \dots, i'_N)$, where $i'_k \in \{0, \dots, n\}$, we have

$$\sum_{\mathbf{V}=(i_1, \dots, i_N) \in \{0, 1, \dots, n\}^N} \langle \mathbf{V}' | \mathfrak{T}(u, u_1) | \mathbf{V} \rangle = 1. \quad (3.3)$$

If the matrix elements of $\mathfrak{T}(u, u_1)$ are nonnegative, then it is a Markov operator. However, considering $\mathfrak{T}(u, u_1)$ as a formal Markov operator with the sum-to-one property suffices for this section.

We have the following stationarity of the queue vertex model under the action of the twisted cylinder Markov operator:

Theorem 3.3. *Let the parameters u, u_1 of the twisted cylinder Markov operator \mathfrak{T} (3.2), as well as the parameters $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{s}^{(v)} = (s_1^{(v)}, \dots, s_n^{(v)})$ in the queue transfer matrix \mathfrak{Q} of Definition 2.12 be arbitrary. Denote $\mathbf{u} := (u_1, u, \dots, u)$. Then we have*

$$\langle \emptyset | \mathfrak{Q}(\mathbf{u}; \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}) \mathfrak{T}(u, u_1) = \langle \emptyset | \mathfrak{Q}(\mathbf{u}; \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}). \quad (3.4)$$

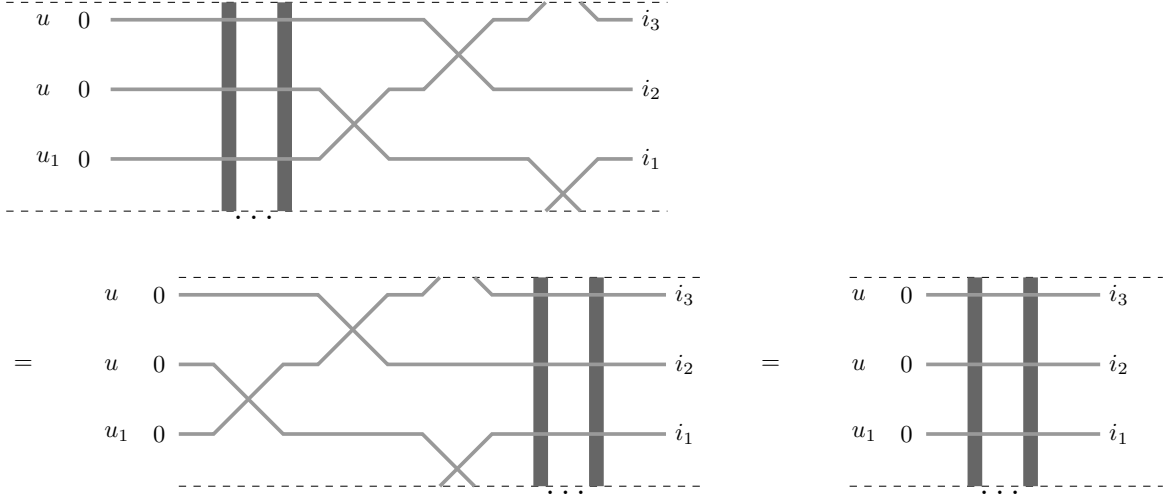


Figure 11: The commutation in the proof of Theorem 3.3 for $N = 3$. Here $i_1, i_2, i_3 \in \{0, 1, \dots, n\}$ are arbitrary.

Proof. We apply the Yang–Baxter equation specialized to the queue vertex weights; see Proposition 2.10. In this equation, the cross vertex weights have the form $W_{\frac{u}{u_1}, 1, 1}$, which are encoded by the operator $\check{\mathcal{R}}_{u_1 u^{-1}}$; see (2.11). We illustrate the argument diagrammatically for $N = 3$ in Figure 11.

Applying the Yang–Baxter equation successively at columns $-n, -n + 1, \dots, -2, -1$ of the queue vertex model, we get the intertwining relation

$$\mathfrak{Q}((u_1, u, \dots, u); \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}) \check{\mathcal{R}}_{u_1 u^{-1}}^{(12)} = \check{\mathcal{R}}_{u_1 u^{-1}}^{(12)} \mathfrak{Q}((u, u_1, \dots, u); \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}) \quad (3.5)$$

Continuing inductively with $\check{\mathcal{R}}_{u_1 u^{-1}}^{(23)}, \dots, \check{\mathcal{R}}_{u_1 u^{-1}}^{(N-1, N)}$ turns the horizontal spectral parameter sequence in \mathfrak{Q} into (u, \dots, u, u_1) . Finally, after applying the last operator $\check{\mathcal{R}}_{u_1 u^{-1}}^{(N, 1)}$, we get back to the original sequence. Here we used the periodicity in the vertical direction of the vertex model defining \mathfrak{Q} , which is crucial to complete the commutation. Therefore,

$$\mathfrak{Q}(\mathbf{u}; \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}) \mathfrak{T}(u, u_1) = \mathfrak{T}(u, u_1) \mathfrak{Q}(\mathbf{u}; \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}).$$

It remains to observe that $\langle \emptyset | \check{\mathcal{R}}_{u_1 u^{-1}}^{(i, j)} = \langle \emptyset |$ for all i, j . This completes the proof. \square

3.2 Straight cylinder Markov operator

Here we define another Markov operator on the cylinder of size $N + 1$ which commutes with the queue transfer matrix on this cylinder when applied to the empty configuration \emptyset at the left boundary of the cylinder. It acts in the space $\mathbb{C}^{n+1} \otimes V^{\otimes N}$, where the first factor \mathbb{C}^{n+1} is an auxiliary space which we identify with the index 0 in the superscripts. The tensor components of the space $V^{\otimes N}$ are indexed by $j = 1, 2, \dots, N$, and have basis vectors $|\mathbf{V}(j)\rangle$, where $\mathbf{V}(j) \in \mathbb{Z}_{\geq 0}^n$ encodes the arrow configuration on site j . That is, in contrast with Section 3.1, we return to the

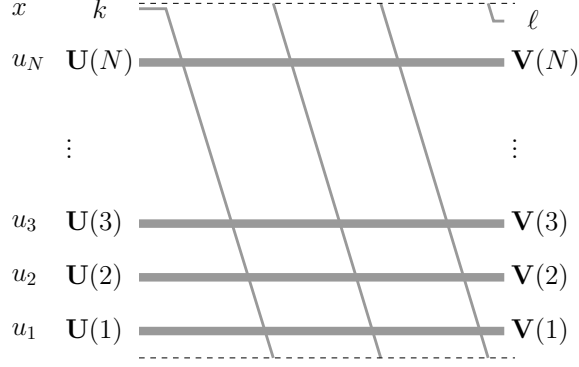


Figure 12: The straight cylinder Markov operator $\mathfrak{S} = \mathfrak{S}(x, \mathbf{u}; \mathbf{s}^{(h)})$, applied three times. The line with spectral parameter x has spin parameter $q^{-\frac{1}{2}}$, so edges along this line can only be occupied by at most one path (hence $k, \ell \in \{0, 1, \dots, n\}$). The partition function of the displayed configuration is the matrix element $\langle k, \mathbf{U} | \mathfrak{S}^3 | \ell, \mathbf{V} \rangle$.

higher spin setting. Later in Sections 5 and 6 we will take limits in which the marginal process corresponding to the factor $V^{\otimes N}$ will lead to the colored q -Boson or the colored q -PushTASEP on the ring of size N .

Recall the Markov operator $\mathcal{L}_{s,z}$ acting in $\mathbb{C}^{n+1} \otimes V$; see (2.3).

Definition 3.4. Fix spectral parameters $(x, \mathbf{u}) = (x, u_1, u_2, \dots, u_N) \in \mathbb{C}^{N+1}$ and horizontal spin parameters $\mathbf{s}^{(h)} = (s_1^{(h)}, \dots, s_N^{(h)}) \in \mathbb{C}^N$. The *straight cylinder Markov operator* denoted by $\mathfrak{S} = \mathfrak{S}(x, \mathbf{u}; \mathbf{s}^{(h)})$ acts in $\mathbb{C}^{n+1} \otimes V^{\otimes N}$ as follows:

$$\mathfrak{S}(x, \mathbf{u}; \mathbf{s}^{(h)}) := \mathcal{L}_{s_N^{(h)}, xu_N^{-1}}^{(0N)} \mathcal{L}_{s_{N-1}^{(h)}, xu_{N-1}^{-1}}^{(0, N-1)} \cdots \mathcal{L}_{s_1^{(h)}, xu_1^{-1}}^{(01)}. \quad (3.6)$$

Here by $\mathcal{L}_{s,z}^{(0,j)}$ we denote the operator $\mathcal{L}_{s,z}$ acting on $\mathbb{C}^{n+1} \otimes V^{\otimes N}$ in the auxiliary space \mathbb{C}^{n+1} and the j -th tensor component of $V^{\otimes N}$. The operator \mathfrak{S} is illustrated in Figure 12.

Remark 3.5. In Definition 3.4 and throughout this subsection, we index the tensor factors of the space $\mathbb{C}^{n+1} \otimes V^{\otimes N}$ by $0, 1, \dots, N$, but put the horizontal strand corresponding to the 0-th factor on top of the cylinder in Figures 12 and 13. This way of indexing is consistent with the action of the operators $\mathcal{L}_{s,z}$ in $\mathbb{C}^{n+1} \otimes V$ in (3.6), while the location of the 0-th strand on top just below the cylinder's cut is convenient for illustrations.

The operator $\mathfrak{S}(x, \mathbf{u}; \mathbf{s}^{(h)})$ satisfies the sum-to-one property similarly to (3.3), and thus is a formal Markov operator. We have the following stationarity of the queue vertex model under \mathfrak{S} :

Theorem 3.6. Let us take the parameters $\mathbf{u}, \mathbf{s}^{(h)}, \mathbf{v}, \mathbf{s}^{(v)}$ as in (2.18), and consider the queue vertex transfer matrix on the ring of size $N + 1$ with the parameters

$$\mathfrak{Q} = \mathfrak{Q} \left((xq^{\frac{1}{2}}, \mathbf{u}); (q^{-\frac{1}{2}}, \mathbf{s}^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right).$$

Then we have

$$\langle \emptyset | \mathfrak{Q} \mathfrak{S} = \langle \emptyset | \mathfrak{Q},$$

where $\mathfrak{S} = \mathfrak{S}(x, \mathbf{u}; \mathbf{s}^{(h)})$ has parameters compatible with those in \mathfrak{Q} .

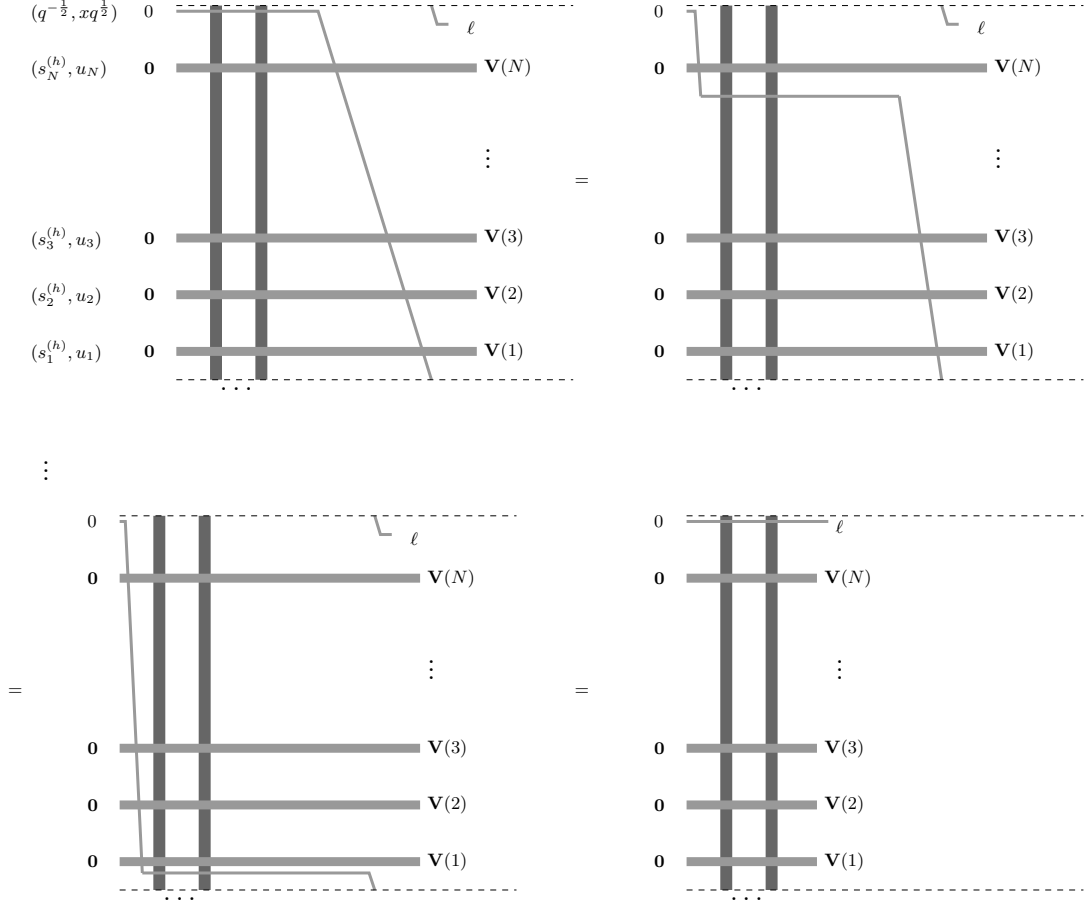


Figure 13: An illustration of the proof of Theorem 3.6. Each move is justified by the Yang–Baxter equation (Proposition 2.10). As usual, the equality of pictures means the equality of the corresponding partition functions with the fixed boundary conditions.

Proof. We employ the Yang–Baxter equation of Proposition 2.10. Let us match our parameters to this equation. The queue vertex model at rows 0 and j has vertex weights

$$\mathbb{W}_{q^{-1/2}, s_m^{(v)}, \frac{xq^{1/2}}{v_m}}^{(-m)}, \quad \mathbb{W}_{s_j^{(h)}, s_m^{(v)}, \frac{u_j}{v_m}}^{(-m)},$$

respectively. The operator \mathfrak{S} has the vertex weight $L_{s_j^{(h)}, xu_j^{-1}}$, which, by (2.11), is the same as the fused weight $W_{xu_j^{-1}/s_j^{(h)}, 1, M}$, where $q^{-M/2} = s_j^{(h)}$. We see that these three weights indeed satisfy the Yang–Baxter equation illustrated in Figure 7.

Applying this Yang–Baxter equation with parameters $(v_1, s_1^{(v)}), \dots, (v_n, s_n^{(v)})$ (that is, at the columns $-1, \dots, -n$ of the queue vertex model), we obtain the following relation between operators acting in the space $\mathbb{C}^{n+1} \otimes V^{\otimes N}$, applied to the empty configuration \emptyset at the left boundary

of the cylinder:

$$\begin{aligned} & \langle \emptyset | \mathfrak{Q} \left((xq^{\frac{1}{2}}, u_1, \dots, u_N); (q^{-\frac{1}{2}}, s_1^{(h)}, \dots, s_N^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right) \mathcal{L}_{s_N^{(h)}, xu_N^{-1}}^{(0N)} \\ &= \langle \emptyset | P^{(0,N)} \mathfrak{Q} \left((u_N, u_1, \dots, u_{N-1}, xq^{\frac{1}{2}}); (s_N^{(h)}, s_1^{(h)}, \dots, s_{N-1}^{(h)}, q^{-\frac{1}{2}}); \mathbf{v}; \mathbf{s}^{(v)} \right) P^{(0,N)}. \end{aligned} \quad (3.7)$$

Here the permutation operator $P^{(0,N)}$ swaps the 0-th and the N -th tensor components in the space $\mathbb{C}^{n+1} \otimes V \otimes \dots \otimes V$, and is required since the operator \mathfrak{Q} in the right-hand side acts in $V \otimes \dots \otimes V \otimes \mathbb{C}^{n+1}$. Identity (3.7) represents the first step of the transformations illustrated in Figure 13. In the next step, the action of $\mathcal{L}_{s_{N-1}^{(h)}, xu_{N-1}^{-1}}^{(0,N-1)}$ results in the following identity:

$$\begin{aligned} & \langle \emptyset | P^{(0,N)} \mathfrak{Q} \left((u_N, u_1, \dots, u_{N-1}, xq^{\frac{1}{2}}); (s_N^{(h)}, s_1^{(h)}, \dots, s_{N-1}^{(h)}, q^{-\frac{1}{2}}); \mathbf{v}; \mathbf{s}^{(v)} \right) P^{(0,N)} \mathcal{L}_{s_{N-1}^{(h)}, xu_{N-1}^{-1}}^{(0,N-1)} \\ &= \langle \emptyset | P^{(N,N-1)} P^{(0,N)} \mathfrak{Q} \left((u_N, u_1, \dots, u_{N-2}, xq^{\frac{1}{2}}, u_{N-1}); (s_N^{(h)}, s_1^{(h)}, \dots, s_{N-2}^{(h)}, q^{-\frac{1}{2}}, s_{N-1}^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right) \\ & \quad \times P^{(0,N)} P^{(N,N-1)}. \end{aligned}$$

Iterating the action of the other operators \mathcal{L} , after N total steps the horizontal parameter sequences \mathbf{u} and $\mathbf{s}^{(h)}$ return back to their original states $(xq^{\frac{1}{2}}, \mathbf{u}); (q^{-\frac{1}{2}}, \mathbf{s}^{(h)})$. Here we employed the periodicity of the vertex model defining \mathfrak{Q} to complete the commutation. This establishes the desired stationarity relation. \square

4 Multi-species ASEP from twisted cylinder

In this section we take a continuous time limit of the twisted cylinder Markov operator \mathfrak{T} defined in Section 3.1, and recover the known descriptions of the stationary distribution of the multi-species TASEP and ASEP on the ring from [FM05], [PEM09], and [Mar20].

4.1 Multi-species ASEP on the ring

Recall that n is the number of particle species (also called “types” or “colors”), and N is the size of the ring $\mathbb{Z}/N\mathbb{Z}$. The state space of the multi-species ASEP consists of particle configurations $\eta = (\eta_1, \dots, \eta_N)$ on the ring, where $\eta_k \in \{0, 1, \dots, n\}$ encodes the type of the particle at site k . The type 0 corresponds to the empty site.

For a configuration η and each pair of neighboring sites $(k, k+1)$ (including $(N, 1)$ for $k = N$), denote by $\eta^{k,k+1}$ the configuration $(\eta_1, \dots, \eta_{k+1}, \eta_k, \dots, \eta_N)$. That is, $\eta^{k,k+1}$ is obtained from η by swapping the types at sites k and $k+1$.

Definition 4.1. The *multi-species ASEP (mASEP)* is a continuous time Markov chain on the space of particle configurations on the ring, with the following transition rates:

$$\text{Rate}(\eta \rightarrow \eta^{k,k+1}) = \begin{cases} q, & \eta_k > \eta_{k+1}; \\ 1, & \eta_k < \eta_{k+1}, \end{cases} \quad (4.1)$$

where k runs over $1, 2, \dots, N$, and in (4.1) we assume that $\eta_k \neq \eta_{k+1}$. The multi-species ASEP depends on a single parameter $q \in [0, 1)$.

Remark 4.2. We use the ordering of colors in which color n has the highest priority (to move from site $j + 1$ to site j on the ring), and color 1 has the lowest priority. Often in the literature on multi-type interacting particle systems, e.g., in [Mar20], a reverse convention is used, in which type 1 has the highest priority. In Section 4.3 below, when this distinction becomes relevant, we recast all the necessary definitions from the existing literature using our color ordering conventions.

Observe that mASEP preserves the number of particles of each type. We denote these type counts by

$$N_m := \sum_{j=1}^N \mathbf{1}_{\eta_j=m}, \quad m = 1, \dots, n. \quad (4.2)$$

We have $N_1 + \dots + N_n \leq N$. In addition, throughout this section, we assume that $N_m \geq 1$ for all $m = 1, \dots, n$. This assumption is very natural since if there are no particles of a given type, then the evolution of the n -type mASEP is the same as that of a $(n - 1)$ -type mASEP, where the missing type is removed entirely. Note also that at the level of queue vertex models, a violation of the assumption that $N_m \geq 1$ for all m leads to problems; see Lemma 2.13 and Remark 2.14.

Definition 4.3 (mASEP stationary distribution). When restricted to a *sector* (namely, the subset of the state space) with fixed type counts (N_1, \dots, N_n) , mASEP becomes an irreducible continuous time Markov chain on a finite state space. Therefore, it admits a *unique stationary distribution* in each sector. We denote this distribution by $\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}(\eta)$.

It is natural to encode the states η as basis vectors in $(\mathbb{C}^{n+1})^{\otimes N}$:

$$|\eta\rangle = |\eta_1, \dots, \eta_N\rangle = |\eta_1\rangle \otimes \dots \otimes |\eta_N\rangle.$$

Since all jumps under mASEP are nearest neighbor, the infinitesimal generator of mASEP written as a sum of local rate matrices as follows. Consider two possible configurations of particles ii', jj' at adjacent lattice sites, say, k and $k + 1$. Define an operator \mathcal{M}_{loc} in $(\mathbb{C}^{n+1})^{\otimes 2}$ such that

$$\begin{aligned} \langle i, i' | \mathcal{M}_{loc} | j, j' \rangle &= (\mathcal{M}_{loc})_{ii', jj'} := \mathbf{1}_{(i, i') = (j', j)} \text{ (jump rate } ii' \rightarrow jj') \\ &\quad - \mathbf{1}_{(i, i') = (j, j')} \text{ (jump rate } ii' \rightarrow j'j). \end{aligned} \quad (4.3)$$

The matrix element (4.3) is nonzero if and only if $(i, i') = (j, j')$ or $(i, i') = (j', j)$. The infinitesimal Markov generator of mASEP then has the form

$$\mathfrak{M}_{\text{mASEP}} = \sum_{l=1}^N (\mathcal{M}_{loc})^{l, l+1}, \quad (4.4)$$

where $(\mathcal{M}_{loc})^{l, l+1}$ acts as \mathcal{M}_{loc} on tensor factors of sites $l, l + 1$, and as the identity on all other factors. Denote by $\{\mathfrak{P}_{\text{mASEP}}(t)\}_{t \in \mathbb{R}_{\geq 0}}$, the continuous time Markov semigroup generated by (4.4). The passage from the infinitesimal generator to this semigroup is straightforward, as the process lives on a finite state space. See the left side of Figure 14 for an illustration of the process.

The next statement identifies mASEP as a Poisson-type limit of the twisted cylinder Markov operators.

Proposition 4.4. *For any $u \in \mathbb{R}$, we have the convergence of Markov operators in $(\mathbb{C}^{n+1})^{\otimes N}$:*

$$\lim_{\epsilon \rightarrow 0} \mathfrak{T}(u, u(1-\epsilon))^{\lfloor (1-q)t/\epsilon \rfloor} = \mathfrak{P}_{\text{mASEP}}(t), \quad t \in \mathbb{R}_{\geq 0}. \quad (4.5)$$

Here \mathfrak{T} is the twisted cylinder Markov operator (3.2). See the right side of Figure 14 for an illustration of the limiting jump rates.

Recall that $q \in [0, 1)$. One readily sees that for any $u \in \mathbb{R}$ and $\epsilon \in (0, 1)$, both sides of the limiting relation (4.5) are Markov operators with nonnegative matrix elements.

Proof outline of Proposition 4.4. This is a standard limit of the stochastic six-vertex model leading to the ASEP, see [BCG16], [Agg17] and also [BW22a, Section 12.3] for its colored version. In short, in the regime $\epsilon \rightarrow 0$, all paths want to follow a “staircase” motion, with occasional deviations that occur in continuous time according to independent exponential clocks. Because of how the cross vertices are organized to form the twisted cylinder Markov operator (Figure 10), the staircase motion corresponds to particles staying in place. For convenience, let us reproduce the main computation.

First, consider the limit of the local operators $\check{\mathfrak{R}}_z$ (3.1), where $z = u_1 u^{-1} = 1 - \epsilon$ because $u_1 = u(1 - \epsilon)$. We have for the matrix elements (2.4):

$$\frac{1-z}{1-qz} = \frac{\epsilon}{1-q} + O(\epsilon^2), \quad 1 - \frac{1-z}{1-qz} = 1 - \frac{\epsilon}{1-q} + O(\epsilon^2),$$

and similarly for $q \frac{1-z}{1-qz}$ and $1 - q \frac{1-z}{1-qz}$. Therefore, the local infinitesimal generator (4.3) of mASEP has the following form:

$$\mathcal{M}_{loc} = (1-q) \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \check{\mathfrak{R}}_{1-\epsilon}. \quad (4.6)$$

See Figure 14, right, for an illustration of how we interpret the operators $\check{\mathfrak{R}}_{1-\epsilon}$ in terms of the hopping rates.

In the $\epsilon \rightarrow 0$ limit, the product of the operators $\check{\mathfrak{R}}_{1-\epsilon}$ over all pairs of neighboring lattice sites (which is equal to the twisted cylinder operator (3.2)) behaves as $Id + \frac{1}{1-q}\epsilon \mathfrak{M}_{\text{mASEP}}$, where Id is the identity matrix, and $\mathfrak{M}_{\text{mASEP}}$ is defined in (4.4). This leads to the desired statement about the convergence to the mASEP Markov semigroup. \square

4.2 Vertex model for the mASEP stationary distribution

Let us now apply Theorem 3.3 to mASEP. Recall that by $\langle \emptyset | \mathfrak{Q} | \mathbf{V} \rangle$ we denote the partition function of the queue vertex model on the cylinder introduced in Definition 2.12. For mASEP, the queue transfer matrix $\mathfrak{Q} = \mathfrak{Q}(\mathbf{u}; \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)})$ has the parameters

$$\mathbf{u} = (u, \dots, u), \quad \mathbf{s}_{\frac{1}{2}}^{(h)} = (q^{-1/2}, \dots, q^{-1/2}),$$

where $\mathbf{v}, \mathbf{s}^{(v)} \in \mathbb{C}^n$ are arbitrary. The N -tuple $\mathbf{V} = \mathbf{V}_\eta := (\mathbf{e}_{\eta_1} \mathbf{1}_{\eta_1 \geq 1}, \dots, \mathbf{e}_{\eta_N} \mathbf{1}_{\eta_N \geq 1})$ encodes the same information as the mASEP state $\eta = (\eta_1, \dots, \eta_N)$. Indeed, this is because each of the

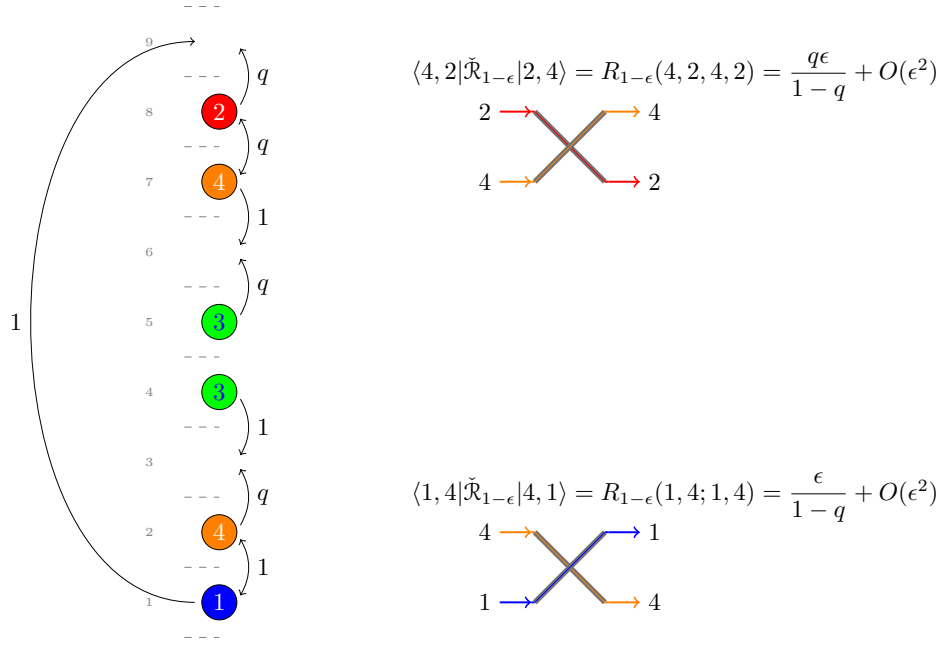


Figure 14: Left: An illustration of a state in mASEP on a ring of size $N = 9$, with all possible jump rates indicated. Right: The interpretation of the mASEP hopping rates as limits of the operators $\check{\mathcal{R}}_{1-\epsilon}$.

subsets $\mathbf{V}_\eta(j) \subset \mathbb{Z}_{\geq 0}^n$, $j = 1, \dots, N$, must have at most one element thanks to the spin- $\frac{1}{2}$ reduction coming from $\mathbf{s}_{\frac{1}{2}}^{(h)}$.

We split this subsection into two parts. First, we show in Proposition 4.5 that the normalized partition functions of the queue vertex model produce the mASEP stationary distribution. Then we present in Proposition 4.9 a slightly modified vertex model for which all partition functions on the cylinder with right boundary η are positive without normalization.

Proposition 4.5. *With the above notation and for generic complex parameters $u, \mathbf{v}, \mathbf{s}^{(v)}$, the queue vertex model partition function with boundary η and vertex weights $\mathbb{W}_{q^{-1/2}, \mathbf{s}_m^{(v)}, u/v_m}^{(-m)}$ is proportional to the stationary probability for the multi-species ASEP on the ring:*

$$\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}(\eta) = \frac{1}{Z_{N_1, \dots, N_n}^{\text{mASEP}}(u; \mathbf{v}; \mathbf{s}^{(v)})} \langle \emptyset | \mathcal{Q}(\mathbf{u}; \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}) | \mathbf{V}_\eta \rangle. \quad (4.7)$$

The normalizing constant $Z_{N_1, \dots, N_n}^{\text{mASEP}}(u; \mathbf{v}; \mathbf{s}^{(v)})$ may depend on the type counts (N_1, \dots, N_n) (4.2), but not on the state η within the sector determined by (N_1, \dots, N_n) .

In Proposition 4.5, by “generic” we mean that the parameters must ensure that $\langle \emptyset | \mathcal{Q} | \mathbf{V}_\eta \rangle$ is finite and nonzero for all η with type counts $N_m \geq 1$, $m = 1, \dots, n$. For fixed N and n , genericity is ensured by excluding zero sets of finitely many polynomials in u, \mathbf{v} , and $\mathbf{s}^{(v)}$ from the parameter space. Later in Proposition 4.9, we present concrete conditions on the parameters producing stationary measures for all N and n .

Proof of Proposition 4.5. Iterating Theorem 3.3, we have

$$\langle \emptyset | \mathfrak{Q}((u_1, u, \dots, u); \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}) \mathfrak{T}(u, u_1)^{\lfloor (1-q)t/\epsilon \rfloor} = \langle \emptyset | \mathfrak{Q}((u_1, u, \dots, u); \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)}).$$

Setting $u_1 = u(1 - \epsilon)$ and sending $\epsilon \rightarrow 0$ turns the power of the twisted cylinder Markov operator in the left-hand side into $\mathfrak{P}_{\text{mASEP}}(t)$; see Proposition 4.4. The vertex weights in the queue vertex model with horizontal spin $\frac{1}{2}$ are given in Figure 15 (recall that \mathfrak{Q} involves the weights $\mathbb{W}_{q^{-1/2}, \mathbf{s}_m^{(v)}, u/v_m}^{(-m)}$). These vertex weights are continuous in the spectral parameter u for generic parameters. Therefore, we can simply take the limit $u_1 \rightarrow u$ in the queue transfer matrix, which implies that $\langle \emptyset | \mathfrak{Q}((u, u, \dots, u); \mathbf{s}_{\frac{1}{2}}^{(h)}; \mathbf{v}; \mathbf{s}^{(v)})$ is the left (row) eigenvector of the Markov semigroup $\mathfrak{P}_{\text{mASEP}}(t)$ with eigenvalue 1 (where here the exchange of the limit and the implicit summation in both sides of the above equation is justified by the fact that the summation defining each partition function is uniformly bounded by a geometric series, and the sum over \mathbf{V}_η on the left hand side has a bounded number of terms, independently of ϵ). Thus, it is proportional to the row vector representing the stationary distribution of mASEP, as desired. \square

$\begin{array}{c} \mathbf{A} \\ \\ 0 \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{1}{1 - su} \end{array}$	$\begin{array}{c} \mathbf{A} \\ \\ k \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{(-su + s^2 q^{A_k}) q^{\mathbf{A}^{[k+1, n]}}}{1 - su} \end{array}$	$\begin{array}{c} \mathbf{A}_k^- \\ \\ 0 \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{-su(1 - q^{A_k}) q^{\mathbf{A}^{[k+1, n]}}}{1 - su} \end{array}$	$\begin{array}{c} \mathbf{A} \\ \\ 0 \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{-su q^{\mathbf{A}^{[m+1, n]}}}{1 - su} \end{array}$
$\begin{array}{c} \mathbf{A}_k^+ \\ \\ k \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{1}{1 - su} \end{array}$	$\begin{array}{c} \mathbf{A}_{k\ell}^{+-} \\ \\ k \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{-su(1 - q^{A_\ell}) q^{\mathbf{A}^{[\ell+1, n]}}}{1 - su} \end{array}$	$\begin{array}{c} \mathbf{A}_{\ell k}^{+-} \\ \\ \ell \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{-s^2(1 - q^{A_k}) q^{\mathbf{A}^{[k+1, n]}}}{1 - su} \end{array}$	$\begin{array}{c} \mathbf{A}_\ell^+ \\ \\ \ell \text{ --- } \mathbf{A} \\ \\ \mathbf{A} \\ \hline \frac{-s^2 q^{\mathbf{A}^{[m+1, n]}}}{1 - su} \end{array}$

Figure 15: Weights $\mathbb{W}_{q^{-1/2}, s, uq^{1/2}}^{(-m)}(\mathbf{A}, k; \mathbf{C}, \ell)$ entering the spin- $\frac{1}{2}$ queue vertex model which represents the stationary distribution of mASEP. Here $m < k < \ell \leq n$, and it suffices to consider only vertices with no colors $\leq m$ entering from the left. Recall that $A_m = +\infty$, so the vertices in the last column also satisfy the arrow conservation property.

Remark 4.6. Proposition 4.5 expresses the stationary probabilities of mASEP, a system depending on a single parameter $q \in [0, 1)$, as normalized partition functions $\langle \emptyset | \mathfrak{Q} | \mathbf{V}_\eta \rangle / Z_{N_1, \dots, N_n}^{\text{mASEP}}$ of the queue vertex model. The latter in addition depends on the parameters u, \mathbf{v} , and $\mathbf{s}^{(v)}$. However, by (4.7), we see that while $u, \mathbf{v}, \mathbf{s}^{(v)}$ enter the weights of the queue vertex model, these parameters do not affect the normalized partition functions.

Definition 4.7. For $s \neq 0$ let us define the m ASEP gauge transformation of the queue vertex weights:

$$\mathbb{W}_{q^{-1/2}, s, uq^{1/2}}^{(-m), \text{mASEP}^+}(\mathbf{A}, k; \mathbf{C}, \ell) := (-1/s)^{\mathbf{1}_{\ell \geq 1}} \mathbb{W}_{q^{-1/2}, s, uq^{1/2}}^{(-m)}(\mathbf{A}, k; \mathbf{C}, \ell). \quad (4.8)$$

That is, we remove the factor $(-s)$ from the weights in the right three columns in Figure 15. The resulting weights make sense for $s = 0$, too. The notation “+” in (4.8) indicates that we will impose conditions on the parameters under which these weights are nonnegative.

Also, denote by

$$\frac{\langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle}{Z_{N_1, \dots, N_n}^{\text{mASEP}^+}} \quad (4.9)$$

the corresponding normalized partition function of the queue vertex model on the cylinder with the right boundary η .

Remark 4.8. The weights (4.8) are the queue limits (as in Section 2.4) of the non-stochastic higher spin colored vertex weights defined in [BW22a, (2.2.2)].

Proposition 4.9. Fix the type counts (N_1, \dots, N_n) with $N_m \geq 1$ for all m . Let the parameters satisfy

$$0 \leq s_m^{(v)} < \frac{u}{v_m q^{1/2}}, \quad s_m^{(v)} \frac{u}{v_m q^{1/2}} < 1, \quad m = 1, \dots, n. \quad (4.10)$$

Then

$$\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}(\eta) = \frac{\langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle}{Z_{N_1, \dots, N_n}^{\text{mASEP}^+}},$$

is the m ASEP stationary distribution, and $\langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle > 0$ for all η .

Note that conditions (4.10) are written for u/v_m entering the vertex weights as $\mathbb{W}_{q^{-1/2}, s_m^{(v)}, u/v_m}^{(-m), \text{mASEP}^+}$, while for displaying the weights in Figure 15 it was convenient to multiply the spectral parameter by $q^{1/2}$.

Proof of Proposition 4.9. Assume first that $s_m^{(v)} > 0$ for all m . Replacing the queue vertex weights with their gauge transformed versions (4.8) multiplies the partition function $\langle \emptyset | \mathfrak{Q} | \mathbf{V}_\eta \rangle$ by

$$\prod_{m=1}^n (-1/s_m^{(v)})^{N_m + \dots + N_n},$$

which depends only on the sector, but not on the configuration η . Therefore, the gauge transformation may be incorporated into the normalizing constant $Z_{N_1, \dots, N_n}^{\text{mASEP}^+}$. One readily sees that under conditions (4.10) and when $s_m^{(v)} > 0$ for all m , all vertex weights (4.8) are positive (see Figure 15). This completes the proof in the case when all the $s_m^{(v)}$'s are positive.

Setting some (or all) of the $s_m^{(v)}$'s to zero in the weights (4.8) is allowed, and leads to a well-defined partition function $\langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle$. In this partition function, some of the vertex weights in Figure 15 vanish. To show that $\langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle$ is still positive and not merely nonnegative for all η , first notice that there exists η for which $\langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle \neq 0$ (this verification is straightforward, and we omit it). Next, observe that

$$\sum_\eta \langle \emptyset | \mathfrak{Q}^{\text{mASEP}^+} | \mathbf{V}_\eta \rangle \langle \mathbf{V}_\eta | \quad (4.11)$$

is a nonzero left (row) eigenvector with eigenvalue 1 of the mASEP semigroup $\mathfrak{P}_{\text{mASEP}}(t)$ corresponding to an irreducible continuous time Markov process on a finite state space. Therefore, (4.11) is proportional to the Perron–Frobenius eigenvector of $\mathfrak{P}_{\text{mASEP}}(t)$, which has all components positive. This completes the proof. \square

4.3 Matching to multiline queues

4.3.1 Original multiline queues

First, we connect the queue vertex model $\mathfrak{Q}^{\text{mASEP}+}$ with special parameters with *multiline queues* introduced by Martin [Mar20]. He showed that the output of the latter produces the stationary distribution of mASEP on the ring.

Setting $u = q^{1/2}$ and $s_m^{(v)} = 0$, $v_m = 1$ for all $m = 1, \dots, N$ in the vertex weights (4.8) leads to the weights given in Figure 16 which we denote by

$$\mathbb{W}^{(-m), \text{mq}} := \mathbb{W}_{q^{-1/2}, 0, q^{1/2}}^{(-m), \text{mASEP}+}. \quad (4.12)$$

By Proposition 4.9, these weights produce positive partition functions of the queue vertex model on the cylinder.

Let us connect the queue vertex model $\mathfrak{Q}^{\text{mASEP}+}$ with these particular parameters to multiline queue diagrams. These diagrams were defined in [Mar20, Sections 1.1 and 3.6], and the vertex model interpretation follows from formula (3.9) in [Mar20]. For the reader’s convenience, in the rest of Section 4.3.1 we reproduce the main definitions and the matching of queues to our vertex models.

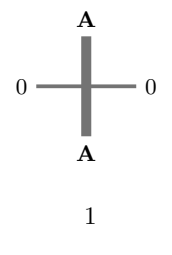
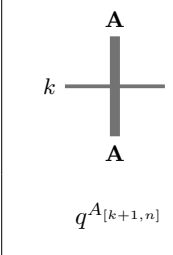
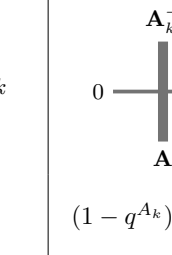
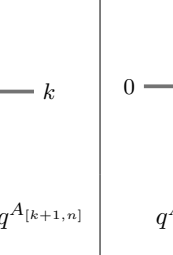
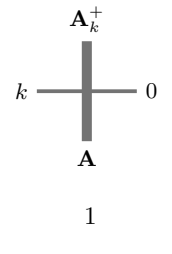
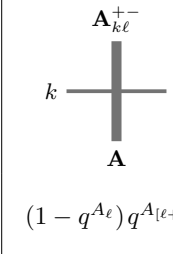
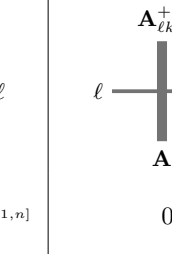
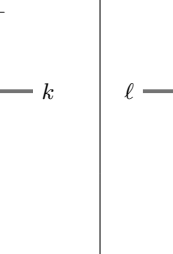
 1	 $q^{A_{[k+1, n]}}$	 $(1 - q^{A_k}) q^{A_{[k+1, n]}}$	 $q^{A_{[m+1, n]}}$
 1	 $(1 - q^{A_\ell}) q^{A_{[\ell+1, n]}}$	 0	 0

Figure 16: Weights $\mathbb{W}^{(-m), \text{mq}}$ (4.12). In Proposition 4.10 we match them to probabilities under multiline queues of [Mar20]. Here $m < k < \ell \leq n$, and recall (Remark 4.2) that our ordering of colors is reversed compared to particle types in [Mar20].

Let us recast [Mar20, Algorithm 2] (called *the Martin algorithm* in what follows) in the language of one column of a vertex model on the cylinder. Fix $m = 1, \dots, n$, type counts

N_m, N_{m+1}, \dots, N_n , and assume that we have an arbitrary fixed configuration η of paths of colors strictly larger than m entering the column $(-m)$ from the left. The configuration η has N_i paths of color i , $i > m$. The Martin algorithm (for color m) samples a random new configuration η' of paths exiting the column $(-m)$ to the right. The configuration η' has N_i paths of color $i \geq m$; it is constructed sequentially, through the following three steps.

1. Start with the empty configuration $\eta' = \{0, \dots, 0\}$ (N zeros). In addition, sample a uniformly random subset $\mathcal{J} \subset \{1, \dots, N\}$ of sites on the cylinder of cardinality $N_m + \dots + N_n$. Next, we randomly update η' such that in the end $\eta'_j > 0$ if and only if $j \in \mathcal{J}$.
2. For each color $i = n, n-1, \dots, m+1$ (in this order), let $a_1^i < \dots < a_{N_i}^i$ be the locations of paths of color i in η .
 - 2a. For $j = 1, \dots, N_i$, if $\eta'_{a_j^i} = 0$ and $a_j^i \in \mathcal{J}$, set $\eta'_{a_j^i} = i$ (if a path of color i can come straight through, it does so).
 - 2b. Otherwise, the j -th path of color i starts from $\eta_{a_j^i}$ and randomly chooses an exit site among yet unoccupied sites in \mathcal{J} as follows. Let $a_j^i < p_1 < p_2 < \dots < p_l$ (here, $p < p'$ means that as we read upwards starting from p , possibly wrapping around in the vertical direction, we observe p' before getting back to a_j^i), where (p_1, \dots, p_l) are all sites in \mathcal{J} for which at this point we have $\eta'_{p_t} = 0$, $t = 1, \dots, l$. Then, set $\eta'_{p_t} = i$ with probability proportional to q^{t-1} .
 - 2b'. Equivalently, instead of step 2b, one can think that the color i path starting from site a_j^i goes up the cylinder and sequentially with probability $1 - q$ picks an unoccupied site from \mathcal{J} to exit, or with probability q skips this site (*accepts or declines the service*, in queueing terminology). The path continues the motion up the cylinder until its exit, and can go around the cylinder an arbitrary number of times. If the path exits at p_t , then after normalization, this produces the same probability proportional to q^{t-1} .
3. Once all paths of all colors strictly larger than m are processed, we have N_m unoccupied sites in \mathcal{J} left. We set $\eta'_j = m$ for all these remaining sites.

To obtain the mASEP stationary distribution $\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}$ (Definition 4.3), one needs to apply the Martin algorithm for color n with input $\eta(0) = \emptyset$, and get a random output $\eta(1)$. Then apply the algorithm for color $n-1$ with input $\eta(1)$, get an output $\eta(2)$, and so on. The final output $\eta(n)$ of the algorithm for color 1 is the random configuration distributed according to the mASEP stationary distribution.

The following statement matches the output of the Martin algorithm to vertex models and essentially coincides with [Mar20, Theorem 3.4]. For convenience, we reproduce it here.

Proposition 4.10. *In each sector determined by the fixed type counts (N_1, \dots, N_n) , the output η of the Martin algorithm has the same distribution as the output of our queue vertex model on the cylinder with the weights $\mathbb{W}^{(-m), \text{mq}}$ given in Figure 16.*

Idea of proof. This follows by matching the vertex weights in Figure 16 to the weights $w_i(Q|A, S)$ given in [Mar20, (3.9)]. The translation from the queueing language to vertex models is straightforward and we omit it. \square

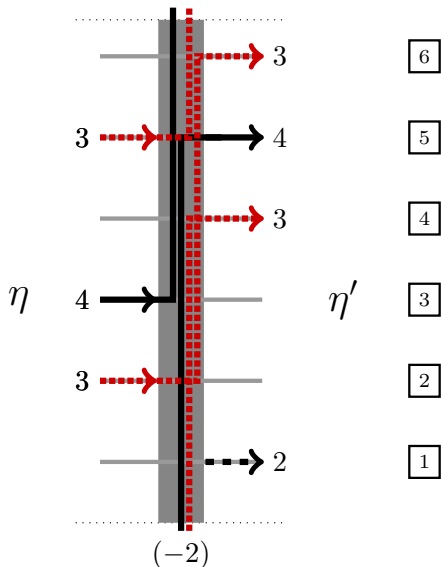


Figure 17: The Martin algorithm from [Mar20] with $N = 6$, $n = 4$, in the column $(-m)$, where $m = 2$. The set \mathcal{J} is $\{1, 4, 5, 6\}$. Given this \mathcal{J} , the conditional probability of the configuration in the figure (according to the description with $\mathbf{2b}'$) is proportional to q^5 (the color 4 path skips five possibilities) times q^2 (the top color 3 path skips two possibilities) times 1 (the bottom color 3 path selects the first available possibility). The boxed numbers indicate the ring sites.

In [Mar20], the stationarity of the output η of the Martin algorithm under mASEP follows from the Matrix Product Ansatz. The connection between the algorithm and the Matrix Product Ansatz is essentially equivalent to Proposition 4.10. We link queue vertex models to Matrix Product Ansatz in Section 4.4 below.

Remark 4.11. While the Martin algorithm and the queue vertex model produce the same output η (in distribution, in each sector), it remains unclear whether one can define appropriate “states” of the queueing system under the Martin algorithm such that these states are in a weight-preserving bijection (possibly up to a common proportionality constant) with states of the queue vertex model. Indeed, tracking each particle’s choices as in step $\mathbf{2b}'$ of the Martin algorithm with the input as in Figure 17 involves the following information:

- Track how many times the path of color 4 wraps around the cylinder.
- Pick a bijection between the color 3 inputs and outputs (there are $2!$ choices in Figure 17).
- For each of the two bijections, track how many times each of the two color 3 paths wraps around the cylinder.

Under the queue vertex model, we do not choose a bijection, and the wrapping arrows of colors $m + 1, \dots, n$ are encoded by the tuple $\mathbf{M}(-m)$. Here $\mathbf{M}(-m)_k$ is the total number of arrows of color k that wrap around the ring (that is, go between the sites N and 1).

To show that summing over all data in the queueing system produces the desired distribution of the output η , one seems to require the intricate argument with compatible queue-length processes; see [Mar20, Section 4.2] for details.

4.3.2 Alternative multiline queues and an interpolation

Let us consider a different specialization of the queue vertex model $\Omega^{\text{mASEP}+}$ introduced in Section 4.2. Namely, set $u = 1$ and $s = s_m^{(v)} = q$, $v_m = 1$ for all $m = 1, \dots, N$ in (4.8), and clear the common denominators $1 - q$. The resulting weights $\mathbb{W}^{(-m), \text{mq alt}} := (1 - q) \mathbb{W}_{q^{-1/2}, q, q^{1/2}}^{(-m), \text{mASEP}+}$ are given in Figure 18. By Proposition 4.9, they lead to positive partition functions on the cylinder.

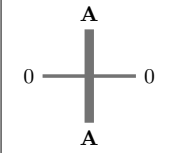
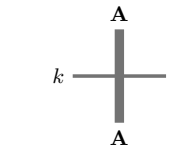
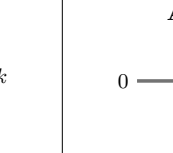
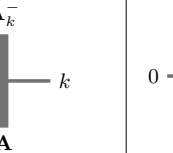
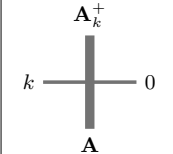
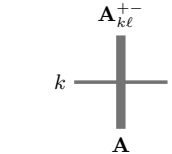
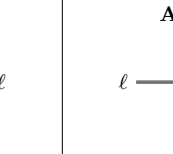
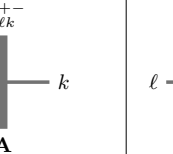
			
1	$(1 - q \cdot q^{A_k}) q^{A_{[k+1, n]}}$	$(1 - q^{A_k}) q^{A_{[k+1, n]}}$	$q^{A_{[m+1, n]}}$
			
1	$(1 - q^{A_\ell}) q^{A_{[\ell+1, n]}}$	$q \cdot (1 - q^{A_k}) q^{A_{[k+1, n]}}$	$q \cdot q^{A_{[m+1, n]}}$

Figure 18: Weights $\mathbb{W}^{(-m), \text{mq alt}}$ for the alternative multiline queue model.

Just as by Proposition 4.10, the weights $\mathbb{W}^{(-m), \text{mq}}$ from Figure 16 produce the same output η (in distribution) as the Martin algorithm, the new weights $\mathbb{W}^{(-m), \text{mq alt}}$ should be related to the *alternative multiline queue model* introduced in [Mar20, Section 7].

By definition, the alternative algorithm for color m consists of the same steps as the algorithm described in Section 4.3.1 above, except step **2a**. Instead, if $\eta_{a_j^i}^i = 0$ and $a_j^i \in \mathcal{J}$, then the entering color i path ($i > m$) from site a_j^i still has probability q to go up the cylinder and not exit immediately through this site.

Arguing similarly to [Mar20, Section 4.2], one can show that in each sector determined by the fixed type counts (N_1, \dots, N_n) , the output η of the alternative multiline queue algorithm has the same distribution as the output of the queue vertex model on the cylinder with the weights $\mathbb{W}^{(-m), \text{mq alt}}$ given in Figure 18. This identification of the alternative multiline queues with the vertex model would resolve the conjecture from [Mar20, Section 7] that the distribution of η is stationary under the mASEP dynamics on the ring. Indeed, this is because the output of the queue vertex model with the weights $\mathbb{W}^{(-m), \text{mq alt}}$ is already stationary under the mASEP dynamics on the ring, thanks to our general Propositions 4.5 and 4.9 which ultimately rely on the Yang–Baxter equation for the twisted cylinder (Theorem 3.3).

If we do not specialize the parameter s to 0 or q , we obtain a family of queue vertex weights

\mathbf{A} $0 \text{ --- } \mathbf{A} \text{ --- } 0$ \mathbf{A} 1	\mathbf{A} $k \text{ --- } \mathbf{A} \text{ --- } k$ \mathbf{A} $(1 - s \cdot q^{A_k}) q^{A_{[k+1, n]}}$	\mathbf{A}_k^- $0 \text{ --- } \mathbf{A} \text{ --- } k$ \mathbf{A} $(1 - q^{A_k}) q^{A_{[k+1, n]}}$	\mathbf{A} $0 \text{ --- } \mathbf{A} \text{ --- } m$ \mathbf{A} $q^{A_{[m+1, n]}}$
\mathbf{A}_k^+ $k \text{ --- } \mathbf{A} \text{ --- } 0$ \mathbf{A} 1	$\mathbf{A}_{k\ell}^{+-}$ $k \text{ --- } \mathbf{A} \text{ --- } \ell$ \mathbf{A} $(1 - q^{A_\ell}) q^{A_{[\ell+1, n]}}$	$\mathbf{A}_{\ell k}^{+-}$ $\ell \text{ --- } \mathbf{A} \text{ --- } k$ \mathbf{A} $s \cdot (1 - q^{A_k}) q^{A_{[k+1, n]}}$	\mathbf{A}_ℓ^+ $\ell \text{ --- } \mathbf{A} \text{ --- } m$ \mathbf{A} $s \cdot q^{A_{[m+1, n]}}$

Figure 19: Weights $(1-s) \mathbb{W}_{q^{-1/2}, s, q^{1/2}}^{(-m), \text{mASEP}^+}$ interpolating between the vertex weights related to the original and the alternative multiline queues of [Mar20].

depending on q and s (see Figure 19). The output of the vertex model with these weights produces the mASEP stationary distribution (this again follows from Propositions 4.5 and 4.9). The (q, s) -dependent weights should be related to a new multiline queue model that interpolates between the original and the alternative multiline queues:

Definition 4.12 (Interpolating multiline queues). Let $s \in [0, 1)$, and modify the Martin algorithm (for a given color m) by changing step 2a as follows. If a color i path ($i > m$) enters at a_j^i and $\eta'_{a_j^i} = 0$ (that is, a service is immediately available), then the path exits (accepts the service) with probability $1 - s$. With probability s , the path turns up the cylinder and skips every successive available exit (service) with probability q (as prescribed by step 2b'). All other parts of the algorithm remain the same.

For $q = 0$ (when mASEP becomes the multi-species TASEP), the interpolating model produces a multiline queue model with *random service assignment*. Note that for $q = 0$, both the original and the alternative multiline queues become the same and are deterministic. For two colors, this deterministic model was constructed in [Ang06] to describe the stationary distribution of the two-color TASEP. It was generalized to n colors in [FM07].

Similarly to the argument in [Mar20, Section 4.2], it should be possible to identify the output of the interpolating multiline queues with that of the (q, s) -dependent queue vertex model on the cylinder. In Section 4.4 below, we outline a possible connection of the latter with the Matrix Product Ansatz.

4.4 Connection to Matrix Product Ansatz

Prior to the multiline queue realization of the mASEP stationary distribution $\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}$ in [Mar20], Prohac–Evans–Mallick [PEM09] showed that $\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}$ can be expressed in a *matrix product form*. For processes on the ring, this expression has the same format as our general

trace formula (2.20), see (4.14) below, and includes matrices $\mathcal{X}_m^{\text{MPA}}$, $m = 0, 1, \dots, n$, indexed by available colors. Matrix product ansatz representations for stationary probabilities of stochastic interacting particle systems date back to [DEHP93]. In the single-species case, the stationary distribution on the ring is uniform, and so the Matrix Product Ansatz becomes nontrivial only for ASEP on an open interval, in which particles can hop in and out at the endpoints. This case was considered in [DEHP93]. For the two- and three-species ASEP on the ring, the matrix product approach was employed, respectively, in [DJLS93] and [MMR99]. A full multi-species solution on the ring appeared about ten years later in [PEM09]. See also [BE07] for an earlier survey of Matrix Product Ansatz applications to particle systems.

In the multi-species case, the matrices $\mathcal{X}_m^{\text{MPA}}$ entering the product ansatz are constructed by recursive tensoring from a few single-species building blocks A, D , and E satisfying quadratic relations

$$AD - qDA = EA - qAE = (1 - q)A, \quad ED - qDE = (1 - q)(E + D). \quad (4.13)$$

The matrices A, D, E , as well as the $\mathcal{X}_m^{\text{MPA}}$'s, are infinite-dimensional, and their products, as well as the trace of A times a finite product of D and E matrices must be well-defined. The tensoring construction of $\mathcal{X}_m^{\text{MPA}}$ resembles the process of horizontally stacking the vertices in columns $-n, -n+1, \dots, -1$ as in Figure 9. We refer to [PEM09, (24)–(33)] or [Mar20, Section 2] for details on the tensoring construction, and omit them here.

Once the matrix product probability distribution is defined through the trace in an appropriate space as

$$\text{Prob}_{N_1, \dots, N_n}^{\text{mASEP}}(\eta) = \frac{\text{Trace}(\mathcal{X}_{\eta_1}^{\text{MPA}} \dots \mathcal{X}_{\eta_N}^{\text{MPA}})}{Z_{N_1, \dots, N_n}^{\text{MPA}}}, \quad \eta = (\eta_1, \dots, \eta_N), \quad (4.14)$$

one must independently check that it is stationary under the mASEP dynamics. A key property in the argument is the existence of the so-called *hat matrices* $\hat{\mathcal{X}}_m^{\text{MPA}}$, $m = 0, 1, \dots, N$, satisfying quadratic relations [PEM09, (68)]:

$$\sum_{i, i'=0}^n \mathcal{X}_i^{\text{MPA}} \mathcal{X}_{i'}^{\text{MPA}} (\mathcal{M}_{loc})_{ii', jj'} = \mathcal{X}_j^{\text{MPA}} \hat{\mathcal{X}}_{j'}^{\text{MPA}} - \hat{\mathcal{X}}_j^{\text{MPA}} \mathcal{X}_{j'}^{\text{MPA}}, \quad (4.15)$$

where \mathcal{M}_{loc} are the local infinitesimal rates of the mASEP, see (4.3). These hat matrices are also constructed in [PEM09] by recursive tensoring procedures. Note that our notation differs from [PEM09] by a transposition (e.g. comparing (4.15) with formula (66) in [PEM09]). See also [AAMP12] for a more general family of matrices $\mathcal{X}_j^{\text{MPA}}$, $\hat{\mathcal{X}}_j^{\text{MPA}}$ satisfying (4.15).

Let us explain how the construction of the hat matrices, and identity (4.15) for vertex model partition functions directly follow from the Yang–Baxter equation. We take a concrete realization of the Matrix Product Ansatz matrices $\mathcal{X}_m^{\text{MPA}}$ using our vertex models.

Namely, let $\mathcal{X}_j^{\text{MPA}}(u)$, $j = 0, 1, \dots, n$, be operators in $V_{-n} \otimes \dots \otimes V_{-1}$, where V_{-m} has basis $|\mathbf{M}(-m)\rangle$, $\mathbf{M}(-m) \in \mathbb{Z}_{\geq 0}^n$, $m = 1, \dots, n$. By definition, the matrix elements of $\mathcal{X}_j^{\text{MPA}}$ are partition functions (with weights $\mathbb{W}_{q^{-1/2}, s, u}$) on the one-row lattice $\{1\} \times \{-n, \dots, -1\}$ with boundary conditions 0 and j on the left and right, respectively. See Figure 9 for an illustration.

$\begin{array}{c} \mathbf{A} \\ \\ 0 \text{ --- } \text{ --- } 0 \\ \\ \mathbf{A} \end{array}$	$\begin{array}{c} \mathbf{A} \\ \\ k \text{ --- } \text{ --- } k \\ \\ \mathbf{A} \end{array}$	$\begin{array}{c} \mathbf{A}_k^- \\ \\ 0 \text{ --- } \text{ --- } k \\ \\ \mathbf{A} \end{array}$	$\begin{array}{c} \mathbf{A} \\ \\ 0 \text{ --- } \text{ --- } m \\ \\ \mathbf{A} \end{array}$
1	$(u - sq^{A_k})q^{A_{[k+1,n]}}$	$u(1 - q^{A_k})q^{A_{[k+1,n]}}$	$uq^{A_{[m+1,n]}}$
$\begin{array}{c} \mathbf{A}_k^+ \\ \\ k \text{ --- } \text{ --- } 0 \\ \\ \mathbf{A} \end{array}$	$\begin{array}{c} \mathbf{A}_{k\ell}^{+-} \\ \\ k \text{ --- } \text{ --- } \ell \\ \\ \mathbf{A} \end{array}$	$\begin{array}{c} \mathbf{A}_{\ell k}^{+-} \\ \\ \ell \text{ --- } \text{ --- } k \\ \\ \mathbf{A} \end{array}$	$\begin{array}{c} \mathbf{A}_\ell^+ \\ \\ \ell \text{ --- } \text{ --- } m \\ \\ \mathbf{A} \end{array}$
1	$u(1 - q^{A_\ell})q^{A_{[\ell+1,n]}}$	$s(1 - q^{A_k})q^{A_{[k+1,n]}}$	$sq^{A_{[m+1,n]}}$

Figure 20: Table of weights $\mathbb{W}_{q^{-1/2}, s, uq^{1/2}}^{(-m), \text{mASEP}^+}$ (4.8) with common denominators $(1 - su)$ cleared, which are used in the construction of the matrices A, D, E satisfying (4.13).

Let $\epsilon > 0$ be small. By the Yang–Baxter equation (Proposition 2.10), the matrices $\mathcal{X}_i^{\text{MPA}}(u)$ and $\mathcal{X}_{i'}^{\text{MPA}}(u(1 - \epsilon))$ satisfy the following identity involving the elements of $R_{1-\epsilon}$ (2.4):

$$\sum_{i, i'=0}^n \mathcal{X}_i^{\text{MPA}}(u) \mathcal{X}_{i'}^{\text{MPA}}(u(1 - \epsilon)) \cdot R_{1-\epsilon}(i, i'; j', j) = \mathcal{X}_j^{\text{MPA}}(u(1 - \epsilon)) \mathcal{X}_{j'}^{\text{MPA}}(u). \quad (4.16)$$

As in the proof of Proposition 4.4, let us differentiate (4.16) with respect to ϵ at $\epsilon = 0$. Denote

$$\widehat{\mathcal{X}}_j^{\text{MPA}}(u) := (1 - q)u \frac{\partial}{\partial u} \mathcal{X}_j^{\text{MPA}}(u), \quad j = 0, 1, \dots, n.$$

Proposition 4.13. *The matrices $\mathcal{X}_i^{\text{MPA}}(u)$, $\widehat{\mathcal{X}}_j^{\text{MPA}}(u)$ defined above with the help of the queue vertex weights $\mathbb{W}_{q^{-1/2}, s, u}$ satisfy the hat matrix identity (4.15).*

Our realization of the matrices $\mathcal{X}_j^{\text{MPA}}$, $\widehat{\mathcal{X}}_j^{\text{MPA}}$ satisfying the hat relation (4.15) as vertex model partition functions highlights the Yang–Baxter structure of the Matrix Product Ansatz for the multispecies ASEP which was previously unknown.

Proof of Proposition 4.13. The ϵ -derivative at $\epsilon = 0$ of the right-hand side of (4.16) is equal to

$$-(1 - q)^{-1} \widehat{\mathcal{X}}_j^{\text{MPA}}(u) \mathcal{X}_{j'}^{\text{MPA}}(u).$$

In the left-hand side, differentiating $\mathcal{X}_{i'}^{\text{MPA}}$ and noticing that $R_1(i, i'; j', j) = \mathbf{1}_{i=j} \mathbf{1}_{i'=j'}$, we obtain $-(1 - q)^{-1} \mathcal{X}_j^{\text{MPA}}(u) \widehat{\mathcal{X}}_{j'}^{\text{MPA}}(u)$. This yields the second summand in the right-hand side of (4.15). Finally, differentiating $R_{1-\epsilon}$ and using (4.6), we recover the local infinitesimal rates of mASEP, also multiplied by $(1 - q)^{-1}$. This completes the proof. \square

Let us also note that one can “recognize” the matrix product building blocks A, D, E in the vertex weights of the queue vertex model on the cylinder. This allows to insert two extra

parameters into the matrices. Namely, in [Mar20], two examples of A, D, E are presented, for the original and for the alternative multiline queue models discussed in Section 4.3 above. Both of these examples satisfy the relations (4.13). In these two examples, the matrices A, D, E are equal to the $n = 2$ species versions of the corresponding \mathcal{X}^{MPA} matrices. Informed by this, and now keeping track of the parameters u and s in the vertex weights, let us define

$$\begin{aligned}
 A &:= \begin{pmatrix} 1 & s & 0 & \dots \\ 0 & q & qs & \dots \\ 0 & 0 & q^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, & D &:= u^{-1} \begin{pmatrix} u-s & 0 & 0 & \dots \\ 1-q & u-sq & 0 & \dots \\ 0 & 1-q^2 & u-sq^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\
 E &:= \begin{pmatrix} 1 & u & 0 & \dots \\ 0 & 1 & u & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
 \end{aligned} \tag{4.17}$$

These matrices can be obtained as one-row partition functions of the two-color queue-specialized vertex model as shown in Figure 21. The relevant vertex weights $\mathbb{W}_{q^{-1/2}, s, uq^{1/2}}^{(-m), \text{mASEP}_+}$ are shown in Figure 20. We use the same parameter u in both columns.

Figure 21: The (M, M') matrix elements of the matrices A, D, E are obtained from the queue-specialized vertex model. The symbol ∞_2 represents $(0, +\infty)$ (infinitely many color 2 paths, and no color 1 paths). Similarly, (∞, M) means a path configuration with M color 2 paths and $+\infty$ color 1 paths.

One can directly check that the matrices (4.17) satisfy (4.13) for all u, s . Alternatively, relations (4.13) can be obtained from the Yang–Baxter equation, satisfied by $A, u^2 D, E$ and the R matrix, in the same way as described in the proof of Proposition 4.13. When $u = 1$ and $s = 0$, the matrices (4.17) become the matrix product building blocks of [PEM09], and lead to the original multiline queue model [Mar20, (2.5)]. For $u = 1$ and $s = q$, they correspond to the alternative model from [Mar20, Section 7]. However, like for the matrices A, D, E for the alternative model, it is not clear how to build product ansatz matrices for mASEP from (4.17) by recursive tensoring in a way analogous to [PEM09]. We do not pursue this question here.

We remark that matrices similar to (4.17) have occurred in the product ansatz for the open ASEP on a bounded interval (for example, see [DEHP93]). There, the extra parameters are tied to the boundary rates.

5 Colored stochastic q -Boson process from straight cylinder

In this section we consider a specialization of the straight cylinder Markov transition operator leading to the colored stochastic q -Boson process on the ring [Tak15], [BW22a, Section 12.4]. It is also called the multi-species totally asymmetric zero range process (mTAZRP) in [AMM22]. The corresponding specialization of the queue vertex model will allow us to recover the stationary distribution of the q -Boson process. In the case of at most one particle of each color, we also match path configurations in the vertex model representing this stationary distribution to states of a multiline queue considered in [AMM22, Section 8].

5.1 Colored q -Boson process on the ring

Let us fix the size of the ring N and the number of colors n . Also let us fix the type counts (N_1, \dots, N_n) , where $N_i \geq 1$ stands for the number of particles of color i in the system. The state space of the colored stochastic q -Boson process consists of configurations of particles at sites of the ring, where at each site there can be an arbitrary number of particles. The configurations are encoded by

$$\mathbf{V} = (\mathbf{V}(1), \dots, \mathbf{V}(N)), \quad \mathbf{V}(j) \in \mathbb{Z}_{\geq 0}^n.$$

Here $\mathbf{V}(j)_i$ denotes the number of particles of color i at site j , and $N_i = \sum_{j=1}^N \mathbf{V}(j)_i$.

Definition 5.1. The *stochastic colored q -Boson process* depends on parameters $q \in [0, 1)$ and $u_1, \dots, u_N > 0$, and evolves in continuous time as follows. A particle of color i hops from site k to site $k - 1$ (cyclically mod N) according to an independent exponential clock with rate

$$u_k^{-1} (1 - q^{\mathbf{V}(k)_i}) q^{\mathbf{V}(k)_{[i+1, n]}}.$$

Here we used the usual notation $\mathbf{V}(k)_{[i+1, n]} = \sum_{r=i+1}^n \mathbf{V}(k)_r$. Denote by $\mathfrak{P}_{\text{qBos}}(t)$, $t \in \mathbb{R}_{\geq 0}$, the continuous time Markov semigroup of this stochastic process.

The colored q -Boson process evolution is of *zero range* kind, that is, the jump from site k depends only on the state of the system at site k . In [AMM22] it is referred to as the *multi-species totally asymmetric zero range process*, or *mTAZRP*.

The q -Boson process preserves the type counts (N_1, \dots, N_n) . For a fixed vector of type counts, this continuous time Markov chain evolves on a finite state space and is clearly irreducible. Thus, it has a unique stationary distribution. We denote it by $\text{Prob}_{N_1, \dots, N_n}^{\text{qBos}}(\mathbf{V})$.

Following [BW22a, Section 12.4.3], we can identify $\mathfrak{P}_{\text{qBos}}(t)$ as a certain Poisson-type continuous time limit of the straight cylinder formal Markov operator $\mathfrak{S}(x, \mathbf{u}; \mathbf{s}^{(h)})$ defined in Section 3.2. Recall that \mathfrak{S} has $N + 1$ sites on the ring. However, in the degeneration to the q -Boson process, the distinguished site corresponding to spectral parameter x (cf. Figure 12) will be empty with probability 1, and the dynamics can be restricted to N sites.

Fix small $\epsilon > 0$, and set the horizontal spin parameters of \mathfrak{S} to $s_j^{(h)} = \epsilon$. Also let $x = -1$. With this specialization, the matrix elements of the operators $\mathcal{L}_{s_j^{(h)}, x u_j^{-1}}$ entering the definition of \mathfrak{S} (formula (3.6), see also Figure 5) become as in Figure 22. The operator \mathfrak{S} is a formal Markov operator acting on states of the form (c, \mathbf{V}) , where $\mathbf{V} \in \mathbb{Z}_{\geq 0}^n$ is a state of the color q -Boson process, and $c \in \{0, 1, \dots, n\}$ corresponds to the auxiliary line (i.e., the one with the spectral parameter x and the spin parameter $q^{-1/2}$; see Figure 12).

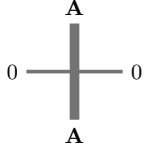
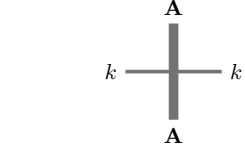
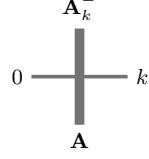
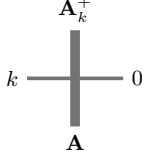
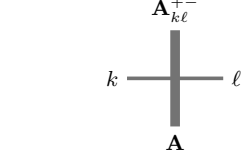
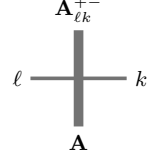
 $1 + O(\epsilon)$	 $\epsilon u^{-1} q^{A_{[k+1, n]}} + O(\epsilon^2)$	 $\epsilon u^{-1} (1 - q^{A_k}) q^{A_{[k+1, n]}} + O(\epsilon^2)$
 $1 + O(\epsilon)$	 $\epsilon u^{-1} (1 - q^{A_\ell}) q^{A_{[\ell+1, n]}} + O(\epsilon^2)$	 $O(\epsilon^2)$

Figure 22: The weights $\mathcal{L}_{\epsilon, -u_j^{-1}}$ employed in the approximation of the colored q -Boson process, Taylor expanded to $O(\epsilon^2)$ or $1 + O(\epsilon)$ depending on whether they go to 0 or 1 as $\epsilon \rightarrow 0$. Here, as usual, $1 \leq k < \ell \leq n$.

Proposition 5.2. Fix $t \in \mathbb{R}_{\geq 0}$. With the parameter specialization as above, have

$$\lim_{\epsilon \rightarrow 0} \langle 0, \mathbf{V} | \mathfrak{S}(-1, \mathbf{u}; (\epsilon, \dots, \epsilon))^{\lfloor t/\epsilon \rfloor} | 0, \mathbf{V}' \rangle = \langle \mathbf{V} | \mathfrak{P}_{\text{qBos}}(t) | \mathbf{V}' \rangle, \quad \mathbf{V}, \mathbf{V}' \in \mathbb{Z}_{\geq 0}^n.$$

Moreover, for any $c \geq 1$ we have $\lim_{\epsilon \rightarrow 0} \langle 0, \mathbf{V} | \mathfrak{S}(-1, \mathbf{u}; (\epsilon, \dots, \epsilon))^{\lfloor t/\epsilon \rfloor} | c, \mathbf{V}' \rangle = 0$.

Not all matrix elements of \mathfrak{S} are nonnegative before the $\epsilon \rightarrow 0$ limit. This is not a problem because the limiting semigroup $\mathfrak{P}_{\text{qBos}}(t)$ is a nonnegative Markov semigroup, and the stationarity result (which we prove in Proposition 5.5 below) is a purely algebraic statement.

Proof of Proposition 5.2. Both statements follow from the expansions in Figure 22, after the identification of the vertices in the cylinder (in Figure 12) with the stochastic q -Boson transitions via

$$\begin{array}{c} k \\ \swarrow \\ \mathbf{A} \longrightarrow \mathbf{C} \\ \searrow \\ \ell \end{array} = k \begin{array}{c} \mathbf{C} \\ \uparrow \\ \mathbf{A} \end{array} \longrightarrow \ell \quad ,$$

where in the right-hand side the time is continuous and increases in the upward direction.

We see that the auxiliary line may become occupied with probability $O(\epsilon)$, and then instantaneously becomes free again with probability $1 + O(\epsilon)$. This means that the vertices of the type $(\mathbf{A}, k; \mathbf{A}, k)$ are not present in the limit. All other probabilities of order $O(\epsilon)$ in Figure 22 give rise to the corresponding colored q -Boson transitions, which leads to the first claim.

To get the second claim, observe that with probability going to 1 in the limit as $\epsilon \rightarrow 0$, the auxiliary line is not occupied. This completes the proof. \square

5.2 Vertex model for the q -Boson stationary distribution

The convergence of Proposition 5.2 together with the general stationarity result (Theorem 3.6) allows us to express the stationary distribution $\text{Prob}_{N_1, \dots, N_n}^{\text{qBos}}(\mathbf{V})$ of the colored q -Boson process as a vertex model partition function.

To get a queue vertex model on the cylinder with nonnegative vertex weights, we take a certain limit in the vertical parameters $\mathbf{v}, \mathbf{s}^{(v)}$. As a first step, let us consider the following degeneration of the queue vertex weights $\mathbb{W}_{s_1, s_2, u}^{(-m)}$ (2.12):

Lemma 5.3. *We have*

$$\begin{aligned} (u; q)_\infty^{-1} \cdot \lim_{s_1 \rightarrow 0} \mathbb{W}_{s_1, s, us_1/s}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \cdot \mathbf{1}_{D_1=\dots=D_{m-1}=0} \\ &\times \sum_{\mathbf{P}} (s^2/u; q)_{|\mathbf{P}|} (s^2)^{|\mathbf{B}|-|\mathbf{P}|} u^{|\mathbf{D}|-|\mathbf{B}||\mathbf{P}|} q^{\sum_{1 \leq i < j \leq n} (B_i - P_i) P_j} \prod_{i=1}^n \frac{(q; q)_{B_i}}{(q; q)_{P_i} (q; q)_{B_i - P_i}} \\ &\times q^{\sum_{m \leq i < j \leq n} D_i (C_j - P_j)} \frac{1}{(q; q)_{D_m}} \prod_{i=m+1}^n \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}}, \end{aligned} \quad (5.1)$$

where the sum is over $\mathbf{P} \in \mathbb{Z}_{\geq 0}^n$ with $0 \leq P_i \leq \min(B_i, C_i)$ for all i , and

$$\begin{aligned} (u; q)_\infty^{-1} \cdot \lim_{s \rightarrow 0} \left(\lim_{s_1 \rightarrow 0} \mathbb{W}_{s_1, s, us_1/s}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \right) &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \cdot \mathbf{1}_{D_1=\dots=D_{m-1}=0} \cdot \prod_{i=1}^n \mathbf{1}_{B_i \leq C_i} \\ &\times u^{|\mathbf{D}|} q^{\sum_{m \leq i < j \leq n} D_i (C_j - B_j)} \frac{1}{(q; q)_{D_m}} \prod_{i=m+1}^n \frac{(q; q)_{C_i - B_i + D_i}}{(q; q)_{C_i - B_i} (q; q)_{D_i}}. \end{aligned} \quad (5.2)$$

Proof. We have from (2.12):

$$\begin{aligned} \mathbb{W}_{s_1, s, us_1/s}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \cdot \mathbf{1}_{D_1=\dots=D_{m-1}=0} \cdot \frac{(u; q)_\infty}{(s_1^2 u; q)_\infty} \\ &\times \sum_{\mathbf{P}} \frac{(s^2/u; q)_{|\mathbf{P}|} (s_1^2 u/s^2; q)_{|\mathbf{B}-\mathbf{P}|}}{(s_1^2; q)_{|\mathbf{B}|}} q^{\sum_{1 \leq i < j \leq n} (B_i - P_i) P_j} \prod_{i=1}^n \frac{(q; q)_{B_i}}{(q; q)_{P_i} (q; q)_{B_i - P_i}} \\ &\times (s^2)^{|\mathbf{B}|-|\mathbf{P}|} u^{|\mathbf{D}|-|\mathbf{B}||\mathbf{P}|} q^{\sum_{m \leq i < j \leq n} D_i (C_j - P_j)} \frac{(s_1^2; q)_{|\mathbf{D}|}}{(q; q)_{D_m}} \prod_{i=m+1}^n \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}}. \end{aligned}$$

Sending $s_1 \rightarrow 0$ immediately leads to (5.1). Further letting $s \rightarrow 0$, we see that $\mathbf{P} = \mathbf{B}$, for otherwise the factor $(s^2)^{|\mathbf{B}|-|\mathbf{P}|}$ vanishes. This eliminates the summation over \mathbf{P} and produces the desired expression (5.2) together with the indicator that $B_i \leq C_i$ for all i . \square

We denote the right-hand side of (5.1) by $\mathbb{W}_{s, u}^{(-m), \text{qBos}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$. The right-hand side of (5.2) is the $s = 0$ degeneration of (5.1). The weights $\mathbb{W}_{s, u}^{(-m), \text{qBos}}$ are nonnegative for $q \in [0, 1)$ and $u > s^2 \geq 0$.

Definition 5.4. Fix parameters $\mathbf{u} = (u_1, \dots, u_N)$, $\mathbf{y} = (y_1, \dots, y_n)$, and $\mathbf{s}^{(v)} = (s_1^{(v)}, \dots, s_n^{(v)})$ such that

$$0 \leq (s_m^{(v)})^2 < u_i y_m, \quad i = 1, \dots, N, \quad m = 1, \dots, n. \quad (5.3)$$

Let $\mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y}; \mathbf{s}^{(v)})$ denote the queue transfer matrix on the $n \times N$ cylinder as in Figure 8, where the vertex weight at each site $(-m, j)$ is $\mathbb{W}_{s_m^{(v)}, u_j y_m}^{(-m), \text{qBos}}$.

The vertex model of Definition 5.4 has nonnegative weights. Note that its partition functions $\langle \emptyset | \mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y}; \mathbf{s}^{(v)}) | \mathbf{V} \rangle$ involve infinite sums over paths winding around the cylinder. Similarly to Lemma 2.13, we see that these sums are convergent when \mathbf{V} has all type counts N_i , $i = 1, \dots, n$, at most 1.

Proposition 5.5. *For any type counts (N_1, \dots, N_n) , $N_i \geq 1$, and the parameters $\mathbf{y}, \mathbf{s}^{(v)}$ satisfying (5.3), the stationary distribution of the colored q -Boson process with parameters \mathbf{u} has the form*

$$\text{Prob}_{N_1, \dots, N_n}^{\text{qBos}}(\mathbf{V}) = \frac{\langle \emptyset | \mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y}; \mathbf{s}^{(v)}) | \mathbf{V} \rangle}{Z_{N_1, \dots, N_n}^{\text{qBos}}}. \quad (5.4)$$

The normalizing constant $Z_{N_1, \dots, N_n}^{\text{qBos}}$ depends on the parameters and the type counts, but not on the state \mathbf{V} within the sector determined by (N_1, \dots, N_n) .

Proof. We use Theorem 3.6 (in particular, recall the queue vertex model on the cylinder interacting with the straight cylinder Markov operator as illustrated in Figure 13). Let us choose the parameters of the queue vertex model

$$\mathfrak{Q} = \mathfrak{Q} \left((xq^{\frac{1}{2}}, \mathbf{u}); (q^{-\frac{1}{2}}, \mathbf{s}^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right)$$

as

$$x = -1, \quad s_j^{(h)} = \epsilon \rightarrow 0, \quad v_m = \frac{s_m^{(v)}}{\epsilon y_m}$$

for all $1 \leq j \leq N$, $1 \leq m \leq n$. By Lemma 5.3, sending $\epsilon \rightarrow 0$ turns the weight at each site $(-m, j)$ of this queue vertex model on the cylinder into $(u_j y_m; q)_\infty \mathbb{W}_{s_m^{(v)}, u_j y_m}^{(-m), \text{qBos}}$. The overall factor $\prod_{j=1}^N \prod_{m=1}^n (u_j y_m; q)_\infty$ is absorbed into the normalizing constant, and thus we can ignore it.

At the sites $(-m, 0)$, before the limit we have the weights $\mathbb{W}_{q^{-1/2}, s_m^{(v)}, -\epsilon q^{1/2} y_m / v_m^{(v)}}^{(-m)}$. Up to reparametrization, these are the same weights as in Figure 15. Sending $\epsilon \rightarrow 0$ (that is, $-su \rightarrow 0$ in the notation Figure 15), we see that

$$\mathbb{W}_{q^{-1/2}, s_m^{(v)}, -\epsilon q^{1/2} y_m / v_m^{(v)}}^{(-m)}(\mathbf{A}, 0; \mathbf{A}, 0) \rightarrow 1, \quad \mathbb{W}_{q^{-1/2}, s_m^{(v)}, -\epsilon q^{1/2} y_m / v_m^{(v)}}^{(-m)}(\mathbf{A}, 0; \mathbf{A}_k^-, k) \rightarrow 0.$$

Since the auxiliary line (i.e., the one with the spin parameter $q^{-1/2}$) begins as initially unoccupied, these convergences imply that this auxiliary line remains unoccupied in the cylindrical queue vertex model under this limit (more specifically, any configuration in which it is occupied has weight going to 0 as $\epsilon \rightarrow 0$). Therefore, we can remove this auxiliary line from the model on the cylinder as follows:

$$\lim_{\epsilon \rightarrow 0} \langle 0, \emptyset | \mathfrak{Q} \left((xq^{\frac{1}{2}}, \mathbf{u}); (q^{-\frac{1}{2}}, \mathbf{s}^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right) | 0, \mathbf{V} \rangle = \langle \emptyset | \mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y}; \mathbf{s}^{(v)}) | \mathbf{V} \rangle, \quad (5.5)$$

where $\mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y})$ is defined before the proposition.

Arguing as in the proof of Proposition 4.5, we can take the limit as $\epsilon \rightarrow 0$ simultaneously in the queue vertex model and in the straight cylinder Markov operator. Before the limit, these operators satisfy the general stationarity relation of Theorem 3.6. By Proposition 5.2, the straight cylinder Markov operator converges as $\epsilon \rightarrow 0$ (in the Poisson-type continuous time limit) to the Markov semigroup $\mathfrak{P}_{\text{qBos}}(t)$. The limit of the general stationarity relation yields

$$\begin{aligned} & \langle \emptyset | \mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y}; \mathbf{s}^{(v)}) \mathfrak{P}_{\text{qBos}}(t) | \mathbf{V} \rangle \\ &= \lim_{\epsilon \rightarrow 0} \langle 0, \emptyset | \mathfrak{Q} \left((xq^{\frac{1}{2}}, \mathbf{u}); (q^{-\frac{1}{2}}, \mathbf{s}^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right) \mathfrak{S}(-1, \mathbf{u}; (\epsilon, \dots, \epsilon))^{\lfloor t/\epsilon \rfloor} | 0, \mathbf{V} \rangle \\ &= \lim_{\epsilon \rightarrow 0} \langle 0, \emptyset | \mathfrak{Q} \left((xq^{\frac{1}{2}}, \mathbf{u}); (q^{-\frac{1}{2}}, \mathbf{s}^{(h)}); \mathbf{v}; \mathbf{s}^{(v)} \right) | 0, \mathbf{V} \rangle \\ &= \langle \emptyset | \mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{y}; \mathbf{s}^{(v)}) | \mathbf{V} \rangle. \end{aligned}$$

Here, the first equality holds by Proposition 5.2 and equation (5.5); the second holds by Theorem 3.6; and the third holds again by (5.5).

This completes the proof. \square

Remark 5.6. While the quantities in the right-hand side of (5.4) seem to depend on \mathbf{y} and $\mathbf{s}^{(v)}$, Proposition 5.5 implies that they are independent of these extra parameters. This observation is parallel to the mASEP situation (see Remark 4.6).

5.3 Matching to multiline queues

In [AMM22, Section 8], a multiline queue model for the stationary distribution of the colored q -Boson process is presented. Let us match this model to our queue vertex model $\mathfrak{Q}^{\text{qBos}}$ on the cylinder, specialized to $s_m^{(v)} = 0$, $1 \leq m \leq n$ (that is, with the simpler product-form weights (5.2)).

As in [AMM22], we restrict our attention to the simpler *strict* case when, by definition, there is at most one particle of each color. First, we recall the definition of a q -Boson multiline queue and its weight. We replace the parameter t from [AMM22] by our q , and adjust the notation of integer indices, spectral parameters, and the direction of the ring to match the conventions used throughout our paper.

Definition 5.7 ([AMM22]). A multiline diagram is an assignment of the labels from $\{1, \dots, n\}$ to the vertices of a cylinder $\{-n, \dots, -1\} \times (\mathbb{Z}/N\mathbb{Z})$, satisfying

- Each vertex $(-m, j)$ is assigned a multiset of labels.
- In column $(-m)$, all labels are from $\{m, m+1, \dots, n\}$.
- The combined multiset of all labels in column $(-m)$ is obtained from the multiset of labels in column $-(m+1)$, together with some new labels of type m .
- (*strict condition*) Each label m , $1 \leq m \leq n$, appears at most once in each of the columns $-m, -(m-1), \dots, -1$.

The weight of a multiline diagram is, by definition, $q^{\mathfrak{R}} u_1^{c_1} \dots u_N^{c_N}$, where c_j is the total number of labels assigned to the row j , and \mathfrak{R} is the *refusal statistic* defined as follows. Let

$$R_m := \sum_{m-1 \leq k < \ell \leq n} \mathbf{1}_{p_\ell(-m-1) > p_k(-m-1) > p_\ell(-m)}, \quad (5.6)$$

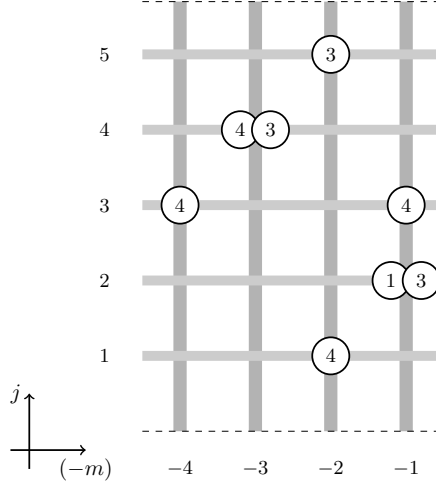


Figure 23: A multiline diagram (Definition 5.7) with weight $q^3 u_1 u_2^2 u_3^2 u_4^2 u_5$. Here the refusal statistic 3 combines $R_3 = 1$ (label 3 in column -2 is “between” the positions of labels 4 in columns -3 and -2 , in the sense described after Equation (5.6)) and $R_2 = 2$ (labels 1 and 3 in column -1 are between the labels 4 in columns -2 and -1). This is the same diagram as in examples in [AMM22, Section 8], but rotated by 90° and with the direction of the ring reversed (to match our vertex model). Here the size of the ring is $N = 5$, and the number of colors is $n = 4$.

where $p_r(-m)$ is the position of the label r in column $-m$, and the event $a > b > c$ means that, reading along the ring in the downward direction (corresponding to decreasing positions j), the label b is strictly between a and c . This includes the case $a = c \neq b$; what this means for a corresponding term in the sum is $p_\ell(-m) = p_\ell(-m - 1)$, and we think of this as p_ℓ making a full loop around the ring to get to its position at $-m - 1$. Then we set $\mathfrak{R} := \sum_{m=2}^n R_m$. See Figure 23 for an illustration.

Given a multiline diagram, associate to it a path configuration on the cylinder with vertex weights $\mathbb{W}_{u_j}^{(-m), \text{qBos}}$ at each vertex $(-m, j)$, and such that the multiset of labels at $(-m, j)$ is exactly the colors of the paths exiting this vertex. Recall that we usually denote the latter multiset of colors by $\mathbf{D} \in \mathbb{Z}_{\geq 0}^n$. Knowing \mathbf{D} at each vertex is enough to reconstruct the whole path configuration on the cylinder, up to unknown windings of paths around the cylinder. In this way, one multiline diagram corresponds to many configurations of the queue vertex model $\mathfrak{Q}^{\text{qBos}}$ on the cylinder.

Proposition 5.8. *Let there be exactly one particle of each color m , $m = 1, \dots, n$. Then the mapping between multiline diagrams and configurations of the queue vertex model $\mathfrak{Q}^{\text{qBos}}(\mathbf{u}; \mathbf{1}; \mathbf{0})$ (that is, $y_i = 1$ and $s_i^{(v)} = 0$ for all i) described before the proposition is weight-preserving. That is, the sum of weights of all vertex model configurations over the winding of the paths around the cylinder is proportional to the weight of the corresponding multiline diagram. The proportionality constant depends on the parameters of the model, but not on the particle configuration.*

If there are no particles of some color, then the sum of the vertex model weights might diverge,

cf. Remark 2.14. On the other hand, we consider the multiline queues for at most one particle of each color. This leads to the restriction in Proposition 5.8.

Proof of Proposition 5.8. It suffices to fix m and consider the behavior in the column $(-m)$. For a configuration of the queue vertex model in this column let the arrow configurations at each vertex $(-m, k)$ be $\mathbf{A}^{(k)}, \mathbf{B}^{(k)}, \mathbf{C}^{(k)}, \mathbf{D}^{(k)}$. The corresponding multiline diagram contains information about $\mathbf{B}^{(k)}, \mathbf{D}^{(k)}$, but not about $\mathbf{A}^{(k)}, \mathbf{C}^{(k)}$. Let us fix $\mathbf{B}^{(k)}, \mathbf{D}^{(k)}$ for all $k = 1, \dots, N$, and sum over $\mathbf{A}^{(k)}, \mathbf{C}^{(k)}$, $k = 1, \dots, N$. The resulting sum must be equal to the weight of column $-m$ in the corresponding multiline diagram.

With this data fixed, out of all allowed configurations of the vertices in column $(-m)$, there is one in which $C_j^{(N)}$ is minimal for each $j > m$. Fixing $\mathbf{C}^{(N)}$ allows one to reconstruct the whole vertex model configuration in column $(-m)$ in a unique way. Denote this minimal configuration by $\mathbf{C}^{(k), \min}$, and let $E_j^{(k)} := C_j^{(k), \min} - B_j^{(k)}$, $j = m+1, \dots, n$.

The product of the vertex weights (5.2) in column $(-m)$, summed over all allowed configurations, is proportional to (using the fact that $D_j^{(k)} \in \{0, 1\}$ for all j, k)

$$\begin{aligned} & \sum_{a_{m+1}=0}^{\infty} \cdots \sum_{a_n=0}^{\infty} \prod_{k=1}^N \left(u_k^{|\mathbf{D}^{(k)}|} q^{\sum_{m \leq r < s \leq n} D_r^{(k)} (E_s^{(k)} + a_s)} \prod_{j=m+1}^n \frac{(q; q)_{a_j + E_j^{(k)} + D_j^{(k)}}}{(q; q)_{a_j + E_j^{(k)}} (q; q)_{D_j^{(k)}}} \right) \\ &= \left(\prod_{k=1}^N u_k^{|\mathbf{D}^{(k)}|} \right) \sum_{a_{m+1}=0}^{\infty} \cdots \sum_{a_n=0}^{\infty} \left(\prod_{k=1}^N q^{\sum_{m \leq r < s \leq n} D_r^{(k)} (E_s^{(k)} + a_s)} \right) \left(\prod_{j=m+1}^n \prod_{k: D_j^{(k)}=1} \frac{1 - q^{1+a_j + E_j^{(k)}}}{1 - q} \right). \end{aligned} \quad (5.7)$$

Observe that $B_j^{(k)} \leq C_j^{(k)}$ for all j and k (see (5.2)). This implies (by arrow conservation, since $A_j^{(k)}, B_j^{(k)}, C_j^{(k)}, D_j^{(k)} \in \{0, 1\}$) that if $D_j^{(k)} = 1$, then $E_j^{(k)} = 0$. For each $j = m+1, \dots, n$, we either have $D_j^{(k)} = 0$ or $D_j^{(k)} = 1$, and there exists exactly one $k = k_j$ for which $D_j^{(k)} = 1$. Thus, inside the summations we have

$$\left(\prod_{k=1}^N \prod_{j=m+1}^n q^{\sum_{m \leq r < j} D_r^{(k)} (E_j^{(k)} + a_j)} \right) \left(\prod_{j=m+1}^n \frac{1 - q^{1+a_j}}{1 - q} \right).$$

As a result, the sum over a_j becomes

$$q^{\sum_{k=1}^N (D_m^{(k)} + \dots + D_{j-1}^{(k)}) E_j^{(k)}} \sum_{a_j=0}^{\infty} q^{a_j \sum_{k=1}^N (D_m^{(k)} + \dots + D_{j-1}^{(k)})} \frac{1 - q^{1+a_j}}{1 - q} = C_{N_1, \dots, N_n}^{[j]} q^{\sum_{k=1}^N (D_m^{(k)} + \dots + D_{j-1}^{(k)}) E_j^{(k)}},$$

where $C_{N_1, \dots, N_n}^{[j]}$ does not depend on the particular multiline diagram but only on the type counts (N_1, \dots, N_n) . Indeed, $\sum_{k=1}^N (D_m^{(k)} + \dots + D_{j-1}^{(k)})$ is the total number of colors i , $m \leq i \leq j-1$, leaving column $(-m)$. Thus, we can continue

$$(5.7) = C_{N_1, \dots, N_n} \left(\prod_{k=1}^N u_k^{|\mathbf{D}^{(k)}|} \right) \prod_{k=1}^N q^{\sum_{m \leq i < j \leq n} D_i^{(k)} E_j^{(k)}}, \quad (5.8)$$

where C_{N_1, \dots, N_n} also depends only on the type counts. Note that $D_i^{(k)}, E_j^{(k)} \in \{0, 1\}$. One can readily verify that each pair $m \leq i < j \leq n$ such that $D_i^{(k)} = E_j^{(k)} = 1$ corresponds to an indicator equal to one in the definition of R_m (5.6). In particular, note that the indicator $\mathbf{1}_{B_j \leq C_j}$ in the weights (5.2) prevents a path from passing straight through without any winding. This behavior is accounted for in (5.7), and corresponds to the fact that the case $a = c \neq b$ counts towards the refusal statistic \mathfrak{R} (see the discussion after its definition (5.6)). Thus, the power of q in (5.8) is exactly the same as the component R_m of the refusal statistic \mathfrak{R} . The powers of the u_j 's also match the ones for the multiline diagrams. This completes the proof. \square

Let us make two final remarks in this section. First, [AMM22] does not explicitly define the weights of general (not necessarily strict) multiline queue diagrams. We may use $\Omega^{\text{qBos}}(\mathbf{u}; \mathbf{1}; \mathbf{0})$ and sum over the winding of the paths around the cylinder to define weights of general multiline diagrams. We conjecture that these weights coming from the vertex model should directly correspond to the tableau process of [AMM22, Section 4], but do not check this here.

Second, the Yang–Baxter equation for the queue vertex model (Proposition 2.10) should allow to directly show the symmetry of the stationary distribution in the parameters u_j . More precisely [AMM22, Proposition 7.2], for any K , the distribution of the configuration at sites $\{1, \dots, K\}$ of the ring is symmetric in the parameters u_{K+1}, \dots, u_N . Moreover, using the Yang–Baxter equation and couplings similarly to [PS22], it should be possible to establish the stronger symmetry of the distributions of the whole trajectories of the colored q -Boson system. This stronger property is proven only for $q = 0$ [AMM22, Theorem 7.14]. We leave these two questions for future work.

6 Colored q -PushTASEP from straight cylinder

This section considers another specialization of the straight cylinder Markov transition operator leading to the colored q -PushTASEP. We also present a vertex model on the cylinder producing its stationary distribution. Our argument here is very similar to Sections 4 and 5 above. The colored q -PushTASEP is a degeneration of the colored stochastic higher spin six-vertex model and was introduced in [BW22a, Section 12.5].

Throughout the section, we assume that $q \in (0, 1)$ and fix a positive integer P . As usual, let N be the size of the ring, and n be the number of colors. The colored q -PushTASEP depends on positive parameters $\mathbf{u} = (u_1, \dots, u_n)$.

Definition 6.1. The state space of the colored q -PushTASEP is the set of particle configurations on the ring. At any site, there can be at most P particles. Particles of the same color are indistinguishable. Let V_P be the vector space with the basis $|\mathbf{V}\rangle$, where $\mathbf{V} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{V}| \leq P$. The states of the colored q -PushTASEP can be identified with the basis vectors of $V_P^{\otimes N}$. The q -PushTASEP evolves in continuous time as follows. Let $\mathbf{A} \in \mathbb{Z}_{\geq 0}^n$ be the configuration of particles at a site k . For each $j = 1, \dots, n$, a particle of type j *activates* and instantaneously leaves the site k (moving toward $k + 1$) with the rate $u_k^{-1}(q^{-A_j} - 1)q^{P - A_{[j+1, n]}}$.

The active particle triggers other instantaneous updates of the configuration according to the following rules. Let $\mathbf{B} \in \mathbb{Z}_{\geq 0}^n$ be the configuration of particles at a site k' . Suppose that an activated particle of type c arrives at k' . Then the following happens:

- It deactivates and stays at k' with probability $1 - q^{P - |\mathbf{B}|}$, then the update ends.

- It deactivates and stays at k' , but causes the activation of another particle from k' (which then moves towards $k' + 1$) of type $d < c$ with probability $(q^{-B_d} - 1)q^{P-B_{[d+1,n]}}$, and the update continues.
- It remains active and moves on to site $k' + 1$ with probability $q^{P-B_{[c,n]}}$, and the update continues.

All particle moves from j to $j + 1$ are considered cyclically mod N . Denote the Markov semigroup of the colored q -PushTASEP by $\mathfrak{P}_{q\text{Push}}(t)$, $t \in \mathbb{R}_{\geq 0}$.

As usual, by (N_1, \dots, N_n) we denote the type counts in the configuration, which are preserved by the q -PushTASEP dynamics. When restricted to a sector determined by (N_1, \dots, N_n) , the colored q -PushTASEP is an irreducible continuous time Markov chain on a finite state space. Therefore, it admits a unique stationary distribution which we denote by $\text{Prob}_{N_1, \dots, N_n}^{q\text{Push}}(\mathbf{V})$.

Remark 6.2 (Frog model). Let us discuss the most degenerate version of the colored q -PushTASEP, namely, when $q = 0$, $P = 1$, and $u_k = 1$ for all k . In this case, each particle at any site k can be activated at rate 1, and moves from k to $k + 1$. Then the instantaneous update proceeds as follows:

- If an active particle arrives at an empty site, it deactivates and stays there, and the update ends.
- If a particle of type c arrives at a site k' with an existing particle of type $d < c$, then the type c particle stays at k' and displaces the type d particle, which now becomes active.
- Finally, if a particle of type c arrives at a site k' with an existing particle of type $d \geq c$, then the type c particle moves through to site $k' + 1$. The update continues with the type c particle.

This process is a particular case of the *frog model* [BC22] related to the problem of the longest common subsequence of a random and a periodic word. Our particular case corresponds to the periodic word with all letters distinct. More general periodic words lead to the simultaneous activation of particles at several sites. The stationary distribution of the frog model was constructed (in the particular case of distinct letters) in [BC22, Section 4].

The colored q -PushTASEP is a degeneration of the straight cylinder formal Markov operator. Thus, its stationary distribution is accessible through the corresponding limit transition from the queue vertex model on the cylinder. These limits are very similar to the q -Boson case (Sections 5.1 and 5.2), so we will only provide pictorial illustrations and brief explanations.

A queue vertex model leading to the q -PushTASEP stationary distribution must have finite spin rows (with the horizontal spin parameters $q^{-P/2}$). We reverse the direction of the straight cylinder operator to match the direction of the particle jumps from k to $k + 1$ (opposite from the q -Boson case). That is, consider the queue vertex model

$$\Omega \left((u_1, \dots, u_N, u_0); (q^{-P/2}, \dots, q^{-P/2}, q^{-1/2}); (v_1, \dots, v_m); (s_1^{(v)}, \dots, s_n^{(v)}) \right) \quad (6.1)$$

with $N + 1$ sites on the ring indexed by $j = 1, \dots, N, 0$. Let the distinguished auxiliary line with $j = 0$ be at the bottom; see Figure 26. In (6.1), the vertex weights at the sites $(-m, j)$, $1 \leq j \leq N$, and at $(-m, 0)$ are, respectively,

$$\mathbb{W}_{q^{-P/2}, s_m^{(v)}, u_j/v_m}^{(-m)} \quad \text{and} \quad \mathbb{W}_{q^{-1/2}, s_m^{(v)}, u_0/v_m}^{(-m)}. \quad (6.2)$$

By the Yang–Baxter equation for the queue vertex model (Proposition 2.10), the vertex weights of the straight cylinder Markov operator must be the fused stochastic weights $W_{q^{1/2-P/2}u_j/u_0, P, 1}$ from Section 2.3. They are given in Figure 24.

Remark 6.3. The weights in Figure 24 are matched to transition probabilities of a discrete time particle system on the ring as

$$\begin{array}{c} k \\ \nearrow \\ \mathbf{A} \text{---} \mathbf{C} \\ \searrow \\ \ell \end{array} = \begin{array}{c} \mathbf{C} \\ \uparrow \\ k \text{---} \mathbf{A} \\ \downarrow \\ \ell \end{array} .$$

The picture in the left-hand side represents vertices in Figure 26. In the right-hand side, the vertical direction corresponds to time, and the states \mathbf{A}, \mathbf{C} encode particle configurations at a given site $j \in \{1, \dots, N\}$ on the ring.

On the right, the horizontal arrow points left because after rotating Figure 26 by 90° counterclockwise, the sites on the ring are cyclically ordered as $(N, N-1, \dots, 1)$. Recall that under the q -PushTASEP, particles move in the direction of increasing j . This direction of the particle motion is opposite to the q -Boson situation, cf. the proof of Proposition 5.2.

$\begin{array}{c} 0 \\ \\ \mathbf{A} \text{---} \mathbf{A} \\ \\ 0 \end{array}$ $\frac{q^{- \mathbf{A} }x^{-1}q - 1}{x^{-1}q - 1}$	$\begin{array}{c} k \\ \\ \mathbf{A} \text{---} \mathbf{A} \\ \\ k \end{array}$ $\frac{q^{-A_{[k+1, n]}}(x^{-1}q - q^{-A_k}s^{-2})}{x^{-1}q - 1}$	$\begin{array}{c} k \\ \\ \mathbf{A} \text{---} \mathbf{A}_k^- \\ \\ 0 \end{array}$ $\frac{x^{-1}q \cdot q^{-A_{[k+1, n]}}(1 - q^{-A_k})}{x^{-1}q - 1}$
$\begin{array}{c} 0 \\ \\ \mathbf{A} \text{---} \mathbf{A}_k^+ \\ \\ k \end{array}$ $\frac{s^{-2}q^{- \mathbf{A} } - 1}{x^{-1}q - 1}$	$\begin{array}{c} \ell \\ \\ \mathbf{A} \text{---} \mathbf{A}_{k\ell}^{+-} \\ \\ k \end{array}$ $\frac{-x^{-1}q(q^{-A_\ell} - 1)q^{-A_{[\ell+1, n]}}}{x^{-1}q - 1}$	$\begin{array}{c} k \\ \\ \mathbf{A} \text{---} \mathbf{A}_{\ell k}^{+-} \\ \\ \ell \end{array}$ $\frac{s^{-2}(1 - q^{-A_k})q^{-A_{[k+1, n]}}}{x^{-1}q - 1}$

Figure 24: The vertex weights $W_{x, P, 1}(\mathbf{e}, \mathbf{A}; \mathbf{e}', \mathbf{A}')|_{q^P = s^{-2}}$. Here \mathbf{e}, \mathbf{e}' are basis vectors corresponding to empty or one-particle configurations in $\mathbb{Z}_{\geq 0}^n$, and $1 \leq k < \ell \leq n$. Note that these weights can be obtained from the stochastic L weights (Figure 5) by reflecting the picture about the diagonal and setting $s^2 \rightarrow s^{-2}, q \rightarrow q^{-1}, sx \rightarrow x^{-1}q$.

Now let us pass to a Poisson-type continuous time limit of the straight cylinder Markov operator to get the continuous time Markov semigroup of the colored q -PushTASEP. Set

$$u_0 := \gamma\epsilon > 0, \quad j = 1, \dots, N; \quad \gamma := q^{P/2-1/2}, \quad (6.3)$$

where $\epsilon > 0$ is small. The $\epsilon \rightarrow 0$ expansions of the vertex weights from Figure 24 are given in Figure 25.

$\begin{array}{c} 0 \\ \\ \mathbf{A} \text{---} \mathbf{A} \\ \\ 0 \end{array}$ $1 + O(\epsilon)$	$\begin{array}{c} k \\ \\ \mathbf{A} \text{---} \mathbf{A} \\ \\ k \end{array}$ $q^{P-A_{[k,n]}} + O(\epsilon)$	$\begin{array}{c} k \\ \\ \mathbf{A} \text{---} \mathbf{A}_k^- \\ \\ 0 \end{array}$ $\gamma \epsilon u_j^{-1} q^{P/2+1/2} q^{-A_{[k+1,n]}} (q^{-A_k} - 1)$
$\begin{array}{c} 0 \\ \\ \mathbf{A} \text{---} \mathbf{A}_k^+ \\ \\ k \end{array}$ $1 - q^{P- \mathbf{A} } + O(\epsilon)$	$\begin{array}{c} \ell \\ \\ \mathbf{A} \text{---} \mathbf{A}_{k\ell}^{+-} \\ \\ k \end{array}$ $\gamma \epsilon u_j^{-1} q^{P/2+1/2} q^{-A_{[\ell+1,n]}} (q^{-A_\ell} - 1)$	$\begin{array}{c} k \\ \\ \mathbf{A} \text{---} \mathbf{A}_{\ell k}^{+-} \\ \\ \ell \end{array}$ $q^{P-A_{[k+1,n]}} (q^{-A_k} - 1) + O(\epsilon)$

Figure 25: Small ϵ expansion of the vertex weights $W_{q^{1/2-P/2}u_j/u_0, P, 1}$ with $u_0 = \gamma\epsilon$.

These expansions imply the convergence as $\epsilon \rightarrow 0$ of the straight cylinder Markov operators $\mathfrak{S}(\mathbf{u}, \gamma\epsilon; (q^{-P/2}, \dots, q^{-P/2}, q^{-1/2}))^{\lfloor t/\epsilon \rfloor}$ to the q -PushTASEP semigroup $\mathfrak{P}_{q\text{Push}}(t)$, in the same way as for the q -Boson process (Proposition 5.2). Indeed, the auxiliary spin 1/2 line becomes occupied at a given instant in time with probability $O(\epsilon)$. Then, with high probability it becomes unoccupied within a finite number of discrete time steps, which corresponds to it becoming unoccupied instantaneously with respect to the macroscopic continuous time t .

The convergence of the straight cylinder Markov operators to the colored q -PushTASEP implies that the stationary distribution of the latter process can be represented as the partition function of a queue vertex model on the cylinder. More precisely, we have the following result:

Proposition 6.4. *Let $q \in [0, 1)$, $P \in \mathbb{Z}_{\geq 1}$ and $u_1, \dots, u_N > 0$. Fix the type counts (N_1, \dots, N_n) with $N_i \geq 1$ for all i . For any v_1, \dots, v_n and $s_1^{(v)}, \dots, s_n^{(v)}$, the stationary measure of the colored q -PushTASEP process on the ring has the form*

$$\text{Prob}_{N_1, \dots, N_n}^{q\text{Push}}(\mathbf{V}) = \frac{\langle \emptyset | \mathfrak{Q}((u_1, \dots, u_N); (q^{-P/2}, \dots, q^{-P/2}); (v_1, \dots, v_m); (s_1^{(v)}, \dots, s_n^{(v)})) | \mathbf{V} \rangle}{Z_{N_1, \dots, N_n}^{q\text{Push}}}. \quad (6.4)$$

Proof outline. This is proven in the same way as Proposition 5.5. The queue vertex model (6.4) on the cylinder has the weights of the first type in (6.2). The weights of the second type have the parameter $u_0 = \gamma\epsilon$. One can check that as $\epsilon \rightarrow 0$, we have

$$\mathbb{W}_{q^{-1/2}, s_m^{(v)}, \gamma\epsilon}^{(-m)}(\mathbf{A}, 0; \mathbf{A}, 0) = 1 + O(\epsilon), \quad \mathbb{W}_{q^{-1/2}, s_m^{(v)}, \gamma\epsilon}^{(-m)}(\mathbf{A}, 0; \mathbf{A}_k^-, k) = O(\epsilon).$$

This means that in the limit $\epsilon \rightarrow 0$, the auxiliary line is unoccupied with probability going to 1. Thus, the weights of the second type in (6.2) do not contribute to the queue vertex model, and we may pass from the model (6.1) on the cylinder with $N + 1$ rows to (6.4) with N rows, in the same way as in (5.5). This completes the proof. \square

Let us now discuss the nonnegativity of the individual vertex weights in the queue vertex model in (6.4). Note that the normalized partition functions are positive as components of the Perron–Frobenius eigenvector of $\mathfrak{P}_{q\text{Push}}(t)$.

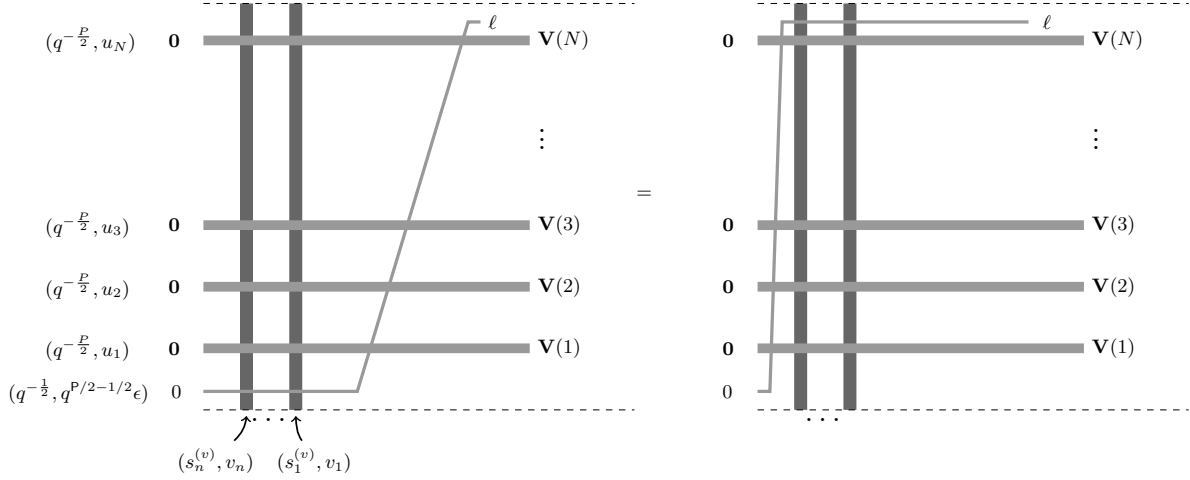


Figure 26: Illustration of the q -PushTASEP stationarity for finite ϵ .

Define

$$\mathbb{W}_{s,u}^{(-m),q\text{Push}(\mathbf{P})_+}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) := (-1/s)^{|\mathbf{D}|} \mathbb{W}_{q^{-P/2}, s, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}), \quad \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geq 0}^n. \quad (6.5)$$

Note that $|\mathbf{B}|, |\mathbf{D}| \leq \mathbf{P}$ due to the finite-spin reduction (see Remarks 2.2 and 2.4). The multiplication by $(-1/s_m^{(v)})^{|\mathbf{D}|}$ in each column of the queue vertex model (6.4) can be absorbed into the normalizing constant, and thus does not affect the normalized partition functions. In other words, we can use the weights (6.5) to represent the stationary distribution of the colored q -PushTASEP.

The weights (6.5) arise from the mASEP queue weights (4.8) by *fusion*. That is, each weight (6.5) is a certain sum of \mathbf{P} -fold products of the weights $\mathbb{W}_{q^{-1/2}, s, uq^i}^{(-m), \text{mASEP}_+}$, where $i = 0, 1, \dots, \mathbf{P} - 1$. We refer to [BW22a, Appendix B] and [BGW22, Theorem 8.5] for details. This implies the following nonnegativity of (6.5):

Proposition 6.5. *Let*

$$0 \leq s_m^{(v)} < \frac{u_j}{v_m} q^{\mathbf{P}-1/2} < \frac{u_j}{v_m} q^{-1/2} \leq 1$$

for all $1 \leq m \leq n, 1 \leq j \leq N$. Then the vertex weights $\mathbb{W}_{s_m^{(v)}, u_j/v_m}^{(-m), q\text{Push}(\mathbf{P})_+}$ (6.5) (entering the queue vertex model on the cylinder representing the stationary distribution of the q -PushTASEP) are nonnegative.

Proof. Under the hypotheses, the weights $\mathbb{W}_{q^{-1/2}, s_m^{(v)}, q^i u_j/v_m}^{(-m), \text{mASEP}_+}$, where i runs from 0 to $\mathbf{P} - 1$, are all nonnegative; see Proposition 4.9. Together with fusion, this implies the desired nonnegativity of the weights $\mathbb{W}_{s_m^{(v)}, u_j/v_m}^{(-m), q\text{Push}(\mathbf{P})_+}$. \square

Remark 6.6. The stationary distribution for the colored q -PushTASEP with equal parameters $u_j = u, 1 \leq j \leq N$, and with $\mathbf{P} = 1$, is the same as for the mASEP. Indeed, this follows by matching the vertex weights (see the discussion before Proposition 6.5). On the other hand, the proofs of the stationarity for mASEP and for the q -PushTASEP require different Markov operators on the cylinder (the twisted and the straight ones, respectively).

7 Stationarity in the quarter plane and on the line

Here we explain how the queue vertex models on the cylinder from Sections 4 to 6 can be used to construct the stationary distributions for mASEP, the colored q -Boson and the colored q -PushTASEP on the line \mathbb{Z} . Instead of passing to the limit as the size of the ring goes to infinity (as in, e.g., [Mar20, Section 5]), our proof of the stationarity on the line passes through applying the Yang–Baxter equation in the quarter plane, which may be viewed as a *colored generalization of Burke’s theorem* for stochastic vertex models. Applications of the latter to single-color stochastic integrable systems were the subject of, e.g., [OY01] (semi-discrete Brownian polymer), [Sep12] (log-gamma polymer). Particular cases of the colored Burke’s theorem (in the language of queues) appeared previously in [FM05], [FM09].

Remark 7.1. We only consider *space-homogeneous* systems on the line ($u_j = u$ for all j for the q -Boson and the q -PushTASEP; there are no known space-inhomogeneous integrable deformations of the ASEP). In contrast with the ring, stationarity of space-inhomogeneous systems on \mathbb{Z} is much more delicate. If the inhomogeneity is smooth in space, we may locally model stationary distributions by the homogeneous ones. In the non-smooth case, inhomogeneity in the q -Boson system may lead to infinite stacks of particles, separating the whole system into independent components. Out-of-equilibrium single-color inhomogeneous models (featuring both smooth and non-smooth inhomogeneity) were considered in, e.g., [BP18b], [BSS17], [KPS19], [Pet20], and we refer to those works for further details.

A special case $P = 1$ of the space-homogeneous colored q -PushTASEP is the same as the colored six-vertex model. Therefore, our constructions immediately produce stationary measures for the colored stochastic six-vertex model on the line. Furthermore, taking the Poisson type limit of the latter model along the diagonal (as described on the ring in Proposition 4.4), we get the mASEP on the line. This implies that the stationary measures for the q -PushTASEP with $P = 1$ coincide with those of the mASEP. In Remark 6.6 we already noticed this connection on the ring. Therefore, in describing stationary measures on the line, we may restrict attention to the q -Boson and the q -PushTASEP.

7.1 Queue steady state

Let $n \geq 1$ be the number of colors, and fix $1 \leq m \leq n$. Fix parameters α, ν such that

$$0 \leq \nu \leq \alpha. \quad (7.1)$$

Consider the queue vertex weights

$$\mathbb{W}_{\alpha, \nu}^{(-m), \text{line}}(\mathbf{A}, k; \mathbf{B}, \ell) := \mathbb{W}_{q^{-1/2}, s, z}^{(-m)}(\mathbf{A}, k; \mathbf{B}, \ell), \quad \alpha = -szq^{-1/2}, \quad \nu = -s^2. \quad (7.2)$$

These weights are given in Figure 15 with $z = uq^{1/2}$, and for convenience we reproduce them with the parameters α, ν in Figure 27. The next statement is straightforward from these expressions and the sum-to-one property (2.10):

Lemma 7.2. *The weights (7.2) sum to one over (\mathbf{B}, ℓ) for any fixed (\mathbf{A}, k) . Moreover, under (7.1), we have $\mathbb{W}_{\alpha, \nu}^{(-m), \text{line}}(\mathbf{A}, k; \mathbf{B}, \ell) \geq 0$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{Z}_{\geq 0}^n$ and $k, \ell \in \{0, 1, \dots, n\}$.*

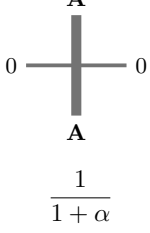
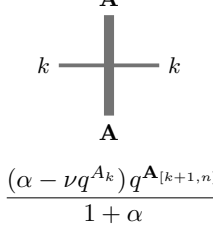
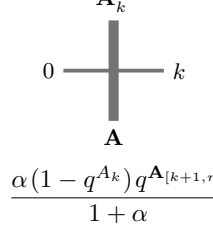
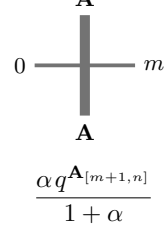
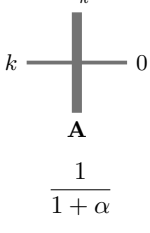
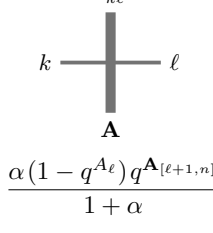
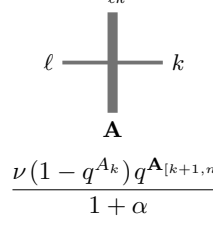
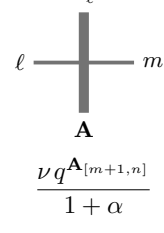
\mathbf{A}  $\frac{1}{1+\alpha}$	\mathbf{A}  $\frac{(\alpha - \nu q^{A_k}) q^{\mathbf{A}_{[k+1, n]}}}{1+\alpha}$	\mathbf{A}_k^-  $\frac{\alpha(1 - q^{A_k}) q^{\mathbf{A}_{[k+1, n]}}}{1+\alpha}$	\mathbf{A}  $\frac{\alpha q^{\mathbf{A}_{[m+1, n]}}}{1+\alpha}$
\mathbf{A}_k^+  $\frac{1}{1+\alpha}$	$\mathbf{A}_{k\ell}^{+-}$  $\frac{\alpha(1 - q^{A_\ell}) q^{\mathbf{A}_{[\ell+1, n]}}}{1+\alpha}$	$\mathbf{A}_{\ell k}^{+-}$  $\frac{\nu(1 - q^{A_k}) q^{\mathbf{A}_{[k+1, n]}}}{1+\alpha}$	\mathbf{A}_ℓ^+  $\frac{\nu q^{\mathbf{A}_{[m+1, n]}}}{1+\alpha}$

Figure 27: The weights $\mathbb{W}_{\alpha, \nu}^{(-m), \text{line}}$ (7.2), where $m < k < \ell \leq n$, and $A_m = +\infty$.

Remark 7.3. On the ring the stochasticity of the queue vertex weights is not essential, and we multiplied them by $(-1/s)^{\mathbf{1}_{\ell > 0}}$ to be nonnegative for $s \geq 0$; see (4.8) and Proposition 4.9. The factors $(-1/s)^{\mathbf{1}_{\ell > 0}}$ were absorbed into the normalizing constant of the stationary distribution on the ring. On the line, this absorption is not possible, so we need to deal with a different range of the parameters s, z as in (7.1) and (7.2).

Fix parameters

$$\alpha_1 > \alpha_2 > \dots > \alpha_n > 0, \quad \nu_i \in [0, \alpha_i], \quad i = 1, \dots, n. \quad (7.3)$$

Our first step is to construct a certain queue steady state vertex model. Fix $K \in \mathbb{Z}_{\geq 1}$ and consider the rectangle

$$\mathbf{R}_K := \{-n, -n+1, \dots, -1\} \times \{-K, -K+1, \dots, 0\}$$

(formed by the bottom K rows and the left n columns in the left side of Figure 29). In this rectangle, define a stochastic vertex model with empty inputs from the bottom and the left, and vertex weights $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$ at each $(-m, -j) \in \mathbf{R}_K$. Denote the outgoing arrow configuration at the top by $\mathbf{M}_K = (\mathbf{M}_K(-n), \dots, \mathbf{M}_K(-1))$, where $\mathbf{M}_K(-m) \in \mathbb{Z}_{\geq 0}^n$, and the outgoing arrow configuration on the right by $\mathbf{d}_K = (d_K(0), d_K(-1), \dots, d_K(-K))$, where $d_K(-j) \in \{0, 1, \dots, n\}$.

Proposition 7.4. Fix arbitrary $c \in \mathbb{Z}_{\geq 0}$. As $K \rightarrow +\infty$, the random tuples

$$\mathbf{M}_K \quad \text{and} \quad (d_K(0), d_K(-1), \dots, d_K(-c))$$

converge in joint distribution to random tuples \mathbf{M} and $\mathbf{d}^{[-c]}$.

We refer to the $K \rightarrow +\infty$ limit of the model in \mathbf{R}_K as to the *queue vertex model in steady state*. The limiting tuples $\mathbf{d}^{[-c]}$ are compatible for $c \geq 0$. Let us denote the corresponding infinite tuple by $\mathbf{d} = (d(0), d(-1), d(-2), \dots)$.

Proof of Proposition 7.4. View the vertical coordinate in \mathbb{R}_K as discrete time $t \in \{-K, -K + 1, \dots, 0\}$. Observe that

$$\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}(\mathbf{A}, k; \mathbf{A}_k^+, 0) = \mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}(\mathbf{A}, 0; \mathbf{A}, 0) = \frac{1}{1 + \alpha_m}.$$

Therefore, if we do not distinguish the colors $\geq m$, then the arrows leaving the column $(-m)$ form a Bernoulli process (in discrete time t) with probability of success $\alpha_m/(1 + \alpha_m)$. This implies that the combined number of arrows of colors $m + 1, \dots, n$ in the column $(-m)$ evolves as a birth and death Markov chain on $\mathbb{Z}_{\geq 0}$ starting from 0, which makes jumps by $-1, 0$, and $+1$. Denote this chain by $A_{[m+1, n]}(t)$. The jumps by $+1$ and -1 have probabilities, respectively,

$$\frac{\alpha_{m+1}}{1 + \alpha_{m+1}} \cdot \frac{1 + \nu_m q^{A_{[m+1, n]}(t)}}{1 + \alpha_m} \quad \text{and} \quad \frac{1}{1 + \alpha_{m+1}} \cdot \frac{\alpha_m(1 - q^{A_{[m+1, n]}(t)})}{1 + \alpha_m}.$$

The jump by 0 occurs with the complementary probability.

Since $\alpha_{m+1} < \alpha_m$, for large $A_{[m+1, n]}(t)$ the probability to go down is strictly larger, which implies that the birth and death chain on $\mathbb{Z}_{\geq 0}$ is recurrent. Thus, in each column $(-m)$, $1 \leq m \leq n$, the number of arrows of color $> m$ does not grow to infinity.

We conclude that the (colored) configurations of arrows in all columns $(-m)$, $1 \leq m \leq n$, jointly form a recurrent Markov chain [Dur19, Chapter 5] — a system of n queues in tandem. The limiting random tuple \mathbf{M} is its steady state. The limiting configuration $\mathbf{d} = (d(0), d(-1), \dots)$ is the steady state (colored) departure process, with time running from $-\infty$ to 0. This completes the proof. \square

Remark 7.5. For $\nu_m = 0$, $1 \leq m \leq n$, the system of n queues in tandem in the proof of Proposition 7.4 was considered in [Mar20, Sections 3.4 and 5].

7.2 Colored Burke's theorem via Yang–Baxter equation

Let us first discuss the general application of the Yang–Baxter equation in the quarter plane without specifying the weights leading to the concrete model. We discuss specializations to our colored stochastic particle systems in Sections 7.3 and 7.4 below. Assume that there exist stochastic, nonnegative vertex weights

$$\mathbb{W}_{\xi, \alpha, \nu}^{(-m), \text{queue}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \quad \text{and} \quad W_{\xi}^{\text{qp}}(k, \mathbf{B}; \ell, \mathbf{D}) \tag{7.4}$$

which together with the weights $\mathbb{W}_{\alpha, \nu}^{(-m), \text{line}}(\mathbf{A}, k; \mathbf{B}, \ell)$ (7.1) satisfy the Yang–Baxter equation given in Figure 28.

In Section 7.1 above we used the weights $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$ to construct the queue steady state vertex model in $\{-n, \dots, -1\} \times \mathbb{Z}_{\leq 0}$ depending on the parameters (7.3). As the output, the steady state model produces the random state $\mathbf{M} = (\mathbf{M}(-n), \dots, \mathbf{M}(-1))$ at the top, and the random departure process $\mathbf{d} = (d(0), d(-1), \dots)$ on the right. Let us define two more stochastic vertex models (see the left side of Figure 29, for an illustration):

- A queue vertex model in $\{-n, \dots, 1\} \times \{1, 2, \dots\}$, with the weight $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ at each vertex $(-m, j)$. This model has no incoming arrows from the left, and the incoming arrow configuration \mathbf{M} from the bottom. Denote by $\mathbf{V}(j)$, $j \in \mathbb{Z}_{\geq 1}$, the outgoing arrow configuration from the j -th horizontal line of this model.

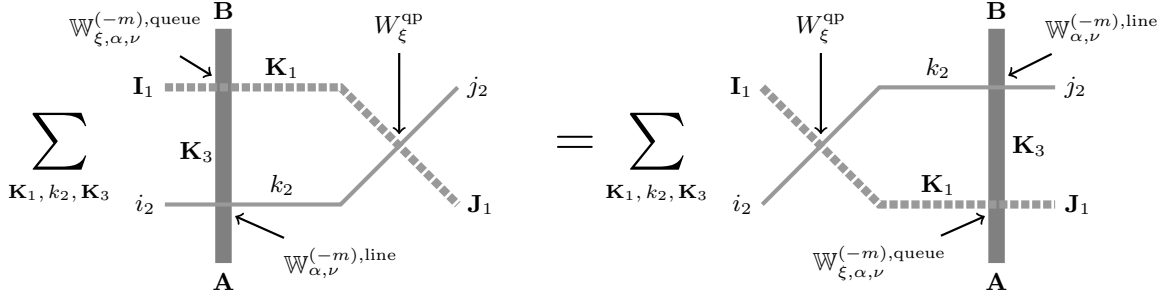


Figure 28: Elementary Yang–Baxter equation for the quarter plane. Here $\mathbf{I}_1, \mathbf{A}, \mathbf{J}_1, \mathbf{B} \in \mathbb{Z}_{\geq 0}^n$ and $i_2, j_2 \in \{0, 1, \dots, n\}$ are fixed, and the sums in both sides are over the internal arrow configurations $\mathbf{K}_1, \mathbf{K}_3 \in \mathbb{Z}_{\geq 0}^n$ and $k_2 \in \{0, 1, \dots, n\}$.

- A vertex model in the quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Let each vertex (i, j) in the quadrant have weight W_ξ^{qp} . Let the incoming arrow configurations for this model be $\mathbf{V}(1), \mathbf{V}(2), \dots$ from the left, and $d(0), d(-1), \dots$ from the bottom.

Let $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Denote by $\mathbf{V}'(j) \in \mathbb{Z}_{\geq 0}^n$ the arrow configuration at the horizontal edge $(0, j) - (1, j)$, and by $d'(-i) \in \{0, 1, \dots, n\}$ the color of the vertical edge $(i, 1) - (i, 2)$.

Theorem 7.6 (Colored Burke’s theorem). *We have equalities of joint distributions:*

$$\left\{ (d(-i))_{i \geq 0}, (\mathbf{V}(j))_{j \geq 1} \right\} \stackrel{d}{=} \left\{ (d'(-i))_{i \geq 1}, (\mathbf{V}(j))_{j \geq 1} \right\} \stackrel{d}{=} \left\{ (d(-i))_{i \geq 0}, (\mathbf{V}'(j))_{j \geq 2} \right\}.$$

In words, the joint distribution of the horizontal and the vertical arrow configurations along the boundary of an arbitrarily shifted quadrant $\mathbb{Z}_{\geq I} \times \mathbb{Z}_{\geq J+1}$ (where $I, J \geq 0$) is the same as for the original quadrant $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Since by our assumption the weights W_ξ^{qp} are stochastic, Theorem 7.6 can be viewed as the statement that the boundary data given by (\mathbf{V}, \mathbf{d}) is stationary for the stochastic vertex model in the quadrant with the weights W_ξ^{qp} .

Proof of Theorem 7.6. The result follows by repeatedly applying the Yang–Baxter equation from Figure 28 to the combination of the three vertex models in the left side of Figure 29. Indeed, to shift the index in $(\mathbf{V}(j))_{j \geq 1}$ up by one, one needs to drag the crosses from the right to the left, and move the dotted line given by $\{y = 1\}$ down to minus infinity. See the top two pictures in the right side of Figure 29. The cross vertices on the left boundary are empty, have probability weight 1, and thus can be removed. In the limit as the dotted line goes down to minus infinity, the distribution of the outputs \mathbf{M} and \mathbf{d} of the queue steady state model becomes the same; in fact, the transition matrix for the system of queues in tandem commutes with the transition matrix for the queues obtained from the stochastic vertex weights on the dotted line, and thus, the dotted line preserves (\mathbf{M}, \mathbf{d}) if (\mathbf{M}, \mathbf{d}) is stationary. The index shift in $(d(-i))_{i \geq 0}$ is performed similarly, but now we need to move the turning line which carried $d(0)$ up to positive infinity (see the bottom picture in the right side of Figure 29 for an illustration). This completes the proof. \square

Remark 7.7. The stochastic vertex model in the quadrant can be made inhomogeneous by letting the parameters α_m and ξ in the queue vertex model part depend on the vertical coordinate

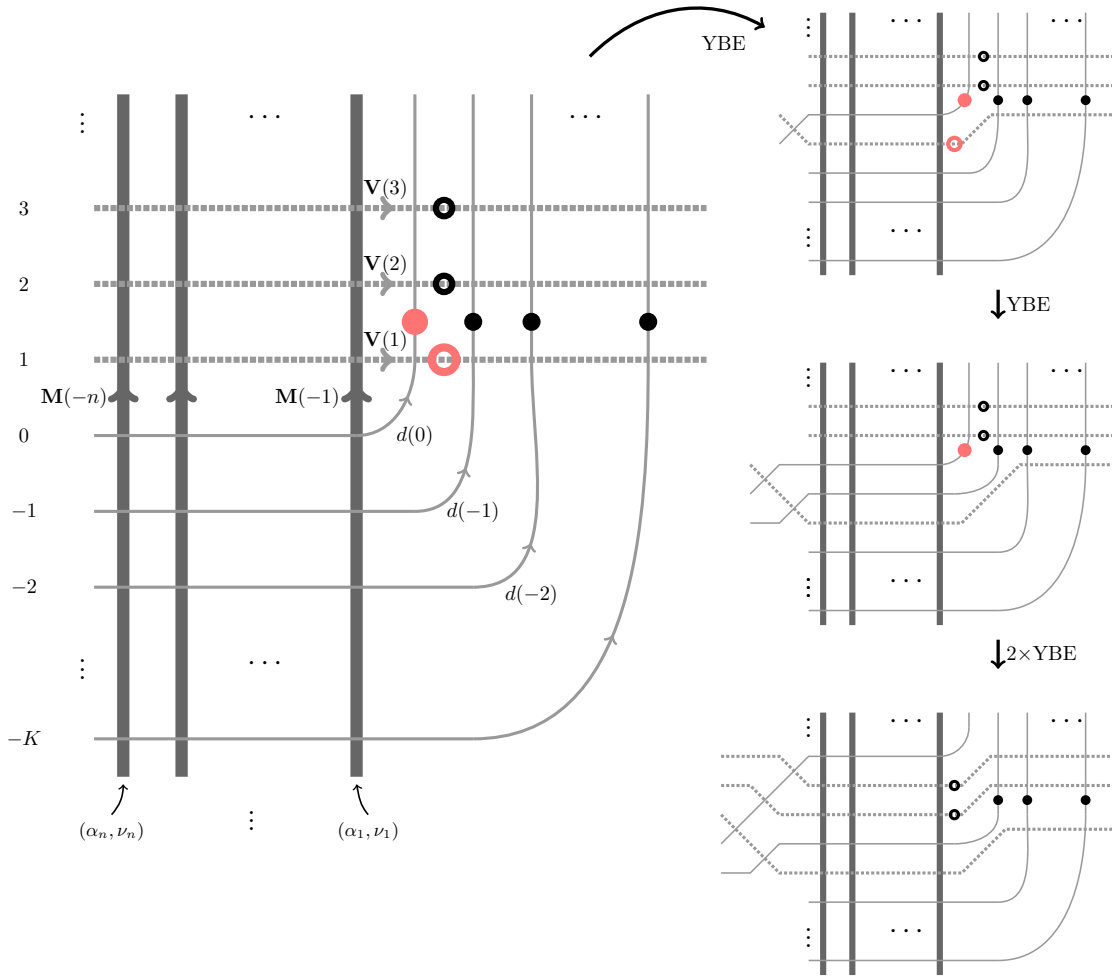


Figure 29: Left: The queue steady state vertex model in $\{-n, \dots, -1\} \times \mathbb{Z}_{\leq 0}$ produces the random state $\mathbf{M}(-n), \dots, \mathbf{M}(-1)$ and the departure process $\mathbf{d} = (d(0), d(-1), \dots)$ (see Proposition 7.4). It has the empty incoming configuration from the left. On top of it we put a queue vertex model in $\{-n, \dots, -1\} \times \mathbb{Z}_{\geq 1}$ (also with no incoming arrows from the left; the choice of the queue weights depends on whether we work with the colored q -Boson or q -PushTASEP). Denote the random output of this model by $\mathbf{V} = (\mathbf{V}(1), \mathbf{V}(2), \dots)$.

The outputs \mathbf{V} and \mathbf{d} form the left and bottom inputs into a third stochastic vertex model in the quadrant. In a limit when the one of the lattice directions turns into the continuous time, the model in the quadrant converges to either the colored q -Boson or the colored q -PushTASEP stochastic particle system.

Right: Consecutive applications of the Yang–Baxter equation to the vertex models on the left which lead to the shift of the quadrant by $(1, 1)$. The cross vertices on the left boundary are empty and can be removed. The black dots on the edges in all pictures represent arrow configurations $d'(-i)$ (solid) and $\mathbf{V}'(j)$ (circled), whose joint distributions are the same in all pictures. Throughout applying the Yang–Baxter equation, the red (larger) dots representing $d'(0)$ and $\mathbf{V}'(1)$ disappear, which corresponds to shifting of the indices in the formulation of Theorem 7.6.

$j \in \mathbb{Z}$. This would lead to the weights $W_{v_i \xi_j}^{\text{qp}}$ at each vertex $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. One can readily formulate an extension of Theorem 7.6 for this situation, but for simplicity we only discussed the homogeneous setting. See also Remark 7.1 on stationarity in the presence of space inhomogeneity.

Let us record a property of the steady state \mathbf{M} which will be useful in Appendix A below:

Lemma 7.8. *Let $\mathbf{M} = (\mathbf{M}(-n), \dots, \mathbf{M}(-1))$ be the steady state of the n -column queue vertex model with the weights $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$ (the bottom n columns in Figure 29, left). Take a one-row vertex model in $\{-n, \dots, -1\} \times \{0\}$ with the weight $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ at each vertex $(-m, 0)$. If there are no arrows incoming from the left, and the configuration \mathbf{M} of arrows incoming from below into this one-row model, then the distribution of the outgoing arrows from the top is the same as that of \mathbf{M} .*

In short, the distribution of \mathbf{M} is preserved by the horizontal action of the n -column queue vertex model with the weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$.

Proof. Consider the rectangle $\{-n, \dots, -1\} \times \{0, -1, \dots, -K\}$. Put a single n -column layer with the weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ at the horizontal coordinate 0. Below it, let us put K layers of the weights $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$. Assume that the incoming arrow configuration at the left boundary is empty, while at the bottom let the input be the random steady state $\mathbf{M} = (\mathbf{M}(-n), \dots, \mathbf{M}(-1))$. Denote the random output at the top by $\mathbf{M}' = (\mathbf{M}'(-n), \dots, \mathbf{M}'(-1))$. See Figure 30, left, for an illustration. Since the solid horizontal lines preserve the distribution of \mathbf{M} (this is the steady state property), it remains to show that \mathbf{M}' and \mathbf{M} have the same distribution.

Attach K cross vertices with the stochastic weights W_{ξ}^{qp} to the rectangle on the right. This does not change the distribution of \mathbf{M}' . Then apply the Yang–Baxter equation (as in Figure 28) to move these cross vertices to the left. On the left, there are no incoming arrows, so the cross vertices in Figure 30, right, can be removed.

Denote by \mathbf{M}'' the random configuration which arises from \mathbf{M} after the single dashed line, that is, the line with the weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$. The Yang–Baxter equation implies that \mathbf{M}' has the same distribution as the random configuration which arises from \mathbf{M}'' after K solid horizontal lines, that is, the lines with the weights $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$. Taking $K \rightarrow +\infty$, we see that the growing number of solid horizontal lines converges to its steady state distributed as \mathbf{M} . In particular, this steady state is independent of the initial configuration \mathbf{M}'' . Therefore, \mathbf{M}' and \mathbf{M} have the same distribution, as desired. \square

From Theorem 7.6, we can construct a stationary random configuration of the vertex model W_{ξ}^{qp} in the whole plane \mathbb{Z}^2 . The model in the plane is characterized as follows (a similar single-color construction appeared in [Agg18, Section A.2]). For any $J \geq 1$, define the map

$$\tau_J: \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}^2, \quad \tau_J: (i, j) \mapsto (i - J, j - J).$$

By Theorem 7.6, the random configurations of colored paths in parts of \mathbb{Z}^2 coming from shifting the configuration in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ by τ_J , $J \in \mathbb{Z}_{\geq 1}$, are compatible in J . Therefore, by the Kolmogorov extension theorem, there exists a distribution on path configurations in the whole plane \mathbb{Z}^2 , which is translation invariant (that is, stationary under the stochastic vertex model W_{ξ}^{qp}).

The parameters (α_m, ν_m) of the queue vertex models attached to the quadrant determine the densities of various types of colors (the “colored slope”) of the resulting translation invariant

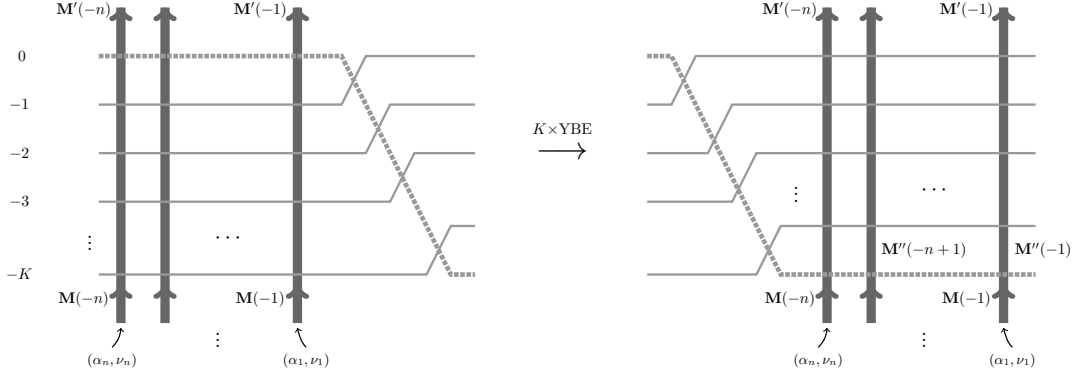


Figure 30: Application of the Yang–Baxter equation in the proof of Lemma 7.8. The dashed and the solid horizontal lines correspond to the vertex weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ and $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$, respectively.

model in \mathbb{Z}^2 . We explore the exact connection between these parameters and the densities of various colors for q -Boson and q -PushTASEP in Sections 7.3 and 7.4 below.

7.3 Specialization to stochastic six-vertex model and q -PushTASEP

Let us specialize Theorem 7.6 to the colored q -PushTASEP (defined on the line in the same way as in Definition 6.1, but with the homogeneous parameters $u_k = u$, $k \in \mathbb{Z}$). Take the weights (7.2), (7.4) to be

$$\begin{aligned}
 \mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}(\mathbf{A}, k; \mathbf{B}, \ell) &= \mathbb{W}_{q^{-1/2}, s_m^{(v)}, z/v_m}^{(-m)}(\mathbf{A}, k; \mathbf{B}, \ell), \\
 \mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &= \mathbb{W}_{q^{-P/2}, s_m^{(v)}, u/v_m}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}), \\
 W_{\xi}^{\text{qp}}(k, \mathbf{B}; \ell, \mathbf{D}) &= W_{q^{1/2-P/2}u/z, P, 1}(\mathbf{e}_k \mathbf{1}_{k \geq 1}, \mathbf{B}; \mathbf{e}_\ell \mathbf{1}_{\ell \geq 1}, \mathbf{D}), \\
 \alpha_m &:= -s_m^{(v)} q^{-1/2} \frac{z}{v_m}, \quad \nu_m := -(s_m^{(v)})^2, \quad \xi := u/z.
 \end{aligned} \tag{7.5}$$

Here $P \in \mathbb{Z}_{\geq 1}$, and α_m, ν_m must satisfy (7.3). The weights in the right-hand sides of (7.5) are given, respectively, in Figure 15, formula (2.12), and Figure 24.

One can check that the weights (7.5) are all nonnegative if

$$\nu_i \in [0, \alpha_i q^{P-1}], \quad i = 1, \dots, n, \quad \xi \geq q^{1/2-P/2}. \tag{7.6}$$

Indeed, the first condition corresponds to the fact that the weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ come from the fusion of $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$ (see Proposition 6.5 for a related nonnegativity property), and the second condition involving the weights W_{ξ}^{qp} is read off from their explicit form given in Figure 24.

Let us first consider the case $P = 1$. Then the vertex model in the quadrant with the weights W_{ξ}^{qp} becomes the colored stochastic six-vertex model (see [BW22a, Figure 1] for a simulation with non-stationary boundary conditions). Theorem 7.6 and the shifting argument after it allows to construct a translation invariant (stationary) colored stochastic six-vertex model in the full plane \mathbb{Z}^2 . By analogy with [NdN95], [Agg22], let us call this path configuration in \mathbb{Z}^2 the *colored*

KPZ pure phase of the stochastic six-vertex model. The colored KPZ pure phase has a finite number n of colors.

From Figure 27, we see that the probability that no paths leave column $(-m)$ under the stochastic weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ and $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{line}}$, respectively, is equal to

$$\frac{1}{1 + \alpha_m \xi q^{-1/2}} \quad \text{and} \quad \frac{1}{1 + \alpha_m q^{-1/2}}. \quad (7.7)$$

By Proposition 7.4, the arrow configurations in the columns $-n, \dots, -1$ are in steady state. This means that the number of colors $i > m$ in the each column $(-m)$ does not grow to infinity. Thus, once a path of color m originates at the column $(-m)$, it must exit the column (-1) at a bounded (random) distance from where it originated. This means that the combined density of paths of colors $\geq m$ exiting the column (-1) (in either the bottom or the top part of the configuration of n vertical columns, see Figure 29, left) is equal to the complementary probability to (7.7).

We conclude that in the colored KPZ pure phase determined by the parameters $\{\alpha_m\}_{m=1}^n$, the horizontal and the vertical densities of each color $m = 1, \dots, n$ are, respectively,

$$\rho_m^{(h)} = \frac{\alpha_m \xi q^{-1/2}}{1 + \alpha_m \xi q^{-1/2}} - \frac{\alpha_{m+1} \xi q^{-1/2}}{1 + \alpha_{m+1} \xi q^{-1/2}}, \quad \rho_m^{(v)} = \frac{\alpha_m q^{-1/2}}{1 + \alpha_m q^{-1/2}} - \frac{\alpha_{m+1} q^{-1/2}}{1 + \alpha_{m+1} q^{-1/2}}. \quad (7.8)$$

Here $\alpha_{n+1} = 0$, by agreement. When $n = 1$, we can solve for α_1 , and get

$$\rho_1^{(h)} = \frac{\rho_1^{(v)} \xi}{1 + (\xi - 1) \rho_1^{(v)}},$$

which agrees with the slope relation $\rho_1^{(h)} = \varphi(\rho_1^{(v)})$ in the single-color KPZ phase (for example, see [Agg22, (2.6)]). For general n , solving for the α_i 's in (7.8) yields the following *colored slope relations*:

$$\rho_m^{(h)} = \frac{\rho_m^{(v)} \xi}{(1 + (\xi - 1) \rho_{[m,n]}^{(v)}) (1 + (\xi - 1) \rho_{[m+1,n]}^{(v)})}, \quad m = 1, \dots, n, \quad (7.9)$$

where $\rho_{[a,b]}^{(v)} = \rho_a^{(v)} + \rho_{a+1}^{(v)} + \dots + \rho_b^{(v)}$.

Under the colored KPZ pure phase, the colors occupying the vertical edges along a given horizontal line induce a random configuration of colors on \mathbb{Z} . This random configuration is a stationary distribution for the mASEP.

Remark 7.9. As shown in [FKS91], for given color densities, a translation invariant stationary distribution for mASEP on \mathbb{Z} is unique. We believe that a similar uniqueness holds for the colored stochastic six-vertex model, but this statement does not seem to be present in the existing literature.

Let us now return to the case of general P, and take a continuous time limit to the colored q -PushTASEP. The limit is achieved by setting $z = q^{P/2-1/2} \epsilon$, and letting $\epsilon \rightarrow 0$. In this limit, the horizontal direction in the quarter plane, scaled by $1/\epsilon$, turns into continuous time $t \in \mathbb{R}_{\geq 0}$. See Figure 25 for the $\epsilon \rightarrow 0$ expansions of the weights W_ξ^{qp} . The continuous time Markov process coming from W_ξ^{qp} lives on configurations on $\mathbb{Z}_{\geq 1}$ (where at each site there are at most

P particles), and coincides with the colored q -PushTASEP (Definition 6.1) with homogeneous parameters $u_k \equiv u$.

After the rescaling, let us further set $\nu_m = 0$, which implies nonnegativity of the remaining weights $\mathbb{W}_{\xi, \alpha_m, 0}^{(-m), \text{queue}}$ and $\mathbb{W}_{\alpha_m, 0}^{(-m), \text{line}}$. Indeed, note that as α_m (containing z as a factor) goes to zero, conditions (7.6) cannot hold unless $\nu_m = 0$.

Remark 7.10. In the constructions on the line, we need to set $\nu_m = 0$, $m = 1, \dots, n$, from the beginning, to ensure the nonnegativity of the queue vertex weights and the corresponding jump rates in the queue columns (occurring as $\epsilon \rightarrow 0$). The nonnegativity is required to ensure the existence of the steady state in the queue columns (see Proposition 7.4).

This should be contrasted to the ring case, where we could initially work with negative probabilities and jump rates formally. Then, when the commutation relation between the queue vertex model transfer matrix and the straight cylinder transfer matrix is established, we can renormalize the queue vertex model on the cylinder to get nonnegative probabilities under the stationary distribution. Thus, we have a whole family of vertex models on the cylinder (depending on the ν_m 's) which produce the same stationary measure, and on the line we have to set $\nu_m = 0$ for all m .

In the remaining vertex models in Figure 29, the weights $\mathbb{W}_{\xi, \alpha_m, 0}^{(-m), \text{queue}}$ in the top part do not depend on z (see (7.5)) and thus do not change in the limit. In the weights $\mathbb{W}_{\alpha_m, 0}^{(-m), \text{line}}$ in the bottom part (given in Figure 27), we have

$$\alpha_m = -\frac{s_m^{(v)}}{v_m} q^{P/2-1} \epsilon =: y_m \epsilon \rightarrow 0. \quad (7.10)$$

Here $y_1 > \dots > y_n > 0$ are the new parameters of the continuous time colored q -PushTASEP. The fact that the α_m 's are proportional to ϵ corresponds to the scaling of the bottom columns $\{-n, \dots, -1\} \times \mathbb{Z}_{\leq 0}$ in Figure 29 to continuous ones, $\{-n, \dots, -1\} \times \mathbb{R}_{\leq 0}$. Note that this scaling does not affect the weights $\mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{queue}}$ since they do not depend on z . The resulting scaled queue steady state model in $\{-n, \dots, -1\} \times \mathbb{R}_{\leq 0}$ runs in continuous time. Let $\mathbf{M} = (\mathbf{M}(-n), \dots, \mathbf{M}(-1))$ be the steady state of these continuous time queues in tandem, and let $\mathbf{d}(t)$, $t \leq 0$, be the (continuous time) departure process. Using $\mathbf{d}(t)$ and the output \mathbf{V} of the top columns $\{-n, \dots, -1\} \times \mathbb{Z}_{\geq 1}$, one can use the Burke's theorem (Theorem 7.6) to construct a stationary version of the colored q -PushTASEP on the whole line.

Let us compute the densities of the colors under the stationary measure for the colored q -PushTASEP. Since more than one arrow may leave the column $(-m)$, we need to take the expectation of the number of arrows. For this expectation, we do not need to distinguish the colors. Employing the color merging discussed in Appendix A, we may assume that $m = n = 1$. By a specialization of (2.12), one can check that the weights have the form (with $a = c = \infty$):

$$\begin{aligned} \mathbb{W}_{\xi, \alpha_m, 0}^{(-1), \text{queue}}(a, b; c, d) &= \frac{\mathbf{1}_{a+b=c+d}}{(q^{-P/2} s_m^{(v)} u / v_m; q)_P} \frac{(q^{P/2} s_m^{(v)} u / v_m)^d (q^{-P}; q)_d}{(q; q)_d} \\ &= \frac{\mathbf{1}_{a+b=c+d}}{(-q^{1-P} u y_m; q)_P} \frac{(-q u y_m)^d (q^{-P}; q)_d}{(q; q)_d}. \end{aligned}$$

Note that these weights sum to 1 over $0 \leq d \leq P$ by the q -binomial theorem [GR04, (1.3.2)]. The expected number of arrows is expressed through the function

$$\phi(\zeta) := \sum_{k=0}^{\infty} \frac{\zeta q^k}{1 - \zeta q^k} \quad (7.11)$$

(up to a change of variables and a linear transformation, this is the q -digamma function). We have

$$\begin{aligned} \sum_{d=0}^P \frac{\zeta^d (q^{-P}; q)_d}{(q; q)_d} &= \frac{(q^{-P} \zeta; q)_{\infty}}{(\zeta; q)_{\infty}} = (q^{-P} \zeta; q)_P, \\ \frac{1}{(q^{-P} \zeta; q)_P} \sum_{d=0}^P d \cdot \frac{\zeta^d (q^{-P}; q)_d}{(q; q)_d} &= \zeta \frac{\partial}{\partial \zeta} \log (q^{-P} \zeta; q)_P = - \sum_{i=0}^{P-1} \frac{q^{i-P} \zeta}{1 - q^{i-P} \zeta}, \end{aligned}$$

and the latter sum is a difference of two functions of the form (7.11) with the arguments differing by the factor q^{-P} . Therefore,

$$\sum_{d=0}^P d \cdot \frac{1}{(-q^{1-P} u y_m; q)_P} \frac{(-q u y_m)^d (q^{-P}; q)_d}{(q; q)_d} = \phi(-q u y_m) - \phi(-q^{1-P} u y_m). \quad (7.12)$$

Thus, the horizontal density of the m -th color is the difference of the above expressions:

$$\rho_m^{(h)} = \phi(-q u y_m) - \phi(-q^{1-P} u y_m) - \phi(-q u y_{m+1}) + \phi(-q^{1-P} u y_{m+1}), \quad (7.13)$$

where, by agreement, $y_{n+1} = 0$. This follows similarly to the case of the colored six-vertex model: since the queues are in steady state, a color m cannot accumulate in any column except $(-m)$. This implies that the expectation (7.12) is equal to $\rho_{[m,n]}^{(h)} = \rho_m^{(h)} + \dots + \rho_n^{(h)}$, which yields (7.13).

We can also compute the currents of the colored q -PushTASEP in stationarity (that is, the vertical densities of the colors), using

$$\mathbb{W}_{\alpha_m, 0}^{(-m), \text{line}}(\mathbf{A}, k; \mathbf{B}, 0) = \frac{1}{1 + \alpha_m} = 1 - y_m \epsilon + O(\epsilon^2),$$

where $\mathbf{B} = \mathbf{A} + \mathbf{e}_k \mathbf{1}_{k \geq 1}$. Therefore, in the continuous time limit, we have

$$\rho_m^{(v)} = y_m - y_{m+1}. \quad (7.14)$$

Expressing the colored currents $(\rho_1^{(v)}, \dots, \rho_n^{(v)})$ in terms of the colored densities $(\rho_1^{(h)}, \dots, \rho_n^{(h)})$ for general P would require finding the y_m 's from (7.13), which is not explicit for general P . However, a reverse expression is essentially given by (7.13)–(7.14):

$$\rho_m^{(h)} = \phi(-q u \rho_{[m,n]}^{(v)}) - \phi(-q^{1-P} u \rho_{[m,n]}^{(v)}) - \phi(-q u \rho_{[m+1,n]}^{(v)}) + \phi(-q^{1-P} u \rho_{[m+1,n]}^{(v)}).$$

Remark 7.11. We believe that for any P , a translation invariant stationary distribution for the q -PushTASEP on \mathbb{Z} with parameter P and with given densities of the colors is unique. However, this statement does not seem to be present in the existing literature.

7.4 Specialization to q -Boson

Let us specialize Theorem 7.6 to the stochastic colored q -Boson process. It is defined the same way on the line as on the ring (Definition 5.1), but we reverse the direction of the particle movement. That is, a particle of color i jumps from k to $k + 1$, $k \in \mathbb{Z}$, at the homogeneous rate $u^{-1}(1 - q^{\mathbf{V}(k)_i})q^{\mathbf{V}(k)_{[i+1,n]}}$, where $\mathbf{V}(k) \in \mathbb{Z}_{\geq 0}^n$ is the arrow configuration at site k .

To obtain the q -Boson process together with its stationary measure from the vertex models in Figure 29, is it convenient to take horizontally fused weights (meaning multiple paths can occupy a horizontal edge, as compared to Section 7.3, when $\mathbb{W}^{(-m),\text{line}}$ were weights for which at most one path can occupy a horizontal edge) in the bottom part $\{-n, \dots, -1\} \times \mathbb{Z}_{\leq 0}$. That is, let us take the following pre-limit weights depending on $\epsilon > 0$:

$$\begin{aligned} \mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{line}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &= \mathbb{W}_{\epsilon, s, u\epsilon y_m/s}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}), \\ \mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{queue}}(\mathbf{A}, k; \mathbf{B}, \ell) &= \mathbb{W}_{q^{-1/2}, s, -\epsilon q^{1/2} y_m/s}^{(-m)}(\mathbf{A}, k; \mathbf{B}, \ell), \\ W_{\xi, \epsilon}^{\text{qp}}(\mathbf{A}, k; \mathbf{B}, \ell) &= W_{-1/(u\epsilon), 1, \mathbb{N}}(\mathbf{A}, \mathbf{e}_k \mathbf{1}_{k \geq 1}; \mathbf{B}, \mathbf{e}_\ell \mathbf{1}_{\ell \geq 1}), \\ \alpha_m &:= \epsilon y_m/s, \quad \nu_m := -s^2, \quad \xi := u, \quad q^{-\mathbb{N}^\epsilon} := \epsilon^2. \end{aligned} \tag{7.15}$$

Note that here we placed the ξ -dependence into the bottom part of the left n columns in Figure 29, left, instead of the top one.

Let us take $\epsilon \rightarrow 0$ and simultaneously rescale the vertical coordinate of the quadrant by $1/\epsilon$. This turns the vertical coordinate into the continuous time $t \in \mathbb{R}_{\geq 0}$. After that, set $s = 0$, which would imply the nonnegativity of the jump rates (the restriction $s = 0$ is parallel to the case of the q -PushTASEP; see Remark 7.10). The results of Sections 5.1 and 5.2 imply that as $\epsilon \rightarrow 0$ and $s = 0$, the weights (7.15) become

$$\begin{aligned} \mathbb{W}_{\xi, \alpha_m, \nu_m}^{(-m), \text{line}} &\rightarrow (uy_m; q)_\infty \mathbb{W}_{0, uy_m}^{(-m), \text{qBos}}, \\ \mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{queue}} &\rightarrow \text{jump rates in a continuous time queue vertex model (Figure 31),} \\ W_{\xi, \epsilon}^{\text{qp}} &\rightarrow \text{colored } q\text{-Boson jump rates in Figure 22,} \end{aligned} \tag{7.16}$$

where $\mathbb{W}_{0, uy_m}^{(-m), \text{qBos}}$ is given by the right-hand side of (5.2), and the limits of $\mathbb{W}_{\alpha_m, \nu_m}^{(-m), \text{queue}}$ to the continuous vertical direction are read off from Figure 15. These limits are given in Figure 31.

One readily sees that the vertex weights (7.16) define Markov processes in discrete and continuous time with nonnegative transitions. Similarly to the proof of Proposition 7.4, for

$$y_1 > y_2 > \dots > y_n > 0$$

one can verify that the queue vertex model with the weights $(uy_m; q)_\infty \mathbb{W}_{0, uy_m}^{(-m), \text{qBos}}$ produces a steady state (that is, it does not run off to infinity). Then, following the proof of Theorem 7.6 and the shifting argument after it, one can construct a stationary version of the colored q -Boson process on the whole line. We omit the details of the construction as they are similar to the ones in Section 7.3.

Remark 7.12. The results of [ABGM21] imply that for $q = 0$, a translation invariant stationary distribution for the q -Boson process on the whole line with given densities of the colors is unique. We believe that a similar result should hold for general $q \in (0, 1)$, but its proof does not seem present in the existing literature.

\mathbf{A} \mathbf{A} $1 - \epsilon y + O(\epsilon^2)$	\mathbf{A} \mathbf{A} $O(\epsilon)$	\mathbf{A}_k^- \mathbf{A} $\epsilon y (1 - q^{A_k}) q^{A_{[k+1, n]}} + O(\epsilon^2)$	\mathbf{A} \mathbf{A} $\epsilon y q^{A_{[m+1, n]}} + O(\epsilon^2)$
\mathbf{A}_k^+ \mathbf{A} $1 - \epsilon y + O(\epsilon^2)$	$\mathbf{A}_{k\ell}^{+-}$ \mathbf{A} $O(\epsilon)$	$\mathbf{A}_{\ell k}^{+-}$ \mathbf{A} 0	\mathbf{A}_ℓ^+ \mathbf{A} 0

Figure 31: Expansions of the weights $\mathbb{W}_{q^{-1/2}, s, -\epsilon q^{1/2} y/s}^{(-m)} \Big|_{s=0}$ as $\epsilon \rightarrow 0$. Here $m < k < \ell \leq n$. The vertices of the types $(\mathbf{A}, k; \mathbf{A}, k)$ and $(\mathbf{A}, k; \mathbf{A}_{k\ell}^{+-}, \ell)$ with probabilities of order ϵ do not occur in the queue vertex model. Indeed, to get a nonempty input from the left, another event of probability $O(\epsilon)$ should have occurred at the same instance of the continuous time.

Arguing similarly to the case of the q -PushTASEP (Section 7.3), we can compute the colored densities and currents in terms of the parameters $y_1 > \dots > y_n > 0$. The colored densities are the vertical densities $\rho_m^{(v)}$ under the steady state vertex model with probability weights $(uy_m; q)_\infty \mathbb{W}_{0, uy_m}^{(-m), \text{qBos}}$ (5.2). Let us compute the expected number of arrows leaving the column $(-m)$. By color merging described in Appendix A, we do not need to distinguish the colors and may assume that $m = n = 1$. We have

$$(uy_m; q)_\infty \mathbb{W}_{0, uy_m}^{(-m), \text{qBos}}(a, b; c, d) = (uy_m; q)_\infty \cdot \frac{(uy_m)^d}{(q; q)_d}, \quad d \in \mathbb{Z}_{\geq 0}.$$

The expected number of arrows is expressed through the function (7.11):

$$\sum_{d=0}^{\infty} d \cdot (uy_m; q)_\infty \cdot \frac{(uy_m)^d}{(q; q)_d} = \phi(uy_m).$$

Thus, we have

$$\rho_m^{(v)} = \phi(uy_m) - \phi(uy_{m+1}). \quad (7.17)$$

Here $y_{n+1} = 0$, by agreement. The colored currents are determined from the continuous time queue vertex model in Figure 31, and have the form

$$\rho_m^{(h)} = y_m - y_{m+1}.$$

Similarly to the q -PushTASEP, expressing the currents in terms of the densities is not an explicit operation. On the other hand, a reverse expression is readily available. It is obtained from (7.17) by replacing each y_j with $\rho_{[j, n]}^{(h)}$.

A Merging of colors in stationary measures on the line

Take a stochastic particle system (mASEP, the q -Boson process or the q -PushTASEP) with n colors on the line \mathbb{Z} . By agreement, holes are viewed as particles of color 0. If we declare that for some $m = 0, 1, \dots, n$, all particles of colors m and $m + 1$ are identified, then we get a stochastic process with $n - 1$ colors. This operation of identifying two colors is a particular case of *color merging* (see Definition A.3 below).

A similar color merging can be performed for a stationary distribution of an n -colored particle system, and as a result, we should get a stationary distribution of a system with $n - 1$ colors and modified densities of the colors. We call this the *color merging property of the stationary distributions*. Here we explain how one can get this property on the line \mathbb{Z} using our queue vertex model constructions from Section 7.

Remark A.1. For some (but not all) of our particle systems, it is proven that a translation invariant stationary distribution with given densities of colors is unique (see Remarks 7.9, 7.11, and 7.12). When this uniqueness is available, it implies the color merging property of the stationary distributions directly, without reference to queue vertex models.

Remark A.2 (Color merging on the ring). For all colored particle systems on the ring (mASEP, the q -Boson process, and the q -PushTASEP), we readily have uniqueness of the stationary distribution in any given sector determined by the number of particles of each color. Thus, on the ring the color merging property holds automatically.

However, it is not clear if this color merging can be seen at the level of queue vertex models on the cylinder. One reason for this is that queue vertex models on a cylinder are *not* stochastic because they involve summing over input and also output path configurations at vertices (see Section 2.5 for more discussion). At the same time, color merging involves summing over outputs only, and the two summations are not readily compatible. In the remainder of Appendix A, we focus only on systems on the line.

Definition A.3 (Color merging). Suppose we have a partition of $\{0, 1, \dots, n\}$ into k disjoint intervals $I_0, \dots, I_k \subset \{0, 1, \dots, n\}$ which are contiguous (that is, $\max(I_j) = \min(I_{j+1}) - 1$ for all j). The *color merging projection* $\pi = \pi_{I_0, \dots, I_k}$ applied to a state $i \in \{0, 1, \dots, n\}$ or $\mathbf{A} \in \mathbb{Z}_{\geq 0}^n$ maps it into $\pi(i) \in \{0, 1, \dots, k\}$ or $\pi(\mathbf{A}) \in \mathbb{Z}_{\geq 0}^k$, respectively, by assigning to all particles (or arrows) with colors in each interval I_j the new color j .

For a probability measure μ on n -color configurations on \mathbb{Z} (where the maximal number of particles at a site is 1, \mathbb{P} , or ∞ , depending on the particle system), denote by $\pi_*\mu$ the pushforward of μ under the color merging projection π .

Fix a partition $\{0, 1, \dots, n\} = I_0 \sqcup \dots \sqcup I_k$ as in Definition A.3. For the densities ρ_1, \dots, ρ_n of the old colors, denote by

$$\rho'_j := \sum_{i \in I_j} \rho_i, \quad j = 1, \dots, k, \quad (\text{A.1})$$

the densities of the new colors. In Sections 7.3 and 7.4, we showed that the densities (ρ_1, \dots, ρ_n) are in one-to-one correspondence with ordered n -tuples $y_1 > \dots > y_n > 0$. More precisely, y_m parametrizes $\sum_{i \geq m} \rho_i$ (the exact parametrization is different for the q -PushTASEP and the q -Boson process, but for color merging we do not need these exact formulas). Therefore, the new

densities $(\rho'_1, \dots, \rho'_k)$ correspond to the ordered k -tuple

$$y' = (y_{\min(I_1)}, \dots, y_{\min(I_k)}). \quad (\text{A.2})$$

Here, by agreement, if I_1 contains the color 0 (corresponding to the hole), then we need to remove the parameter $y_{\min(I_1)} = y_0$ from the tuple y' .

Denote by μ_y and $\mu_{y'}$ the stationary measures, respectively, for the n - and the k -colored particle systems on the whole line \mathbb{Z} .

Let us record the color merging properties of the vertex weights which appeared in Section 2. We have:

$$\sum_{i_2, j_2: \pi(i_2)=i'_2, \pi(j_2)=j'_2} R_z(i_1, j_1; i_2, j_2) = R_z(\pi(i_1), \pi(j_1); i'_2, j'_2), \quad (\text{A.3})$$

$$\sum_{\mathbf{B}, \ell: \pi(\mathbf{B})=\mathbf{B}', \pi(\ell)=\ell'} L_{s,x}(\mathbf{A}, k; \mathbf{B}, \ell) = L_{s,x}(\pi(\mathbf{A}), \pi(k); \mathbf{B}', \ell'), \quad (\text{A.4})$$

$$\sum_{\mathbf{C}, \mathbf{D}: \pi(\mathbf{C})=\mathbf{C}', \pi(\mathbf{D})=\mathbf{D}'} W_{x,L,M}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = W_{x,L,M}(\pi(\mathbf{A}), \pi(\mathbf{B}); \mathbf{C}', \mathbf{D}'), \quad (\text{A.5})$$

$$\sum_{\mathbf{C}, \mathbf{D}: \pi(\mathbf{C})=\mathbf{C}', \pi(\mathbf{D})=\mathbf{D}'} \mathbb{W}_{s_1, s_2, u}^{(-m)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \mathbb{W}_{s_1, s_2, u}^{(-\pi(m))}(\pi(\mathbf{A}), \pi(\mathbf{B}); \mathbf{C}', \mathbf{D}'). \quad (\text{A.6})$$

Identity (A.3) is immediate. Vertical fusion or a direct verification leads to (A.4). Then by horizontal fusion, (A.4) leads to (A.5). Finally, we get (A.6) from (A.5) by the queue specialization defined in Section 2.4. Note that in (A.6), both m and $\pi(m)$ must be strictly positive.

In probabilistic language, identity, say, (A.6), can be interpreted as follows. Starting from $(\pi(\mathbf{A}), \pi(\mathbf{B}))$, to sample $(\mathbf{C}', \mathbf{D}')$ under the k -color stochastic weight $\mathbb{W}_{s_1, s_2, u}^{(-\pi(m))}$, we may choose any representatives (\mathbf{A}, \mathbf{B}) for the input, sample (\mathbf{C}, \mathbf{D}) under the n -color stochastic weight $\mathbb{W}_{s_1, s_2, u}^{(-m)}$, and then project the output (\mathbf{C}, \mathbf{D}) back to k colors using π . The projection π “forgets” some of the information about the colors, and this operation is the same as the summation in the left-hand side of (A.6). The other identities (A.3)–(A.5) have a similar probabilistic interpretation.

Stacking vertices vertically or horizontally results in a Markov mapping which commutes with the projection π in the same way as described in (A.3)–(A.6). We need an instance of this stacking for queue vertex models on the whole line. Let us take a queue vertex model in $\{-n, \dots, -1\} \times \mathbb{Z}$ with empty input from the left and vertex weights $\mathbb{W}_{s_1, s_2, u}^{(-m)}$ in the column $-m$, $m = 1, \dots, n$. Assume that the parameters s_1, s_2, u make these weights nonnegative. More precisely, we assume that the weights are of the form $\mathbb{W}_{q^{-P/2}, s_m^{(v)}, u/v_m}^{(-m)}$ for the q -PushTASEP (7.5), or $(uy_m; q)_\infty \mathbb{W}_{0, uy_m}^{(-m), q\text{Bos}}$ for the q -Boson process (7.16). In both cases, we can define the steady state distribution of this model as in Proposition 7.4. Denote the corresponding random tuple by $\mathbf{M} = (\mathbf{M}(-n), \dots, \mathbf{M}(-1))$.

Lemma A.4. *Fix a partition $\{0, 1, \dots, n\} = I_0 \sqcup \dots \sqcup I_k$ as in Definition A.3, and let π be the corresponding projection.*

Let the random configurations $\mathbf{V} = (\mathbf{V}(1), \mathbf{V}(2), \dots)$ and $\tilde{\mathbf{V}} = (\tilde{\mathbf{V}}(1), \tilde{\mathbf{V}}(2), \dots)$ be sampled from the vertex models shown on the left and right in Figure 32, respectively. That is, \mathbf{V} is the output of the original n -color queue vertex model run in steady state. The model for $\tilde{\mathbf{V}}$ has input $\pi(\mathbf{M})$, and k -color queue vertex weights $\mathbb{W}_{s_1, s_2, u}^{(-\pi(m))}$ in the column $-m$, $m = 1, \dots, n$. Then

$$\pi(\mathbf{V}) \stackrel{d}{=} \tilde{\mathbf{V}}.$$

Moreover, the distribution of the n -tuple $\pi(\mathbf{M})$ is the steady state for the system of n k -colored queues in tandem in the right side of Figure 32.

Proof. Both claims follow from an inductive application of the color merging property (A.6). \square

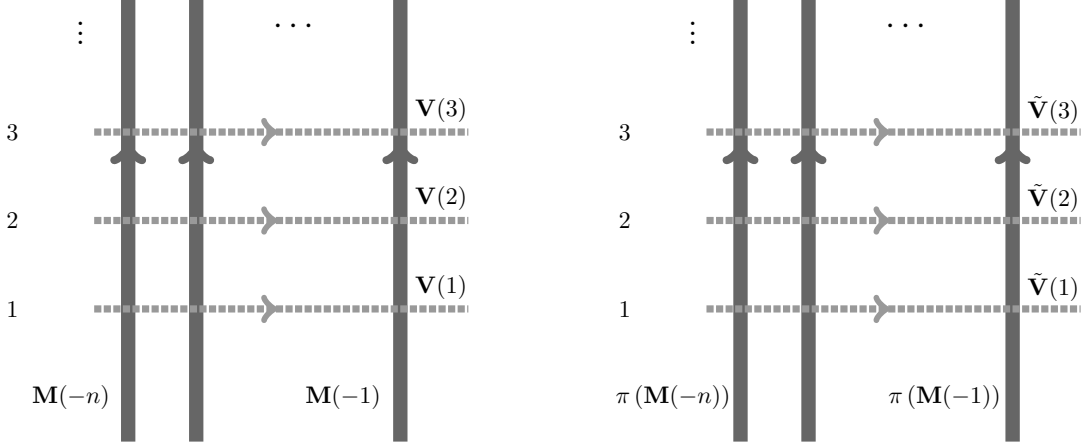


Figure 32: Color merging applied to queue vertex models; see Lemma A.4. There are n columns in both figures.

Lemma A.5. *Let π be a projection which merges colors 1 and 0. Under π , the n -color queue vertex weights $\mathbb{W}_{s_1, s_2, u}^{(-1)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ in the rightmost column turn into the $(n-1)$ -color fused stochastic weights (2.9)*

$$W_{s_1 s_2^{-1} u, \mathbf{L}, \mathbf{M}}(\mathbf{A}', \mathbf{B}'; \mathbf{C}', \mathbf{D}') \Big|_{q^{-\mathbf{L}}=s_1^2, q^{-\mathbf{M}}=s_2^2}, \quad (\text{A.7})$$

where $\mathbf{A}' = (A_2, \dots, A_n) \in \mathbb{Z}_{\geq 0}^{n-1}$, and similarly $\mathbf{B}', \mathbf{C}', \mathbf{D}'$, and

$$\mathbf{A} = (\infty, \mathbf{A}'), \quad \mathbf{B} = (0, \mathbf{B}'), \quad \mathbf{C} = (\infty, \mathbf{C}'), \quad \mathbf{D} = (D_1, \mathbf{D}').$$

Here $B_1 = 0$ because in a queue vertex model, no arrows of color 1 can enter the column (-1) .

Proof. We use the explicit expression (2.12) for $\mathbb{W}_{s_1, s_2, u}^{(-1)}$. Let $\mathbf{P}' = (P_2, \dots, P_n)$, which is a part of the summation index in $\mathbb{W}_{s_1, s_2, u}^{(-1)}$. We have $P_1 = 0$ because $P_1 \leq B_1$. The projection π involves the summation over D_1 from 0 to ∞ . The latter reduces to the q -binomial theorem [GR04, (1.3.2)]:

$$\begin{aligned} \sum_{D_1=0}^{\infty} \mathbb{W}_{s_1, s_2, u}^{(-1)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) &= \frac{(s_1^{-1} s_2 u; q)_{\infty}}{(s_1 s_2 u; q)_{\infty}} \sum_{\mathbf{P}'} \frac{(s_1 s_2 / u; q)_{|\mathbf{P}'|} (s_1 u / s_2; q)_{|\mathbf{B}' - \mathbf{P}'|}}{(s_1^2; q)_{|\mathbf{B}'|}} \\ &\times q^{\sum_{2 \leq i < j \leq n} (B_i - P_i) P_j} (s_1 s_2 u^{-1})^{|\mathbf{B}'| - |\mathbf{P}'|} (s_1^{-1} s_2 u)^{|\mathbf{D}'|} \prod_{i=2}^n \frac{(q; q)_{B_i}}{(q; q)_{P_i} (q; q)_{B_i - P_i}} \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}} \\ &\times q^{\sum_{2 \leq i < j \leq n} D_i (C_j - P_j)} (s_1^2; q)_{|\mathbf{D}'|} \sum_{D_1=0}^{\infty} q^{D_1 (|\mathbf{C}'| - |\mathbf{P}'|)} (u s_2 / s_1)^{D_1} \frac{(s_1^2 q^{|\mathbf{D}'|}; q)_{D_1}}{(q; q)_{D_1}} \end{aligned}$$

$$\begin{aligned}
&= \frac{(s_1^{-1}s_2u; q)_\infty}{(s_1s_2u; q)_\infty} \sum_{\mathbf{P}'} \frac{(s_1s_2/u; q)_{|\mathbf{P}'|} (s_1u/s_2; q)_{|\mathbf{B}'-\mathbf{P}'|}}{(s_1^2; q)_{|\mathbf{B}'|}} \\
&\quad \times q^{\sum_{2 \leq i < j \leq n} (B_i - P_i) P_j} (s_1s_2u^{-1})^{|\mathbf{B}'| - |\mathbf{P}'|} (s_1^{-1}s_2u)^{|\mathbf{D}'|} \prod_{i=2}^n \frac{(q; q)_{B_i}}{(q; q)_{P_i} (q; q)_{B_i - P_i}} \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}} \\
&\quad \times q^{\sum_{2 \leq i < j \leq n} D_i (C_j - P_j)} (s_1^2; q)_{|\mathbf{D}'|} \frac{(s_1s_2uq^{|\mathbf{C}'| - |\mathbf{P}'| + |\mathbf{D}'|}; q)_\infty}{(s_1^{-1}s_2uq^{|\mathbf{C}'| - |\mathbf{P}'|}; q)_\infty}.
\end{aligned}$$

Canceling out the infinite q -Pochhammer symbols, we continue the computation as follows:

$$\begin{aligned}
&= \sum_{\mathbf{P}'} \frac{(s_1^{-1}s_2u; q)_{|\mathbf{C}'| - |\mathbf{P}'|} (s_1^2; q)_{|\mathbf{D}'|}}{(s_1s_2u; q)_{|\mathbf{C}'| - |\mathbf{P}'| + |\mathbf{D}'|}} \frac{(s_1s_2/u; q)_{|\mathbf{P}'|} (s_1u/s_2; q)_{|\mathbf{B}' - \mathbf{P}'|}}{(s_1^2; q)_{|\mathbf{B}'|}} (s_1s_2u^{-1})^{|\mathbf{B}'| - |\mathbf{P}'|} (s_1^{-1}s_2u)^{|\mathbf{D}'|} \\
&\quad \times q^{\sum_{2 \leq i < j \leq n} (B_i - P_i) P_j} q^{\sum_{2 \leq i < j \leq n} D_i (C_j - P_j)} \prod_{i=2}^n \frac{(q; q)_{B_i}}{(q; q)_{P_i} (q; q)_{B_i - P_i}} \frac{(q; q)_{C_i - P_i + D_i}}{(q; q)_{C_i - P_i} (q; q)_{D_i}}.
\end{aligned}$$

From (2.9), one readily sees that the resulting expression matches (A.7), and we are done. \square

Now, we can formulate and prove our main statement about the color merging property of the stationary distributions.

Proposition A.6. *We have $\pi_* \mu_y = \mu_{y'}$. Here μ_y is the stationary distribution of the n -colored q -PushTASEP or the q -Boson process on \mathbb{Z} with the densities of the colors depending on the parameters $y_1 > \dots > y_n > 0$ via (7.13) or (7.17), respectively. The distribution $\mu_{y'}$ is stationary for the corresponding k -colored system, and the parameters y' are obtained from y by merging as in (A.2).*

Proof. Arguing inductively, it suffices to consider the merging any two consecutive colors m and $m + 1$. Case (1) with $m = 0$ is special, we treat it separately first. For $m \geq 1$, we need to show that the output of the column $-(m + 2)$ (distributed as $\mu_{(y_{m+2}, \dots, y_n)}$), passed through two consecutive columns with infinitely many arrows of color m and parameters y_{m+1} and y_m , is distributed as $\mu_{(y_m, \dots, y_n)}$. After that, we can pass the output of the column $(-m)$ into further columns, and the final output will be distributed as $\mu_{y'}$ by the very definition. Thus, in Case (2) it suffices to take $m = 1$ and merge the colors 1 and 2.

Case (1). Step 1. Consider the system of n columns of queue vertex models (“queues in tandem”) which produces the n -color stationary distribution μ_y . Let us denote the output of this system by $\mathbf{V} = (\mathbf{V}(1), \mathbf{V}(2), \dots)$. We know that the color merged output $\pi(\mathbf{V})$ is the same as the output of n queues with $n - 1$ colors, where the columns $-2, \dots, -n$ do not change, and π is applied to the rightmost column (-1) . Indeed, π affects only the column (-1) , and erases the color 1 which does not appear in columns $(-m)$, $m \geq 2$. It thus suffices to show that $\pi(\mathbf{V})$ is distributed as $\mu_{y'}$, that is, it is stationary for the $(n - 1)$ -color interacting particle system.

Case (1). Step 2. Applying Lemma A.5, we see that the weights in the rightmost column (-1) become the q -PushTASEP or q -Boson specializations of the $(n - 1)$ -color weights (A.7). More precisely, for the q -PushTASEP, we have the following weights in the column (-2) and the column (-1) :

$$\mathbb{W}_{q^{-P/2}, s, -q^{1-P/2} s^{-1} u y_2}^{(-2)} \Big|_{s=0} \quad \text{and} \quad W_{-q^{1-P} s^{-2} u y_1, P, M} \Big|_{s=0}, \quad (\text{A.8})$$

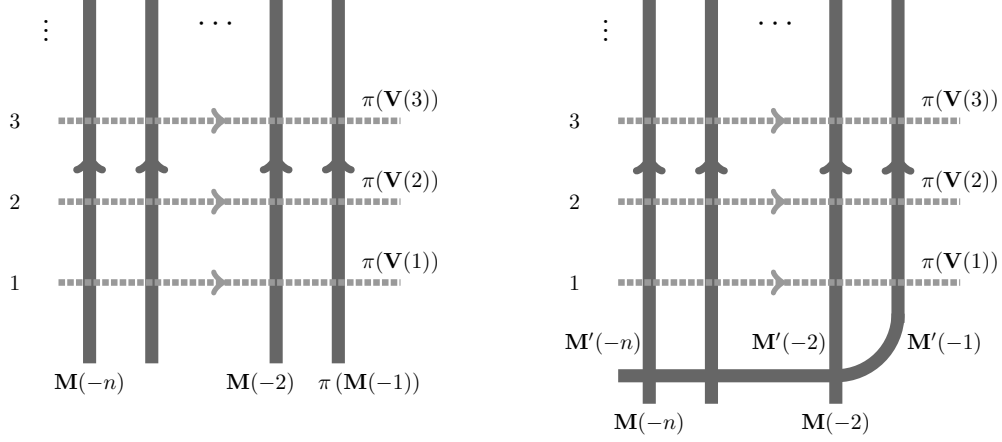


Figure 33: Two ways to sample the random configuration $\pi(\mathbf{V})$ in Case (1) in the proof of Proposition A.6. Left: $\pi(\mathbf{V})$ is the output of a system of n queues with $(n-1)$ colors in the steady state, where π is applied only in the rightmost column. Right: We add auxiliary vertices with the weights $\mathbb{W}_{0,0,y_m/y_1}^{(-2)}$ at the bottom of each column $(-m)$, $m = 2, \dots, n$. The incoming arrow configurations are empty on the left, and the $(n-1)$ -color steady state $(\mathbf{M}(-n), \dots, \mathbf{M}(-2))$ at the bottom. The partition function on the right satisfies the Yang–Baxter equation at the triangular intersection of the lines.

where $q^{-M} = s^2$ (see Section 7.3). Note in particular that for the q -PushTASEP, the queue weights $\mathbb{W}_{\xi,\alpha_m,\nu_m}^{(-m),\text{queue}}$ (see (7.5)) do not scale with ϵ which entered the weights $\mathbb{W}_{\alpha_m,\nu_m}^{(-m),\text{line}}$ in the quadrant through the scaling (7.10).

For the q -Boson process, these weights are, respectively,

$$\mathbb{W}_{\epsilon,s,\epsilon s^{-1}uy_2}^{(-2)} \Big|_{\epsilon \rightarrow 0 \text{ then } s=0} \quad \text{and} \quad W_{(\epsilon/s)^2uy_1,L,M} \Big|_{\epsilon \rightarrow 0 \text{ then } s=0}, \quad (\text{A.9})$$

where $q^{-L} = \epsilon^2$ and $q^{-M} = s^2$ (see Section 7.4).

Let us now choose the auxiliary weights with which (A.8) or (A.9), respectively, satisfy the Yang–Baxter equation. They are found from Proposition 2.10. The auxiliary weights for the q -PushTASEP and the q -Boson are exactly the same, and they have the form

$$\mathbb{W}_{s,s,y_2/y_1}^{(-2)} \Big|_{s=0}. \quad (\text{A.10})$$

One can check that the weights (A.8), (A.9), and (A.10) are nonnegative under our restrictions on the parameters (in particular, recall that $y_1 > y_2 > 0$).

Case (1). Step 3. Let us show that the random output $\pi(\mathbf{V})$ may be sampled using another vertex model which is displayed in Figure 33, right. We claim that

$$(\mathbf{M}(-n), \dots, \mathbf{M}(-2), \pi(\mathbf{M}(-1))) \stackrel{d}{=} (\mathbf{M}'(-n), \dots, \mathbf{M}'(-2), \mathbf{M}'(-1)). \quad (\text{A.11})$$

By Lemma 7.8, the distribution of $(\mathbf{M}(-n), \dots, \mathbf{M}(-2))$ in Figure 33, right, can be generated by infinitely many dashed horizontal lines below the picture (carrying the corresponding queue

vertex weights). Using the Yang–Baxter equation, we may bring an arbitrary number, say, K , of these dashed horizontal lines into the space between the auxiliary line (solid horizontal line) and the output $(\mathbf{M}'(-n), \dots, \mathbf{M}'(-2), \mathbf{M}'(-1))$. See Figure 34 for an illustration. Taking the limit as $K \rightarrow \infty$, we get (A.11) because its left-hand side is the steady state of the system of n queues with $(n - 1)$ colors.

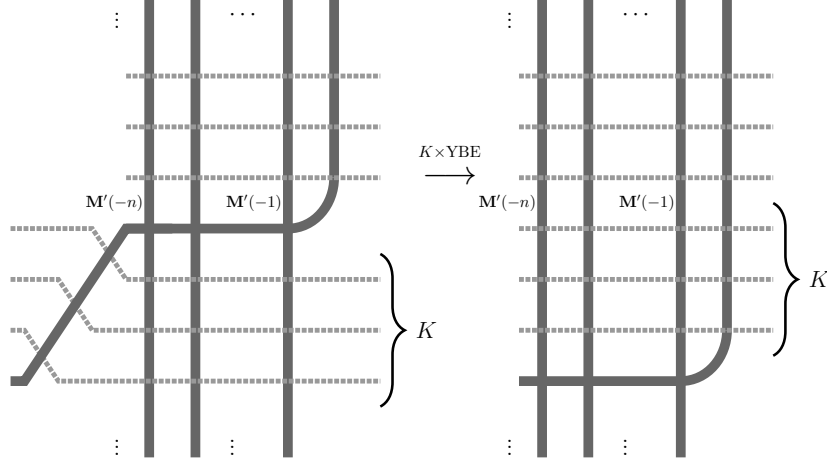


Figure 34: Moving of K dashed horizontal lines in step 3 of the proof of Proposition A.6. In the picture, $K = 3$. We add K empty cross vertices to the left, and drag them through to the right. In the limit as $K \rightarrow +\infty$, the distribution of \mathbf{M}' becomes the same as of \mathbf{M} .

Case (1). Step 4. Now, using the Yang–Baxter equation, we may move the rightmost vertical line in Figure 33, right, all the way to the left of the column $-n$. There, this vertical line can be removed because it carries no arrows. This application of the Yang–Baxter equation transforms the lattice from the right panel of Figure 34 to the left one. The resulting output configuration from the $(n - 1)$ -column system is distributed as $\pi(\mathbf{V})$, and we are done. This shows that $\pi(\mathbf{V})$ has the same distribution as the output of the system of $n - 1$ queues with $(n - 1)$ colors and parameters $y_2 > \dots > y_n > 0$. Thus, $\pi(\mathbf{V})$ is distributed as $\mu_{(y_2, \dots, y_n)} = \mu_{y'}$, as desired.

The applications of the Yang–Baxter equation in Steps 3 and 4 are similar to what we used in the colored Burke’s theorem in Section 7.2.

Case (2). Now let us consider the merging of colors 1 and 2. Consider the system of $n - 1$ queues in tandem, which have $n - 1$ colors, and parameters $y_2 > \dots > y_n$. Denote its output by $\mathbf{V}' = (\mathbf{V}'(1), \mathbf{V}'(2), \dots)$; it is distributed as the $(n - 1)$ -colored stationary distribution $\mu_{(y_2, \dots, y_n)}$. We need to replace the color 2 by 1 in \mathbf{V}' and pass it as an input into the column (-1) with parameter y_1 . To complete the proof, it suffices to show that the output \mathbf{V} of the column (-1) in this scenario has the distribution $\mu_{(y_1, y_3, \dots, y_n)}$, see (A.2).

Notice that by Lemma 2.9, the queue vertex weights in the column (-1) do not depend on the arrows of color 1 incoming from the left. Therefore, instead of replacing the color 2 by 1 in \mathbf{V}' , we may replace the color 2 by 0, and pass the result into the column (-1) . Denote by π° the projection which merges the colors 2 and 0. By Case (1), $\pi^\circ(\mathbf{V}')$ has $n - 2$ colors $3, \dots, n$ and is distributed as $\mu_{(y_3, \dots, y_n)}$. Passing $\pi^\circ(\mathbf{V}')$ through the column (-1) with the parameter

y_1 adds the color 1 and, by the very definition of the queue vertex model, produces \mathbf{V} with the distribution $\mu_{(y_1, y_3, \dots, y_n)}$. This completes the proof. \square

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