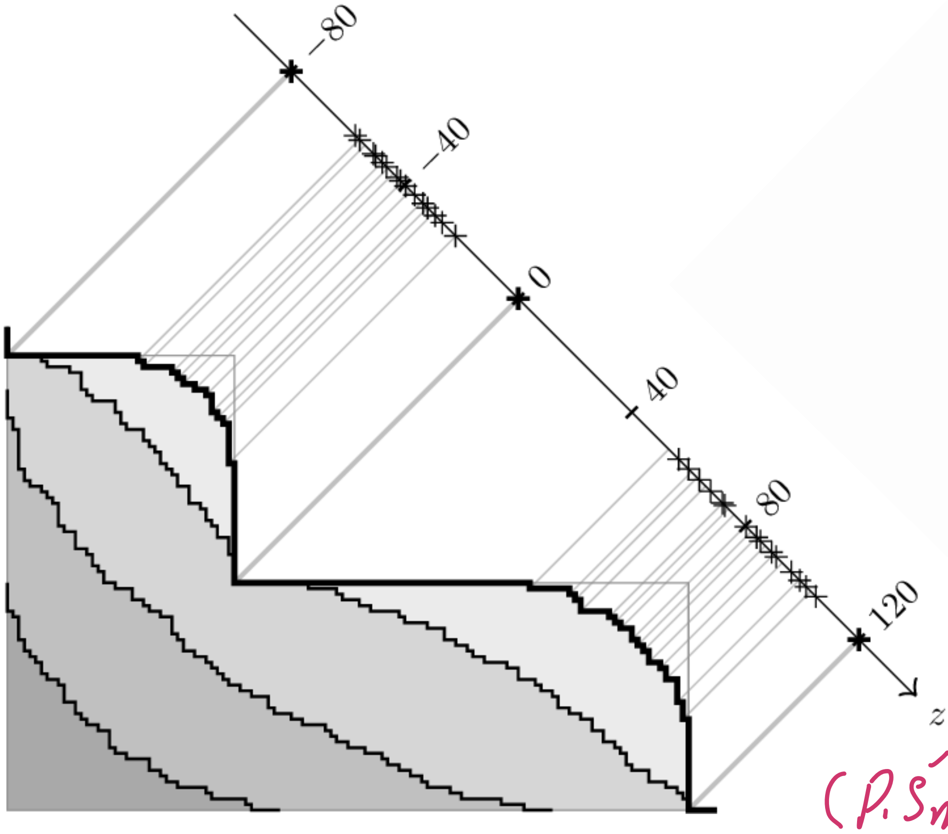
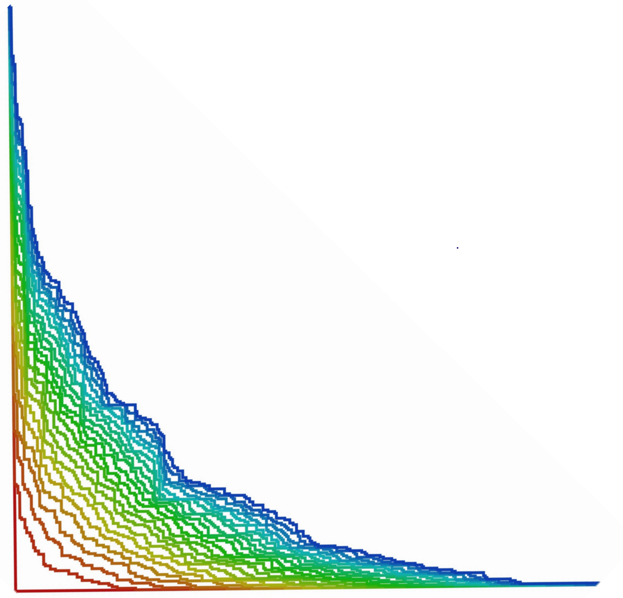
Welcome to ART! (asympt. rep. th.)

lpetrov.cc/art2022/



(P. Śniady 2014)



(LP – Haoyu Li 2019)

Leonid Petrov   office: Ker 209

office hours: M 2:30 – 3:30   zoom (link on web)

W 2:30 – 3:30   in person (Ker 209)

by appointment (schedule online)

Plan for next 4-6 weeks

1. Basic RT of finite groups
2. Inductive limit $S(\infty)$, approximation of characters
3. Combinatorial formulation via Gibbs measures on branching graphs
4. Simpler branching graphs Pascal, ballot, q-Pascal
5. Young graph
6. Symmetric functions
7. Edrei–Thoma's theorem on irred. ch. of $S(\infty)$

Note: I'm thinking of adding an optional reading seminar once a week – any interest? Talk to me after the class

1. Basic representation theory

(Note: some facts w/o proofs)

1.1. Definitions

G – (finite or f.d. compact Lie group)

e, g^{-1}

Examples. $S(n)$

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \downarrow & \downarrow & & \\ b_1, & b_2 & \cdots & b_n \end{pmatrix}$$

(Linear)

<u>Representation</u> $\quad T : G \to \mathrm{End}(V)$ fd vector space/$\mathbb{C}$

$\left(\simeq \mathrm{Mat}_{n\times n}(\mathbb{C})\right)$

$$T(e) = Id$$

$$T(g^{-1}) = T(g)^{-1}$$

$$T(gh) = T(g)\,T(h)$$

<u>Note</u>: In fact, $T : G \to GL(V)$

Extends to $T : \mathbb{C}[G] \longrightarrow \mathrm{End}(V)$

Examples.

$S(n)$ $\qquad\qquad\qquad$ $\mathbb{Z}/n\mathbb{Z}$ $\qquad\qquad$ $\mathbb{Z}$

$$S(n) \to GL_1 \qquad T(\sigma) = \operatorname{sgn}(\sigma)$$

$$S(n) \to GL_1 \qquad T(\sigma) = 1$$

$$S(n) \to GL_n \,, \quad T(\sigma) = \left[1_{j=\sigma(i)} \right]_{i,j=1}^{n}$$

$$T\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Equivalence of representations

$$\begin{tikzcd} G \arrow[r, "T_1"] \arrow[dr, "T_2"'] & V \arrow[d, "\varphi"] \\ & W \end{tikzcd}$$

φ — linear bijection

& diagram commutes

1.2. More definitions

Regular representation.

G - finite, consider the space

$V = \mathbb{C}[G]$, $\dim V = |G|$

$\sum_g c_g \cdot g$

G acts on V by

$T_{reg}(g)\, v = g v, \quad v \in V$ (we could also multiply from the right)

Invariant subspace

V - rep. of G

$W \subseteq V$ is called invariant

$\Leftrightarrow$ all matri. $T(g)$ look like

$$\begin{array}{cc} & \begin{array}{cc} W & \end{array} \\ W & \left(\begin{array}{c|c} & 0 \\ \hline & \end{array}\right) \end{array}$$

Irreducible representation ("irrep"), $\hat{G}$

V is irr. if it doesn't have nontrivial invar. subspaces

G-equivar. φ

$$\varphi(T_V(g))v = T_W(g)\varphi(v)$$

$$\begin{tikzcd} G \arrow[r, "T_V"] \arrow[d, equal] & V \arrow[d, "\varphi"] \\ G \arrow[r, "T_W"] & W \end{tikzcd}$$

linear

1.3. Complete reducibility

Schur's Lemma. T_V, T_W irrep of G

into $GL(V)$, $GL(W)$ resp.

1) V, W not equiv. $\Rightarrow$ no non trivial G-equivar. maps between V, W

2) $V = W$ $\Rightarrow$ all G-equivar. maps are scalar, $v \mapsto \lambda v$, $\lambda \in \mathbb{C}$ fix.

Cor. Abelian finite group $\Rightarrow$ only 1d reps.
(any element $g \in G$ intertwines $V \to V$)

$$\begin{tikzcd} G \arrow[r, "T_V"] \arrow[d, "*h"'] & V \\ G \arrow[ur, "\tilde{T}_V"'] & \end{tikzcd}$$

irred.

$$\tilde{T}_V(g) = T_V(gh) = T_V(hg) = T_V(h)\, T_V(g)$$

dim $V = 1$ $\Leftarrow$ must be $\lambda_h \circ Id$

Proof. $\ker\varphi \subset V$, invariant under T_V

$v \in \ker\varphi$

$$\begin{tikzcd} G \arrow[r, "T_V"] \arrow[d, equal] & V \arrow[d, "\varphi"] \\ G \arrow[r, "T_W"] & W \end{tikzcd}$$

$\varphi(T_V(g)v) =$

$= T_W(g)\varphi(v)$

$= 0$

so $\ker\varphi = 0$ b/c V - irred.

Similarly, $\mathrm{Im}\,\varphi \subseteq W$ is invar.

$w = \varphi(v)$

$T_W(g)w$

$= \varphi(T_V(g)v)$

Proves part 1

Part 2. $V = W$. (Exercise)

Exercise

Prop. G – finite (or ~~compact Lie~~)

$\Rightarrow \exists$ unitary sesquilinear form on V s.t.

$$T : G \longrightarrow U(V) \quad ; \quad T(g^{-1}) = T(g)^*$$

$$(\bar{A})^t = A^* = A^{-1}$$

Proof. Any form $\underline{\langle v, w \rangle}$

Define $$(v, w) = \frac{1}{|G|} \sum_g \langle T(g)v, T(g)w \rangle$$

want

$$(T(h)v, w) = (v, T(h^{-1})w) \quad \forall h$$

$$\| $$

$$\frac{1}{|G|} \sum_g \langle T(\underbrace{gh}_{\tilde{h}})v, T(g)w \rangle \qquad g = \tilde{h} h^{-1} \qquad gh = \tilde{h}$$

$$= \frac{1}{|G|} \sum_{\tilde{h}} \langle T(\tilde{h})v, T(\tilde{h}h^{-1})w \rangle$$

$$= (v, T(h^{-1})w).$$

$\square$

Cpt Lie: avg over G by Haar probab. measure.

Theorem ("Maschke"). $T: G \to U(V)$,

finite or cpt

f.d.

$W \subseteq V$ subrepresentation

$\Rightarrow \exists\, U$ s.t. $V = U \oplus W$

also invar. under action of G

orthogonal direct sum

Cor. $\forall$ f.d. V is $= \bigoplus_{i=1}^{k} V_i$.

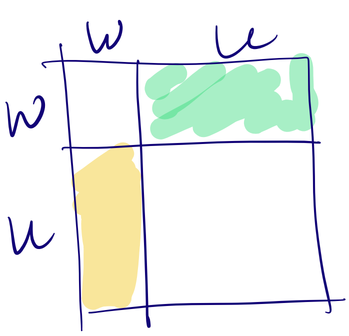
irred.

Proof. Easiest for unitary (but true more generally)

$W \subseteq V$, let $V = W \oplus U$ as vector spaces / unitary form

(i.e. take basis aligned w. W, U)

$T(g)W \subseteq W \Rightarrow T(g)^* W \subseteq W \quad \forall g$



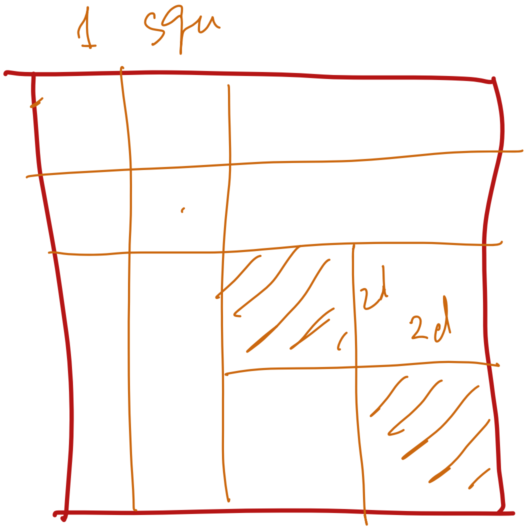
$\Downarrow$

$T(g)U \subseteq U \quad \forall g$

$\square$

1.4. Regular rep. & picture for $S(n)$

T_{reg} $\wr\wr$



S_3

(Peter-Weyl thm.)

Fact. (w/o proof)

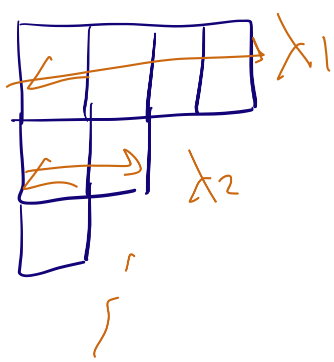
Each irrep. λ appears $\dim V_\lambda$ times

$$\Rightarrow \quad |G| = \sum_{\lambda \in \hat{G}} (\dim V_\lambda)^2 .$$

(Burnside thm)

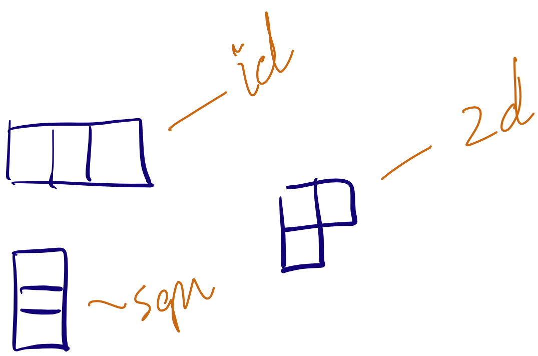
$\boxed{\hat{G} = \text{set of all irreps}}$

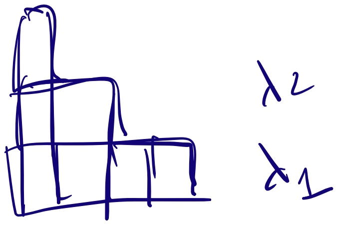
$$\widehat{S(n)} = \{ \text{Partitions } \lambda \text{ of } n \}.$$



$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0, \quad \sum \lambda_i = n$

$S(3)$:





1.5. Example with asymptotics.

($\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, no p-adics here)

$\mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \mathbb{Z}_{16} \subset \cdots$

let $G = \varinjlim \mathbb{Z}_{2^n}$, $\hat{G} = [0,1)$

Also, $S(1) \subset S(2) \subset S(3) \subset \cdots$

Define $S(\infty) = \varinjlim S(n)$

$S(\infty)$ acts on $\mathbb{N}$

On the history...
http://www.math.stonybrook.edu/~aknapp/pdf-files/notices2.pdf

Finite Groups

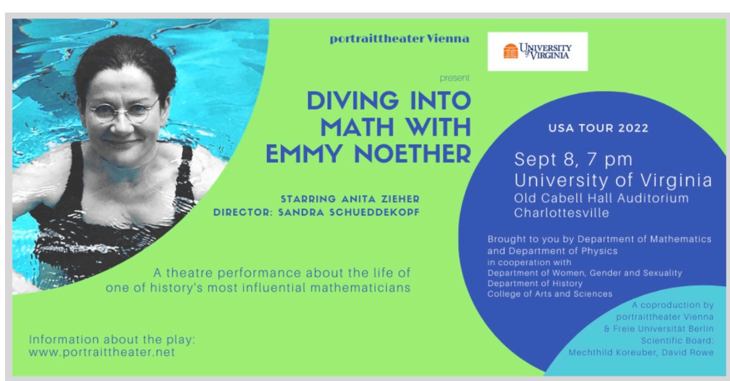
In his work on algebraic number theory, Dedekind noticed a curious thing about finite abelian groups. Let $G = \{g_1 = 1, g_2, \ldots, g_h\}$ be a finite group of order h, and let $x_{g_1}, \ldots, x_{g_h}$ be commuting independent variables parametrized by the elements of G. Dedekind worked with the determinant $\theta(x_{g_1}, \ldots, x_{g_h})$ of the matrix $(x_{g_i g_j^{-1}})$, and in the abelian case he proved that θ admits a factorization

$$\theta(x_{g_1}, \ldots, x_{g_h}) = \prod_{\chi} \Big(\sum_{j=1}^{h} \chi(x_{g_j}) x_{g_j} \Big),$$

the product being taken over all multiplicative characters of G.

Dedekind wondered to Frobenius how this result might generalize to the nonabelian case, and Frobenius ([4], vol. III) began his work in representation theory in 1896 by introducing (irreducible) characters for arbitrary finite groups and solving Dedekind's problem. Today a character is the trace of a representation, but Frobenius did not introduce representations right away. Instead, doing mathematics that looks strange today, he initially worked directly with characters, introducing finite-dimensional representations only in a later paper.

Burnside, starting in 1904, and the young I. Schur, ([13], vol. I), starting in 1905, each redid the theory, the primary objects of each study being matrix representations (homomorphisms into the group of invertible matrices of some size). According to E. Artin ([1], p. 528), "It was Emmy Noether who made the decisive step. It consisted in replacing the notion of a matrix by the notion for which the matrix stood in the first place, namely, a linear transformation of a vector space." Noether's definition was thus essentially the modern general definition of representation given above. For Burnside and Schur the spaces of representations were spaces $V = \mathbb{C}^n$ of column vectors, and the linear transformations were viewed as matrices. Later when representation theory was extended to Lie groups and when quantum mechanics forced infinite-dimensional representations into the study, it would have been awkward to proceed without Noether's viewpoint.



→ Reading Seminar?

→ Mailing List — let me know if you'd like updates

L2 August 25, 2022

1. Basic Representation theory

1.6. Characters

Character of a representation T

$T: G \to \mathrm{End}(V)$

$\boxed{\chi(g) = \mathrm{Tr}\, T(g)}$

$\boxed{\det(1 - zAB) = \det(1 - zBA)}$

$\chi_V(e) = \dim V$;

$\chi_V(g^{-1}) = \overline{\chi_V(g)}$ by unitarity

Central functions (= class functions)

$\chi(gh) = \chi(hg)$

$\boxed{\mathrm{Tr}(AB) = \mathrm{Tr}(BA)}$

$\chi(g_1 g_2 g_3)$ vs $\chi(g_2 g_1 g_3)$

not necessarily equal

Recall g_1, g_2 conjugate if $\exists h$
$g_1 = h g_2 h^{-1}$.

χ only dep. on the conjugacy class.

$V = W \underset{G}{\oplus} U$ (as reps of G)

$\Rightarrow \chi_V = \chi_W + \chi_U$

$\mathrm{Hom}_G(V,W) = \{\varphi$ s.t.

$$\begin{matrix} G \xrightarrow{T_V} V \\ \quad\quad\quad \downarrow \varphi \\ G \xrightarrow{T_W} W \end{matrix} \qquad \begin{matrix} \varphi(T_V(g)v) \\ \| \\ T_W(g)\varphi(v) \end{matrix} \}$$

Schur
<u>Orthogonality</u>. (as class functions)

$$\boxed{\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_g \alpha(g) \overline{\beta(g)}}$$

<u>Thm</u>. χ_λ, χ_μ – irred. ch.

$$\langle \chi_\lambda, \chi_\mu \rangle = 1_{\lambda=\mu}.$$

Schur's Lemma

More generally, $\boxed{\langle \chi_W, \chi_V \rangle = \dim \mathrm{Hom}_G(V,W)}$

<u>Proof</u> Let $P = \frac{1}{|G|} \sum_g g \in \mathbb{C}[G]$,

acts in every rep. V.

P is a projector onto space

$V^G = \{v : T(g)v = v \ \forall g\}$.

<u>Ex</u>. $\mathrm{Tr}\, P = \dim V^G$.

Let $M = \mathrm{Hom}_{\mathbb{C}}(V, W)$, space of linear maps $V \to W$

$\to$ G acts on M by

$$\varphi \longmapsto T_W(g)\, \varphi\, T_V(g^{-1})$$

$\to$ $P \in \mathbb{C}[G]$ acts by

$$\frac{1}{|G|} \sum_g T_W(g)\, \varphi\, T_V(g^{-1})$$

& the image of M under P is $\mathrm{Hom}_G(V,W)$, the G-equivariant maps

$\to$ Now compute the trace of P, using characters

$\mathrm{Hom}_{\mathbb{C}}(V,W)$ basis $E_{ij} = \begin{array}{cc} & \begin{array}{cc} & j \end{array} \\ i & \begin{pmatrix} 0 & 0 \\ & 1 \\ 0 & 0 \end{pmatrix} \end{array}$

$$\mathrm{Tr}\, P = \sum_{ij} (P E_{ij}, E_{ij})$$

V basis e_i $\quad$ W basis f_j

$T_W(g) f_j$

$$T_W(g)\, E_{ij}\, T_V(g^{-1}) = \begin{array}{cc} & j \\ i & \begin{pmatrix} 0 & & 0 \\ & & \\ 0 & & 0 \end{pmatrix} \end{array}$$

$T_V(g^{-1}) e_i$

Trace: of P,

$$\sum_{g,i,j} \left(T_V(g^{-1}) e_i, e_i\right)\left(T_W(g) f_j, f_j\right) / |G|$$

$$= \frac{1}{|G|} \sum_g \chi_W(g)\, \overline{\chi_V(g)} = \langle \chi_W, \chi_V \rangle$$

$\square$

Prop. irreducible characters form a linear basis in the space of all class functions.

(Main property of char.)

$\oplus$ dec. of Treg in $\mathbb{C}[G]$

It is orthonormal wrt the inner product

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_g \alpha(g) \overline{\beta(g)}$$

($\Rightarrow$ # of irreducible reps equals # of conjugacy classes = dim of that space)

<u>Proof</u>. Take T_{reg}, χ_{reg} (rep. space $\mathbb{C}[G]$)

Easy, $\begin{cases} \chi_{reg}(e) = |G|, \\ \chi_{reg}(g) = 0, \quad g \neq e \end{cases}$

By Schur orthog., for any irrep λ,

$$\langle \chi_\lambda, \chi_{reg} \rangle = \dim \mathrm{Hom}_G(V_\lambda, \mathbb{C}[G])$$

But

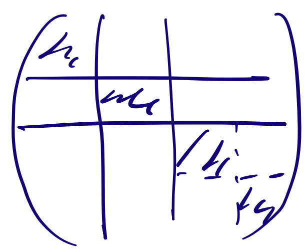
$$\langle \chi_\lambda, \chi_{reg} \rangle = \frac{1}{|G|} \sum_g \chi_\lambda(g) \overline{\chi_{reg}(g)}$$

$$= \dim V_\lambda$$

$\Rightarrow$ V_λ arises $\dim V_\lambda$ times

(Proved Peter-Weyl theorem for finite groups from last time, that

$$T_{reg} = \bigoplus_{\lambda \in \hat{G}} V_\lambda^{\dim V_\lambda}$$



Note: $\chi_{reg}(g)$ is indicator of the conj. class of e, so we're close to showing that χ_λ span all class functions on G.

Next, center of $\mathbb{C}[G]$ is (exercise)

$$\left\{ \sum_g f(g)\, g \;\middle|\; f \text{ - central on } G \right\}$$

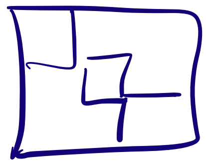
$\dim V_\lambda$

We show: $\mathbb{C}[G] = \bigoplus_{\lambda \in \hat{G}} \mathrm{Mat}_{\dim V_\lambda}(\mathbb{C})$

follows by taking $\boxed{\mathrm{End}_G}$ of:

$$T_{reg} = \bigoplus_{\lambda \in \hat{G}} V_\lambda^{\dim V_\lambda}$$

$\mathrm{Mat}_{\dim V_\lambda}$ — comes from

mapping  $\}$ $\dim V_\lambda$ copies of V_λ, but different elements

by Schur's lemma,
each V_λ maps to another V_λ
with a scalar

$\Rightarrow$ total of $\mathrm{Mat}_{\dim V_\lambda}$
elements.

& $\mathrm{End}_G\, \mathbb{C}[G] = \mathbb{C}[G]$, right mult.

$$\begin{tikzcd} G \arrow[r] & V \arrow[d, "\varphi"] \\ G \arrow[r] & W \end{tikzcd}$$

$\varphi(T_V(g)v)$
$= T_W(g)\varphi(v)$,

$\varphi(v) = vh$,
$h \in \mathbb{C}[G]$.

Taking centers,

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \hat{G}} \mathrm{Mat}_{\dim V_\lambda}(\mathbb{C})$$

$\downarrow$ class functions (dim = # of conjugacy classes in G)

$\downarrow$ scalar matrix for each λ, so # of irreps of G.

dim's coincide
$\Rightarrow$ spaces coincide.

$\square$

Character table of $S(3)$.

	e	c_1	c_2	$\cdots$
χ_1				
χ_2				
χ_3				
$\vdots$				

$c_i =$ conjugacy classes

Recall conjugacy classes of $S(n)$:

— are cycle structures.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 1 & 4 & 6 & 2 & 3 \end{pmatrix}$$

$\downarrow$

cycle struct. is $(3,3,1)$

$1 \to 7 \to 3$

$2 \to 5 \to 6$

4

For $S(3)$:

		e	(12)	(123)
		0	$2,1$	3
	χ_e	1	1	1
1d	χ_{sgn}	1	-1	1
2d	χ_{2d}	2	*	*

$$n! = \sum_\lambda (\dim \lambda)^2$$

$$\frac{1}{6}\Big(\chi_1(e)\,\chi_2(e) + 3\,\chi_1(12)\,\chi_2(12)$$
$$+\,2\,\chi_1(123)\,\chi_2(123)\Big)$$

1.7. Fourier transform / Ex. $\mathbb{Z}/n\mathbb{Z}$

$$f(g) \longmapsto \hat{f}(\lambda)$$

$$\lambda \in \mathrm{Irreps}(G)$$

Def

$$\hat{f}(\lambda) = \sum_{g \in G} f(g)\chi_\lambda(g)$$

multiplication of functions

Fact. $\widehat{f * g} = \hat{f} \circ \hat{g}$.

(exercise)

convolution in G

$$f * g(b) \sum_{h \in G} f(bh^{-1})g(h)$$

Fourier transform for $\mathbb{Z}/n\mathbb{Z}$

$$\hat{G} = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\} \qquad \omega = e^{2\pi i/n}$$

$$\chi_{\omega^j}(i) = \omega^{ij} \qquad (ij \text{ taken mod } n)$$

$$\hat{f}(\omega^j) = \sum_{i=0}^{n} f(i) (\omega^j)^i \quad , \quad |\omega| = 1.$$

coefficients of a Fourier series $\Longleftrightarrow$ function on $\mathbb{Z}_n$ (or $\mathbb{Z}$)

$\Downarrow$

$$\hat{f}(z) = \sum_{i=0}^{n} f(i) z^i$$

(looks familiar?)

Asymptotics.

$\mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \cdots$

$G = \varinjlim \mathbb{Z}_{2^n} \subseteq [0,1]$

dyadic numbers

irred. ch. of G (all reps still 1d)

later we will prove that $\chi(x)$, $x \in G$ is a limit of restrictions of $\chi(x_n)$, $n\to\infty$, in the following sense

(Vershik's ergodic theorem).

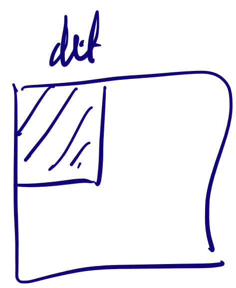
So, <u>characters are</u> :

& Fourier transform limits:

1.8. Positive definiteness

A pos-def if $(Av,v)\geq 0$ $\forall v$

$\Updownarrow$ all princ. minors ≥ 0



$f(x)$, $x \in \mathbb{R}$ pos-def if

$A_{ij} = (f(x_i - x_j))$ pos-def.

---

f - function on G.

Is pos-def. if $\forall\, c_i \in \mathbb{C}$, $g_i \in G$,

$$\sum_{i,j=1}^{k} c_i \bar{c}_j f(g_i g_j^{-1}) \geq 0.$$

<u>Prop.</u> Characters of reps. are pos-def

<u>Proof.</u> Let χ be char. of V, not necess. irred.

$(\cdot,\cdot)$ - unitary form on V

$$\chi(h) = \sum_\alpha (T(h) e_\alpha, e_\alpha)$$

$$\sum_{ij} c_i \overline{c_j}\ \chi(g_i g_j^{-1}) = \sum_{ij\alpha} c_i \overline{c_j} \left(T(g_i) T(g_j^{-1}) e_\alpha, e_\alpha \right)$$

$$= \sum_{ij\alpha} c_i \overline{c_j} \left(T(g_j^{-1}) e_\alpha, T(g_i^{-1}) e_\alpha \right) \ominus$$

Def. $v_\alpha = \sum_i \overline{c_i} T(g_i^{-1}) e_\alpha$

$$\Rightarrow \ominus \sum_\alpha (v_\alpha, v_\alpha) \geq 0 \qquad \square$$

G - finite $\rightsquigarrow$ space $\boxed{\gamma(G)}$

$\mathcal{P}(G)$ = space of funct. on G:

→ central (= class) funct.

→ positive definite

→ normalized, $f(e) = 1$.

Note: $f \in \mathcal{P}(G)$ does not necessarily correspond to actual characters

(only if expands with integer coefficients)

Prop. $\gamma(G)$ is convex.

Proof. $f, g \in \gamma(G) \Rightarrow \alpha f + (1-\alpha) g \in \gamma(G)$

$0 \leq \alpha \leq 1$ $\square$

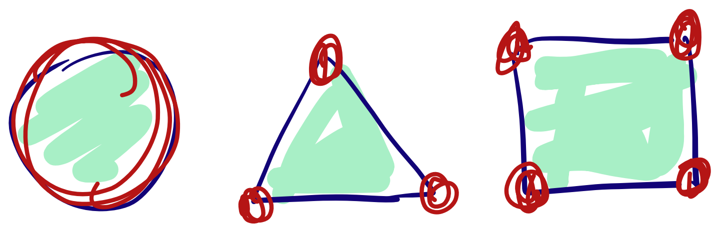
Convex space $\rightsquigarrow$ extreme points

f extreme if $f = \alpha f_1 + (1-\alpha) f_2$

$\alpha \in (0,1)$

$\Rightarrow f_1 = f_2 = f$

Ex



simplex

Simplex:

Def: $\forall$ pt is a unique conv. combin. of extremes

Prop. $\Upsilon(G) \subset Z(\mathbb{C}[G])$ — simplex

$$\mathrm{Ex}\,\Upsilon = \hat{G},$$

extreme points

normalized irr. ch. $\dfrac{\chi_\lambda}{\dim V_\lambda}$

Proof. Enough: $\dfrac{\chi_\lambda}{\dim V_\lambda}$ — extremes

$$= \tilde{\chi}_\lambda$$

$$\tilde{\chi}_\lambda = \alpha f_1 + (1-\alpha) f_2 , \quad \alpha \in (0,1)$$

f_1, f_2 - pos-def.

If f_1, f_2 - actual ch. of repr.

$\Downarrow$

$$V_\lambda = V_1^{d_1} \oplus V_2^{d_2}$$

contradicts irreducibility.

particular case of convex comb. is fine

(General case - next time).

Lecture 3. 8/30

Recall last time:

1.8. Positive definiteness

$Z(\mathbb{C}[G])$

$\Downarrow$

G - finite $\rightsquigarrow$ space $\boxed{\mathcal{P}(G)}$

$\mathcal{P}(G)$ = space of funct. on G:

$\to$ central (= class) funct.

$\to$ positive definite

$\to$ normalized, $f(e) = 1$.

Prop. $\Upsilon(G) \subset Z(\mathbb{C}[G])$ – simplex

$\operatorname{Ex} \Upsilon = \hat{G}$.,

extreme points

normalized irr. ch. $\tilde{\chi}_\lambda := \dfrac{\chi_\lambda}{\dim V_\lambda}$

$\sum_g \chi_\lambda(g)\overline{\chi_\mu(g)}$

Lemma. $\boxed{\chi_\lambda * \chi_\mu = 1_{\lambda=\mu} \dfrac{|G|}{\dim V_\lambda} \chi_\lambda}$

(recall $f_1 * f_2(g) = \sum_h f_1(hg^{-1}) f_2(g)$)

Proof ① T_λ, T_μ two irreps, different.

$\Rightarrow \sum_g T_\lambda(g)_{ik} \, T_\mu(g^{-1})_{\ell j} = 0$

$\hookrightarrow Y = \frac{1}{|G|} \sum_g T_\lambda(g) E_{k\ell} T_\mu(g^{-1})$

$\Rightarrow T_\lambda(h) Y = Y T_\mu(h) \quad \forall h \in G$

because - - -

$g = h^{-1}\tilde{g}$
$g^{-1} = \tilde{g}^{-1} h$

$T_\lambda(h) Y =$

$$= \frac{1}{|G|} \sum_g T_\lambda(\underbrace{hg}_{=\tilde{g}}) E_{kl} T_\mu(g^{-1})$$

$$= Y T_\mu(h)$$

So Y intertwines $T_\lambda, T_\mu \Rightarrow Y = 0$

The ij-th element of Y is

$$\sum_g T_\lambda(g)_{ik} \, T_\mu(g^{-1})_{lj} = 0$$

(many orthogonal poly's come from rep. th. like this)

$$\Rightarrow \quad \chi_\lambda * \chi_\mu = 0 \qquad \lambda \neq \mu.$$

because

$$\boxed{\sum_g \chi_\lambda(hg)\chi_\mu(g^{-1})}$$

$$\sum_{ijk} \sum_g T_\lambda(h)_{ij} \overline{T_\lambda(g)_{ij}} \circ$$

$$\circ \, T_\mu(g^{-1})_{kk}$$

② $\chi_\lambda * \chi_\lambda\ (h) = ?$

$$y = \frac{1}{|G|} \sum_g T_\lambda(g)\, E_{kl}\, T_\lambda(g^{-1}),$$

$$T_\lambda(h)\, y = y\, T_\lambda(h)$$

$$\Rightarrow \quad y = z \cdot \mathrm{Id}$$

$$\mathrm{tr}\, y = \mathrm{tr}\, E_{kl} = 1_{k=l}$$

$\Rightarrow$ for $k = l$, $\quad y = \mathrm{Id} / \dim V_\lambda$

$$\Rightarrow \quad \frac{1_{ab}}{\dim V_\lambda} = \frac{1}{|G|} \sum_g \left(T_\lambda(g)\, E_{ii}\, T_\lambda(g^{-1}) \right)_{ab}$$

$$= \frac{1}{|G|} \sum_g T_\lambda(g)_{ai}\, T_\lambda(g^{-1})_{ib}$$

$\forall i, a, b$

$\chi_\lambda * \chi_\lambda(h)$

$$= \sum_{g, ijk} T_\lambda(h)_{ij} \, \overline{T_\lambda(g)_{ij}} \, T_\lambda(g)_{kk}$$

$i = j = k$ must be

$$\| $$

$$\frac{|G|}{\dim V_\lambda} \sum_i T_\lambda(h)_{ii} \qquad \square$$

Now, $f \in \mathcal{P}^0(G)$

$$f = \sum_\lambda c_\lambda \tilde{\chi}_\lambda$$

if we show $c_\lambda \geq 0$, $\sum c_\lambda = 1$

$\Rightarrow$ we get

$\mathcal{P}^0(G) \simeq$ simplex $\left\{ \begin{array}{c} c_{\lambda_1} + \cdots + c_{\lambda_N} = 1 \\ c_{\lambda_i} \geq 0 \end{array} \right\}$

& $\tilde{\chi}_\lambda$ - extreme

$$(\chi_\lambda * f * \chi_\lambda)(e) =$$

must be ≥ 0

by pos-def.

$$= \Big(\chi_\lambda * \sum_\mu c_\mu \tilde{\chi}_\mu * \chi_\lambda\Big)(e)$$

$$\boxed{= c_\lambda \, \chi_\lambda(e).} \quad \text{non neg. const.}$$

$$= \sum_{g,h} \underbrace{\boxed{\chi_\lambda(g)}}_{= c_g} f(g^{-1}h) \underbrace{\boxed{\chi_\lambda(h^{-1})}}_{= \bar{c}_h} = \sum_{g,h} c_g \bar{c}_h f(g^{-1}h)$$

≥ 0

by pos-def.

To conclude, $\mathcal{V}(G)$ is a simplex with coordinates

$$\boxed{\left\{ c_\lambda \geq 0, \quad \sum_{\lambda \in \hat{G}} c_\lambda = 1 \right\}} \quad (*)$$

$\|$

Space of probab. measures on $\hat{G}$ = space $\mathcal{V}(G)$ of characters of G

Extreme pts $\quad Ex(\mathcal{V}(G))$

$\|$

delta measures from $(*)$

1.9 $S(n)$ representations (w/o proof)

$\longrightarrow$ conjugacy classes

= cycle structures

$$\rho = (\rho_1, \rho_2, \dots, \rho_k)$$

$$\rho_1 \geq \rho_2 \geq \dots \geq \rho_k \geq 0 \quad (k \text{ arb.})$$

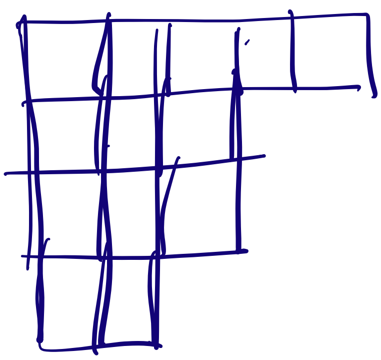
$$\sum \rho_i = n$$

$\longrightarrow$ $\widehat{S(n)}$ (partitions, $|\lambda|$)

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$$

number of boxes $\longrightarrow$ $|\lambda| = \sum \lambda_i = n$

$\ell(\lambda)$ = length, number of nonzero parts

$\lambda =$ 

λ_1

λ_2

λ_3

λ_4

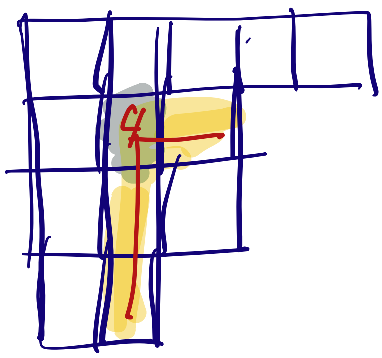
$\ell(\lambda) = 4$

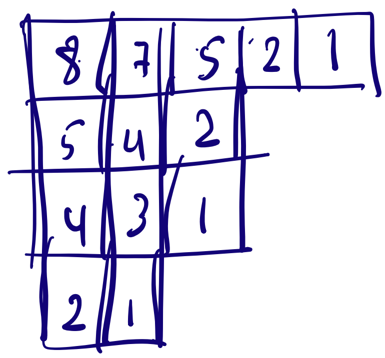
$\lambda = (5, 3, 3, 2)$

Schpecht modules $\sim \lambda$

→ dimensions (SYT, hook)

$$\dim \lambda = \text{hook formula} = n! \Big/ \prod_{\square \in \lambda} h(\square)$$





$$\frac{13!}{(2^3\cdot 3\cdot 4^2\cdot 5^2\cdot 7\cdot 8)}$$

→ characters

$$\boxed{p_\rho = \sum_\lambda \chi_\lambda(\rho)\, s_\lambda}$$

irrep

conj. class

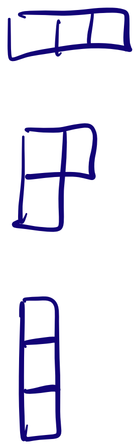
Symm. poly's in $x_1, x_2 \dots x_n$

$$p_\rho = \prod_i \Big(\sum_j x_j^{\rho_i} \Big)$$

$$s_\lambda = \frac{\det \left[x_i^{\lambda_j + n - j} \right]_{i,j=1}^{n}}{\prod_{1 \le i < j \le n} (x_i - x_j)}$$

Schur

$S(3)$



1

$\frac{3!}{3\cdot 1\cdot 1} = 2$

1

$e, \ (12), \ (123)$

$\rho = (111), \ (21), \ (3)$

$$\sum_{\lambda} \chi_{\lambda}((12))\, s_{\lambda}$$

$$\| $$

$$(x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3)$$

2. Character theory of $S(\infty)$ (reduction to branching graphs)

2.1. The group $S(\infty)$ and its conjugacy classes

$S(\infty) = \varinjlim S(n)$ permutes $\mathbb{N} = \{1, 2, \dots\}$

$$S(1) \subset S(2) \subset S(3) \subset S(4) \subset \cdots$$

$\forall \sigma \in S(\infty)$ $\exists n$: $\sigma \in S(n)$

<u>Fact</u>: $\forall$ f.d. rep. of $S(\infty)$ is a $\oplus$ of id & sign rep's.

$h \in S(\infty)$

ghg^{-1}

<u>Conj. classes of $S(\infty)$</u>

$\rho = (\rho_1 \geq \rho_2 \geq \dots \geq 2)$

finite strings ρ.

2.2 Space $\mathcal{X}(S(\infty))$ of characters

<u>Def.</u>

$\mathcal{X}(S(\infty))$

→ χ on $S(\infty)$, class funct. $\chi(gh)=\chi(hg)$

→ pos-def. $g_1 \dots g_n \in S(\infty)$, c_i

$$\sum_{ij} c_i \bar{c}_j \, \chi(g_i^{-1} g_j) \geq 0$$

→ $\chi(e)=1$.

$$\text{"} \quad \chi(g) = \frac{\operatorname{Tr} T(g)}{\dim V} \quad \text{"}$$

$\leftarrow \infty$

$\leftarrow \infty$

<u>Irr. ch. of $S(\infty)$</u>

$\updownarrow$ by def.

<u>$\operatorname{Ex}(\mathcal{X}(S(\infty)))$.</u>

2.3 Vershik's ergodic theorem (1974) and its corollary for $S(\infty)$

Доклады Академии наук СССР
1974. Том 218, № 4

УДК 517.39 **МАТЕМАТИКА**

А. М. ВЕРШИК

ОПИСАНИЕ ИНВАРИАНТНЫХ МЕР ДЛЯ ДЕЙСТВИЙ НЕКОТОРЫХ БЕСКОНЕЧНОМЕРНЫХ ГРУПП

(Представлено академиком А. Н. Колмогоровым 4 III 1974)

(4pp.)

proof, §3

ERGODIC UNITARILY INVARIANT MEASURES ON THE SPACE OF INFINITE HERMITIAN MATRICES

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January 1996

<u>Theorem</u> (V.E.T.) X – compact Polish (complete, separable, metric)

$G = \varinjlim G_n$, G acts on X by continuous maps $X \to X$

all G_n's – finite (or compact)

Let μ – ergodic G-invar. probab. meas. X

$$\mu(gA) = \mu(A) \quad \forall g.$$

$\forall A$ – Borel subset of X

(ergodic means $A \subseteq X$; $\boxed{gA = A}$ $\forall g$
$\Rightarrow \mu(A) = 0$ or 1)

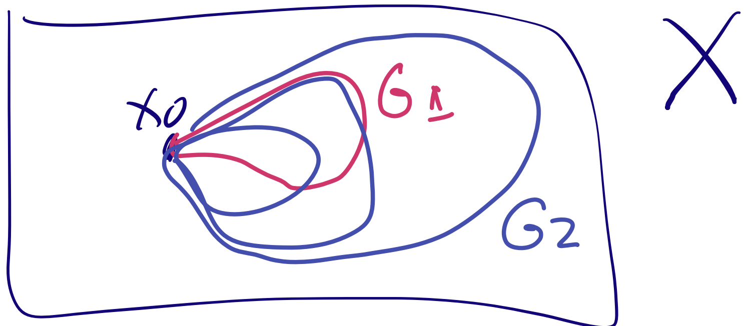
$\Updownarrow$

it is an extreme point, exercise

Then $\exists$ $x_0 \in X$ s.t.

$$\mu = \lim_{n \to \infty} \mu_{x_0}^{(n)}$$ (weak limit of meas.)

where $\mu_{x_0}^{(n)}$ are normalized measures on x_0-orbits under G_n. (these are G_n-invar.)



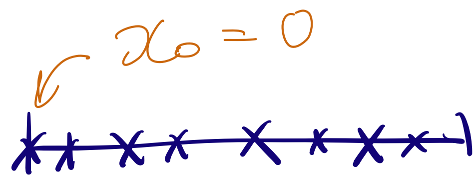
Ex. $X = [0,1]$ (torus)

$G =$ dyadic shift mod 1

$G = \varinjlim \mathbb{Z}/2^n\mathbb{Z}$

μ — uniform on $[0,1]$

$\mu =$ lim. of



$G_3 = \mathbb{Z}/8\mathbb{Z}$

$\mu = \lim.$

$\forall f$

$$\int_0^1 f(x)\, d\mu(x) = \lim_{N=2^n} \frac{1}{N} \sum_{i=1}^{N} f\left(\frac{i}{N}\right)$$

Example. ~~de Finetti's setup, action of~~ ~~$S(\infty)$ on $X=\{0,1\}^{\mathbb{N}}$.~~

V. E. T. implies for $S(n)$:

Thm. 1 $\chi \in \mathrm{Ex}\,\gamma(S(\infty)) \iff \chi$ is a limit of $\chi_n \in \mathrm{Ex}\,\gamma(S(n))$, where the limit is pointwise on $S(\infty)$

2 In other words, $\chi \in \mathrm{Ex}\,\gamma(S(\infty))$

$\iff \exists\ \lambda(n)$, $|\lambda(n)| = n$, s.t.

$\forall\ \rho$ – conj. class in $S(\infty)$ $\boxed{\chi_{\lambda(n)}(\rho) \to \chi(\rho).}$

③ In expansion of restriction to some fixed $S(k)$:

χ on $S(\infty)$

$$\chi\big|_{S(k)} \in \mathcal{R}(S(k))$$
$$\| $$
$$\sum_{\lambda : |\lambda| = k} \chi_\lambda^{S(k)} \cdot \boxed{c_\lambda^k}$$

Do the same to $\underline{\chi_n \text{ of } S(n)}$ $n > k$

$$\chi_n\big|_{S(k)} = \sum_{\lambda : |\lambda| = k} \chi_\lambda^{S(k)} \cdot c_\lambda^k(n)$$

want $c_\lambda^k(n) \longrightarrow c_\lambda^k \quad \forall k.$

<u>Next</u>: Properties of c_λ^k ?

$S(\infty)$

$S(n)$

$S(k+1)$

$S(k)$

$S(k-1)$?

L4. 9/1.

! Colloquium today (on Rep-Th.) 3:45, Clark 102

Notation: $|\lambda| = n \iff \lambda \in \mathbb{Y}_n = \widehat{S(n)}$

(step back from ergodic stuff)

2.4 Restrictions for $S(n)$ (w/o proof) & properties of $\{M_k(\lambda)\}$

(finite $S(n)$'s fact)

Fact. Restrict χ_λ to $S(n)$, $\lambda \in \mathbb{Y}_{n+1}$

(for char.) $\tilde{\chi}_\lambda(\underset{S(n)}{\rho} 1) = f(\rho) \in \mathcal{X}(S(n))$

$\tilde{\chi}_\lambda(e) = 1$

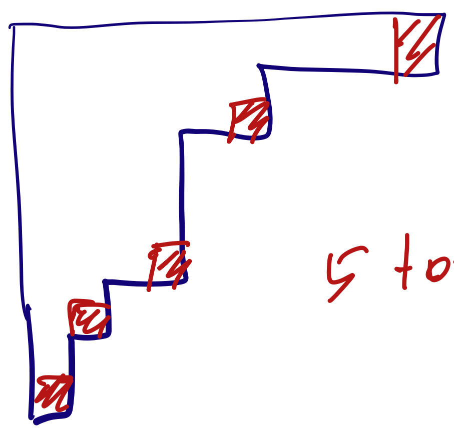
(for repr.) $\boxed{T_\lambda(\rho 1)}$ in $\operatorname{End}(V_\lambda)$

as a rep of $S(n)$, no longer irreducible

Fact w/o proof

$$T_\lambda^{S(n+1)}\Big|_{S(n)} = \bigoplus_{\mu = \lambda - \square} T_\mu^{S(n)}$$

Ex.



λ

5 total μ's $\quad \mu = \lambda - \square$

__Rmk.__ $\lambda = \boxed{|||}$ $\leftarrow$ id

or $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$ $\leftarrow$ sgn

$$\tilde{\chi}_\lambda \Big|_{S(n+1)} = \sum_{\mu = \lambda - \square} \tilde{\chi}_\mu \frac{\dim \mu}{\dim \lambda}$$

$$\dim \lambda = \sum_{\mu = \lambda - \square} \dim \mu$$

__Now:__ $\underline{S(\infty)} \rightsquigarrow \underline{S(k)}, \; S(k-1)$

$$\chi \in \Upsilon(S(\infty)) \longrightarrow \{M_k(\lambda)\}_{\lambda \in \mathbb{Y}_k}$$

coherent measures (def. & properties)

$M_k(\lambda)$ prob meas. on $\mathbb{Y}_k$

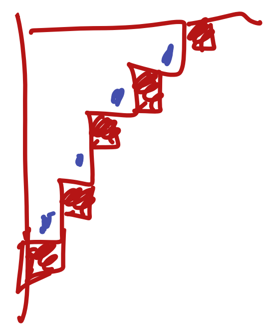
$$\boxed{\chi\big|_{S(k)} = \sum_{\lambda \in \mathbb{Y}_k} M_k(\lambda)\, \tilde{\chi}_\lambda}$$

Prop. (coherent)

$$\boxed{M_{k-1}(\mu) = \sum_{\lambda = \mu + \square} M_k(\lambda)\, \frac{\dim \mu}{\dim \lambda}}$$

Proof.

$$\sum_{\lambda \in \mathbb{Y}_k} M_k(\lambda)\, \tilde{\chi}_\lambda \Big|_{S(k-1)}$$



$$= \sum_{\lambda} \sum_{\mu = \lambda - \square} M_k(\lambda)\, \frac{\dim \mu}{\dim \lambda}\, \tilde{\chi}_\mu$$

$$= \sum_{\mu \in \mathbb{Y}_{k-1}} M_{k-1}(\mu)\, \tilde{\chi}_\mu \qquad \square$$

$\boxed{\Upsilon(S(\infty))}$ $=$ $\boxed{\text{Space of coherent prob. meas. } \{M_n \text{ on } \mathbb{Y}_n\}_{n=1,2,\dots}}$

Coh

isomorphic
as convex sets
$\Downarrow$
Extreme
coh. meas.
$=$ irred.. ch. of $S(\infty)$

We are after $\operatorname{Ex}(\Upsilon(S(\infty)))$.

$\Upsilon(S(\infty)) = \text{Coh}$, the space of coherent measures

(& need ergodicity to approximate Coh)

2.3. Vershik's ergodic theorem (a gentler discussion)

1) Usual ergodic theorem (Birkhoff)

$G = \mathbb{Z}$ acts on (X, μ) prob measr

($\Leftrightarrow$ one invertible operator T & its powers)

Space: X - cpt. sep. metric
μ - prob. Borel measure

T preserves measure: $\forall A$

$$\mu(TA) = \mu(A)$$

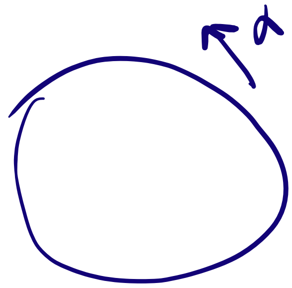
[note: usually $\mu(T^{-1}A) = \mu(A)$]

μ - ergodic: If $TA = A$
then $\mu(A) = 0$ or 1

$\Leftrightarrow$ μ is extreme among all T-inv. measures

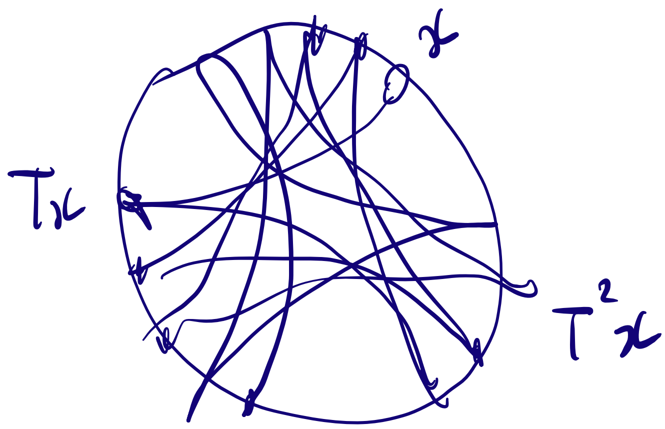
Ergodic theorem: μ-a.every $x \in X$, $f \in L^2(\mu)$

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \xrightarrow[N\to\infty]{} \int_X f\,d\mu$$

Example $X =$ 

, $T =$ irrational rotation α

$\mu =$ Lebesgue



$X = \{0,1\}^{\mathbb{Z}}$, $\mu =$ Product measure

$P(1) = p$, $P(0) = 1-p$

$(T\vec{x})_n = x_{n+1}$

Bernoulli shift

2) $G = \varinjlim G_n$, Vershik's ergodic thm.
→ similar

$$\int_X f \, d\mu = \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{g\in G_n} f(g x_0) \quad (\forall f)$$

(set of such x_0 is μ-a.e.)

G acts on (X, μ)
& ergodic

2.5 Application / example

$S(\infty)$ acting on $X = \{0,1\}^{\mathbb{N}}$ by permutations.

- exchangeability
- Pascal triangle
- action on the space of paths
- approximation

de Finetti. thm

1. Ergodic meas on X wrt $S(\infty)$

=

Bernoulli product measures μ_p (iid coin flips $\sim p$, $p \in [0,1]$)

---

2. Exchangeable. $\xi_1 \xi_2 \xi_3 \ldots$ $\quad \xi_i \in \{0,1\}$

distri is the same as for

$\xi_{\sigma_1} \xi_{\sigma_2} \xi_{\sigma_3} \ldots \quad \forall \sigma \in S(\infty)$

de F. = $\exists\ \nu$ on $[0,1]$ s.t.

$\vec{\xi}$ is obtained as:

① sample random p from ν on $[0,1]$

② Given p, flip iid coins $\sim p$.

③ If μ - exch. then $\exists! \, \nu$

μ - convex comb. of μ_p

$$\mu = \int_0^1 \mu_p \, \nu(dp)$$

$\mathcal{P}^\nu = \{\text{exch. measures}\}$

$\mathrm{Ex}(\mathcal{P}^\nu) = [0,1]$.

V.E.T. applied here.

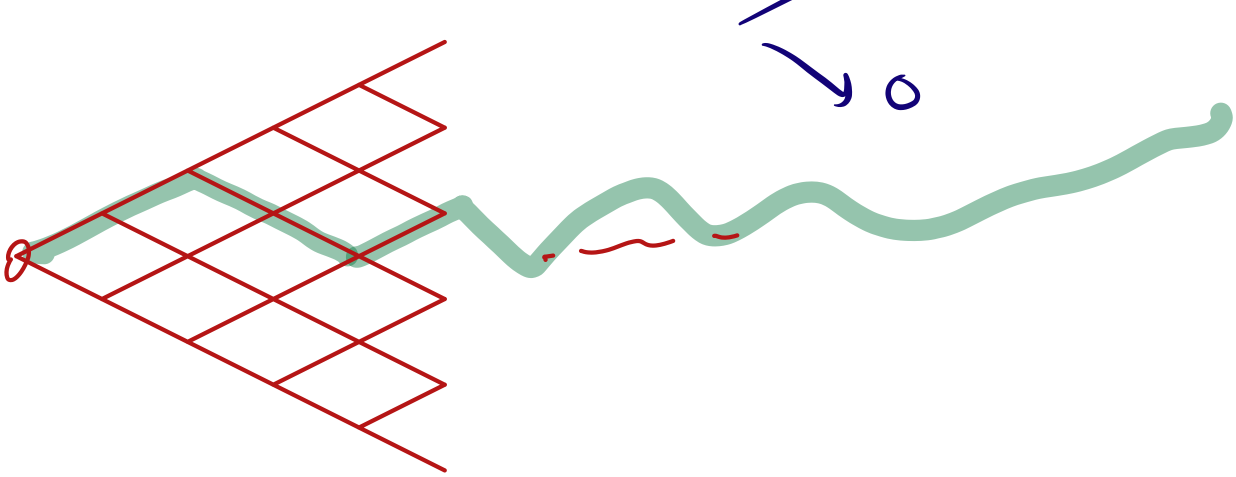
If μ - ergodic for $S(\infty)$ $\curvearrowright$ $\{0,1\}^{\mathbb{N}} = X$

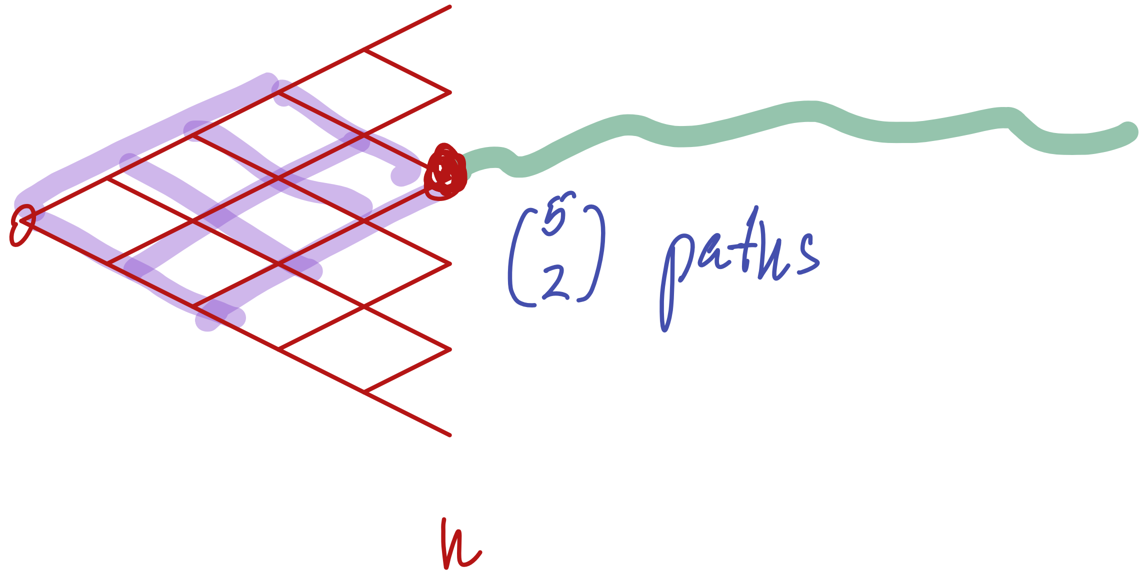
Then $\exists$ $\vec{x}^0 \in X$

s.t. $\mu = \lim_{n\to\infty} \mu_n^{\vec{x}^0}$ ← ?

$\mu_n^{\vec{x}^0}$ = uniform measr on all sequences of length n with # of 1's $= x_1^0 + \dots + x_n^0$.

$\vec{x}^0$ = path





$$\mu_n^{\vec{x}^0} \longrightarrow \mu \qquad n \to \infty$$

means joint convergence of

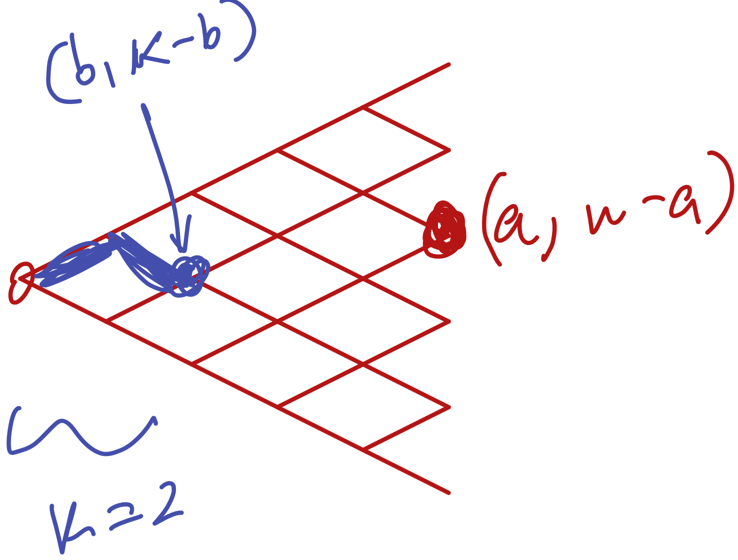
$$\left(\xi_1^{(n)}, \dots, \xi_k^{(n)}\right) \longrightarrow \left(\xi_1, \dots, \xi_k\right)$$

$\forall k$, fixed

$$\mathbb{P}\left(\xi_1^{(n)} = \alpha_1, \dots, \xi_k^{(n)} = \alpha_k\right) = ? \qquad (*)$$

$$\alpha_1 + \dots + \alpha_k = b$$

$$x_1^0 + \dots + x_n^0 = a$$



a dep on $\vec{X}^0$

$$(\ast) = \frac{\binom{n-k}{a-b}}{\binom{n}{a}}$$

μ-ergodic $\iff$ $\exists\, a(n)$ s.t.

$$\frac{\binom{n-k}{a(n)-b}}{\binom{n}{a(n)}}$$

have a limit

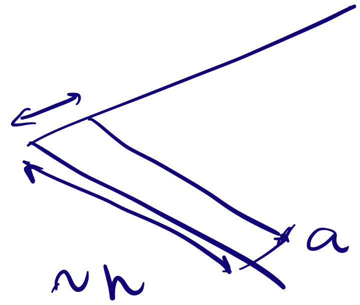
$= \mu(\xi_1 = d_1 \ldots \xi_k = d_k)$

$$\| \quad \frac{(n-k)!}{n!} \cdot \frac{a(n)!}{(a(n)-b)!} \cdot \frac{(n-a(n))!}{(n-k+b-a(n))!}$$

Case 1. $a/n \to 0$

$$\Downarrow$$

$$\mu(\xi_1 = \dots = \xi_k = 0) = 1$$



$\left(\text{or } \frac{a(n)}{n} \to 1\right)$

Case 2. $a/n \not\to 0$, $\frac{n-a}{n} \not\to 0$

$$= \frac{1}{n^k}(a(n))^b\,(n-a(n))^{k-b}$$

$$= \left(\frac{a(n)}{n}\right)^b \left(1 - \frac{a(n)}{n}\right)^{k-b}$$

$$\Downarrow$$

$\frac{a(n)}{n}$ must $\longrightarrow p \in (0,1)$

$\Rightarrow$ μ-ergodic $\Rightarrow$ $\mu = \mu_p$.

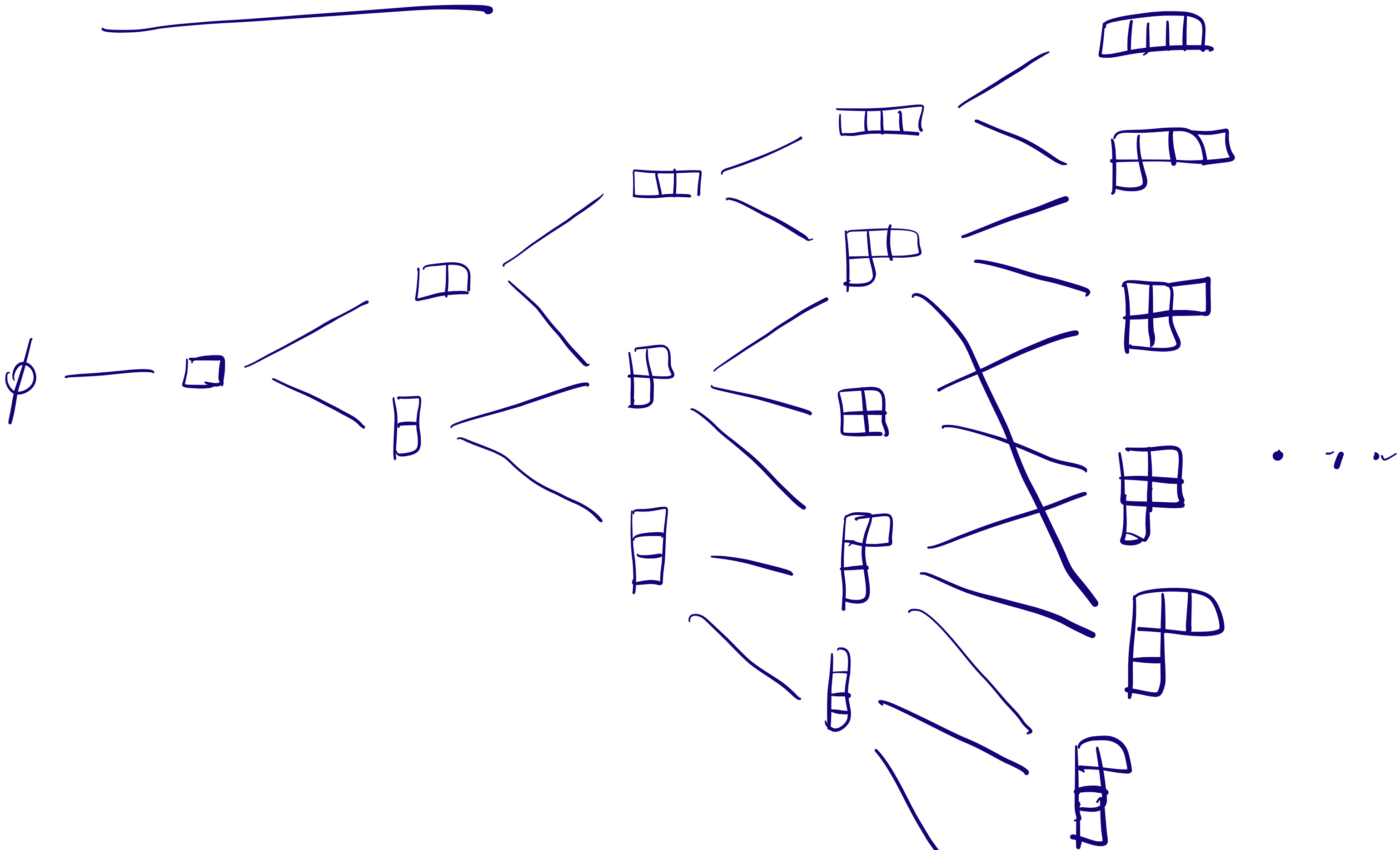
Bernoulli

2.6 Branching graph associated with $S(\infty)$, and $\mathrm{Ex}(\mathcal{M}(S(\infty)))$

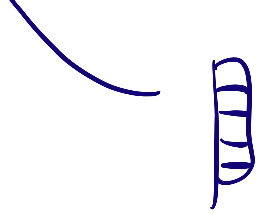
Ex as space of ergodic measures.
V.E.T. $\Rightarrow$ approximat.

(analogy with Pascal triangle as the problem, but approximation is $\chi_{\lambda(n)} \to \chi$ on the group level) ...)

Young graph (lattice)



ϕ



2.7 Proof of Vershik's ergodic theorem

Schur - Weyl duality (after the colloquium on 9/1)

$V = \mathbb{C}^N$

$V^{\otimes n} = W \qquad \dim W = N^n$

$S(n)$ permutes vector factors

$v_1 \otimes \cdots \otimes v_n \longmapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$

$GL(N, \mathbb{C}) \ni A$ acts by

$v_1 \otimes \cdots \otimes v_n \mapsto A v_1 \otimes \cdots \otimes A v_n$

$\{T_\sigma : \sigma \in S(n)\}' =$ def all operat. on W commuting w. all T_σ.

$\{B : BT_\sigma = T_\sigma B \ \forall \sigma\}$

SW-duality: this is generated by $GL(N)$ action

Similarly in the other direction

$\{GL_N\text{-operators}\}' = \operatorname{span}\{T_\sigma : \sigma \in S(n)\}$

$$W = \bigoplus_{\lambda} V_\lambda^{S(n)} \otimes V_\lambda^{GL_N}$$

as a bimodule

all part. $|\lambda| = n$
& $\leq N$ rows

$$N^n = \sum_\lambda \dim \lambda \cdot \dim_{GL_N} \lambda$$

Schur-Weyl random partitions.

Summary so far

(Reminder: please interrupt me if unclear!)

→ finite $S(n)$ representations (λ, $\dim\lambda$, branching)

(Y. Zhao's notes)

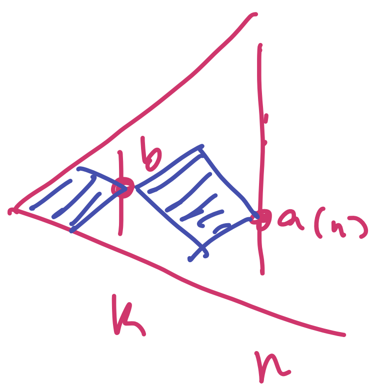
→ Ergodic method, that ∞-level objects are approximated by finite ones

→ Example with $\{0,1\}^{\mathbb{N}}$
& action of $S(\infty)$

$\forall$ ergodic infinite exchangeable $\vec{\xi}$
there exists $a(n)$ s.t. $\forall k$,

$$\mathbb{P}(\xi_1 + \dots + \xi_k = b) =$$

$$= \lim_{n\to\infty} \frac{\binom{k}{b}\binom{n-k}{a(n)-b}}{\binom{n}{a(n)}} \qquad (*)$$



Showed: For limit $(*)$ to exist,
it must be $\dfrac{a(n)}{n} \longrightarrow p \in [0,1]$.

$\Rightarrow$ $\vec{\xi} \sim$ iid coin flip sequence
with $p = \mathbb{P}(1)$.

→ Next: proof* of Vershik ergodic theorem & application to $S(\infty)$ & more general branching graphs.

a bit of „real analysis"...

L5 2.7. Proof of the ergodic theorem

X – cpt separable metric (example: Paths in Pascal Δ)

$C(X)$ cont. funct.

Def. prob. meas. ν_n on X weakly converge to ν if $\forall f \in C(X)$,

$$\langle f, \nu_n \rangle \longrightarrow \langle f, \nu \rangle$$

Notation $\langle f, \nu \rangle = \int_X f \, d\nu$

Lemma (not proving) $\exists$ countable family $\Psi \in C(X)$ determining weak convergence

$\nu_n \to \nu$ if $\forall f \in \Psi$,

$$\langle f, \nu_n \rangle \to \langle f, \nu \rangle.$$

Def. G_n finite group, $x_0 \in X$

$\mu_n^{(x_0)}$ is by def.

$$\langle f, \mu_n^{x_0} \rangle = \frac{1}{|G_n|} \sum_{g \in G_n} f(g x_0)$$

$\mu(gA) = \mu(A)$

Note. (also works for compact G_n & noncompact X)

Thm. $G = \varinjlim G_n$, G_n finite*

μ - ergodic G-inv. Borel meas on X

$\Rightarrow \exists\ x_0$ s.t. $\mu = \lim_n \mu_n^{x_0}$ (weak limit)

& the set of such x_0 is of full μ-measure

Proof. Let $f \in C(X)$

$$C(X) \ni f_n(x) := \langle f, \mu_n^x \rangle = \frac{1}{|G_n|} \sum_{g \in G_n} f(gx)$$

$$\bar{f} = \langle f, \mu \rangle \cdot 1 \qquad \text{(constant)}$$

Exercise: enough to show $f_n(x) \to \bar{f}(x)$ for ν-a.e. x (uses lemma about Ψ)

Step 1. $f_n \longrightarrow f$ in $L^2(\mu)$

Step 2. $\exists$ μ-a.e. limit $f_n \to f_\infty$.

($\Rightarrow f_\infty = \bar{f}$ and we're done)

$$\langle f, \mu \rangle = \int_X f(x)\mu(dx) = \int_X f(gx)\mu(dx)$$

$$gf(x) = f(gx)$$

Proof of step 1.

G, G_n act in $L^2(X, \mu)$

Let V_n, $V \subset L^2(X, \mu)$ be spaces of G_n or G invar. funct.

$$V_n = \{ f : \quad f(gx) = f(x) \quad \forall\, g \in G_n \}.$$

$\boxed{\dim V = 1}$ by def. of egrodicity of μ.

Let P_n $\checkmark$ be orthog projector onto V_n $\checkmark$, P, V

$(P^2 = P$ range of P is $V)$ $\quad Pf = \langle f, \mu \rangle \cdot \mathbb{1}$

Since $G = \varinjlim G_n$,

$$P_n f \to \underbrace{Pf}_{} \quad \text{in } L^2 \qquad (\forall f)$$

it is constant, equal to $\bar{f}$

this is $f_n(x) = \frac{1}{|G_n|} \sum_{g \in G_n} f(gx)$

(Proves step 1)

f_n have a.e. limit

Proof of step 2.

Let $E_N = \{ x : \sup_{1 \le n \le N} f_n(x) > 0 \}$

$E_\infty = \bigcup_{N=1}^{\infty} E_N = \{ x : \sup_n f_n(x) > 0 \}$

$E_{mN} = \{ x : f_m(x) > 0,\ f_i(x) \le 0,\ m+1 \le i \le N \}$

G_m invariant

$E_N = E_{1N} \cup \dots \cup E_{NN}$

so $\int_{E_{mN}} f \, d\mu = \int_{E_{mN}} f(gx) \, \mu(dx)$ $\forall g \in G_m$

$= \int_{E_{mN}} f_m(x) \mu(dx)$ (by averaging over G_m)

We have $f_m > 0$ on E_{mN}

$\Rightarrow \int_{E_{mN}} f \, d\mu \geq 0 \quad \Rightarrow \int_{E_N} f \, d\mu \geq 0$

$\Rightarrow \int_{E_\infty} f \, d\mu \geq 0$. $(*)$

Finally let $X_{ab} = \{ x : \underline{\lim} f_n < a < b < \overline{\lim} f_n \}$
$(a<b)$

$\rightarrow X_{ab}$ is G-invariant

$\rightarrow$ by ergodicity, $\mu(X_{ab}) = 0$ or 1
we want 0

$(*) \Rightarrow$ (exercise)

$$a\mu(X_{ab}) \geq \int_{X_{ab}} f \, d\mu \geq b\mu(X_{ab})$$

we know $a<b \Rightarrow \mu(X_{ab}) = 0$
$\forall a<b$

$\Rightarrow \underline{\lim} f_n = \overline{\lim} f_n$ & step 2 done $\square$

3. Branching graphs (with finite floors)

3.1. General definitions

- graph
- coherent measures
- harmonic functions

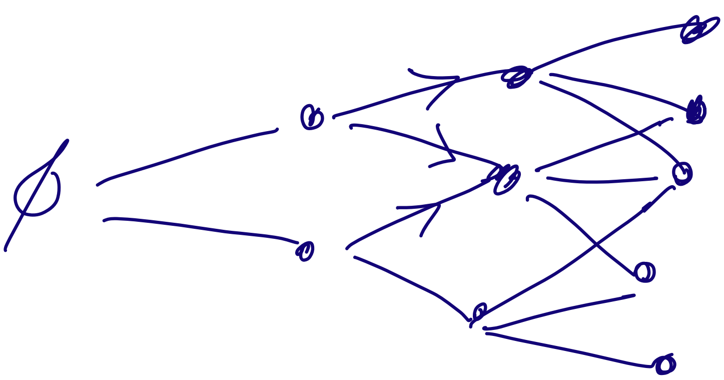
$\mathbb{G}_n$ — finite sets, $n \geq 0$

$\mathbb{G}_0 = \{\varnothing\}$, $\boxed{\mathbb{G} = \bigcup_{n=0}^{\infty} \mathbb{G}_n}$

edges connect $\mathbb{G}_n, \mathbb{G}_{n+1}$ $\forall n$

directed $n \to n+1$

$\mu \nearrow \lambda$ $\mu \in \mathbb{G}_{n-1}$, $\lambda \in \mathbb{G}_n$, connected



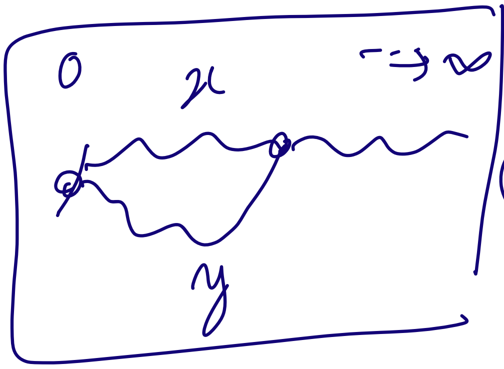
$\forall \lambda \in \mathbb{G}_n$, $\exists$

$\mu \nearrow \lambda$

$\nu \not\nearrow \lambda$

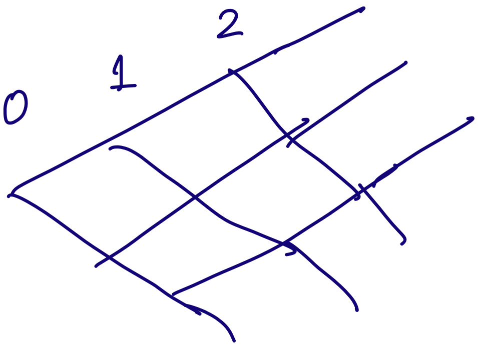
$\mathbb{G} \rightsquigarrow X(\mathbb{G})$ of ∞ paths in $\mathbb{G}$.

compact space



x, y close if x, y are eventually equal

$\lambda \in G_n$, $\quad \underline{\dim_G \lambda} \overset{\text{def}}{=}$ $\boxed{\#\text{ of paths } \varnothing \longrightarrow \lambda}$



Pascal, $\quad \boxed{\dim(a, n-a) = \binom{n}{a}}$

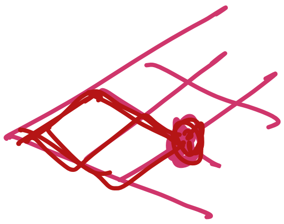
Want random paths, compatible with graph structure.

Probab. μ on $X(G)$ is called <u>central</u>

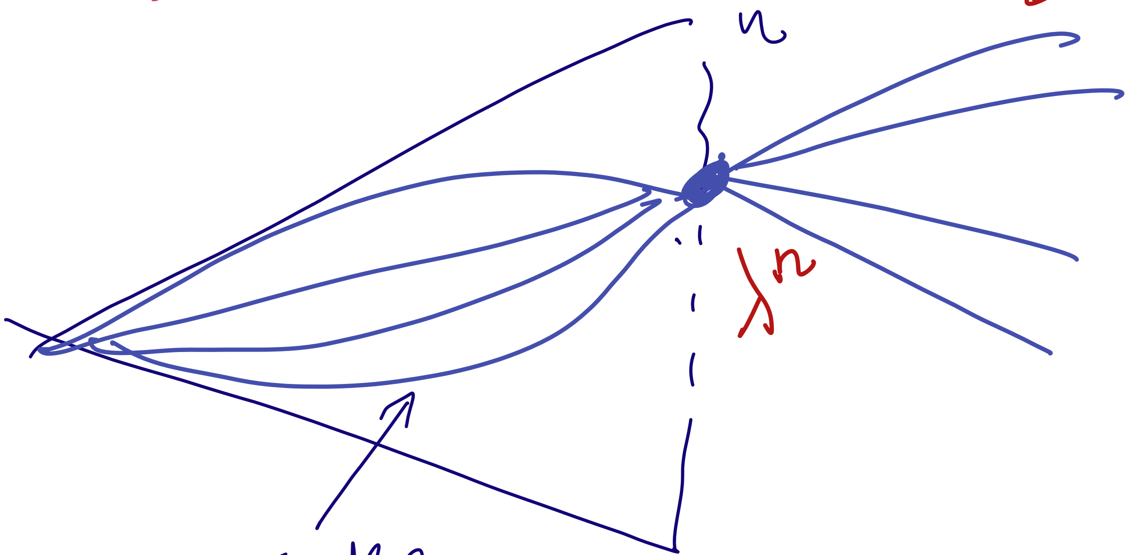
$\forall$ fixed path $\varnothing \nearrow \lambda^1 \nearrow \lambda^2 \nearrow \cdots \nearrow \lambda^n \nearrow \cdots$

$$\mu(x^1 = \lambda^1, x^2 = \lambda^2, \ldots, x^n = \lambda^n)$$

$$= \frac{1}{\dim_G \lambda^n} \cdot f(\lambda^n)$$



ignores



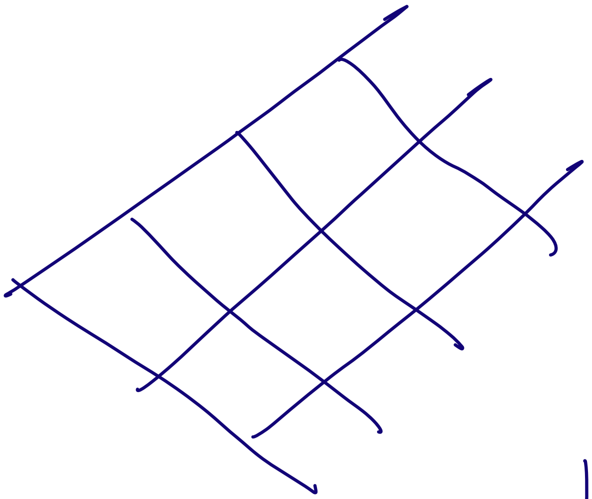
indep.
of how you reach λ

$\{\mu\text{-central}\} \xleftrightarrow{1-1}$

Def. $\{$coherent measures on $\mathbb{G}_n\}$

$$M_n(\lambda) = \mu(\text{path passes through } \lambda)$$

$\lambda \in \mathbb{G}_n$



μ_p = iid coin flip sequence

$$M_n\big((a, n-a)\big) = \binom{n}{a} p^a (1-p)^{n-a}$$

$$\frac{M_n(a, n-a)}{\dim (a, n-a)}$$

$$\frac{M_n(\lambda)}{\dim \lambda} = \mu\left(\begin{array}{c}\text{any particular} \\ \text{path from } \emptyset \\ \text{to } \lambda\end{array}\right)$$

Lemma μ - central $\Rightarrow$ M_n

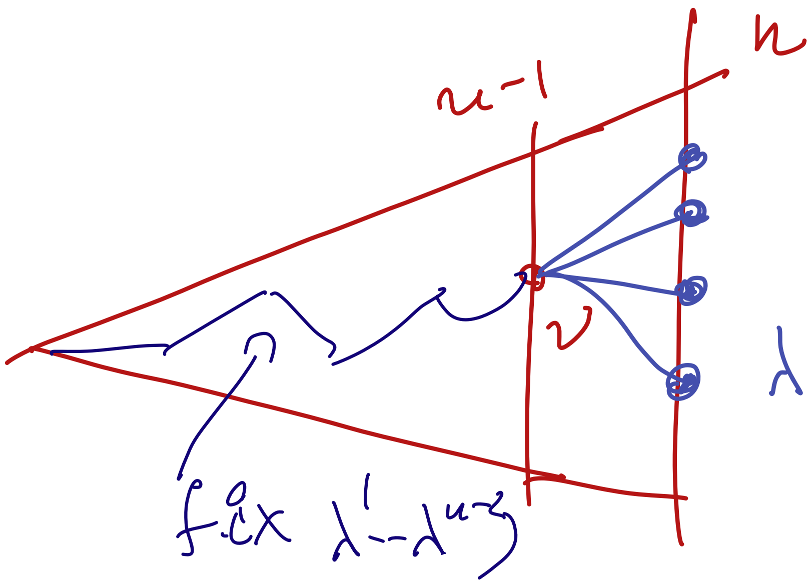
defined by

$$M_n(\lambda) = \mu(\text{path passes through } \lambda)$$

satisfy

$$M_{n-1}(\nu) = \sum_{\lambda : \lambda \searrow \nu} M_n(\lambda) \frac{\dim \nu}{\dim \lambda}$$

Proof



$$\frac{M_{n-1}(\nu)}{\dim \nu}$$

$$\mu(x^1 = \lambda^1, \ldots, x^{n-2} = \lambda^{n-2}, x^{n-1} = \nu)$$

$$\|$$

$$\sum_{\lambda} \mu(x^1 = \lambda^1, \ldots, x^{n-2} = \lambda^{n-2}, x^{n-1} = \nu, x^n = \lambda)$$

$$// \quad \mu_n(\lambda) / \dim \lambda$$

Boundary of $\mathbb{G}$ $= Ex(\mathbb{G})$

$\overset{def}{\parallel}$

space of all ergodic (extreme) central measures

$\forall \mu \in Ex(\mathbb{G})$,

$\mu = \lim_{n \to \infty}$ of finite central measures

L6. 9/8.

3.2. Example of a branching graph – Pascal triangle

| all def's in this example + why it is called „boundary" |

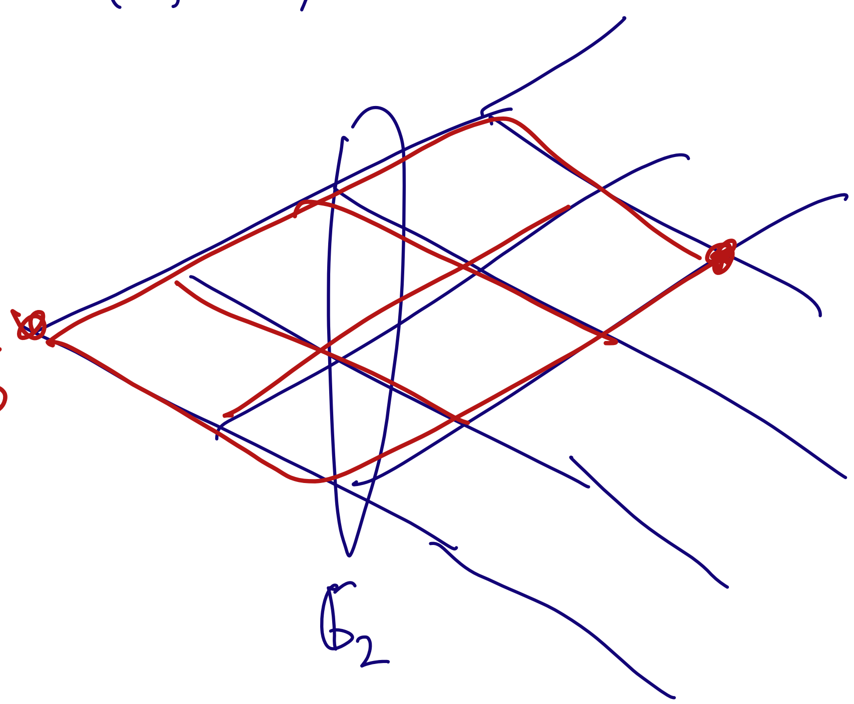
$\mathbb{G}_n = \{ (a, n-a) =\lambda \quad a = 0 \dots n \}$

$\mathbb{G}_0 = \{ (0,0) = \varnothing \}$

$\lambda \nearrow \mu$ if or $\boxed{\begin{matrix} b = a \\ b = a+1 \end{matrix}}$

$\lambda = (a, n-a)$, $\mu = (b, n+1-b)$

$d := (a, n-a) = \binom{n}{a}$



$\varnothing$

$\mathbb{G}_2$

Central meas. μ on paths

path $= (\xi_1, \xi_2, \xi_3, \dots) \in \{0,1\}^{\mathbb{N}}$

$= \left\{ (\emptyset \to \lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \lambda^{(3)} \nearrow \dots) \right\}$

$\lambda^{(n)} = (a_n, n - a_n)$

$\xi_n = a_n - a_{n-1}$

μ central if μ invar. under $S(\infty)$ on $\{0,1\}^{\mathbb{N}}$

Def. function φ on $\mathbb{G} = \bigcup_{n=0}^{\infty} \mathbb{G}_n$ is called harmonic if

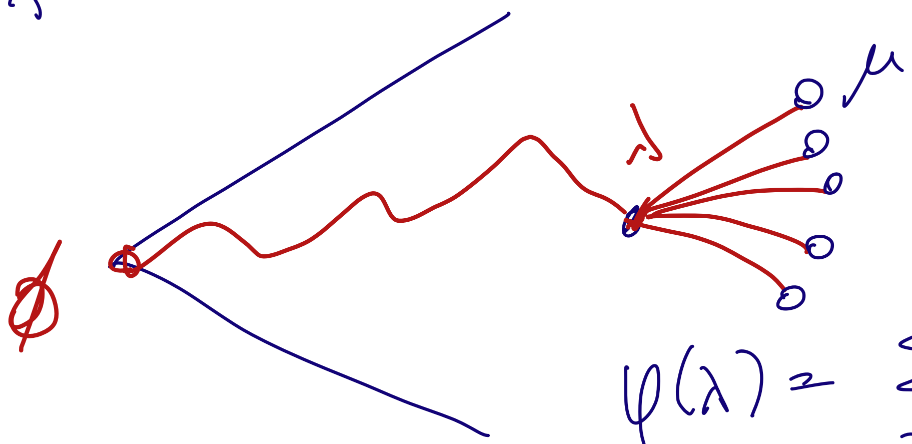
in the sense of Vershik-Kerov

$$\varphi(\lambda) = \sum_{\mu : \mu \searrow \lambda} \varphi(\mu)$$

Lemma $\{\text{central } \mu \text{ probab.}\} \longleftrightarrow \{\text{harmonic } \varphi \geq 0 \ \& \ \boxed{\varphi(\emptyset) = 1}\}$

Proof. $\varphi(\lambda) = \mu$ (path starts as $\varnothing \to \lambda^{(1)} \to \dots \to \lambda^{(n-1)} \to \lambda$)

$\lambda \in \mathbb{G}_n$



$$\varphi(\lambda) = \sum_{\mu \searrow \lambda} \varphi(\mu)$$

Pascal. iid coin flips $\sim p$

$$\varphi((a, n-a)) = p^a (1-p)^{n-a}$$

$$p^a (1-p)^{n-a} = p^{a+1} (1-p)^{n-a} + p^a (1-p)^{n+1-a}$$

Coherent $\overset{\text{prob.}}{\text{measures}}$ $\{M_n(\lambda)\}$ prob. meas. on $\mathbb{G}_n$

$\updownarrow$

(norm. harm. f.)

$$\varphi(\lambda) = \frac{1}{\dim \lambda} \underbrace{M_n(\lambda)}_{\mu(\text{a path goes through } \lambda)} , \quad \lambda \in \mathbb{G}_n$$

Pascal, iid (p)

dim λ

$$M_n(a, n-a)) = \binom{n}{a} p^a (1-p)^{n-a}$$

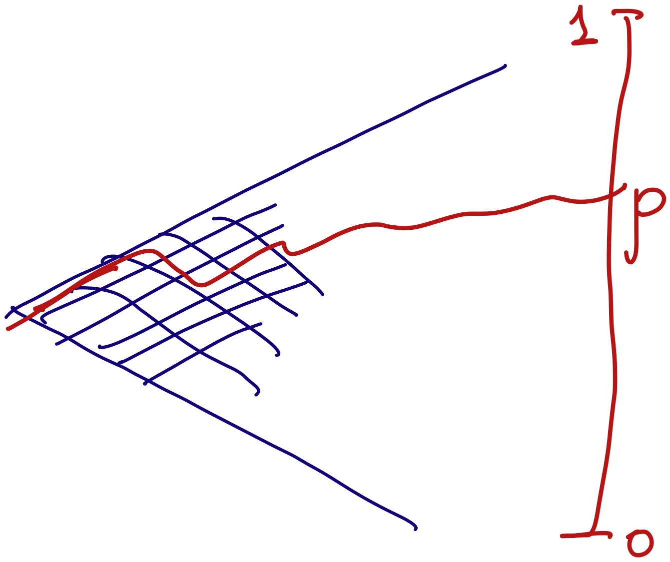
$$\mathcal{P}(\mathbb{G}) = \{\text{central prob } \mu\}$$

$$= \{\text{normalized } \varphi \ \& \ \varphi \geq 0\}$$

$$= \{\text{coh. syst. } \{M_n\}\}$$

<u>Boundary</u> of $\mathbb{G}$ $\stackrel{\text{def}}{=}$ $\mathrm{Ex}(\mathcal{P}(\mathbb{G}))$

$\mathrm{Ex}\, \mathcal{P}(\mathbb{G}) \longleftrightarrow [0,1]$



$$\{(a, n-a)\} = G_n$$

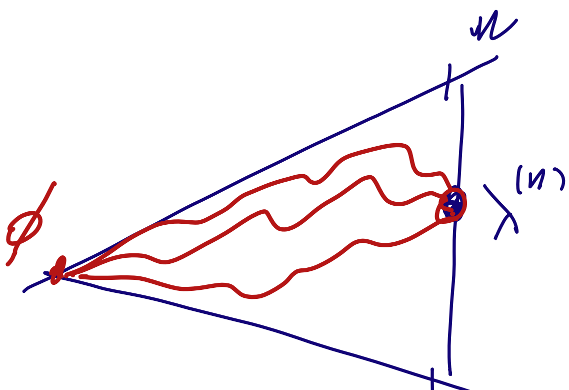
$$\Updownarrow$$

$$\left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$$

3.3* Application of ergodic theorem to branching graphs

<u>want:</u> $\forall \ \mu \in Ex(\mathcal{V}(G))$, $\exists \ \lambda^{(n)} \in G_n$

s.t. $\mu = \lim_{n\to\infty}$ of finite extreme meas. coming from $\lambda^{(n)}$?



$\boxed{\text{just unif. meas. on all paths } \varnothing \longrightarrow \lambda^{(n)}}$

limit is in restrictions to any fixed level K.

μ ergodic $\longleftrightarrow \{M_n^{(\mu)}\}$, then $\forall K, \forall \nu \in \mathcal{B}_K$

$$M_K^{(\mu)}(\nu) = \lim_{n\to\infty} \frac{\dim \nu \cdot \dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}$$

$$\boxed{\dim(\nu, \lambda^{(n)}) = \#\text{ of paths in between}}$$

„adic shift"

- paths in a graph = cpt space X

- $x \overset{n}{\sim} y \qquad x_i = y_i \quad i > n \quad \longleftarrow \boxed{\mathfrak{Z}_n}$

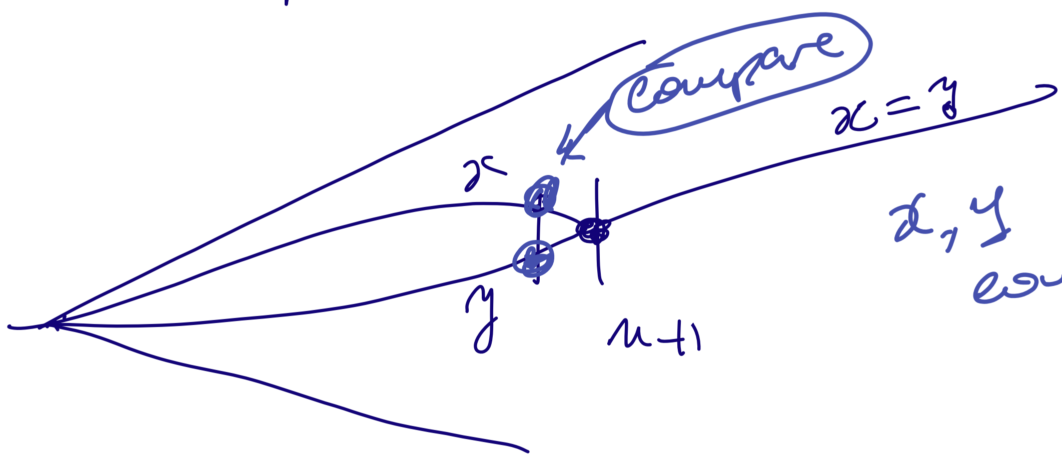
  $x \sim y \qquad$ if eventually $x \overset{n}{\sim} y$

- $\mathfrak{Z}$, tail equiv. relation

- Linear order on $\mathfrak{S}_n$

- $x > y$ if $x \overset{n}{\sim} y$ & $x_n > y_n$,

  partial order on X



x, y are comparable

$T : x \rightsquigarrow$ next minimal path

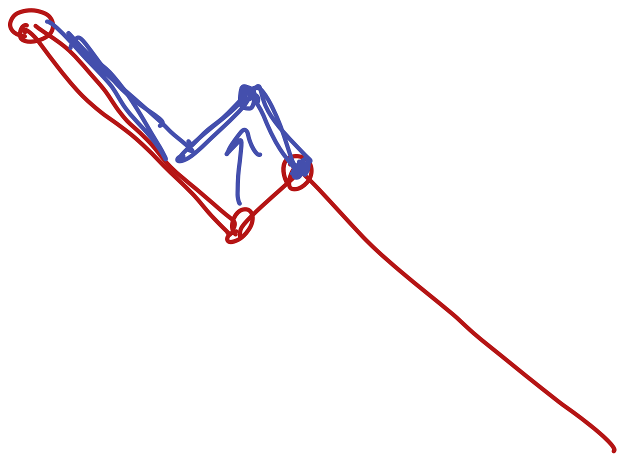
Example

$\xi = (0, 0, 0, 0 \ldots 0)$

minimal

$\xi = (0, 0, 0, 0, 1, 0, 0, \ldots)$

ϕ



— $\overline{X}$, $\underline{X}$ set of min/max pts
(assume these are single-pt sets)

— $T : X \setminus \overline{X} \longrightarrow X \setminus \underline{X}$

$x \longrightarrow y$

$y > x$ s.t. y is minimal among such $y > x$

— any μ invariant under T (& under S)
corresponds to a coherent syst. on $\mathbb{G}$

— ξ is a limit of ξ_n, &
T is a sort of "inductive limit"
So Vershik's theorem applies.

$\Rightarrow$ $\forall$ coherent system on $\mathbb{G}$
is a limit of "finite coherent syst."
(define)

T is approx. by T_n's

T_n are just mixing all paths $\emptyset \to \lambda^{(n)}$

T_n-invariant meas. on paths

$\Longleftrightarrow$ meas. which are central up to level n

<u>Prop</u> (w/o proof)

μ-central $\Longleftrightarrow$ μ is T-invar.

3.4. $\boxed{\text{Ex } \mathcal{X}(S(\infty))}$ and the Young graph (+ S.Y.T.)

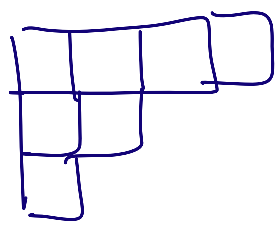
rep. th.

Remember χ – ex. ch. of $S(\infty)$

normalized

$$\chi\big|_{S(n)} = \sum_{\lambda \in \mathbb{Y}_n} M_n^{(\chi)}(\lambda) \cdot \hat{\chi}_\lambda^{S(n)}$$

$\hat{\chi}(e) = 1$

$\mathbb{Y}_n = \{$  with n boxes $\}$

$\{M_n^{(\chi)}\}$ – prob. measures $\forall n$

Also, we can restrict $S(n+1)$ to $S(n)$

$\lambda \in \mathbb{Y}_{n+1}$

not normalized

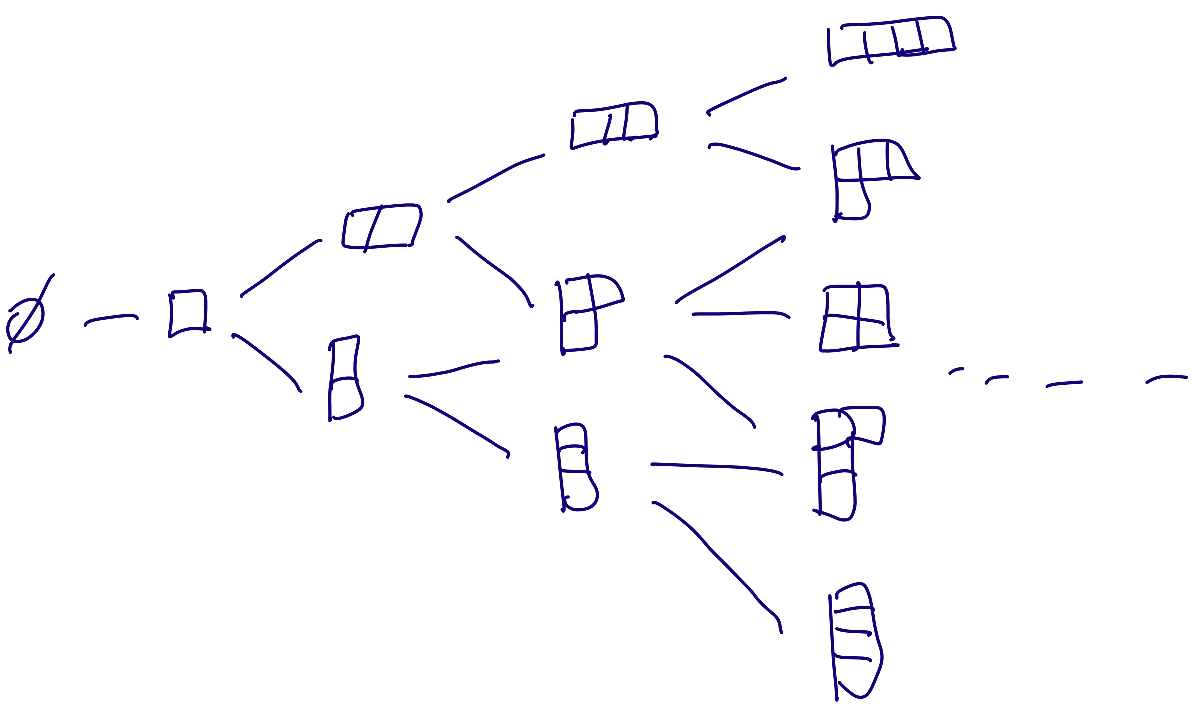
$$\chi_\lambda^{S(n+1)}\Big|_{S(n)} = \sum_{\mu = \lambda - \square} \chi_\mu^{S(n)}$$

$S(n+1) \downarrow S(n)$ implies that

$\{ M_n^{(\chi)}(\lambda) \}$ is coherent on the Young graph.

$$\mathbb{Y} = \bigcup_{n=0}^{\infty} \mathbb{Y}_n$$

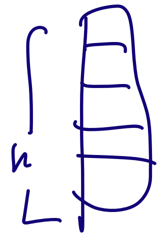
$\mu \nearrow \lambda$ if $\lambda = \mu + \square$



Examples $\chi = id$

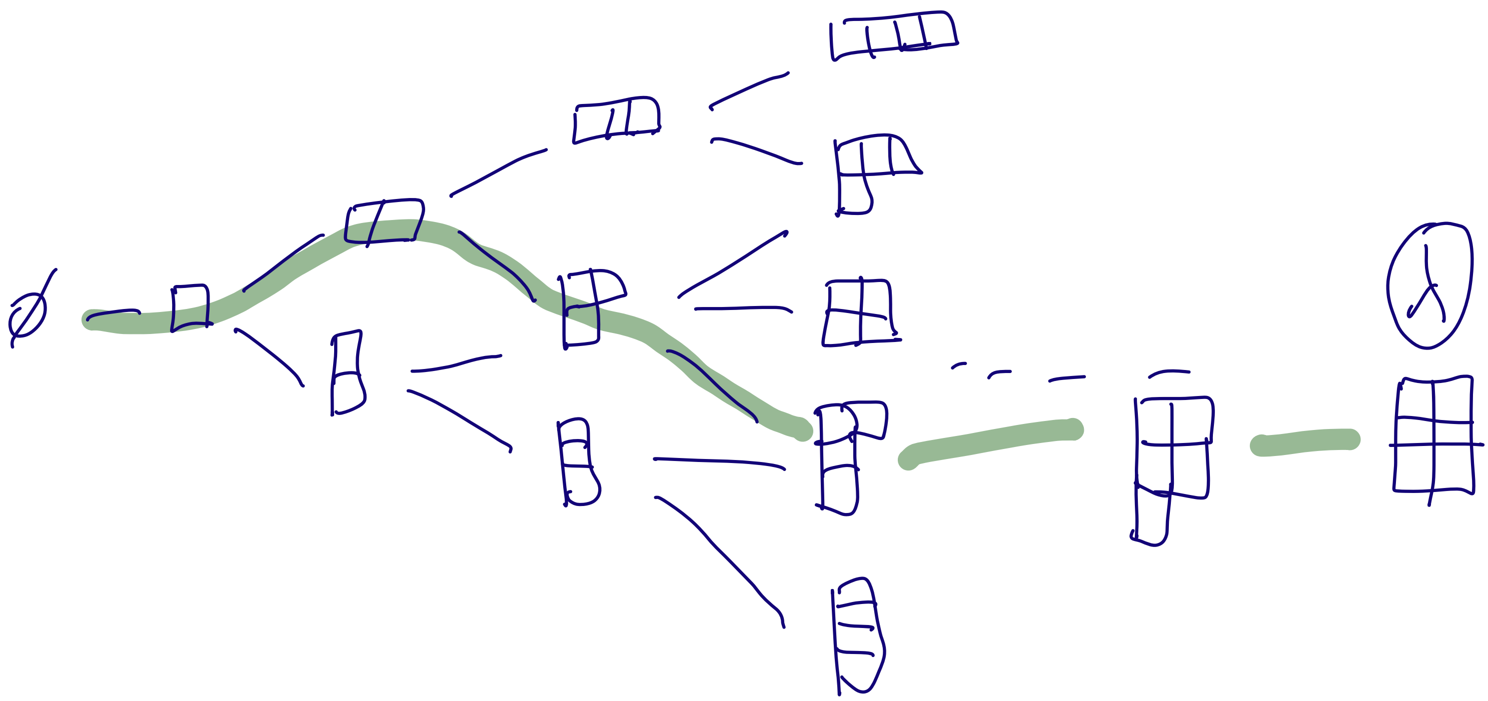
$\Rightarrow M_n(\lambda) = \delta$ at $\overbrace{\square\square\square\square}^{n}$

$\chi = \mathrm{sgn}$

$\Rightarrow \quad M_n(\lambda) = \delta$ at 

Paths in $\mathbb{Y}$.

$\boxed{\emptyset \longrightarrow \lambda} \in \mathbb{Y}_n$



1	2
3	5
4	6

strictly incr. in both directions

i	j
k	

$i < j$

$i < k$

Standard Young tableau

$$\dim \lambda = \#\text{ of paths}$$
$$= \#\text{ of } SYT(\lambda)$$
$$= \frac{n!}{\prod_{\square \in \lambda} h(\square)} \qquad \text{(hook formula)}$$

$$\dim \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array} = \frac{\cancel{6}\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{3}\cdot\cancel{4}} = 5$$

3.5 The problem of relative dim asymptotics

(determine all finite ergodic meas. & their limits)

$$\dim \lambda \quad \& \quad \dim(\mu, \lambda)$$

$$= \#(\mu \qquad \lambda)$$

$\forall$ ex. coh. measure, $\exists\ \lambda^{(n)}$

s.t. $\forall$ fixed ν,

the limit

$$\cancel{(\dim \nu)} \cdot \frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}$$

exists.

So, the goal is to describe all possible limits of

$$\boxed{\frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}}$$

$\lambda^{(n)} \in \mathbb{Y}_n$

$n \to \infty$

Boundary looks like:

$$\Omega = \left\{ \begin{array}{l} \vec{\alpha} = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0) \\ \vec{\beta} = (\beta_1 \geq \beta_2 \geq \dots \geq 0) \\ \text{s.t. } \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1 \end{array} \right\}$$

Thoma simplex

$$\subset \mathbb{R}^{2\infty+1}$$

$$\gamma = 1 - \sum_{i=1}^{\infty} \alpha_i + \beta_i \geq 0$$

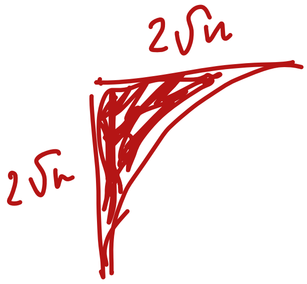
Ω = compact inf. simplex

Example

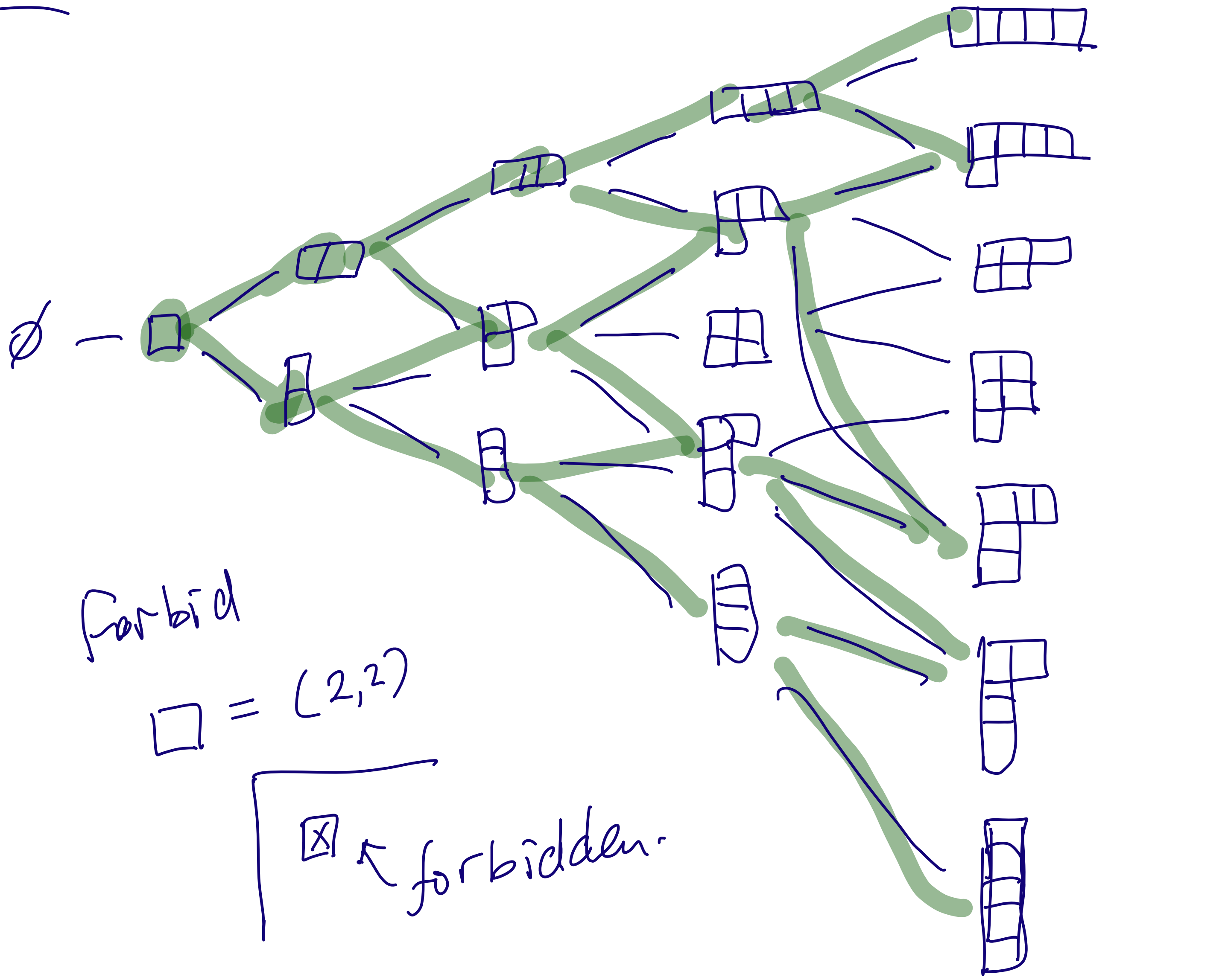
$\alpha_i = \beta_i = 0, \ \delta = 1$

$\longrightarrow$ $$M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n$$

(Plancherel)



Note: Pascal sits inside $\mathbb{Y}$



forbid $\square = (2,2)$

$\boxtimes$ forbidden.

$$\Omega_{\text{Pascal}} \subset \Omega$$

$$\| \quad \{(\alpha_1, \beta_1) : \alpha_1 + \beta_1 = 1$$
$$\alpha_2 = \dots = 0 = \beta_2 = \dots$$
$$\gamma = 0 \}$$

$\Rightarrow$ From Ω_{Pascal}, we get χ of $S(\infty)$ corresp. to iid coin flips.

4. Pascal triangle & polynomial algebra

4.1 Coherent measures / harmonic functions & $\mathbb{R}[x,y]$

4.2 Relative dimension & a "shifted basis" in $\mathbb{R}[x,y]$

Recall. $\mathbb{G} = \bigcup_{n=0}^{\infty} \mathbb{G}_n$ branching graph

① $\dim \lambda = \dim(\emptyset, \lambda)$ $\qquad \dim(\nu, \lambda)$

(numbers of paths)

$\mathcal{M}(\mathbb{G}) = \{$ central probab. μ on paths of $\mathbb{G} \}$

$\mu(\emptyset \to \lambda^{(1)} \to \dots \to \lambda^{(n)})$ depends only on $\lambda^{(n)}$

$\simeq \{$ nonneg. normalized harm. $\varphi \}$

$$\varphi(\emptyset) = 1, \quad \varphi \geq 0$$

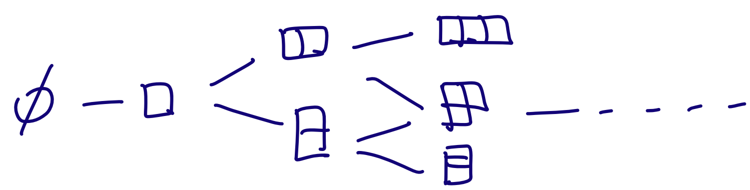
$$\varphi(\lambda) = \sum_{\nu : \nu \searrow \lambda} \varphi(\nu)$$

$$\varphi(\lambda^{(n)}) = \mu(\emptyset \to \lambda^{(1)} \to \dots \to \lambda^{(n)})$$

$\simeq \{$ coherent syst. of probab. meas. $M_n(\lambda)$ on $\mathbb{G}_n \}$

$$M_n(\lambda) = \dim \lambda \cdot \varphi(\lambda)$$

$$M_n(\lambda) = \sum_{\nu : \nu \searrow \lambda} M_{n+1}(\nu) \frac{\dim \lambda}{\dim \nu}$$

② $\mathbb{Y} =$ 

$$\mathcal{M}(\mathbb{Y}) \cong \underbrace{\mathcal{M}(S(\infty))}$$

normalized characters

→ χ – central

→ $\chi(e) = 1$

→ χ pos-def.

$$\chi\Big|_{S(n)} = \sum_{\lambda \in \mathbb{Y}_n} M_n(\lambda)\, \tilde{\chi}_\lambda^{S(n)}$$

3 Ex $\mathcal{M}(\mathbb{G})$ $\Longleftrightarrow$

all possible limits

of $\dfrac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}$, $\lambda^{(n)} \in \mathbb{G}_n$

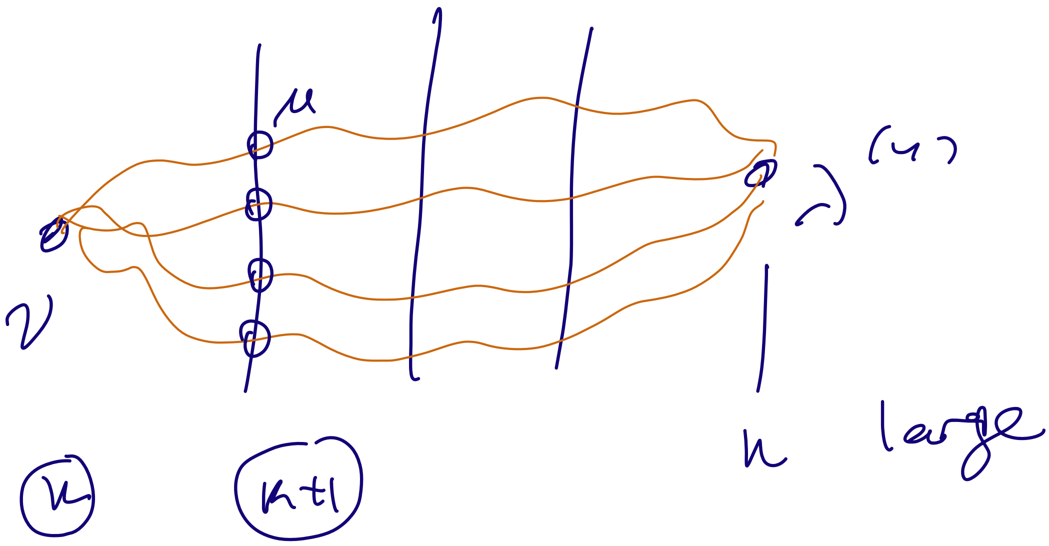
(ν fixed) $n \to \infty$

Lemma. $\varphi(\nu) = \dfrac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}$ is ~~unnorm~~ normalized harmonic in ν, $|\nu| < n$

$\varphi_\lambda(n)$

Proof $\varphi_{\lambda^{(n)}}(\emptyset) = \dfrac{\dim \lambda^{(n)}}{\dim \lambda^{(n)}} = 1, \quad \varphi \geq 0$

$$\boxed{\varphi_{\lambda^{(n)}}(\nu) = \sum_{\mu : \mu \searrow \nu} \varphi_{\lambda^{(n)}}(\mu)}$$



$\square$

$$\varphi(\nu) = \frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}$$

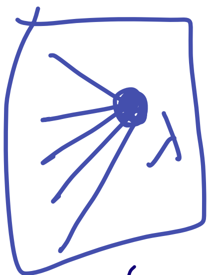
$$\varphi_{\lambda^{(n)}}(\mu) = \begin{cases} \dfrac{1}{\dim \lambda^{(n)}}, & \mu = \lambda^{(n)} \\ 0, & \text{else} \end{cases}$$

$$\boxed{\begin{aligned} |\lambda^{(n)}| &= n \\ |\mu| &= n \end{aligned}}$$

④ Adic shift on paths of G.

X = space of inf. paths, compact

"Adic" order: paths are comparable iff cofinal

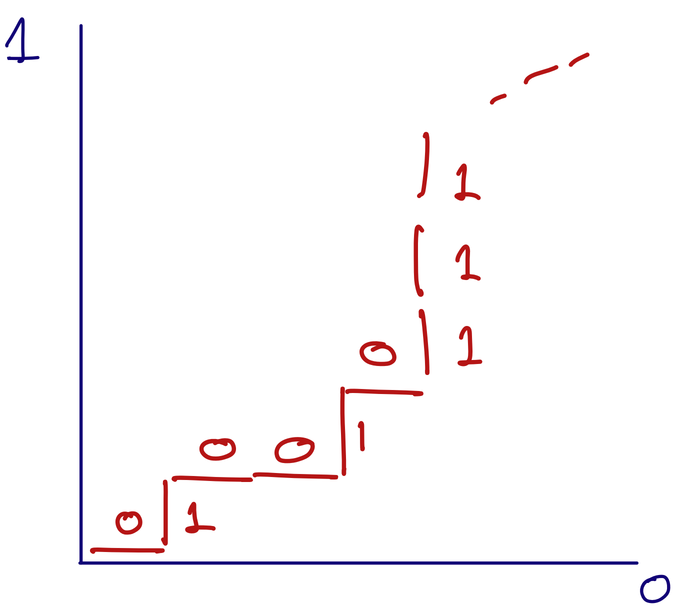


$x < y$ if $x_i = y_i$, $i > j$

$x_j < y_j$

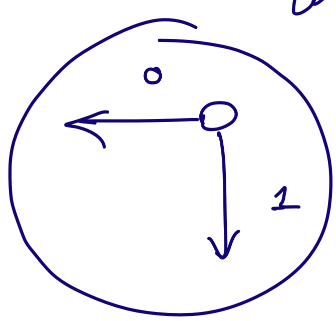
(Need total order on all outgoing down edges from each vertex)

$Tx = y$ if y is the immediate successor

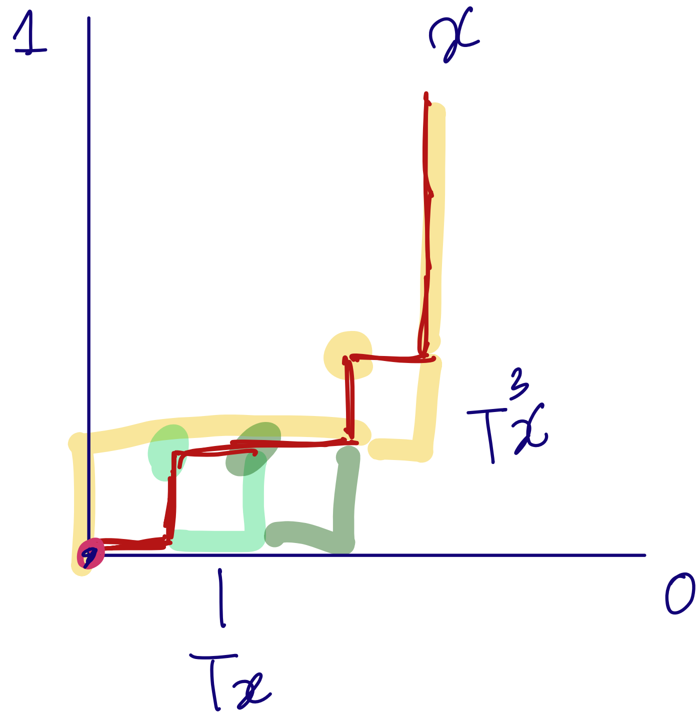


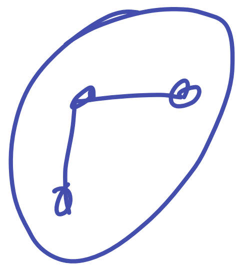
(Pascal)

We say $0 < 1$



go from $\emptyset$ up the path & find first place you can switch



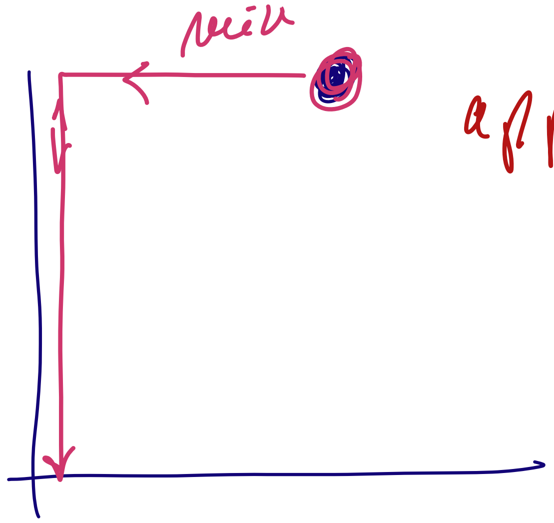


$$T(0^p 1^q 1 0 *)$$

$$= 1^q 0^p 0 1 * \quad (p, q \geq 0)$$

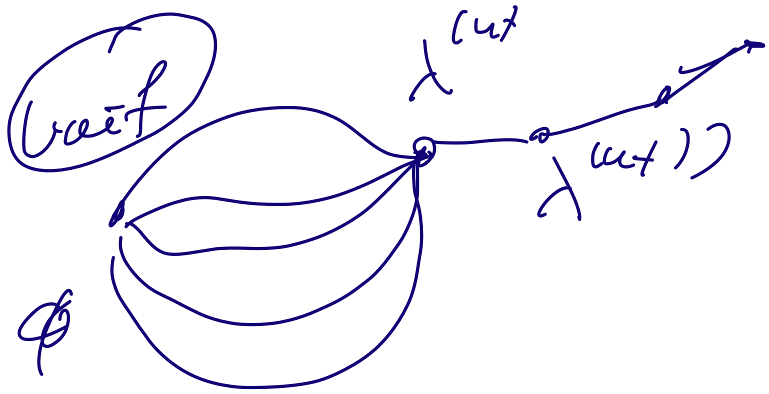
Fact: central on paths $\iff$ invariant wrt the adic shift.

why?



apply T, cycle through all paths.

$\lambda^{(n)}$

$\mu_n =$ 

4. Pascal graph via algebra

4.1 Identification with algebra morphisms

Let $A = \mathbb{R}[x, y]$.

$p_1 = x + y$;

$\lambda = (a, n-a)$

$f_\lambda = x^a y^{n-a}$ ← linear basis in A

$$\Rightarrow \boxed{p_1 f_\lambda = \sum_{\nu : \nu \searrow \lambda} f_\nu}$$

$\underline{(x+y)\, x^a y^{n-a}}$

$$\varphi(\lambda) = \sum_{\nu \searrow \lambda} \varphi(\nu)$$

So, $\varphi(\lambda) = f_\lambda$ looks like harmonic f. on the Pascal triangle if $x + y = 1$

associated to an algebra

Theorem (Ring theorem) ← works $\forall$ G

$$\mathrm{Ex}\left[g^{\nu}(\text{Pascal})\right] = $$ algebra homomorphism

$$F : A \to \mathbb{R}$$

1) F vanishes on $(p_1 - 1)A$

2) $F(f_\lambda) \geq 0 \quad \forall \lambda.$

$$A = \mathbb{R}[x, y]$$

$$F(fg) = F(f)F(g)$$

Proof. $\{f_\lambda\}$ is a basis for A

$$p_1 f_\lambda = \sum_{\nu : \nu \searrow \lambda} f_\nu$$

$x^a y^{n-a}$

0)

First,

$$\mu(\text{Pascal}) = \left\{ \begin{array}{l} F : A \to \mathbb{R} \text{ linear, normalized } F(1)=1 \\ - F \text{ vanishes on } (p_1 - 1)A \\ - F(f_\lambda) \geq 0 \quad \forall \lambda \end{array} \right\}$$

normalized nonneg. harmonic

$$\varphi(\lambda) = F(f_\lambda)$$

$$\varphi(\emptyset) = F(1) = 1$$

$$F(p, f) = F(f)$$

Reason to match extreme harm. f
to algebra homomorphisms

(Thm. 4.3 in [B-O] book)

1) $A_+ \subset A$ nonneg. lin comb. of f_λ
$F(A_+) \subseteq \mathbb{R}_{\geq 0}$

closed under mult

$f_\lambda f_\mu = \sum_\nu c^\nu_{\lambda\mu} f_\nu, \quad c^\nu_{\lambda\mu} \geq 0$

2) $p_1^n - \dim\lambda \cdot f_\lambda \in A_+ \quad \forall \lambda$

$$p_1^n = \sum_\lambda \dim\lambda \cdot f_\lambda$$

3) If F - linear s.t. $F(f) > 0$.

def. $F_f(g) = \dfrac{F(fg)}{F(f)}$. — also nonneg normalized & linear in g

4) Let F extreme.

4a) if $F(f_\lambda) = 0$ then

$$F(f_\lambda f_\mu) = 0 \quad \forall \mu \text{ , indeed}$$

$(\mu \in \text{Pascal}_n)$ $\quad 0 \leq F(f_\lambda f_\mu) \leq F(p_1^n f_\lambda) = F(f_\lambda) = 0$

4b) $F(f_\lambda) > 0$, define

Let $f_1 = \frac{1}{2} \dim \lambda \circ f_\lambda$ $\qquad$ $F(f_1), F(f_2) > 0$

$f_2 = p_1^n - f_1$ $\qquad \lambda = (a, n-a)$

$\Rightarrow$ F_{f_1}, F_{f_2} exist, $\forall g$

we have

$$F(g) = F(p_1^n g) = F(f_1 g) + F(f_2 g)$$

$$= F(f_1) F_{f_1} + F(f_2) F_{f_2}$$

F extreme $\Rightarrow$ $F_{f_1} = F$ so

F is multipl.

$$\frac{F(f_1 g)}{F(f_1)} = F_{f_1}(g) = F(g) \Rightarrow \forall g,\ F(f_1 g) = F(f_1) F(g)$$

5) F - mult, show it is extreme

$$F(f) = \int G(f)\, P(dG)$$

P-probab.

$G \in$ Extreme nonnegative normalized linear maps

$f \in A$

Abstract Choquet's Theorem

$$F(f^2) = (F(f))^2 \Rightarrow$$

$$\int_{G\in E_x} \underbrace{G(f^2)}_{\text{G is extreme}} \, dP = \left(\int_{G\in E_x} G(f) \, dP \right)^2$$

$$G(f^2) = (G(f))^2$$

$\Rightarrow$ variance of f is 0:

$$\int (G(f))^2 \, dP(G) = \left(\int G(f) \, dP(G) \right)^2$$

$$\forall f$$

$$\operatorname{Var} X = E\left((X - EX)^2 \right) = 0$$

$$\Rightarrow \quad X = EX \quad \text{a.e.}$$

$\square$

So,

Thm. Boundary of the Pascal Δ is $F : A = \mathbb{R}[x,y] \to \mathbb{R}$

$F(x) = p$, $F(y) = 1-p$, $p \in [0,1]$.

(another proof of de Finetti: classif. of ergodic exchangeable random seq.)

$$\frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}} \longrightarrow ?$$

4.2 Relative dimension in Pascal via algebra $A = \mathbb{R}[x,y]$

$$\frac{\dim(\underbrace{\nu}_{(b,k-b)}, \underbrace{\lambda}_{(a,n-a)})}{\dim \lambda} = \frac{\binom{n-k}{a-b}}{\binom{n}{a}} =$$

let $a_1 = a$, $a_2 = n - a_1$, $b_1 = b$, $b_2 = k - b_2$

$$= \frac{(n-k)!}{n!} \cdot \frac{a_1!\, a_2!}{(a_1-b_1)!\,(a_2-b_2)!}$$

$\boxed{n, a_1, a_2 \text{ large} \quad b, k \text{ fixed}}$

Define $z^{\downarrow m} = z(z-1)\ldots(z-m+1)$

$$= \frac{a_1^{\downarrow b_1}\, a_2^{\downarrow b_2}}{n^{\downarrow k}} \quad \text{polynomial in } a_1, a_2$$

$n^{\downarrow k} \simeq n^k$ as $n \to \infty$

$A = \mathbb{R}[x,y]$

<u>Define</u> $f^*_\lambda(x,y) = x^{\downarrow b_1} y^{\downarrow b_2}$ $\quad \lambda = (b_1, b_2)$

$f^*_\lambda \in A$ — inhomogeneous elements of degree $b_1 + b_2$

still a basis, because

$f^*_\lambda = f_\lambda +$ lower ord. terms

$$\Rightarrow \frac{\dim(\nu, \lambda)}{\dim \lambda} = \frac{1}{n^{\downarrow k}} f^*_\nu(\lambda)$$

$\deg f^*_\nu = k$

$\nu = (b, k-b)$

$$= \frac{1}{n^{\downarrow k}} \left[f_\nu(\lambda) + g(\lambda) \right]$$

$g \in A$,

$\deg g \le k-1$

Clearly $\frac{g(\lambda)}{n^k} \to 0$ if $\lambda = (a, n-a)$

because $g(\lambda) \le \text{Const} \cdot n^{k-1}$

$$\frac{\dim(\nu, \lambda)}{\dim \lambda} = \frac{1}{n^{\downarrow k}} \cdot f_\nu^*(\lambda)$$

$$\simeq \frac{1}{n^k} f_\nu(\lambda) + O\left(\frac{1}{n}\right)$$

as $n \to \infty$
unif. in a
where $\lambda = (a, n-a)$

$$f_\nu\left(\frac{\lambda}{n}\right)$$

$$\boxed{\lambda^{(n)} \text{ is s.t. } \frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}} \text{ has a limit } (\forall \nu)}$$

$\Uparrow$

$f_v\left(\frac{\lambda}{n}\right)$ has a limit $\forall v$

$v = (1,0) \Rightarrow \frac{x_1}{n} = \frac{a}{n}$ has a limit

& also $\frac{n-a}{n}$ has a limit.

Second proof (Today) of de Finetti

(the same as the original one in Lect 3 (?), but now with algebra on top)

4.2. Pascal via $A = \mathbb{R}[x,y]$.

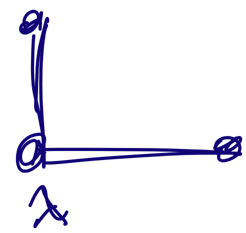
$\boxed{\lambda = (a, n-a)}$ graded by deg

$$f_\lambda = x^a y^{n-a} \qquad f_\lambda^* = x^{\downarrow a} y^{\downarrow (n-a)}$$

$x(x-1)\ldots(x-a+1)$

$$p_1 = x + y$$

$$p_1 f_\lambda = \sum_{\mu \searrow \lambda} f_\mu$$



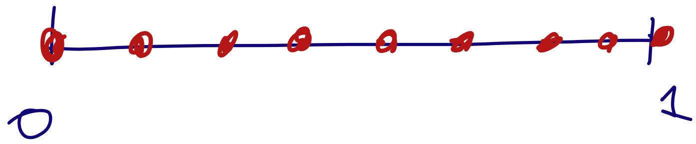
skew (relative) dimensions

$$\frac{\dim(\nu, \lambda)}{\dim \lambda} = \frac{1}{n^{\downarrow k}} f_\nu^*(\lambda)$$

$\lambda = (a, n-a) \in \mathbb{P}_n$
$\nu = (b, k-b) \in \mathbb{P}_k$

$$\mathbb{P}_n = \{(0,n), (1,n-1), \ldots, (n,0)\}$$

Let $\Omega = [0,1]$ (the boundary)



$$\mathbb{P}_n \hookrightarrow \Omega$$

$$\lambda = (a, n-a)$$

$$\lambda \longmapsto \boxed{\frac{a}{n} \in [0,1]}$$

Then
$$\frac{\dim(\nu,\lambda)}{\dim \lambda} = \frac{1}{n^{\downarrow k}} f^*_\nu(\lambda) \simeq f^o_\nu\left(\frac{\lambda}{n}\right)$$

$n \to \infty$, up to $O(\frac{1}{n})$

$f^o_\nu \in C[0,1]$

$$f^o_{(b_1,b_2)}(x) = x^{b_1}(1-x)^{b_2}$$

f^o_ν is a function on $\Omega = [0,1]$

$\nu = (b_1, b_2)$

$\nu = (b_1, b_2)$

(1) $x^{b_1} y^{b_2} = f_v(x,y) \in A$

(2) f_v^*

(3) $f_v^0 = x^{b_1}(1-x)^{b_2}$

which is at the same time

the image of $f_v \in A$

in $A^0 = A/(p_1 - 1)A$

$$A \longrightarrow A^0$$

$$f(x,y) \longmapsto f(x, 1-x)$$

$$A^0 = \mathbb{R}[x].$$

$$p_1 = x+y \longmapsto 1.$$

$$(p_1 - 1)A \longrightarrow 0.$$

So: $A \longleftrightarrow$ harmonic f. recursion

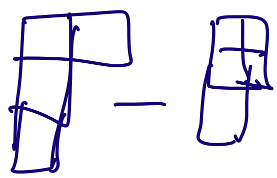
$$A^0 = A/(P_1 - 1)A \subset C(\Omega)$$

boundary

$$\boxed{\frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}} \simeq f^{\circ}_{\nu}\left(\frac{\lambda^{(n)}}{n}\right)}$$

(aim to replicate for $\mathbb{Y}$)

5 Combinatorics of $\mathbb{Y}$



$$\mathbb{Y}_n = \{\lambda \text{ with } n \text{ boxes}\}$$

5.1 Recursion for $\dim \lambda$

$\dim \lambda = \dim(\varnothing, \lambda) = \#$ of std. y.t. of shape λ

1	2	6	7
3	5	8	
4			

$<$

$\wedge$

$$\{\text{paths } \emptyset - \lambda \text{ in } \mathbb{Y}\} = SYT(\lambda)$$

Remark

$\lambda \rightsquigarrow$ irrep. of $S(n)$ corresp to λ

$\boxed{V}$ $\quad S(n) \supset S(n-1) \supset S(n-2) \supset \cdots$

$$V = \bigoplus_{\mu \nearrow \lambda} V_\mu^{S(n-1)} = \bigoplus_{\mu \nearrow \lambda} \bigoplus_{\nu \nearrow \mu} V_\nu^{S(n-2)}$$

$$= \bigoplus \cdots$$

$$= \bigoplus_{\text{paths } \emptyset - \lambda} V_1^{S(1)} = \mathbb{C}$$

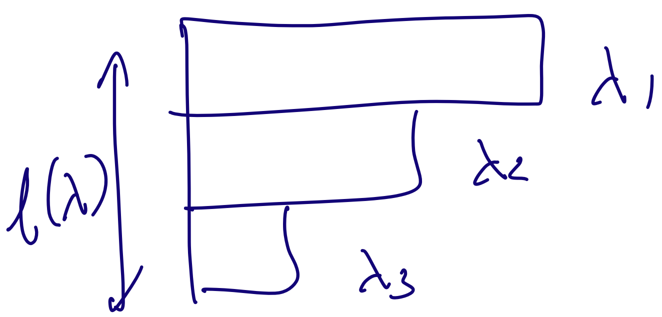
$\Rightarrow$ $\exists$ basis in V ~ restrict. called the Gelfand-Tsetlin basis

& action of $S(n)$ is very explicit there

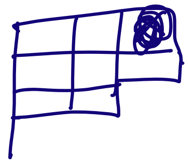
$$\dim \lambda = \sum_{\mu : \mu \nearrow \lambda} \dim \mu \qquad \text{(recursion)}$$

follows from def.

1	2	3	6
4	5		



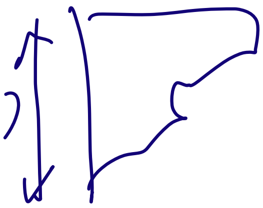
5.2 Formulas for $\boxed{\dim \lambda}$



$\lambda_1 - \lambda_2 + 2 - 1$

① $$\dim \lambda = \boxed{n! \, \frac{\prod_{i<j} (\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{N} (\lambda_i + N - i)!}} \quad (\text{HW } 2)$$

$\ell(\lambda)$



$\prod_{1 \le i < j \le N}$

\# of parts

$N \geq \boxed{\ell(\lambda)}$

② $$\dim \lambda = n! \, \det \left(\frac{1}{(\lambda_i + j - i)!} \right)_{i,j=1}^{N}$$

(would follow from symm. f.)

$N \geq \ell(\lambda)$ arbitrary

③ Hook formula

$$\dim \lambda = \frac{n!}{\prod_{\square \in \lambda} h(\square)}$$

9	6	3	1
7	4	1	
5	2		
4	1		
2			
1			

5.3. Probabilistic proof of hook formula

$$\dim \lambda \overset{?}{=} \frac{n!}{\prod_{\square \in \lambda} h(\square)} = F(\lambda)$$

Want.

$$\boxed{F(\lambda) = \sum_{\mu \nearrow \lambda} F(\mu)}$$

$$1 = \sum_{\mu \nearrow \lambda} \underbrace{F(\mu)/F(\lambda)}_{}$$

is a probability of event A_μ

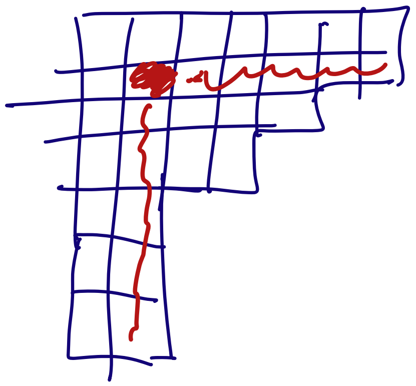
if A_μ are disjoint

& $\bigcup_\mu A_\mu = \Omega$

then done.

Hook walk algorithm

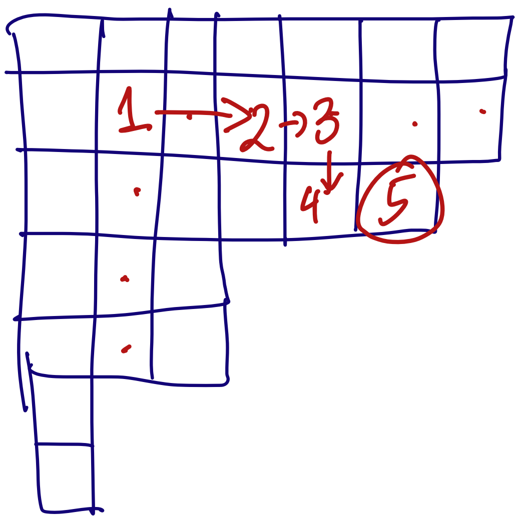
λ



1) pick a box $\square_1$ at random

$\boxed{1/n}$

2) Recursively, pick a box $\square_{j+1}$ unif. from the hook of $\square_j$, $\square_{j+1} \neq \square_j$



stop when get to the boundary of λ

Prop. (HW3)

$$\boxed{\frac{F(\mu)}{F(\lambda)} = \mathrm{Prob}\left(\lambda - \square_{\text{final}} = \mu\right)}$$

$\Rightarrow$ hook formula.

Note:

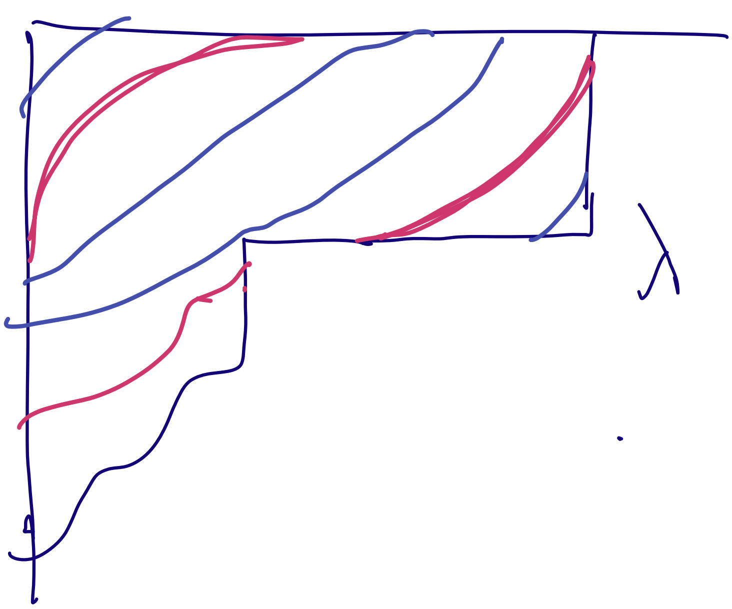
$$\underbrace{\lambda \xrightarrow{HW} \mu \xrightarrow{HW} \nu \xrightarrow{HW} \cdots \to \emptyset}_{\text{path } \emptyset \to \lambda \text{ in } \mathbb{Y}}$$

$\frac{\dim \mu}{\dim \lambda}$ $\qquad$ $\frac{\dim \nu}{\dim \mu}$

& uniformly random

$\Leftrightarrow$ unif. random $SYT(\lambda)$

(Dan Romik 2008 $\underline{\text{MacTableaux}}$)



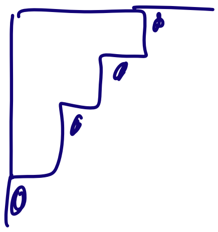
$1 \ldots n$

$\frac{n}{4}$ $\quad$ $\frac{n}{2}$ $\quad$ $\frac{3n}{4}$

5.4 Operators D and U & the second recursion for dim λ

Thm.

$$\dim\lambda = \frac{1}{n+1}\sum_{\nu:\,\nu\searrow\lambda}\dim\nu$$



$|\lambda| = n$

$\ell^2(\mathbb{Y})/\mathbb{R}$ $\qquad \{\underline{\lambda}\}$ basis, $\qquad (\cdot,\cdot)$

vector

$$(\underline{\lambda},\underline{\mu}) = \mathbb{1}_{\lambda=\mu}$$

$$U\underline{\lambda} = \sum_{\nu:\,\nu\searrow\lambda}\underline{\nu}$$

$$D\underline{\lambda} = \sum_{\mu:\,\mu\nearrow\lambda}\underline{\mu}$$

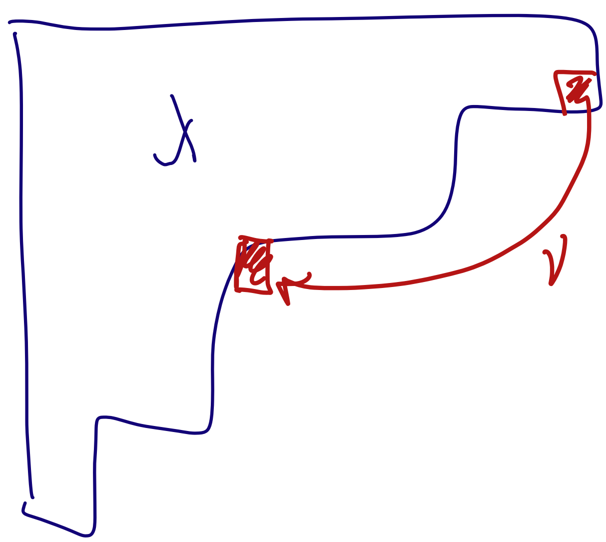
nonzero if $\mu = \lambda - \square$

(1) $\quad (D\underline{\lambda},\underline{\mu}) = (\underline{\lambda}, U\underline{\mu})$

$D = U^*$, $U = D^*$

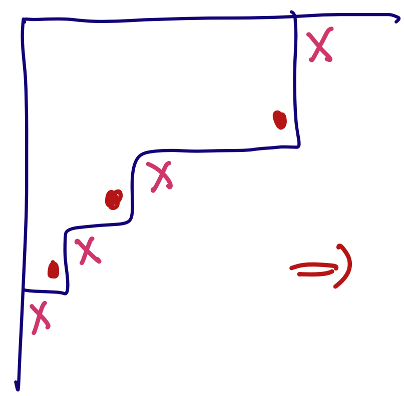
② $[D, U] = DU - UD.$

$DU\underline{\lambda} - UD\underline{\lambda} =$ lin comb of $\underline{\lambda}$ & $\underline{\nu}$, $\nu \neq \lambda$, $|\nu| = |\lambda|$



1) $\nu \neq \lambda$ does not participate (have 0 coeff)

2) $\underline{\lambda}$



$\Rightarrow$ $\boxed{[D, U] = Id}$

③ $\lambda \in \mathbb{Y}_n$,

$$\dim \lambda = (U^n \underline{\varnothing}, \underline{\lambda})$$
$$= (D^n \underline{\lambda}, \underline{\varnothing})$$

④ Proof of second recursion

$$\sum_{\nu : \nu \searrow \lambda} \dim \nu \qquad |\lambda| = n$$

$$= \sum_{\nu} (U^{n+1} \underline{\varnothing}, \underline{\nu})$$

$$= \Big(U^{n+1} \underline{\varnothing}, \underbrace{\sum_{\nu : \nu \searrow \lambda} \underline{\nu}}_{U \underline{\lambda}} \Big)$$

$$= \left(D U^{n+1} \underline{\phi}, \underline{\lambda} \right)$$

$$\begin{aligned}
D U^{n+1} &= DU \; U^n \\
&= (UD + 1) U^n \\
&= \boxed{U^n} + U D U^n \\
&= U^n + U (U^{n-1} + U D U^{n-1}) \\
&= \cdots = \underline{(n+1) U^n} + \\
&\qquad + U^{n+1} D
\end{aligned}$$

$$D \underline{\phi} = 0$$

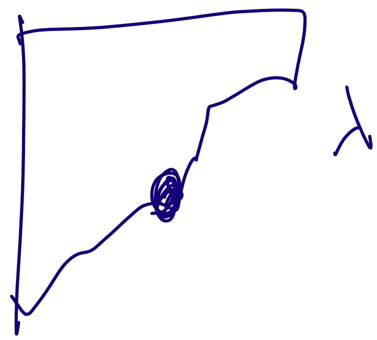
$$\sum_{\nu : \nu \vdash \lambda} \dim \nu = (n+1) \left(U^n \underline{\phi}, \underline{\lambda} \right)$$

$$= (n+1) \dim \lambda$$

$\square$

Note:

Hook walk



1) $\lambda \to \mu$

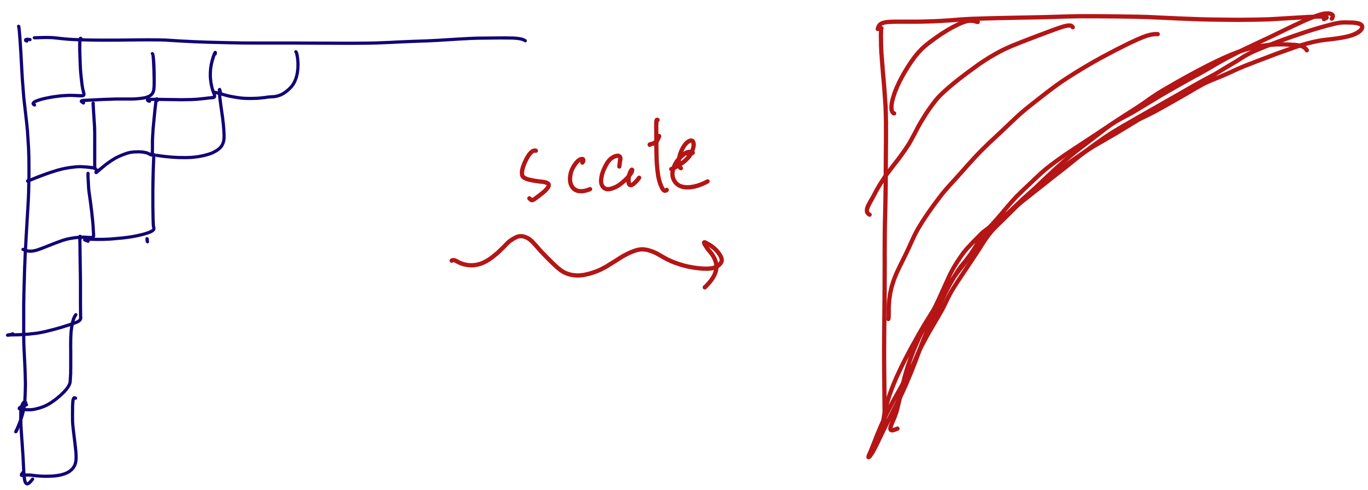
$\mu = \lambda - \square$

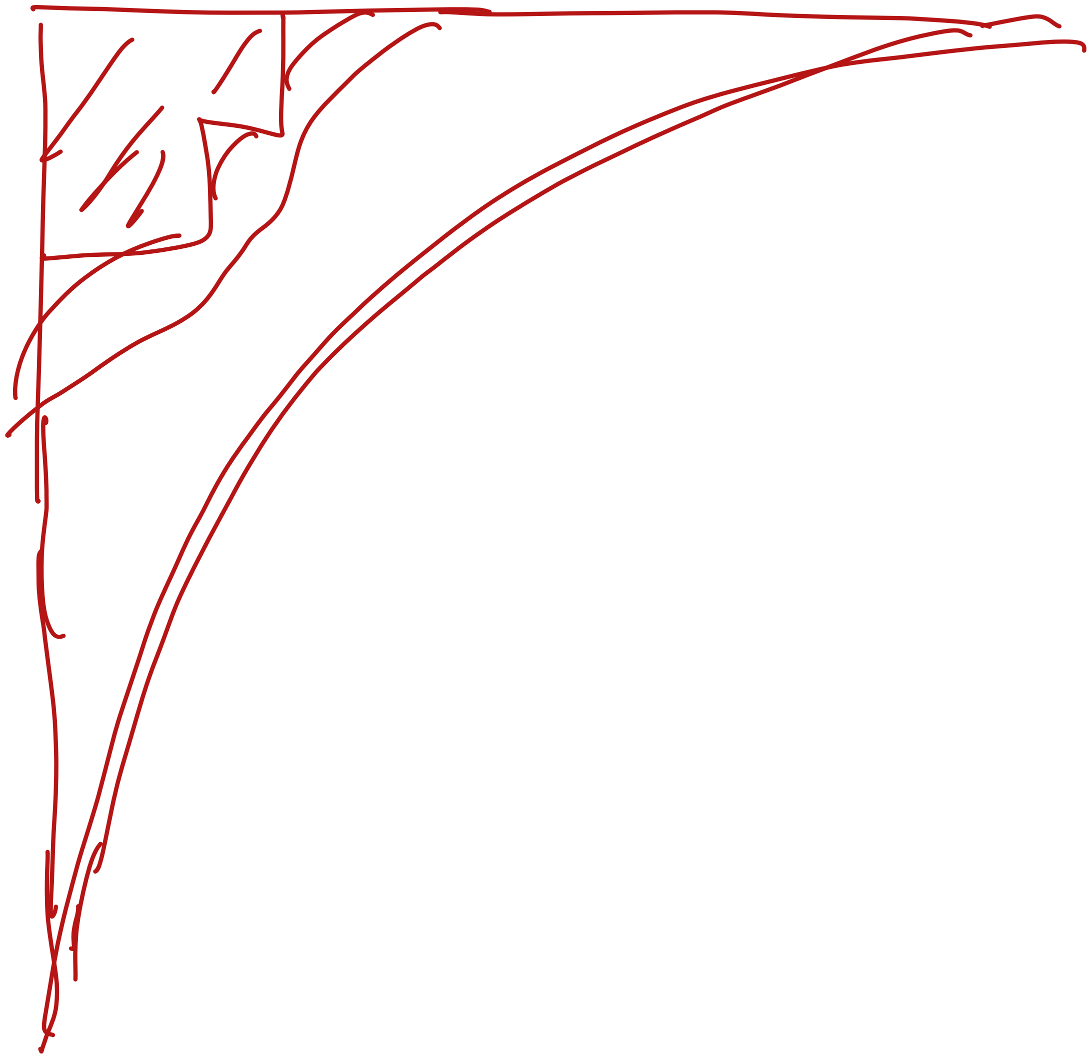
$$P(\lambda \to \mu) = \frac{\dim \mu}{\dim \lambda}$$

2) $\lambda \to \nu$, $\nu = \lambda + \square$

Plancherel growth process

$$P(\lambda \to \nu) = \frac{\dim \nu}{(n+1) \dim \lambda}$$





Last time

$\dim \lambda$, ...

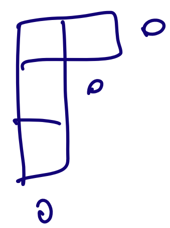
$S(\infty)$

harm. φ on $\mathbb{Y}$

$$\varphi(\lambda) = \sum_{\nu = \lambda + \square} \varphi(\nu)$$

$\varphi \geq 0$

$\varphi(\emptyset) = 1$



$$x^a y^{n-a}(x+y) = \;---$$

$$f_\lambda = \sum_{\nu \searrow \lambda} f_\nu$$

6. Symmetric functions

$f_n(x)$ $\qquad$ $\int f_n f_m \, d\mu = 1_{m=n}$

$1, x, x^2, x^3, \ldots$ $\qquad$ Gram - Schmidt

Symmetric functions

6.1. Algebra Λ

$$\Lambda_n = \mathbb{R}[x_1 \ldots x_n]^{S(n)} = \bigoplus_{k \geq 0} \Lambda_n^k$$

Symm pol in $x_1 \ldots x_n$

$\Lambda_n^k =$ homogeneous degree k poly.

Examples,

$$x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + \ldots$$

degree 3.

Def.

$$\Lambda_{n+1} \longrightarrow \Lambda_n$$

$$f(x_1 \ldots x_n, x_{n+1}) \longmapsto f(x_1 \ldots x_n, 0)$$

Inverse limit

$$\Lambda^k = \varprojlim_n \Lambda_n^k$$

$\in$

$(f_1, f_2, f_3, \ldots)$

$$\boxed{\Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k}$$

(sometimes Sym)

$\deg f_i = k \quad \forall i, \quad f_i$ - homogeneous

$$f_{n+1}\Big|_{x_{n+1}=0} = f_n \quad \forall n$$

Examples. $\quad p_1 = e_1 = h_1 = x_1 + x_2 + x_3 + \cdots$

$\deg = 1$

$$\left(f_n = x_1 + \cdots + x_n \right)$$

Non-example $\quad \prod_{i=1}^{\infty} (1 + x_i) \notin \Lambda$

need bounded degree.

6.2 e_k, h_k & Generating functions

$e_0 = h_0 = 1$

$$e_k = \sum_{1 \le i_1 < \dots < i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$$

$\deg e_k = k = \deg h_k$

elementary sym. poly

$e_k(x_1, \dots, x_n) = 0$ if $k > n$

complete homogeneous

$$h_k = \sum_{1 \le i_1 \le \dots \le i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$$

(every possible monomial of deg k)

$$\begin{cases} e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots \\ h_2 = e_2 + x_1^2 + x_2^2 + \dots \end{cases}$$

$e_k = e_k(x_1, x_2, \dots)$

(Viet) $$\sum_{k=0}^{\infty} e_k t^k = \prod_{i=1}^{\infty} (1 + x_i t) = \boxed{E(t)}$$

$$1 + (x_1 + x_2) t + x_1 x_2 t^2 = (1 + x_1 t)(1 + x_2 t)$$

$n = 2$ var's

$$e_k(\underbrace{1, \dots, 1}_{n}) = \binom{n}{k}$$

$$\sum_{k=0}^{\infty} h_k t^k = \prod_{i=1}^{\infty} \left(1 + tx_i + (tx_i)^2 + (tx_i)^3 + \cdots\right)$$

$$= \prod_{i=1}^{\infty} \frac{1}{1 - tx_i} = H(t)$$

$$h_k(\overbrace{1 \ldots 1}^{n}) = \binom{n+k-1}{n} \qquad \text{(exercise)}$$

$$E(t) H(-t) = 1.$$

$$\sum e_k t^k \qquad \sum h_k (-t)^k$$

$1 + e_1 t + e_2 t^2 + \cdots$

$1 - h_1 t + h_2 t^2 + \cdots$

$$(e_0 = h_0 = 1)$$

$$e_0 h_0 = 1$$

$$e_1 - h_1 = 0$$

$$e_2 - e_1 h_1 + h_2 = 0$$

etc.

$\vdots$

$$\Rightarrow \quad e_k \in \mathbb{R}[h_1, \ldots, h_k]$$

$$h_k \in \mathbb{R}[e_1, \ldots, e_k]$$

6-3 p_k, m_λ, more relations

$\underline{\underline{p_k}}$ & $\underline{\underline{e_n}}$, h_n

$p_k = x_1^k + x_2^k + x_3^k + \cdots$

power sums

$$P(t) = \sum_{k=1}^{\infty} \frac{p_k}{k} t^k = \log\left(\prod_{i=1}^{\infty} \frac{1}{1-x_i t}\right)$$

$H(t)$

$|x_i t| < 1$

$$\sum_{k\geq 1} \frac{x^k t^k}{k} = \log(1-xt)^{-1}$$

$$\boxed{H(t) = e^{P(t)} = \frac{1}{E(-t)}}$$

$\Downarrow$

relation for coeffs $\quad h \leftrightarrow p$

$$1 + h_1 t + h_2 t^2 + \cdots = \exp\left(p_1 t + p_2 t^2/2 + p_3 t^3/3 + \cdots\right)$$

$$= 1 + p_1 t + p_2 t^2/2 + \cdots$$
$$+ \frac{1}{2}\left(p_1 t + p_2 t^2/2 + \cdots\right)^2$$
$$+ \cdots$$

$x_1^2 + x_1 x_2 + \cdots$

$\uparrow$

$$h_2 = \frac{p_2}{2} + p_1^2/2$$

Def. $m_\lambda = \sum\limits_{\substack{\text{sum over}\\ \text{all distinct}\\ \text{monomials}}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$

$\lambda = \ydiagram{3,3,1}$

$$m_{\ydiagram{2,2}} = x_1^2 x_2^2 + x_1^2 x_3^2 + \cdots$$

$$m_{\ydiagram{1}} = p_1 = e_1 = h_1$$

$$m_{\underbrace{\ydiagram{3}}_{k}} = p_k \qquad m_{\left.\ydiagram{1,1,1,1}\right\}k} = e_k$$

Prop. $(m_\lambda)_{\lambda \in \mathbb{Y}}$ is linear basis in Λ

$\square$

Orthogonality. (in Λ_n)

$$\langle f,g\rangle = \frac{1}{n!}\oint_{|z_1|=\dots=|z_n|=1} f(z)\,\overline{g(z)}\,\frac{dz_1\dots dz_n}{\prod_j (2\pi i z_j)}$$

$\overline{z} = \frac{1}{z}$ if $|z|=1$

$$\oint z^k \frac{dz}{2\pi i z} = \begin{cases} 1, & k=0 \\ 0, & \text{else} \end{cases}$$

$k\in\mathbb{Z}$

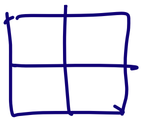
$$\langle m_\lambda, m_\mu\rangle = \frac{1}{n!}\oint\dots\oint \sum_{i_1,\dots,i_n} z_{i_1}^{\lambda_1}\dots z_{i_n}^{\lambda_n}\, z_{i_1}^{-\mu_1}\dots z_{i_n}^{-\mu_n}\,\frac{d\vec{z}}{(2\pi i)^n\,\vec{z}}$$

$= 0$ if $\lambda\neq\mu$.

if $\lambda=\mu$

$$= \frac{1}{n!}\cdot\frac{n!}{(\text{combinatorial factor})}.$$

Orthogonal basis in each Λ_n.

x_1, x_2  $\frac{2!}{2} = 1$

$x_1^2x_2^2 + \cancel{x_1^2x_2^2}$

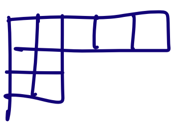
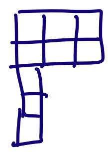
6.4 Fundamental theorem

$$\Lambda_n = \mathbb{R}[e_1, \dots, e_n]$$

$$\Lambda = \underbrace{\mathbb{R}[e_1, e_2, \dots]}_{\text{finite linear comb.'s of monomials in } e_j\text{'s}}$$

(every symm. f. is a polynomial in e_i's)

Idea: $\lambda \rightsquigarrow \lambda' = $ transpose

$\lambda =$  $\quad \lambda' =$ 

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

triangular change of var's.

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu \qquad (*)$$

partial order

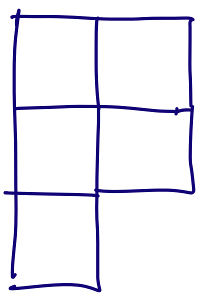
$\mu<\lambda$: $\quad \mu_1+\dots+\mu_k \le \lambda_1+\dots+\lambda_k \ \forall k$

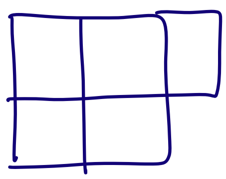
& $\mu \neq \lambda$

λ is obtained from μ by moving some boxes up.

Proof of (*)

$e_{\lambda'}$ = monomial expansion

λ 

$\lambda' =$ 

$e_3 \quad e_2$

$x_1 x_2 x_3 \quad x_1 x_2$

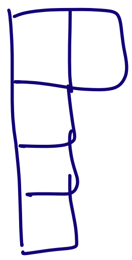
$m_\lambda \qquad x_1^2 x_2^2 x_3$

leading coeff

Something else?

$x_1 x_2 x_3 \quad * \quad x_1 x_4$

$\Downarrow$

μ  $\longleftarrow \quad x_1^2 x_2 x_3 x_4$

$\mu < \lambda$ (move box down)

$\square$

<u>So;</u> $e_{\lambda'}$ also a linear basis of Λ

$e_{\lambda'} \longleftrightarrow m_\lambda$ related by unitriangular change of variables.

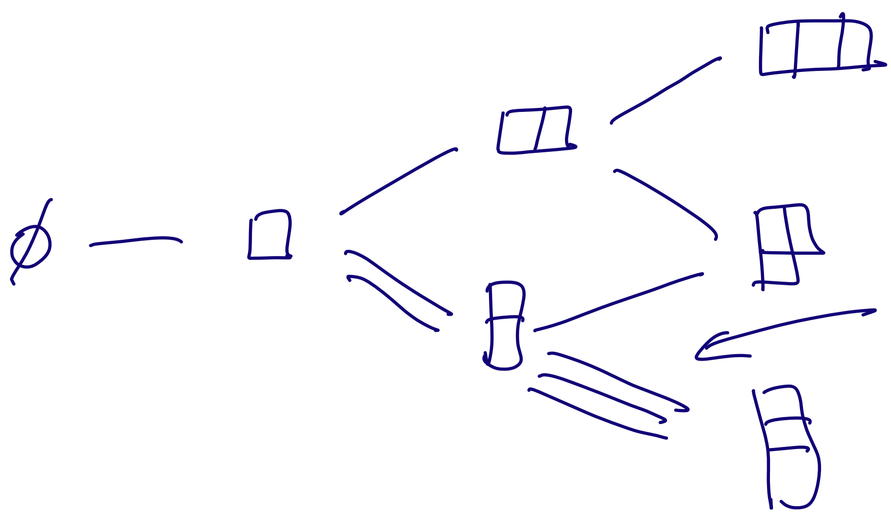
$\Downarrow$

m_λ = linear comb. of $e_{\mu'}$, so a polynomial in e_j's

$\Rightarrow$ Fundam theorem is done.

$$p_1 m_\lambda = \sum_{\mu = \lambda + \square} x_{\lambda\mu} \, m_\mu$$

$x_1 x_2 (x_1 + x_2 + x_3 + \cdots$



$x_{\lambda\mu}$ edge mult.

So, $\{m_\lambda\}_{\lambda \in \mathbb{Y}}$ is not the "right" basis for the Young graph

6.5 Antisymm. functions & Schur pol.

→ next lecture

6.6. Pieri rule

$$p_1 s_\lambda = \sum_{\nu = \lambda + \square} s_\nu$$

Proof In Λ_n

$$a_{\lambda+\delta}\, p_1 = \sum_{k=1}^{n} a_{\lambda+\delta+\underbrace{e_k}_{\text{basis vector}}}$$

$$\lambda+\delta+e_k = (\lambda_1+n-1,\ \lambda_2+n-2,\ \dots,\ \lambda_k+n-k+1,\ \dots,\ \lambda_{n-1}+1,\ \lambda_n)$$

$a_{\lambda+\delta+e_k}$ vanishes if ---

Recall symmetric functions Λ

mult. gener.

$$p_k = x_1^k + x_2^k + \dots$$

e_k = elem. poly $\quad x_1 x_2 \dots x_k + \dots$

h_k = sum of all deg k monomials

$$m_\lambda = \sum_{\substack{\text{all distinct} \\ \text{monomials}}} x_{i_1}^{\lambda_1} \dots x_{i_n}^{\lambda_n}$$

linear basis

6.5 Antisymm. polynomials & Schur poly.

$A_n \subseteq \mathbb{R}[x_1 \dots x_n]$ — def. & basis $\{a_\alpha\}$

$$f(x_{\sigma_1}, \dots, x_{\sigma_n}) = (-1)^{\operatorname{sgn} \sigma} f(x_1 \dots x_n)$$

$$\forall \sigma \in S(n)$$

$$a_\alpha(x_1 \dots x_n) = \sum_{\sigma \in S(n)} (-1)^\sigma x_{\sigma_1}^{\alpha_1} \dots x_{\sigma_n}^{\alpha_n}$$

$$\alpha_i = \alpha_j \quad \Rightarrow \quad a_\alpha = 0.$$

$$\Rightarrow \quad \alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0$$

$\{a_\alpha\}$ is a basis in A_n

$$a_\alpha = \det \left[x_i^{\alpha_j} \right]_{i,j=1}^n$$

$$\rho = (n-1, n-2, \dots, 1, 0)$$

$$a_\rho = \det[x_i^{j-1}] = V(\vec{x}) = \prod_{1\le i<j\le n} (x_i - x_j)$$

Ex. $f \in A_n \Rightarrow f/V \in \Lambda_n$

where $V(x) = \prod_{1\le i<j\le n} (x_i - x_j)$

$f \longmapsto f/V$ is linear isomorphism

Def. $s_\lambda(x_1 \ldots x_n)$ & basis in Λ_n

Schur poly's.

$$\frac{a_\alpha}{a_\rho} = s_\lambda, \qquad \alpha = \lambda + \rho$$

form a basis

strictly $\quad$ weakly

$$= (\lambda_1 + n-1, \ldots, \lambda_{n-1}+1, \lambda_n)$$

$$A_n \simeq \Lambda_n$$

$$a_\alpha \longleftrightarrow a_\alpha / a_\rho$$

$$\boxed{s_\lambda(x_1 \ldots x_n) = \det\left[x_i^{\lambda_j + n-j}\right]_1^n \Big/ V(\vec{x})}$$

Ex. s_λ - homog. &

- $\deg s_\lambda$
$= |\lambda|$
$= \lambda_1 + \cdots + \lambda_n$

$$\rho \longleftrightarrow \lambda = \emptyset$$

$$S_{\emptyset} = \frac{a_\rho}{a_\rho} = 1$$

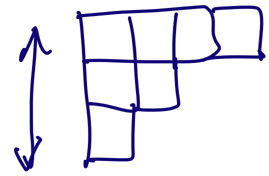
$$S_{(k,0)}(x,y) = \frac{\det\begin{bmatrix} x^{k+1} & y^{k+1} \\ 1 & 1 \end{bmatrix}}{x-y}$$

$$= \frac{x^{k+1} - y^{k+1}}{x-y} = x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k$$

$$S_{\Box} = x_1 + \cdots + x_n$$

$$S_{(1,0,0,0,0,\dots)}(x_1, x_2, \dots x_n)$$

$$= \det\begin{bmatrix} x_1^n & x_2^n & \cdots & x_n^n \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ & & \vdots & \\ 1 & & \cdots & 1 \end{bmatrix} \Big/ V(\vec{x})$$

$\ell(\lambda)$ 

Prop. $$S_\lambda(x_1,\dots,x_n,0) = \begin{cases} S_\lambda(x_1,\dots,x_n), & \text{if } n \geq \ell(\lambda) \\ 0 \ (*), & n < \ell(\lambda) \end{cases}$$

Proof. (*) 0 part: if $n < \ell(\lambda)$,

$$\det\left[x_i^{\lambda_j+n+1-j}\right]_1^{n+1}$$

$n+1 \leq \ell(\lambda) \Rightarrow \lambda_{n+1} > 0$

set $x_{n+1} = 0 \Rightarrow$ column of 0's in det.

Let $n \geq \ell(\lambda)$

$\Rightarrow \lambda_{n+1} = 0$

$x_i^{\lambda_j+n+1-j}$

$$\det\left[x_i^{\lambda_j+n+1-j}\right]_1^{n+1}\Bigg|_{x_{n+1}=0} = \det \begin{array}{c} \left[\begin{array}{ccc|c} & & & 0\\ & \boxed{} & & 0\\ & & & 0\\ 1 & \cdots & 1 & 1 \end{array}\right] \\ n+1 \end{array} \; n+1$$

$$= \det\left[x_i^{\lambda_j+n-j}\right]_1^{n} \cdot (x_1 x_2 \cdots x_n)$$

$$V_{n+1}(\vec{x})\Big|_{x_{n+1}=0} = V_n(\vec{x}) \cdot (x_1 \cdots x_n)$$

$\square$

$\Rightarrow$ S_λ are $\in \Lambda$

(compatible with projections $\Lambda_{n+1} \xrightarrow[x_{n+1}=0]{} \Lambda_n$)

$\{S_\lambda\}_{\lambda\in\mathbb{Y}}$ — basis of Λ

Note:
(w/o proof)

S(n) characters

$\mathrm{Tr}\, T_\lambda(\sigma), \ \sigma \in C_\mu$

$$p_\mu = \sum_\lambda \chi^\lambda(\mu)\, s_\lambda$$

$|\lambda| = |\mu| = n$

$\| \ \ p_{\mu_1} p_{\mu_2} \ldots$

$\| \ \ $ Schur

$$C_\mu := \left\{ \sigma \in S(n) : \begin{array}{l} \sigma \text{ has} \\ \text{cycle} \\ \text{lengths} \\ \mu_1, \mu_2, \mu_3, \ldots \end{array} \right\}$$

6.6. Pieri rule

$p_1 = s_\square = e_1 = h_1$
$= x_1 + x_2 + \cdots$

<u>Thm.</u> $p_1 s_\lambda = \sum_{\nu = \lambda + \square} s_\nu$

(Pascal: $(x+y) \underbrace{f_{(a,n-a)}}_{x^a y^{n-a}} = \cdots$)

<u>Proof</u> In Λ_n (n - large)

$$a_{\lambda+\delta}\, p_1 = (x_1 + \cdots + x_n) \sum_{\sigma} (-1)^{\sigma} x_{\sigma_1}^{\lambda_1 + n - 1} \cdots x_{\sigma_n}^{\lambda_n}$$

expand, get $\sum_{i=1}^{n}$ of:

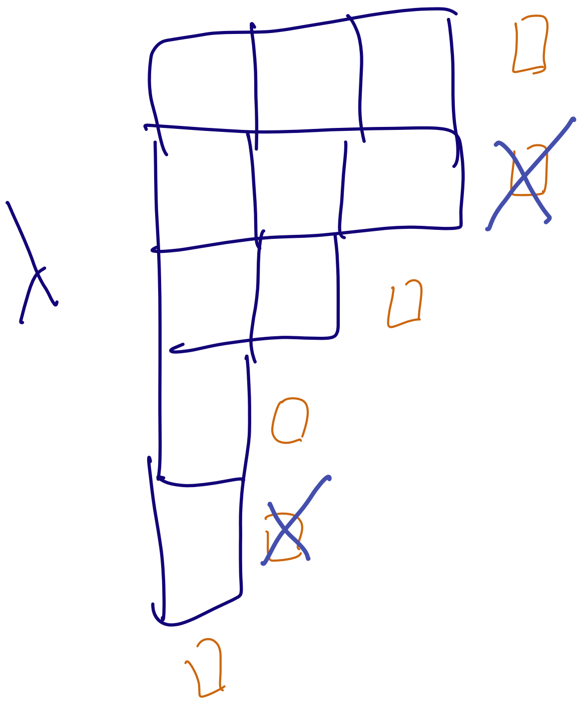
$$\sum_{\sigma} (-1)^{\sigma} x_{\sigma_1}^{\lambda_1 + n - 1} \cdots x_{\sigma_{i-1}}^{\lambda_{i-1} + n - (i-1)} \cdot x_{\sigma_i}^{\lambda_i + n - i + 1} \cdots x_{\sigma_n}^{\lambda_n}$$

if $\lambda_{i-1} > \lambda_i$

$\Rightarrow$ $\lambda_{i-1} + n - (i-1) > \lambda_i + n - i + 1$

$\Rightarrow$ only if $\lambda_{i-1} > \lambda_i$,
we can add a box
to row i

$\square$



6.7 Ring theorem again

& characters of $S(\infty)$

If a branching graph $\longleftrightarrow$ polynomial algebra

Then extreme harmonic funct. $\longleftrightarrow$ mult. funct. on the algebra

Implies

$$\left\{ \begin{array}{l} \varphi \text{ - harm on } \mathbb{Y} \\ \quad \varphi(\lambda) = \sum\limits_{\nu = \lambda + \square} \varphi(\nu) \\ \varphi(\emptyset) = 1 \\ \varphi(\lambda) \geq 0 \end{array} \right\} = \left\{ \begin{array}{l} \text{multiplicative} \\ \text{funct. } F \\ \Lambda \longrightarrow \mathbb{R} \\ F((p_1 - 1)\Lambda) = 0 \\ F(S_\lambda) \geq 0 \\ \qquad \forall \lambda \end{array} \right\}$$

& extreme

$$\varphi(\lambda) = F(s_\lambda)$$

we know: LHS = extreme characters of $S(\infty)$

— $\chi(e) = 1$

— χ noneg. def

— χ class funct.

$$\boxed{\chi(\mu) \longleftrightarrow F(?)}$$

Computation. F - mult. on Λ

abstractly

χ - character of $S(\infty)$.

$$\chi\Big|_{S(n)}(\mu) = \sum_{\lambda} \underbrace{M_n(\lambda)}_{= \dim\lambda \cdot \underbrace{\varphi(\lambda)}_{\text{harmonic}}} \frac{\chi^\lambda(\mu)}{\dim\lambda}$$

conj. class

normalized char. of $S(n)$

$$\varphi(\lambda) = F(s_\lambda)$$

$$F(p_\mu) = \sum_\lambda \chi^\lambda(\mu)\, F(s_\lambda)$$

$$\Rightarrow \quad \chi\Big|_{S(n)}(\mu) = F(p_\mu)$$

$$|\mu| = n$$

$$= F(p_{\mu_1})F(p_{\mu_2})\cdots F(p_{\mu_n})$$

$F(p_1) = 1$ (normalization)

<u>Conclusion.</u> μ – conj class of $S(\infty)$

$\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_k \geq 2)$

then $\chi(\mu) = \chi(\mu_1)\chi(\mu_2)\cdots\chi(\mu_k)$

$= F(p_{\mu_1}) \cdots F(p_{\mu_k})$

Nice!

indep. proof

A priori: characters of $S(\infty)$ should be multiplicative!?

(HW!)

$$\frac{1}{n!}\sum_{h} \tilde{\chi}(g_1 h g_2 h^{-1}) = \tilde{\chi}(g_1)\tilde{\chi}(g_2)$$

$\Longleftrightarrow$ $\tilde{\chi}$ - normalized irr. char. of $S(n)$

If χ – irr. ch. of $S(\infty)$,

$\chi = \lim_{n\to\infty} \tilde{\chi}^{\lambda(n)}$ of irr. norm. ch. of $S(n)$

$\Downarrow$

$$\lim_{n\to\infty} \frac{1}{n!}\sum_{h\in S(n)\subset S(\infty)} \chi(g_1 h g_2 h^{-1}) = \chi(g_1)\chi(g_2)$$

Let $g_1 \sim$ conj class μ

$g_2 \sim$ ν

wts $\chi(g_1)\chi(g_2) = \chi(\underbrace{\mu \cup \nu}_{\text{union of cycles}})$

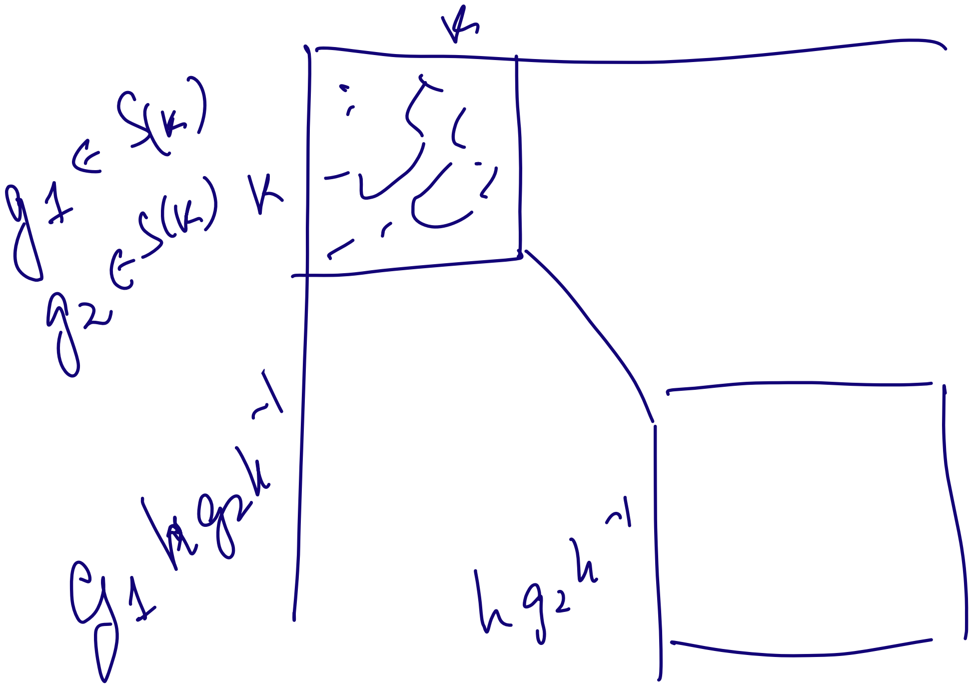
$g_1 \sim \mu$

$g_2 \sim \nu$

$\Rightarrow$ $g_1 \boxed{h g_2 h^{-1}}$

$h \sim$ random in $S(n)$

n – large



$\Rightarrow$ $h g_2 h^{-1}$ with high prob. fixes $\{1, \dots, k\}$

$\Rightarrow$ cycle structure of $g_1 h g_2 h^{-1}$ is w.h.p. $\mu \cup \nu$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n!} \sum_{h} \chi(g_1 h g_2 h^{-1})$$

$$= \chi(\mu \cup \nu)$$

$$\Rightarrow \boxed{\chi(\mu)\chi(\nu) = \chi(\mu \cup \nu)}$$

$\square$

7. Relative dimension in $\mathcal{D}$.

Recall what we want:

7.1. $\dim(\mu, \lambda)$ & $p_1^{n-m} s_\mu$

Aitken's formula

Recall

1) Symm. functions Λ

$$e_k,\ h_k,\ p_k;\ m_\lambda$$

N var.

$$s_\lambda(x_1 \dots x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_1^N}{\prod_{1 \le i < j \le N} (x_i - x_j)}$$

$$\det\left[x_i^{N-j}\right]_1^N$$

Note: s_λ vs $s_\lambda(x_1, \dots, x_N)$

2) $\Lambda \longleftrightarrow \mathbb{Y}$ graph

$$s_\lambda p_1 = \sum_{\nu = \lambda + \square} s_\nu$$

multiplicative graphs

$\updownarrow$

Algebra + basis

3) Characters of $S(\infty)$ (extreme, normalized)

$$\{\chi\} \longleftrightarrow \left\{ \begin{array}{l} \text{algebra homomorphisms } \Lambda \to \mathbb{R} \\ F((p_1 - 1)\Lambda) = 0 \\ F(s_\lambda) \geq 0 \quad \forall \lambda \end{array} \right\}$$

Then χ(cycle structure $\rho_1 \geq \rho_2 \geq \dots \geq \rho_\ell \geq 2$)

$$= F(p_{\rho_1}) F(p_{\rho_2}) \dots F(p_{\rho_\ell}).$$

Followed from general Ring Theorem & indep. from the functional equation for characters

$$F(p_k) = \begin{cases} 1, & k=1 \\ \dots, & k \geq 2 \end{cases}$$

Our goal: to classify $\{\chi\}$.

Via ergodic method, need to look at

$$\lambda^{(n)} \in \mathbb{Y}_n, \quad n \to \infty,$$

s.t. $\forall \nu$ – fixed,

$$\boxed{\frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}}$$ has a limit in n

Thoma (1964), Edrei (1953)

– classification of irred. χ of $S(\infty)$

Vershik – Kerov (1981)

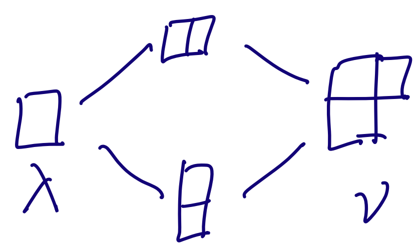
– asymptotic (ergodic) approach

Going along sec. 6 of [BO] book.

7. Relative dimension & proofs

7.1. det formula for $\dim(\mu,\lambda)$

$$p_1 s_\lambda = \sum_{\nu=\lambda+\square} s_\nu$$

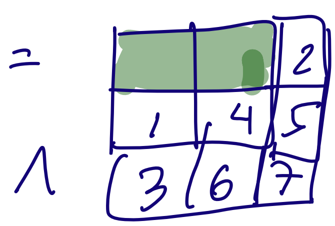


$p_1 = x_1 + x_2 + x_3 + \dots$

$$p_1^k s_\lambda = \sum_{\nu,\ |\nu|=n+k} \dim(\lambda,\nu)\, s_\nu$$

$|\lambda| = n$

$\lambda = \square\square$ $\nu =$



Note $\dim(\mu, \lambda) = f^{\lambda/\mu}$ in comb.

$= \#$ of SYT
of skew shapes

(recent progress,
- Naruse hook length formula,
- special cases &
asymptotics)

HOOK FORMULAS FOR SKEW SHAPES III. MULTIVARIATE AND PRODUCT FORMULAS

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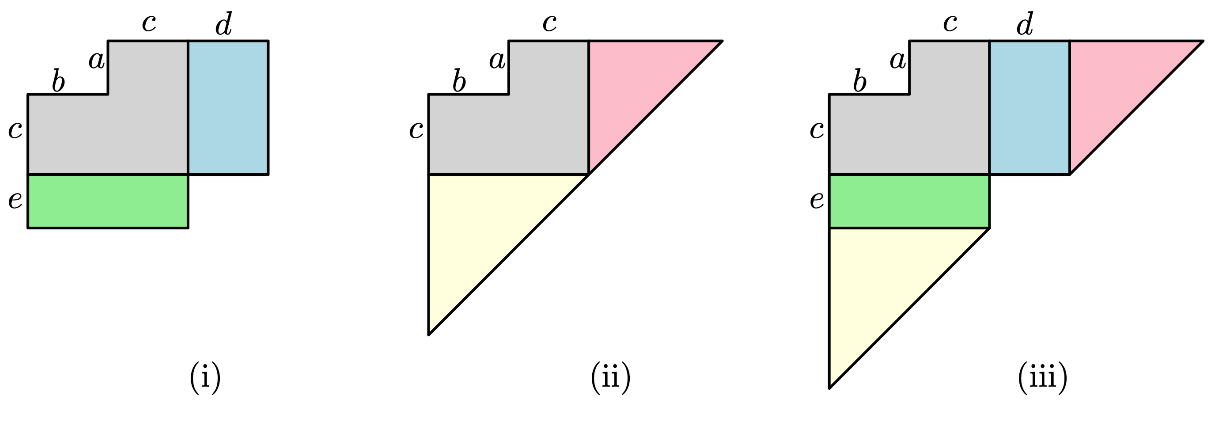


FIGURE 1. Skew shapes with product formulas for the number of SYT.

$$\sum_{\lambda} \frac{(\dim\lambda)^2}{n!} = 1$$

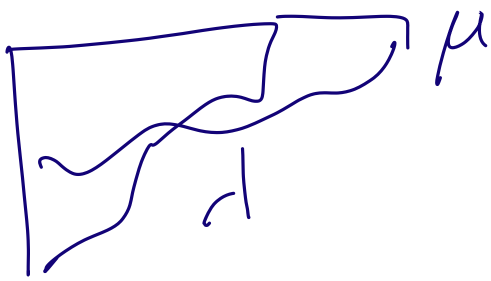
Prop. $N \geq \ell(\lambda)$, $|\lambda| = n$, $|\mu| = m$

$$\frac{\dim(\mu,\lambda)}{(n-m)!} = \det\left[\frac{1}{(\lambda_i - \mu_j + j - i)!}\right]_1^N$$

$\Gamma(n+1) = n!$, $\Gamma(-k) = \infty$

$$p_1^{n-m} s_\mu = \sum_{\lambda} \dim(\mu,\lambda)\, s_\lambda$$

Proof. 1) Vanishing $\mu \not\subset \lambda$



$\mu_i > \lambda_i$

2) $\lambda = \mu$

$$\lambda_i - \mu_j + j - i = 0$$

$$\det \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1$$

3) $\mu \subset \lambda$, $\mu \neq \lambda$, $\ell(\mu) \leq \ell(\lambda)$,

$m < n$

$\Downarrow$ $\quad s_\mu = \dfrac{a_{\mu+\delta}}{a_\delta}$,

coeff in $\boxed{a_{\mu+\delta}(x_1 + \dots + x_N)^{n-m}}$

by $\boxed{x^{\lambda+\delta}}$

$\delta = (N-1, N-2, \dots, 2, 0)$

$a_\alpha = \det[x_i^{\alpha_j}]_1^N$

$$a_{\mu+\delta}(x_1+\dots+x_N)^{n-m} = \sum_\lambda \dim|\mu,\lambda\rangle \circ a_{\lambda+\delta}$$

$\Rightarrow$ Follows from binomial theorem.

Let $\boxed{\begin{aligned} l_i &= \lambda_i + N - i \\ m_i &= \mu_i + N - i \end{aligned}}$

$$\sum_{\sigma\in S_N} (-1)^\sigma \cdot \prod x_i^{m_{\sigma i}} (x_1+\dots+x_N)^{n-m}$$

coeff. by $x_1^{l_1} \cdots x_N^{l_N}$

Fixed $\sigma \Rightarrow$ coeff.

$$\begin{pmatrix} n-m \\ l_1 - m_{\sigma_1} \cdots l_N - m_{\sigma_N} \end{pmatrix}$$

$\boxed{\binom{N}{k_1 \dots k_l} = \frac{N!}{k_1! \dots k_l!}}$ (multinomial)

$$\sum_{b} (n-m)! \, (-1)^{b} \prod_{i} \frac{1}{(\ell_i^{\circ} - m_{b_i})!}$$

$$\lambda_i + N - i - (\mu_{b_i} + N - b_i)$$

$\Rightarrow$ determinant $\square$.

7.2. Shifted Schur polynomials

$$\dim(\nu, \lambda^{(n)}) \,/\, \dim \lambda^{(n)} = \frac{f_\nu^*(\lambda^{(n)})}{n^{\downarrow m}}$$

$$|\lambda| = n, \quad |\nu| = m$$

$$\lambda = (a, n-a) \qquad \nu = (b, m-b)$$

$$f_\nu^*(x,y) = x^{\downarrow b} \, y^{\downarrow (m-b)}$$

$$x^{\downarrow k} = x(x-1)(x-2)\cdots(x-k+1)$$

Recall Pascal : relative dim. belongs to the same algebra (not the case for $\mathbb{Y}$).

$$s_\lambda \hookleftarrow \frac{\det[x_i^{\lambda_j+N-j}]}{\det[x_i^{N-j}]} \quad = V(\vec{x})$$

Okounkov–Olshanski

Sh. Sch. Perly

$$S^*_\mu(x_1 \dots x_N) = \begin{cases} \dfrac{\det\left[(x_i+N-i)^{\downarrow \mu_j+N-j}\right]_1^N}{\det\left[(x_i+N-i)^{\downarrow N-j}\right]_1^N} \\ 0, \quad N < \ell(\mu) \end{cases}$$

- $S^*_\mu(x_1 \dots x_N)$ not symm. in $x_1 \dots x_N$ <u>is</u> symm. in $x_1-1, \dots, x_N-N$

- Denominator

$$\det\left[x_i^{j-1}\right] = \text{Vandermonde}$$

$$\underline{\det\left[p_{j-1}(x_i)\right]}$$

$p_j \leftarrow$ poly of deg. j

$p_j(x) = x^j + \cdots$

$$\begin{matrix} \overset{1}{p_0(x_1)} & \cdots & p_0(x_N) & \\ p_1(x_1) & \cdots & p_1(x_N) & \leftarrow x+\cancel{c} \\ p_2(x_1) & \cdots & p_2(x_N) & \leftarrow x^2+\cancel{b}x+\cancel{c} \\ \vdots & & & \end{matrix}$$

$$\det\left[(\underbrace{x_i+N-i}_{\text{var's}})^{\overset{N-j}{\downarrow N-j}}\right]_1^N = \prod_{i<j}(x_i-i-x_j+j)$$

shifted Vandermonde

- Top degree term in $x_1, \dots x_N$:

$$S^*_\mu(x_1 \dots x_N) = S_\mu(x_1, \dots x_N) + \text{L.o.T.}$$

lower degree

- Stability: $x_{N+1} = 0$ (exercise)

$$S^*_\mu(x_1 \dots x_N, 0) = S^*_\mu(x_1, \dots, x_N)$$

(just as s_λ's)

- $S^*_\mu(\lambda)$ is well def $\forall \lambda$

$$\lambda = (\lambda_1, \dots, \lambda_n, 0, 0, \dots)$$

Theorem. $\forall\ \mu, \lambda$ $\quad |\lambda| = n,\ |\mu| = m$

$$\boxed{\frac{\dim(\mu, \lambda)}{\dim \lambda} = \frac{s^*_\mu(\lambda)}{n^{\downarrow \mu}}}$$

(Recall Pascal) $\quad \frac{f^*_\mu(\lambda)}{n^{\downarrow m}}$

$$x^{\downarrow b} y^{\downarrow (m-b)} = x^b y^{m-b} + \dots$$

Proof.

$$\frac{\dim(\mu, \lambda)}{(n-m)!} = \det\left(\frac{1}{(\lambda_i - \mu_j + j - i)!}\right)$$

$$\frac{\dim \lambda}{n!} = (HW) = \frac{\overset{*}{\prod_{i<j}} (\lambda_i - \lambda_j + j - i)}{\prod_i (\lambda_i + N - i)!}$$

① * – shifted Vandermonde

② $n! / (n-m)! = n^{\downarrow m}$

$$\det\left(\frac{1}{(\lambda_i-\mu_j+j-i)!}\right)\prod_i(\lambda_i+N-i)!$$

$$=\det\left(\frac{(\lambda_i+N-i)!}{(\lambda_i-\mu_j+j-i)!}\right)$$

$$\| \quad (\lambda_i+N-i)^{\downarrow \mu_j+N-j}$$

$\square$

7.3. Shifted symm. functions. Λ^*

(Not the same algebra)

$s_\mu^*(x_1, \dots, x_N) \in \Lambda^*_N$

Λ^*_N : polynomials symm. in $x_1 - 1, \dots, x_N - N$

Ex. $p^*_{k,c}(x_1 \dots x_N) = \sum_{i=1}^{N} \left((x_i - i + c)^k - (-i+c)^k \right)$

$\cap \; \Lambda^*_N$

$[p^*_{k,c}] = p_k$, top degree term

(so, $\Lambda^*_N \longrightarrow \Lambda_N$, $f \to [f]$)

always symmetric

filtered by degree

graded by degree

$\Lambda^{*,k}_N = \{$all sh. symm. of deg $\leq k\}$

$\Lambda^*_{N+1} \longrightarrow \Lambda^*_N$, $x_{N+1} = 0$

& $\Lambda^{*k} = \varprojlim_N \Lambda^{*k}_N$

$\Lambda^* = \bigcup_{k \geq 0} \Lambda^{*,k}$ | $\Lambda = \bigoplus_k \Lambda^k$

Filtered / graded , $\Lambda^{*k} / \Lambda^{*k-1} = \Lambda^k$ homog. symm. pol. of deg. k

& Shifted Schur functions $s^*_\mu \in \Lambda^*$

- basis in Λ^*
- $[s^*_\mu] = s_\mu$

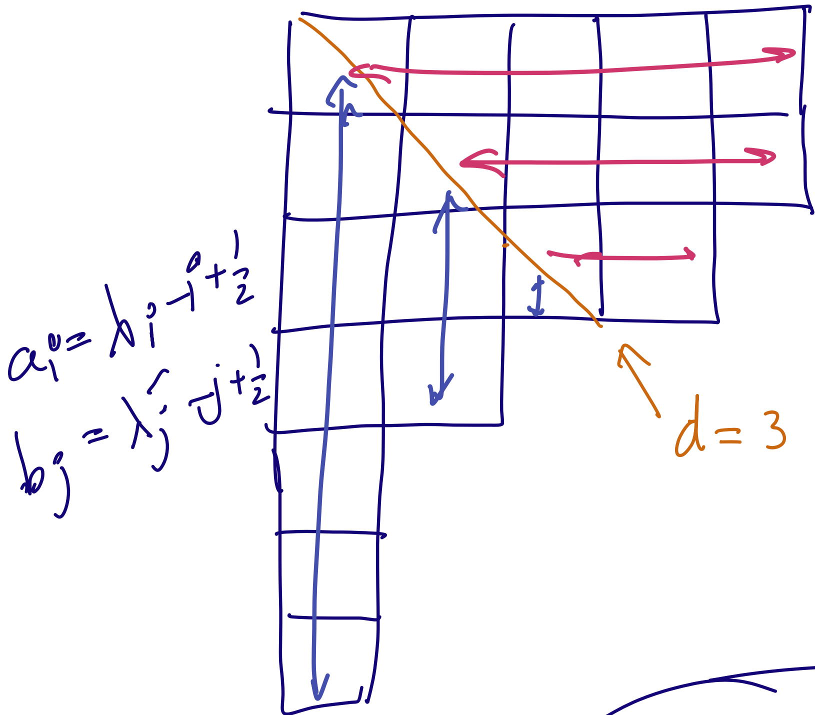
$$p^*_{k,c}(\underbrace{x_1, x_2, \ldots}_{\text{finitely many nonzero}}) = \sum_{i=1}^{\infty} \left((x_i - i + c)^k - (-i+c)^k \right) \in \Lambda^*$$

$$\frac{s^*_\mu(\lambda^{(n)})}{n^{\downarrow m}}$$

$p^*_{k,c}$ — algebraically indep in Λ^*

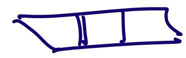
$$[p^*_{k,c}] = p_k$$

7.4. Modified Frobenius Coord.



λ

$\lambda = (a_1, \ldots a_d \mid b_1, \ldots b_d)$

lengths of  ,  $\in \mathbb{Z} + \frac{1}{2}$.

$$|\lambda| = \sum a_i + b_i$$

$$p^*_{k, \frac{1}{2}}(\lambda_1, \lambda_2, \ldots) = \sum_{i=1}^{\infty} \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^k - \left(-i + \tfrac{1}{2}\right)^k \right)$$

Proposition.

$$= \sum_{i=1}^{d} \left(a_i^k - (-b_i)^k \right)$$

Lemma.

$$\prod_{i=1}^{\infty} \frac{u+i-1/2}{u+i-1/2-\lambda_i} = \prod_{i=1}^{d} \frac{u+b_i}{u-a_i}$$

Proof

$$\frac{u+i-1/2}{u+i-3/2} \cdot \frac{u+i-3/2}{u+i-5/2} \cdots \frac{u+i-\lambda_i+1/2}{u+i-\lambda_i-1/2}$$

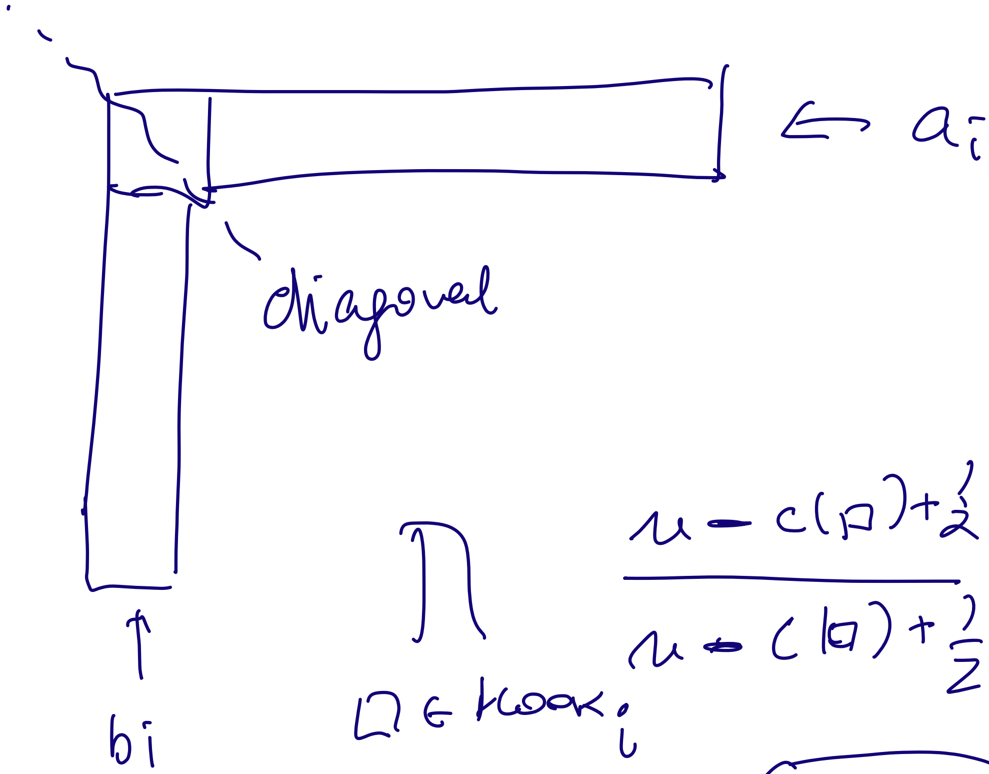
$i=2$

Content $c(\square) = j-i$

	0	1	2	3	4
λ_2	-1	0	1	2	3
	-2	-1	0	1	
	-3	-2			
	-4				
	-5				
	-6				

$$\prod_{\square \in \lambda_i} \frac{u-c(\square)+\frac{1}{2}}{u-c(\square)-\frac{1}{2}}$$

$$\text{LHS} = \prod_{\square \in \lambda} \frac{u-c(\square)+\frac{1}{2}}{u-c(\square)-\frac{1}{2}}$$



$$\prod_{\square \in \mathrm{Hook}_i} \frac{u - c(\square) + \frac{1}{2}}{u - c(\square) + \frac{1}{2}} = \frac{u + b_i}{u - a_i}$$

$\square$

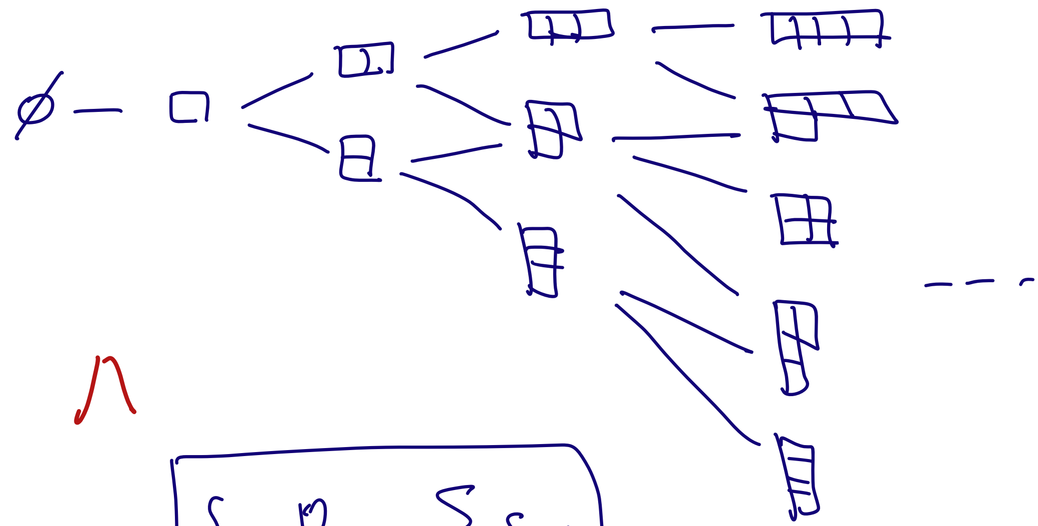
Next, p_k^* & Frobenius coord.

NO class on 10/6 (thu)

- have a good break
  & see you on 10/11

- HW5 just posted,
  others are being graded

Recall.



$$\mathbb{Y} \longleftrightarrow \Lambda$$
$$\updownarrow$$
$$\Lambda^*$$

$$\boxed{s_\lambda p_1 = \sum_{\nu=\lambda+\square} s_\nu}$$

$$\boxed{\frac{\dim(\mu,\lambda)}{\dim\lambda} = \frac{s^*_\mu(\lambda)}{n^{\downarrow m}}}$$

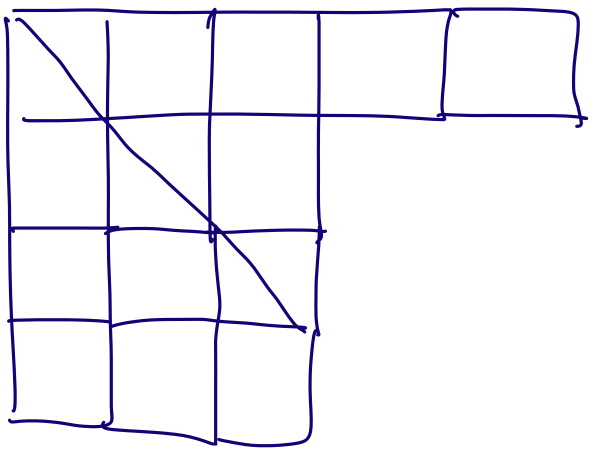
$|\lambda| = n$
$|\mu| = m$

$$\underbrace{s^*_\mu \in \Lambda^*}_{\text{symm. in } \{x_i - i\}}, \qquad [s^*_\mu] = s_\mu$$

top homog. component

$$p_k^* = \sum_{i=1}^{\infty}\left(\left(\lambda_i - i + \tfrac{1}{2}\right)^k - \left(-i + \tfrac{1}{2}\right)^k\right)$$

7.4. Modified Frobenius coordinates



$$\lambda = \left(4+\tfrac{1}{2},\ 1+\tfrac{1}{2},\ \tfrac{1}{2} \,\middle|\, 3+\tfrac{1}{2},\ 2+\tfrac{1}{2},\ 1+\tfrac{1}{2}\right)$$

$f \in \Lambda^*$, $f(\lambda)$ is nice in a_i, b_i

Proved

$$\prod_{i=1}^{\infty} \frac{u+i-1/2}{u+i-1/2-\lambda_i} = \prod_{j \geq 1}^{d} \frac{u+b_j}{u+a_j} \qquad (*)$$

$$p_k^*(x_1, x_2, \dots) = \sum_{i=1}^{\infty} \left[\left(x_i - i + \tfrac{1}{2}\right)^k - \left(-i+\tfrac{1}{2}\right)^k\right]$$

Prop. $p_k^*(\lambda) = \sum_{i=1}^{d} \left(a_i^k - (-b_i)^k \right)$

Proof. Expand log of $(*)$ into powers of $1/u$ at $\underline{\underline{u=\infty}}$.

$$\log\left(\frac{u+i-\frac{1}{2}}{u+i-\frac{1}{2}-\lambda_i}\right) = \log\left(\frac{1+u^{-1}(i-1/2)}{1+u^{-1}(i-\frac{1}{2}-\lambda_i)}\right)$$

$-\log(1-z)$

$$= \sum_{k=1}^{\infty} \frac{(\lambda_i+\frac{1}{2}-i)^k - (-i+\frac{1}{2})^k}{k}$$

$$\sum_i \sum_{k=1}^{\infty} \cdots \Rightarrow \boxed{\sum_k \frac{p_k^*(\lambda)}{k}}$$

$\|$

$\log(\mathrm{LHS}(*))$

$$\prod_{i=1}^{\infty} \frac{u+i-1/2}{u+i-1/2-\lambda_i} = \underbrace{\prod_{j=1}^{d} \frac{u+b_j}{u+a_j}}_{} \quad (*)$$

expand this & get

$$p_k^* = \sum a_i^k - (-b_i)^k$$

$\square$

7.5. Thoma Simplex Ω

$$\Omega \subset [0,1]^{\infty} \times [0,1]^{\infty}$$

closed, compact

(Exercise)

$$\Omega = \left\{ \alpha_i, \beta_i \quad \begin{array}{l} \alpha_1 \geq \alpha_2 \geq \dots \geq 0 \\ \beta_1 \geq \beta_2 \geq \dots \geq 0 \\ \sum_{i=1}^{\infty} \alpha_i + \beta_i \leq 1 \end{array} \right\}$$

$\omega \in \Omega$

$C(\Omega)$ = cont. funct.

Pascal $\qquad \Omega = [0,1]$

$$x^a y^b \in \mathbb{R}[x,y] \longmapsto p^a(1-p)^b$$

Morphism $\Lambda \longrightarrow \Lambda^0 \subset C(\Omega)$

Def. $\boxed{\Lambda^0 = \Lambda / (p_1 - 1)\Lambda}$ $\quad p_1 \mapsto 1$

$f \mapsto f^0$ $\quad \omega = (\alpha; \beta)$

$\sum \alpha_i + \beta_i \leq 1$

$p_1^0(\omega) = 1$

$k \geq 2$ $\quad p_k^0(\omega) = \sum_{i=1}^{\infty} \left(\alpha_i^k - (-\beta_i)^k \right)$

Pascal Analogy: $\left(\mathbb{R}[x,y]\right)^0 \subset C[0,1]$, $\left(x^a y^b\right)^0 = x^a (1-x)^b$

Note. $\sum \alpha_i + \beta_i$ is not continuous.,

$p_k^0(\omega)$ $\quad k \geq 2$ $\quad$ are cont.

$$\boxed{\begin{array}{l} \omega(n) = \left(\frac{1}{n}, \dots \frac{1}{n}, 0 \dots ; 0 \right) \\ \sum \alpha_i(n) + \beta_i(n) = 1 \\ \omega(n) \longrightarrow 0. \end{array}}$$

$\sum \alpha_i + \beta_i \leq 1$

$\Rightarrow \alpha_i, \beta_i \leq \frac{1}{i}$

$\Rightarrow \sum \alpha_i^2 \leq \sum 1/i^2$

$$\boxed{\Lambda^0 \subset C(\Omega)}$$

Prop.

$$\boxed{\begin{array}{l} - \ 1^0 = 1 \\ - \ \left((p_1 - 1)\Lambda\right)^0 = 0 \end{array}}$$

$-$ $\Lambda^0 \subset C(\Omega)$ is $\boxed{\text{dense}}$

Proof.

Λ^0 – algebra, $1 \in \Lambda^0$
+ separates pts.

$\Rightarrow$ (Stone – Weierstrass), dense

separates points, as $p_k^0 = \sum \alpha_i^k + (-\beta_i)^k$

$(\alpha, -\beta)$: poles of

$$\sum_{k=1}^{\infty} \frac{p_k^0(\alpha,\beta)}{u^k} = \sum_i \frac{\alpha_i}{u - \alpha_i} + \sum_i \frac{\beta_i}{u + \beta_i} + \frac{1}{u}\left(1 - \sum(\alpha_i + \beta_i)\right)$$

$k=1:$ $\quad \frac{1}{n} = \frac{1-\sum(\alpha_i+\beta_i)}{n} + \frac{\sum \alpha_i+\beta_i}{n}$

Another way? $\quad (\beta_i \equiv 0)$

$$\alpha_1 = \lim_{k\to\infty} \sqrt[k]{\sum_{i=1}^{\infty} \alpha_i^k}$$

$\sqrt[k]{C\,\alpha_1^k}$

$$\sqrt[k]{\alpha_1^k \left[C + \sum_j \left(\frac{\alpha_j}{\alpha_1}\right)^k \right]}.$$

smaller than 1

finite

$p^0_k(w) = \ldots$

Recall
$$\boxed{H(t) = E(-t)^{-1} = e^{\sum_{k\geq 1} \frac{p_k t^k}{k}}}$$

$\underline{h^0_k(w)}$, $\underline{e^0_k(w)} = ?$

$$\boxed{\sum_{k=0}^{\infty} h^0_k(w)\, t^k} = \exp\left[\sum_k \frac{t^k}{k} \sum_i \alpha_i^k - (-\beta_i)^k\right]$$

let $\sum \alpha_i + \beta_i = 1$

$$= \exp\left(\sum_i \left(-\log(1-\alpha_i t) + \log(1+\beta_i t)\right)\right)$$

$$= \boxed{\prod_i \frac{1+\beta_i t}{1-\alpha_i t}}$$

If $\gamma = \underline{1} - \sum(\alpha_i + \beta_i) > 0$

then you add $e^{\gamma t}$

7.6 Proof of Thoma's theorem & Vershik – Kerov's theorem for $S(\infty)$

Thoma: Extremes are param. by $\omega \in \Omega$
(1964)

– harm. f. on $\mathbb{Y}$, $\boxed{\varphi_\omega(\lambda) = s^o_\lambda(\omega)}$

– coherent systems $\boxed{M_n^{(\omega)}(\lambda) = \dim\lambda \cdot s^o_\lambda(\omega)}$

– char. of $S(\infty)$,

$$\chi_\omega(\rho) = \boxed{p^o_{\rho_1}(\omega)\, p^o_{\rho_2}(\omega)\, p^o_{\rho_3}(\omega) \cdots}$$

$$p^o_k(\omega) = \sum\left(\alpha_i^k - (-\beta_i)^k\right)$$
$k \geq 2$

Vershik – Kerov (1981)

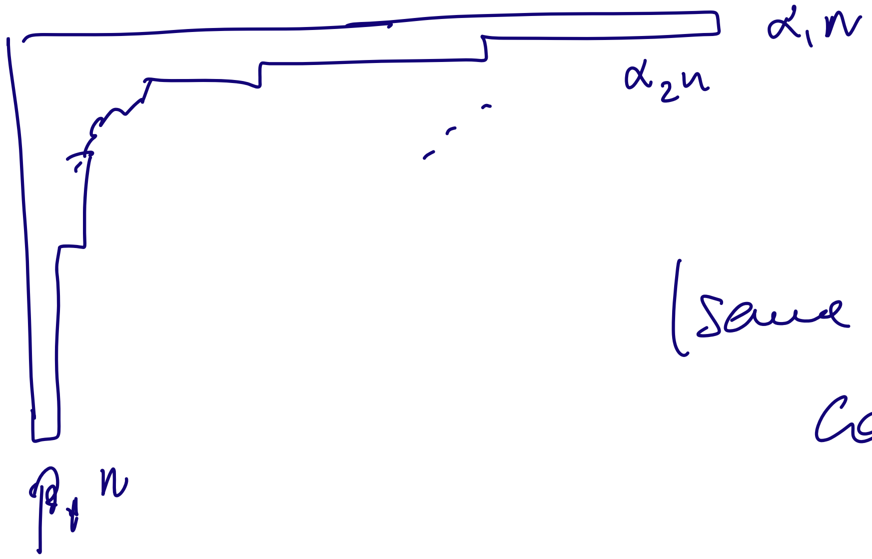
$$\lim_{n\to\infty} \chi^{S(n)}_{\lambda^{(n)}} \Big/ \dim \lambda^{(n)} \longrightarrow \chi$$

extreme ch. of $S(\infty)$

$$\Updownarrow$$

$\lambda_i - i + 1/2$

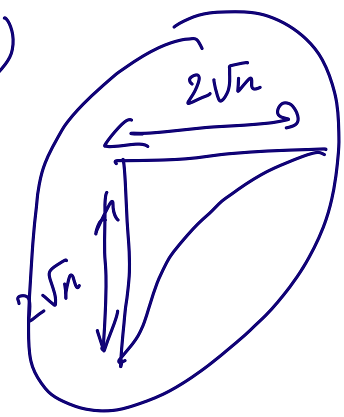
$$\frac{a_i(\lambda^{(n)})}{n} \to \alpha_i \,, \qquad \frac{b_i(\lambda^{(n)})}{n} \to \beta_i$$



(same as row/column convergence)

$$\gamma = 1 - \sum(\alpha_i + \beta_i)$$

$$\gamma = 1, \ \alpha_i = \beta_i = 0 \longrightarrow$$



sublinear

Proof of both by algebraic approximation

Need $\lambda^{(n)}$ s.t. $\dfrac{\dim(\mu, \lambda^{(n)})}{\dim \lambda^{(n)}}$ has a limit, $\forall \mu$ fixed

1

$$\mathbb{Y}_n = \{\lambda : |\lambda| = n\} \hookrightarrow \Omega$$

$$\lambda \longmapsto \boxed{\frac{1}{n} \omega_\lambda} ; \qquad \alpha_i = a_i/n \qquad \beta_i = b_i/n$$

$$\left(\text{note } \sum \alpha_i + \beta_i = 1\right)$$

$$\textcircled{2}\quad f^* \in \Lambda^* \rightsquigarrow \begin{array}{c} f = [f^*] \in \Lambda \\ \big\downarrow \\ f^\circ \in C(\Omega) \end{array}$$

If $\deg f^* = m$, $|\lambda| = n$

$$\Rightarrow \quad \frac{f^*(\lambda)}{n^m} = f^\circ\left(\frac{1}{n}\,\omega_\lambda\right) + \underbrace{O\left(\frac{1}{n}\right)}_{\text{uniform in } \lambda}$$

Indeed,

$$\frac{1}{n^k}\, p_k^*(\lambda) = \frac{1}{n^k} \sum_{i=1}^{d} \left(a_i^k - (-b_i)^k \right)$$

$$= p_k^\circ\left(\frac{1}{n}\,\omega_\lambda\right)$$

(exact identity, continues to

all functions by linearity
in p_μ^*'s.)

$f^* \in \Lambda^*$, $\underline{\underline{f^*}} = \sum_\nu \boxed{p_{\nu_1}^* \cdots p_{\nu_m}^*}$

(3) $$\frac{\dim(\mu, \lambda)}{\dim \lambda} = s_\mu^0\left(\frac{1}{n}\omega_\lambda\right) + O\left(\frac{1}{n}\right)$$

$|\lambda| = n$

$$\frac{s_\mu^*(\lambda)}{n^{\downarrow m}}$$

(4) $\dfrac{\dim \, l_{\mu, \lambda^{(n)}}}{\dim \lambda^{(n)}}$ (fixed μ)

(informal)

has a limit as $n \to \infty$

iff $s^{\circ}_{\mu}\left(\frac{1}{n} \omega_{\lambda^{(n)}}\right)$ has a lim.

iff $\frac{1}{n} \omega^{\circ}_{\lambda} \in \Omega$ has a lim.

iff $\left(\frac{a_i}{n}, \frac{b_i}{n}\right) \to (\alpha_i, \beta_i)$

$\square$

$$e^{\gamma t} \prod \frac{1+\beta_i t}{1-\alpha_i t} = H(t)$$

$\alpha_i = \beta_i \equiv 0$

$$H(t) = e^t = \sum t^k/k!$$

$$h_k = \frac{1}{k!}$$

HW5: Jacobi – Trudi

$$S_\lambda = \det\left[h_{\lambda_i - i + j}\right]_1^N$$

$$S_\lambda^0(\alpha=\beta=0, \gamma=1) = \det\left[\frac{1}{(\lambda_i - i + j)!}\right]$$

$$= \boxed{\frac{\dim\lambda}{n!}}$$

(was a formula)

$$\frac{\dim(\mu,\lambda)}{(n-m)!} = \det\left(\text{———} \right)$$

Plancherel meas.

$\gamma = 1$, coherent system is

$$M_n(\lambda) = \dim\lambda \cdot s_\lambda^0 = \boxed{\frac{(\dim\lambda)^2}{n!}}$$

→ we classified extreme objects for $S(\infty)$

---

what next? (any feedback appreciated!)

several options

→ representations of $\underline{\underline{S(\infty)}}$

→ non-extreme measures for $S(\infty)$

→ q-deformations

→ Young-Fibonacci

→ more probability

(still deciding ...)

(office hours

M 2:30–3:30

W 2:00–3:00)

Changed

2-3 L: Next: construction of irreps of $S(\infty)$

q-analogues

Fibonacci / cont. fractions

Recall Thoma's theorem

$S(\infty)$ characters $\chi \in \mathcal{X}$,

$\chi : S(\infty) \to \mathbb{C}$

→ central $\chi(ab) = \chi(ba)$

→ normalized $\chi(e) = 1$

→ pos-def. $\sum_{ij} c_i \bar{c}_j \chi(g_i g_j^{-1}) \geq 0$

$\forall g_i, c_i$

Extreme points & classification

$$\mathrm{Ex}(\mathcal{X}^\nu) = \left\{ \underset{\mathcal{X}^\nu}{\overset{\uparrow}{\varphi}} \text{ s.t. } \begin{array}{l} \varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2 \\ 0 < \alpha < 1 \\ \Rightarrow \varphi = \varphi_1 = \varphi_2 . \end{array} \right\}$$

Thus. $E(\mathcal{X}^\nu) = \Omega = \left\{ \begin{array}{l} \alpha_1 \geq \alpha_2 \geq \dots \geq 0 \\ \beta_1 \geq \beta_2 \geq \dots \geq 0 \\ \sum_i (\alpha_i + \beta_i) \leq 1 \end{array} \right\}$

$$\chi_{\alpha\beta}(\sigma) = \prod_{i=1}^{k} \underbrace{\left(\sum_j \alpha_j^{\rho_i} - (-\beta_j)^{\rho_i} \right)}_{\boxed{p^0_{\rho_i}(\alpha;\beta)}}$$

σ cycles $\rho_1 \geq \rho_2 \dots \geq \rho_k \geq 2.$

Goal: construct representations for these irred. characters.

8. Construction of $S(\infty)$ representations

(we need ∞-dim representations!)

⟨see [BO-book, ch. 8–10]⟩

8.1 Unitary representations

Idea: Consider H – complex Hilbert (Banach, $(\cdot,\cdot)$)

$\boxed{H^* = H}$

$B(H)$ = bounded ops

$$U(H) = \{ A - \text{bdd}, \quad AA^* = A^*A = 1 \}$$

$\boxed{\text{Def of } A^*:}$ $(Au, v) = (u, A^* v)$

$$A \in U(H) \Rightarrow (Au, Av) = (u, v).$$

$T: G \to \underline{\underline{U(H)}}$ is a repr. if... $\boxed{T(g)\,T(h) = T(gh).}$

e.g. $S(\infty)$

irreducible T : $\boxed{\text{H has no nontrivial } T\text{-invariant subspaces}}$

(Note: if $F \subseteq H$ subspace $T(G)F \subset F \Rightarrow F^{\perp}$ is also invariant)

$\xi \in H$ is called cyclic if $\boxed{\text{span}\{T(g)\xi,\ g \in G\}}$ is dense in H

Fact. if T - irred. $\Rightarrow$ any nonzero vector ξ is cyclic

Proof. If $\eta \neq 0$

$F = \overline{\text{span}\{T(g)\eta\}}$ ← invar. subspace

ξ - cyclic $\Longrightarrow$ spherical funct of ξ

$\varphi : G \longrightarrow \mathbb{C}$

$$\boxed{\varphi(g) = (T(g)\xi, \xi)}$$

Note $\varphi(g)$ may not be central, i.e.

$$\varphi(gh) \neq \varphi(hg)$$

Let $\mathcal{P}'(G)$ = pos-def. central normalized

$\Phi(G)$ = pos-def.

$\Phi_1(G)$ = pos-def normalized

(GNS)

Theorem (Gelfand – Naimark – Segal)

(1) T – rep of G, $0 \neq \xi \in \mathcal{H}$

$\Rightarrow$ $\varphi(g) = (T(g)\xi, \xi)$ pos-def

(2) φ - pos. def. $\neq 0$ $\Rightarrow$

$\exists!$ rep. T with cyclic vector

s.t. $\varphi(g) = (T(g)\xi, \xi)$

Proof of (1)

$$\varphi(g^{-1}h) = (T(h)\xi, T(g)\xi)$$

so φ pos. def because of

Gram matrix of

$\{ \xi_g = T(g)\xi \}$.

Gram matr. $\{(v_i, v_j)\}_{i,j}$ always pos-def.

$$\boxed{\sum c_i \bar{c}_j (v_i, v_j) = \|\sum c_i v_i\|^2}$$

$\boxed{\psi(g,h) = \varphi(g^{-1}h)}$

kernel

(2) idea pos-def $\Rightarrow$ Gram matrix

$\Rightarrow$ construct H [i.e. $(\cdot,\cdot)$] from a family of vectors whose Gram matrix is given

(exercise)

$$\|\xi\| = \sqrt{(\xi, \xi)} = 1 \iff \varphi(e) = (T(e)\xi, \xi) = 1$$

(normalized)

<u>Theorem</u> (see [BO] section 8 for proof)

$\varphi \in \Phi_1(G)$ extreme (as a point in the convex set)

$\iff$ T corresp. to φ is <u>irreducible</u>

(Irred. unitary T with $\|\xi\| = 1$ cyclic $\longleftrightarrow$ $\mathrm{Ex}\,\Phi_1(G)$)

Def. Commutant of T in $\mathcal{U}(\mathcal{H})$

— all bdd op. in $\mathcal{H}$
which commute with $T(g)$

Schur's Lemma.

T - irred. $\iff$ commutant are
scalar operators

Proof. Projection to inver. subspace
commutes with T.

If $A \in$ Comm.(T), nonscalar

$\Rightarrow$ $A+A^*$, $i(A-A^*)$ $\in$ Comm(T)

hermitian

$\Rightarrow$ spectral projection
associated to $A+A^*$
or $i(A-A^*)$ (at least
one is
nonzero)

is also $\in$ Comm.

8.2. Motivation: connection to the classical theory of reps & characters as Traces

Let $T: G \to \mathrm{End}(V)$ be a usual f.d. repr. of a finite group.

$\mathcal{H} = \mathrm{End}(V)$ is Hilbert if

$(A, B) = \mathrm{Tr}(AB^*)$

Then define $\tilde{T}: G \to \mathcal{U}(\mathcal{H})$,

$$\tilde{T}(g)A = T(g)A.$$

It is unitary:

$$(\tilde{T}(g)A, B) = \mathrm{Tr}(\tilde{T}(g)AB^*)$$

$$= \mathrm{Tr}(AB^*\tilde{T}(g^{-1})^*)$$

$$= \mathrm{Tr}(A(\tilde{T}(g^{-1})B)^*)$$

$$= (A, \tilde{T}(g^{-1})B)$$

Let $\xi = \mathrm{Id} \in \mathrm{End}(V)$

$$\varphi(g) = (\tilde{T}(g)\xi, \xi)$$

$$= \mathrm{Tr}(T(g)) = \chi_T(g),$$

the character.

T - irrep. $\Rightarrow \xi = \mathrm{Id}$ is cyclic because $T(\mathbb{C}[G]) = \mathrm{End}(V)$

<u>Note</u>: Unitary rep. give „fractional rep's"

$$\varphi_\xi(g) = (T(g)\xi, \xi).$$

$$\xi \to \eta = \alpha\xi \qquad \alpha \in \mathbb{C}$$

$$\varphi_\eta(g) = \varphi_\xi(g) \cdot |\alpha|^2.$$

$\psi(g)$ — for f.d. repres.
is central (because character)

This relies on the special property

$$(A, B) = \mathrm{Tr}(AB^*) = \mathrm{Tr}(B^*A) =$$
$$= (B^*, A^*)$$

For general setting, we will need a replacement of the notion of centrality

If $\tilde{T}$ acts on H as

$$\tilde{T}(g)\, A = T(g)\, A \qquad \left(\begin{array}{l} G - \text{finite}, \\ H = \mathrm{End}(V) \end{array} \right),$$

then there is a commutant action T':

$$T'(g)\,(A) = A\, T(g^{-1}),$$

which commutes with T.

Consider $(G \times G,\ \underbrace{\overset{\text{denote } K}{\overbrace{\mathrm{diag}\, G}}}_{= \{(g,g)\}})$

& define spherical function on $G \times G$:

$\xi = \mathrm{Id}$

$$\varphi(g,h) = \left(T \otimes T'(g,h)\, \xi,\ \xi \right)$$

$$= \mathrm{Tr}\left(T(g)\, T(h^{-1}) \right) \left(= \chi_T(g h^{-1}) \right)$$

The function φ is K-biinvariant, i.e.

$$\varphi(k_1 g k_2,\ k_1 h k_2) = \varphi(g,h) \quad \forall\, k_1, k_2 \in K$$

Indeed,

$$\chi_T(k_1 g k_2 k_2^{-1} h k_1^{-1}) =$$
$$= \chi_T(k_1 g h^{-1} k_1^{-1}) = \chi_T(gh)$$

$\square$

Next, we consider a more general setting of Gelfand pairs where there is K-biinvariance & not just centrality

---

Correction wrt what was in class

— Not $\mathcal{Y}(G)$ but $\Phi(G)$

— class (= central) functions are replaced by biinvariant

Recall. [I updated notes for 10/11 for better presentation]

$T: G \to U(H)$, $\underbrace{\varphi(g) = (T(g)\xi, \xi)}_{\text{spherical f.}}$, ξ – cyclic

$$\left\{\begin{array}{c}\text{pos-def. funct.}\\ \text{on } G\end{array}\right\} \underset{(GNS)}{\longleftrightarrow} \left\{\begin{array}{c}\text{spherical f.}\\ \text{of unitary}\\ \text{represent. of } G\end{array}\right\}$$

Not necessarily $\varphi(gh) = \varphi(hg)$, not like characters

Notations for functions on the group

$\mathcal{P}(G)$ ⟵ $\varphi(e)=1$, $\varphi(gh) = \varphi(hg)$

$\boxed{\Phi(G)}$ } pos-def.

$\Phi_1(G)$ ⟵ $\varphi(e)=1$

$\Phi_1(G//K)$

finite

Ex. $\pi : K \to End(V)$ (unitary) $\quad H = End(V)$

Hilbert: $\quad (A,B) = Tr(AB^*)$

$$\boxed{\sum_{k,i} a_{ik} \overline{b_{ik}}}$$

$T = \pi \otimes \bar{\pi}$ represent. of $\quad G = K \times K$

$$T(g,h) A = \pi(g) A \, \pi(h^{-1})$$

$$\xi = Id/_{\dim V} \qquad \|\xi\| = 1$$

$$\varphi(g,h) = Tr(\pi(gh^{-1}))/_{\dim V}.$$

sph. funct. on $\boxed{K \times K}$

$$K \times K \supset \operatorname{diag} K = \{(g,g)\}$$

$\varphi(g,h) = \mathrm{Tr}(\pi(gh^{-1}))/\dim V.$

is $\boxed{\text{diag } K - \text{biinvariant}}$

$\varphi(k_1 g k_2, k_1 h k_2) = \varphi(g,h)$

$$k_1 g \boxed{\cancel{k_2 k_2^{-1}}} h^{-1} k_1^{-1} = k_1 g h^{-1} \cdot k_1^{-1}$$

This is more general setup $\boxed{G \supset K,}$

we discuss it now

8.3 Biinvariant functions

$\boxed{K \subset G \quad \text{subgroup}}$

Examples

$$\boxed{\begin{aligned} G &= S(a+b) \\ K &= S(a) \times S(b) \end{aligned}}$$

$$\boxed{G = U(N) \supset K = O(N)}$$

$\mathcal{H}$

$T : G \to U(\mathcal{H})$ rep.

$\mathcal{H}^K = \{\text{invar. under } K\} \subseteq \mathcal{H}$

Prop.:

$\xi \in H^K$

$\Rightarrow \varphi(g) = (T(g)\xi, \xi)$ is K-biinvar.

Proof. $\varphi(k_1 g k_2) = \varphi(g) \quad \forall k_1, k_2 \in K$

$$\varphi(k_1 g k_2) = (\underbrace{T(k_1)}\, T(g)\, \underbrace{T(k_2)\xi}_{\xi}, \xi)$$

$\square$

GNS

$\varphi \in \Phi(G)$ pos-def

$\Downarrow$

$\exists$ repr. in $\mathcal{U}(\mathcal{H})$, ξ.

$\varphi(g) = (T(g)\xi, \xi)$

Prop.

pos-def

Let $\varphi \in \overbrace{\Phi(G)}$, K - biinvariant,

T - corresp. representation

$\Rightarrow$ cyclic vector ξ belongs to H^K

Proof. $\left(T(k)\xi, T(g^{-1})\xi\right) = \varphi(gk) = \varphi(g)$

$\Rightarrow$ $T(k)\xi$ does not dep. on k

$\Rightarrow$ $T(k)\xi = \xi \quad \forall k \in K$

$\square$

$\Phi(G)$

$\subseteq$

Let $\boxed{\Phi_1(G//K)}$ = subspace of

K - biinv., pos-def., normalized

$\varphi(e) = 1$.

(Analogue : $\mathcal{P}^0(G)$)

double quotient

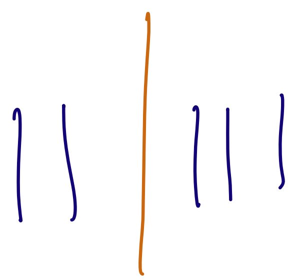
$K\backslash G/K$

$KgK \circ KhK = KghK$

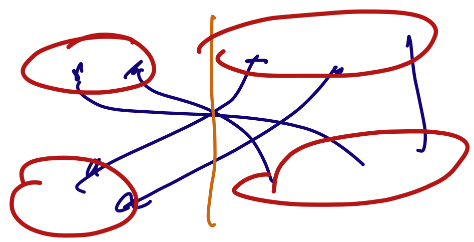
$\Phi_1(G//K) \longleftrightarrow$ functions on $K\backslash G/K$

Ex. $S(a)\times S(b)\backslash S(a+b)/S(a)\times S(b)$

$a=2$ $\qquad$ $b=3$



allowed to permute

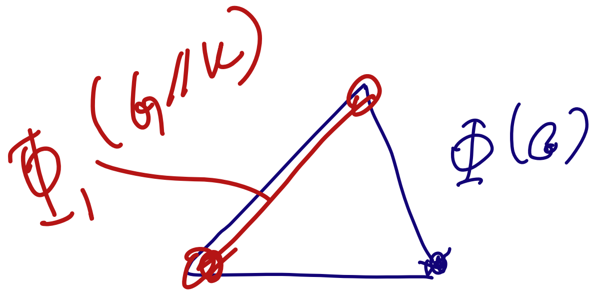


$$\boxed{\Phi_1(G/\!/K) \quad - \text{convex set}}$$

Fact. ([BO])

$\varphi \in \Phi_1(G/\!/K)$ extreme in this convex set

$\Leftrightarrow$ φ extreme as a point in $\Phi_1(G)$



$\mathbb{C}[G] \supset \mathbb{C}[G/\!/K]$

8.4. Gelfand pairs

G - finite group, $K < G$

(G, K) — Gelfand pair if

Def: $\underbrace{\mathbb{C}[G//K]}_{K\text{-bi-inv. funct. on } G}$ is commutative

under convolution

$\boxed{T: G \to U(H)}$

Prop. (∗) If (G, K) — G.p.

$\iff$ $\forall$ irrep. T, $\dim H^K = 0$ or 1

Recall ergodic measure-pres. transformations, analogy

(Analogy)

(X, μ) $\quad \mu(X) = 1$

$T: X \to X$ meas. pres.

$\mu(T^{-1}A) = \mu(A)$

T is ergodic

$\iff$ dim of the space of T-inv. funct. is 0 or 1.

$f(Tx) = f(x)$ μ-a.e. x

Proof of $(*)$ $p = \frac{1}{|K|} \sum_{k \in K} k \in \mathbb{C}[G]$

$P = T(p) =$ projection onto H^K

convolution

$PT(f)P = T(p * f * p)$

$\Rightarrow P\,T(\mathbb{C}[G])P = T(\mathbb{C}[G//K])$

T - irr. $\Rightarrow$ $T(\mathbb{C}[G]) = \mathrm{End}(H)$

(Burnside thm for inf-dim, too)

$\Downarrow$

$T(\mathbb{C}[G//K])$

$=$

$\left(P\,T(\mathbb{C}[G])P\right|_{H^K} = \mathrm{End}(H^K)$

If G.p. $\Rightarrow$ $\mathrm{End}(H^K)$ is commut. so dim 0 or 1

If $\dim H^K = 0$ or 1 $\Rightarrow$ $T(\mathbb{C}[G//K])$ is commutative $\forall T$

$\Rightarrow$ $\mathbb{C}[G//K]$ is commutative (as it holds for all T's)

$\square$

Prop. ① $\sigma : G \to G$ anti-automorph.

$(\sigma(gh) = \sigma(h)\sigma(g))$

② $\sigma(K) = K$ ③ $\boxed{K\sigma(g)K = KgK}$

③ $\forall g, \exists k_1, k_2$ s.t.

$$g k_2 = k_1 \sigma(g)$$

$\Rightarrow (G, K)$ is G-p.

Proof. σ : anti-aut of $\mathbb{C}[G//K]$

that leaves any element invar

$gh = \sigma(gh) = \sigma(h)\sigma(g) = hg$.

as $KgK = K\sigma(g)K$

$\Rightarrow \mathbb{C}[G//K]$ commutative.

$\square$

Prop. K - finite. $\Rightarrow$

$$\underbrace{(K \times K, \operatorname{diag} K)}_{\text{"double" of the group}} \text{ is G-P.}$$

Use previous.

$$\sigma : K \times K \longrightarrow K \times K$$

$$(k_1, k_2) \longmapsto (k_2^{-1}, k_1^{-1})$$

$$\sigma((g_1, g_2)(h_1, h_2)) = \sigma((g_1 h_1, g_2 h_2))$$

$$= (h_2^{-1} g_2^{-1}, h_1^{-1} g_1^{-1})$$

$$\sigma(g_1, g_2) = (g_2^{-1}, g_1^{-1})$$
$$\sigma(h_1, h_2) = (h_2^{-1}, h_1^{-1}) \qquad \square$$

Spherical rep. of G.p. (G, K)

is a unitary rep. of T

with cyclic K-invar. vector ξ.

$$\varphi(g) \qquad \frac{1}{2}\varphi(g) = \underline{\underline{\left(T(g)\frac{\xi}{\sqrt{2}}, \frac{\xi}{\sqrt{2}}\right)}}$$

$\varphi(g) = (T(g)\xi, \xi)$ — spherical function.

$$\left((T(g) z\xi, z\xi) = (T(g)\xi, \xi) \right)$$

as $\dim H^K = 0$ or 1

irred. spherical $\Rightarrow$ unique spherical vector ξ, $\|\xi\| = 1$ up to multiple by $|z| = 1$, $z \in \mathbb{C}$

Comments (from last time)

1) Fractional rep's

2) irrep. of K - finite

$\Longleftrightarrow$

irrep. sph. of $(K \times K, \operatorname{diag} K)$

2 equiv. approaches to rep. th. of finite groups K.

irred. sph. T of $(K \times K, \operatorname{diag} K)$

$$\iff \quad T(g,h)A = \pi(g)\,A\,\pi(h^{-1})$$

$$A \in \mathcal{H}$$

8.5 Gelfand pairs for ∞ groups

G - any group, $K \subset G$

$\boxed{\text{Def}}$ (G, K) is G.p. if $\boxed{P = \text{proj. onto } \mathcal{H}^K}$

$\forall$ T - irrep, $PT(g)P$ commute $\forall g$:

$$PT(g)PT(h)P = PT(h)PT(g)P.$$

(there is no $\mathbb{C}[G//K]$ but there are „$T(\mathbb{C}[G//K])$" $\forall T$)

$\boxed{\text{Fact}}$ G.p.

$\Leftrightarrow$ $\forall$ irrep. T, $\dim \mathcal{H}^K = 0$ or 1.

Olshanski pairs. (Proposition):

= Gelfand pairs which are inductive limits

Fact.

$G = \varinjlim G(n)$ inductive lim

$K = \varinjlim K(n)$

$K(n), G(n)$ finite

$(G(n), K(n))$ — G.p. $\forall n$

$\Rightarrow$ (G, K) is G.p.

Facts (finally)

① $K = S(\infty)$ and
$(K \times K, \operatorname{diag} K)$ is a G.p.

② If K – any group then
there is an isomorphism

$$\mathcal{X}(K) \longleftrightarrow \Phi_1(G // \operatorname{diag} K)$$

normalized characters χ

$$G = K \times K$$

$$\chi(h^{-1}g) = \varphi(g, h)$$

$g, h \in K.$

Abstractly, χ do not corresp. to representations but $\varphi \in \Phi_1(G//K)$ do by the G.N.S. construction.

I have a very related problem about this

(8.6) Realizations of rep's

$G = S(\infty)$, G.p. $(G \times G, \operatorname{diag} G)$

— need action of $G \times G$ on H

$$\varphi(g,h) = \chi(\underbrace{h^{-1}g}) = \prod_{j=1}^{k} \left(\sum \alpha_i^{\rho_j} - (-\beta_i)^{\rho_j} \right)$$

$\rho = (\rho_1 \geq \dots \geq \rho_k \geq 2)$ cycle str.

$k_1, k_2 \in S(\infty)$

$$\varphi(k_1 g k_2, k_1 h k_2) = \varphi(g, h) = \boxed{\varphi(h^{-1}g, e)}$$

① $\boxed{\mathring{B}\text{iregular}}$ is $\boxed{\text{irred.}}$ $\Longleftrightarrow$ $\boxed{\alpha_i = \beta_j = 0}$

$$\mathcal{H} = \ell^2(S(\infty))$$

$$\boxed{T(g,h)} : f(x) \longmapsto f(g^{-1}xh)$$

ξ = delta function at e;

ξ is diag $S(\infty)$ invariant

$$\varphi(g,e) = (T(g,e)\xi, \xi) = \begin{cases} 1, & g = e \\ 0, & \text{else} \end{cases}$$

exactly $\chi(g)$ on $S(\infty)$

corresp. to $\alpha_i = \beta_j = 0$

Note G - finite $\Rightarrow$ regular repr.

G on $\mathbb{C}[G]$ is <u>not</u> <u>irreducible</u>

Let $\sum \alpha_i = 1$, $\beta_j = 0$

$E = \ell^2(\mathbb{Z})$

(2) E, $\overline{E}$ Hilbert space & dual

$$v = \sum_{n=1}^{\infty} \sqrt{\alpha_n}\; e_n \otimes \overline{e_n},$$

(e_n - orth. basis)

$\|v\| = 1$

$v \in E \otimes \overline{E}$

Let $$H = \bigotimes_{i=1}^{\infty} (E \otimes \overline{E})$$

$e_{i_1} \otimes \overline{e_{j_1}} \otimes \cdots \otimes e_{i_k} \otimes \overline{e_{j_k}} \otimes \cdots$

$S(\infty) \times \{e\}$ permutes E,

$\{e\} \times S(\infty)$ permutes $\overline{E}$

$\xi = v \otimes v \otimes v \otimes \cdots$ – cyclic & Invariant under diag $S(\infty)$

$$v = \sum_{n=1}^{\infty} \sqrt{\alpha_n}\, e_n \otimes \overline{e_n}$$

Let us compute $\boxed{\begin{aligned}&\varphi(g,h)\\ &= (T(g,h)\xi, \xi)\end{aligned}}$

$g \in S(n)$

$$T(g^{-1}, e)\xi = \sum_{i_1 \dots i_n} \sqrt{\alpha_{i_1} \dots \alpha_{i_n}}\, (e_{i_{g(1)}} \otimes \overline{e_{i_1}})$$

$$\otimes \dots \otimes (e_{i_{g(n)}} \otimes \overline{e_{i_n}}) \otimes v \dots$$

Product with ξ $\Rightarrow$ $i_{g(1)} = i_1$,

$i_{g(2)} = i_2$

$\vdots$

$i_{g(n)} = i_n$

& role of the cycle structure.

---
1. Recall spherical representations and what we are about to do
2. Tensor products of Hilbert spaces
3. Biregular representation with alpha_i = beta_i = 0
4. Realization of representations with only alphas
5. Realization of representations with alphas and betas summing to 1

Next, q-analogues of Pascal and Young graphs. Start with graphs with edge multiplicities.

$\mathcal{H}$ $\qquad T: G \to U(\mathcal{H})$

$$\boxed{\varphi(g) = (T(g)\xi, \xi)}$$

$K \subset G$, $\quad \xi \in \mathcal{H}^K \iff \varphi(g)$ is K-biinvariant

$$\varphi(k_1 g k_2) = \varphi(g)$$

$G = K \times K \supset \operatorname{diag} K$

$$\chi(gh^{-1}) = \varphi(g, h) \qquad g, h \in K$$

$\boxed{S(\infty)}$

(1) $\alpha_i = \beta_j \equiv 0$ $\qquad$ BiRegular rep.

$$H = \ell^2(S(\infty))$$

$$T: f(x) \longmapsto f(g^{-1} x h) \qquad g, h \in S(\infty)$$

$$\xi = \delta\text{-funct. at } e$$

Irred. character

$\varphi(g,h) = \mathbb{1}_{g=h}$

$\chi(g) = \mathbb{1}_{g=e}$

$\Omega = \{ \sum \alpha_i + \beta_i \le 1 \}$

(2) $\beta_j \equiv 0$ $\qquad \alpha_i \ge 0,$ $\qquad \sum \alpha_i = 1$

$G = S(\infty)$ acts in a ∞ tensor product of Hilbert spaces

Before $S(\infty)$:

$H \rightsquigarrow H^{\otimes \infty} = ?$

$H = L^2([0,1])$ $\quad e_1, e_2, e_3 \dots$ $\quad f(x)g(y)$

$\underbrace{H \otimes H \otimes \cdots \otimes H}_{d} = L^2([0,1]^d)$

$$\underline{\underline{H^{\otimes\infty}}} = L^2\Big(\underbrace{[0,1]^\infty}\Big)$$

$\boxed{\sigma\text{-algebra is cylindric,}}$

generated by

$$f_1(x_1)\,f_2(x_2)\cdots f_k(x_k)$$

$$\forall k$$

$$e_{i_1}\otimes e_{i_2}\otimes e_{i_3}\otimes\cdots\otimes e_{i_k}\otimes \underbrace{1\otimes 1\otimes 1\otimes\cdots}_{\text{distinguished vector.}}$$

In general, H with basis $\{e_i\}$

$\rightsquigarrow$

$H^{\otimes\infty}$ with basis

$$e_{i_1}\otimes\cdots\otimes e_{i_k}\otimes\xi\otimes\xi\otimes\cdots$$

$\xi \in H$ distinguished vector

$$\|\xi\| = 1.$$

$$(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \xi \otimes \xi \otimes \cdots, \ e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes \xi \cdots)$$

$$= \delta_{i_1 j_1} \cdots \delta_{i_k j_k}$$

(2) $\alpha_i \geq 0$, $\sum \alpha_i = 1$ $\qquad \beta_j = 0$

$$H = (E \otimes \bar{E}) \otimes (E \otimes \bar{E}) \otimes \cdots$$

$E \longrightarrow$ basis e_i

$\bar{E} \longrightarrow$ basis $\bar{e}_i$

$\xi \in E \otimes \bar{E}$

$$\xi = \sum_{i=1}^{\infty} \sqrt{\alpha_i}\, e_i \otimes \bar{e}_i \qquad \|\xi\| = 1$$

$\xi \in H$ is $\xi \otimes \xi \otimes \xi \otimes \xi \cdots$

Ex. Is H a probab. space? ?

$$G = \underbrace{S(\infty)}_{\text{perveres } E \text{ copies}} \times \underbrace{S(\infty)}_{\text{permutes } \bar{E} \text{ copies}} \quad \text{on} \quad H = \bigotimes_i (E \otimes \bar{E})$$

$$K = \operatorname{diag} S(\infty) \subset S(\infty) \times S(\infty)$$

preserves $\xi \otimes \xi \otimes \xi \otimes \cdots$

$$\varphi(g,h) = \left(T(g,h)\,\xi,\ \xi\right)$$
$$= \left(T(\underbrace{gh^{-1}}_{\text{any element in } S(\infty)}, e)\,\xi,\ \xi\right)$$

$$g \in S(n) \subset S(\infty)$$

$$T(g,e)\,\xi = \sum_{i_1 \ldots i_n = 1}^{\infty} \sqrt{\alpha_{i_1} \ldots \alpha_{i_n}}\,\left(e_{i_{g_1}} \otimes \bar{e}_{i_1}\right) \otimes

$$\cdots \otimes \left(e_{i_{gn}} \otimes \bar{e}_{i_n}\right) \otimes \xi \otimes \xi \otimes \cdots$$

disting. vector in $E \otimes \bar{E}$

$$\left(T(g,e)\xi, \xi\right) = \sum_{i_1 \dots i_n} \alpha_{i_1} \cdots \alpha_{i_n} \cdot$$

$$\cdot \left(e_{i_{g1}}, e_{i_1}\right) \cdots \left(e_{i_{gn}}, e_{i_n}\right)$$

Ex. $g =$
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend left] \end{tikzcd}$$

$$\left(e_{i_2}, e_{i_1}\right)\left(e_{i_3}, e_{i_2}\right)\left(e_{i_1}, e_{i_3}\right)$$

$i_1 = i_2 = i_3$

$$\Rightarrow \boxed{\sum_{i=1}^{\infty} \alpha_i^3}$$

Ex. $g =$ 

2 cycles lengths 4, 3

$$\left(\sum \alpha_i^4\right) \cdot \left(\sum \alpha_i^3\right)$$

etc.

$\Downarrow$

If $g \sim$ cycles $\rho_1 \geq \rho_2 \geq \dots \rho_\ell \geq 2$

$$\left(T(g,e)\, \xi_1\, \xi\right) = \underbrace{\prod_{j=1}^{\ell} \left(\sum_{i=1}^{\infty} \alpha_i^{\rho_j} \right)}$$

Irr. Character of $S(\infty)$

$\sim$ α's

③ β-part. $\rightsquigarrow$ want

$\sum \alpha_i + \beta_i = 1$

$\sum \left(\alpha_i^{s_j} - (-\beta_i)^{\beta_j} \right)$

2-cycle: $\sum (\alpha_i^2 - \beta_i^2)$

$H = E \otimes \bar{E}$, $\quad E = E^{(0)} \oplus E^{(1)}$

$\bar{E} = \bar{E}^{(0)} \oplus \bar{E}^{(1)}$

odd parts

$E^{(0)} \sim e_i$

$E^{(1)} \sim f_j$

$\xi = \sum_i \sqrt{\alpha_i}\, e_i \otimes \bar{e}_i + \sum_j \sqrt{\beta_j}\, f_j \otimes \bar{f}_j$

$H = \bigotimes_1^{\infty} (E \otimes \bar{E})$, distring. vector ξ.

Action of $\underbrace{S(\infty)} \times S(\infty)$

$$\bigotimes \left(\underbrace{E^{(0)} \oplus E^{(1)}}_{\text{permuted by } S(\infty)}\right) \otimes \left(\bar{E}^{(0)} \oplus \bar{E}^{(1)}\right)$$

$$f_1 \otimes f_2 \otimes f_3 \longrightarrow (-1)\, f_2 \otimes f_1 \otimes f_3$$

when permute f - vectors,
multiply by sign.

$g = (12)$

$$T(g,e)\xi = \sum_{i_1 i_2} \sqrt{\alpha_{i_1}\alpha_{i_2}}\; e_{i_2} \otimes \bar{e}_{i_1} \otimes e_{i_1} \otimes \bar{e}_{i_2} \otimes \cdots$$

$$+ \sum_{j_1 j_2} (-1)\sqrt{\beta_{j_1}\beta_{j_2}}\; f_{j_2} \otimes \bar{f}_{j_1} \otimes f_{j_1} \otimes \bar{f}_{j_2} \otimes \cdots$$

$$+ \cdots \qquad \underline{e_{i_2} \otimes \bar{e}_{i_1} \otimes f_{j_1} \otimes \bar{f}_{j_2}}$$

disappears

If $\mathcal{H} = L^2(X, \mu) \circlearrowleft S(\infty) \times S(\infty)$

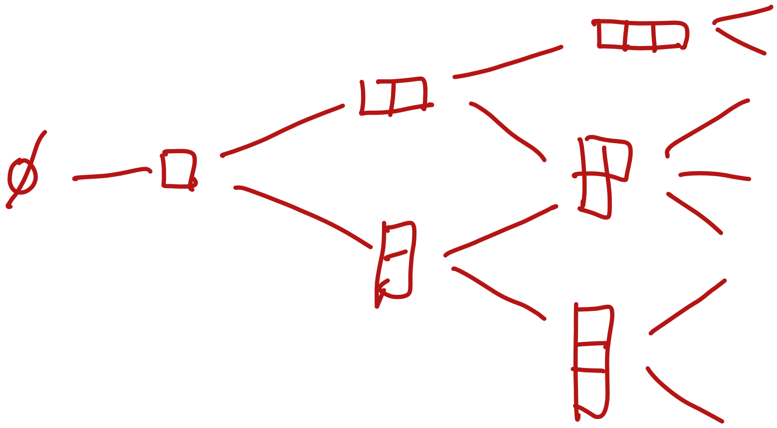
$f(T(g,e)x) \longrightarrow$ ― ― ―

q- Analogues (of combinatorics, not R.T.)

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

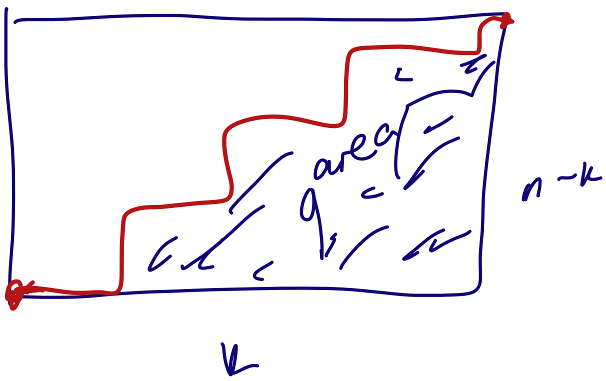
$q \to 1$, $[n] \to n$

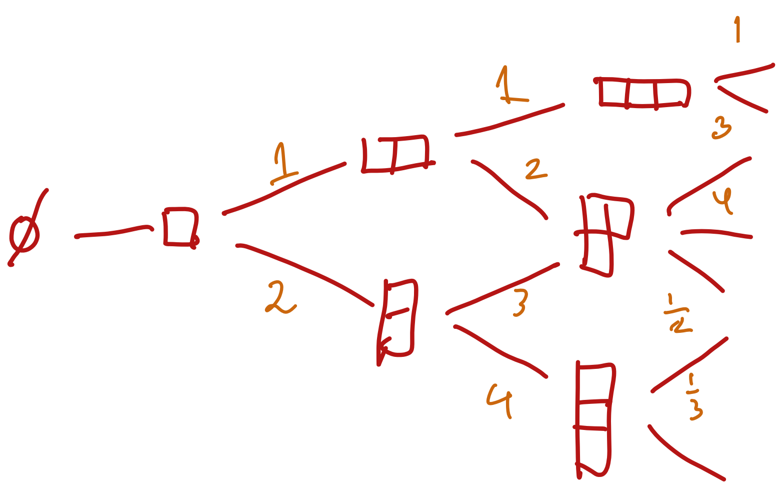
q-analogue?



$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \longrightarrow \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[1][2]\cdots[n]}{[1][2]\cdots[k]\cdot[1][2]\cdots[n-k]}$$

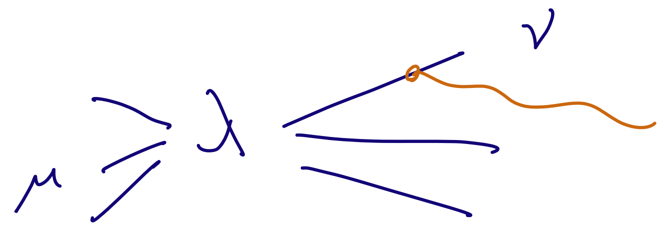
q-binomial





Branching

Graphs with edge multiplicities



$\varkappa(\lambda, \nu) > 0$

- harmonic functions

$$\varphi(\lambda) = \sum_{\nu \searrow \lambda} \varphi(\nu) \cdot \varkappa(\lambda, \nu)$$

- $\dim \lambda$ & recursion

$\dim \lambda$ = weighted Σ over paths from $\varnothing$ to λ

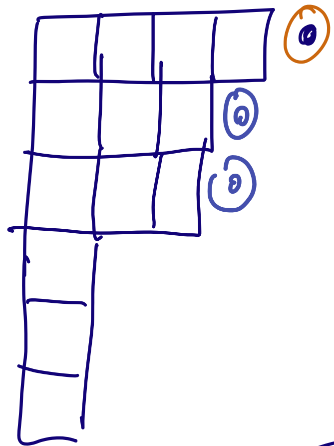
$$\dim \lambda = \sum_{\mu \nearrow \lambda} \dim \mu \cdot \varkappa(\mu, \lambda)$$

- example: Kingman graph

branching of $\{m_\lambda\}$

$$m_\lambda = \sum_{\substack{\text{all} \\ \text{distinct} \\ \text{monomials}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$$

(433111)

$\lambda =$ 

$x_{i_2}^3 x_{i_3}^3,$

$(x_{i_2}+x_{i_3})$

$\parallel$

$x_{i_2}^4 x_{i_3}^3 + x_{i_2}^3 x_{i_3}^4$

$$m_\lambda \cdot \left(\sum x_i\right)$$

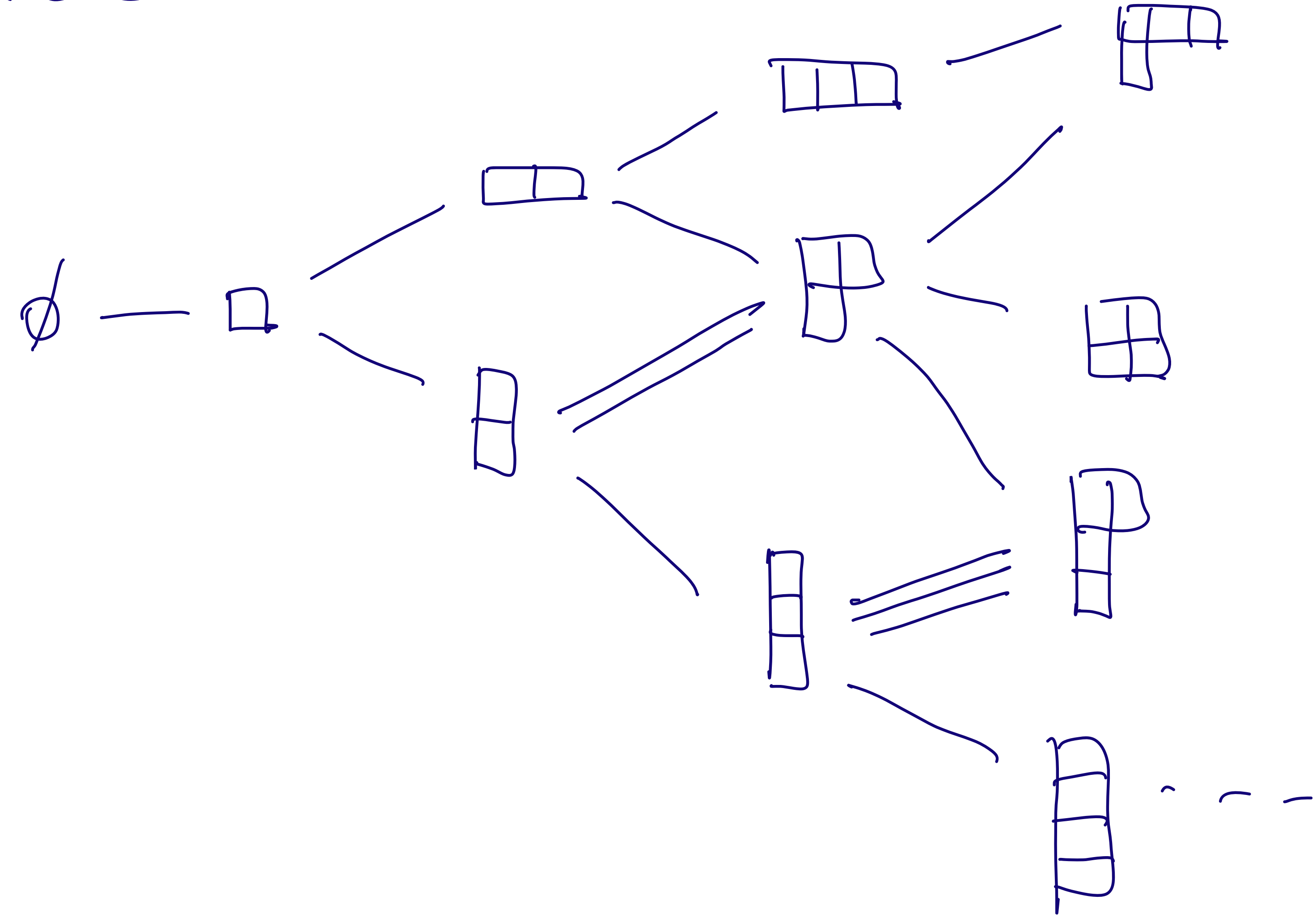
$\parallel$

$$m_{(533111)} + 2m_{(443111)} + 3m_{(433211)}$$

Define $\chi(\mu,\lambda)$ by

$$\boxed{m_\mu p_1 = \sum_{\lambda \gtrdot \mu} m_\lambda \cdot \chi(\mu,\lambda)}$$

Kingman graph:



basis $\quad s_\lambda \longrightarrow$ Young

1953, 1964
1981

$m_\lambda \rightsquigarrow$ Kingman

1975

(prev. examples
1997)

$P_\lambda^{(q,t)} \rightsquigarrow$ most complicated example

2 params

2017

Pascal $\triangle$

q-Pascal $\triangle$

exchangeability & q-exchangeability

Random: $(x_1, x_2, x_3, \ldots) \in \{0,1\}^\infty$

$\varepsilon_i \in \{0,1\}$ fixed

(1) Exchangeable: $\mathbb{P}((x_1, \ldots x_n) = (\varepsilon_1 \ldots \varepsilon_n))$ indep. of order of ε_i's

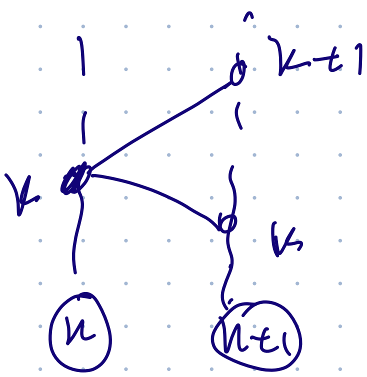
$\lambda = (n-k, k)$

$$\varphi(\lambda) = \mathbb{P}\Big((x_1 \ldots x_n) = (\underbrace{1 \ldots 1}_{k}\ \underbrace{0 \ldots 0}_{n-k})\Big)$$

Harmonic on Pascal graph

$$\varphi(n-k, k) = \mathbb{P}((x_1 \ldots x_{n+1}) = (\underbrace{1 \ldots 1}_{k}\ \underbrace{0 \ldots 0}_{n-k+1}))$$

$$+ \ \mathbb{P}((x_1 \ldots x_{n+1}) = (\underbrace{1 \ldots 1}_{k}\ \underbrace{0 \ldots 0}_{n-k}\ 1))$$



$$\| \qquad \varphi(n-k+1, k) + \varphi(n-k, k+1)$$

$$\boxed{\varphi(\lambda) = \sum_{\nu \searrow \lambda} \varphi(\nu)}$$

Thm. (de Finetti)

$\forall$ exch. random sequence,

$\exists\, \mu$ on $[0,1]$ s.t.

$$\varphi(k, n-k) = \int_0^1 p^k (1-p)^{n-k} \mu(dp)$$

Extreme $\varphi \iff p \in [0,1]$, $\mu = \delta_p$

$\boxed{[0,1] = \text{the boundary of Pascal } \Delta}$

$$S(n) \text{ "}=\text{" } GL(n, \mathbb{F}_1)$$

"field w. 1 element"

Note. Pascal $\subset$ Young

$(a,b) \in$ Pascal

$(0,0) = \square$

$$\left.\begin{array}{l} \frac{a}{n} \longrightarrow p = \alpha_1 \\ \frac{b}{n} \longrightarrow 1-p = \beta_1 \end{array}\right\} \text{Young graph parameters}$$

$\boxed{\alpha_1 + \beta_1 = 1}$

$\Omega = \{0,1\}^\infty$, cylindric σ-alg.

② q - exchangeable $\qquad x_i \in \{0,1\}$ $\qquad \boxed{q>0}$

(often $q<1$)

$$\mathbb{P}\left((x_1 \ldots x_n) = (\ldots 1 0 \ldots)\right) =$$

$$= q \cdot \mathbb{P}\left((x_1 \ldots x_n) = (\ldots 0 1 \ldots)\right)$$

$q \mapsto \frac{1}{q} \iff$ replace $0 \leftrightarrow 1$

$$\varphi(\lambda) = \mathbb{P}\left((x_1 \ldots x_n) = (\underbrace{1 \ldots 1}_{k}, \underbrace{0 \ldots 0}_{n-k})\right)$$

$\lambda = (n-k, k) = n-k$ "0", k "1".

Lemma. (q-Harmonicity)

edge weight...

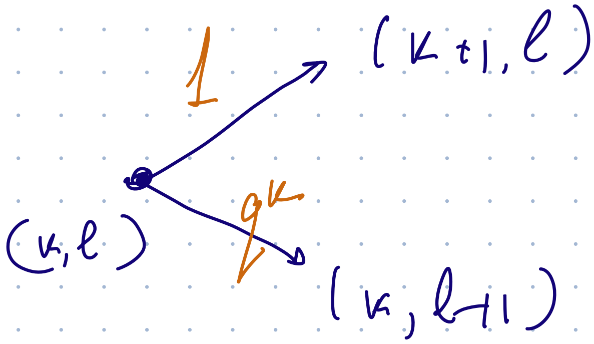
$$\varphi(n-k, k) = \varphi(n+1-k, k) + \boxed{q^{n-k}}\, \underline{\varphi(n-k, k+1)}$$

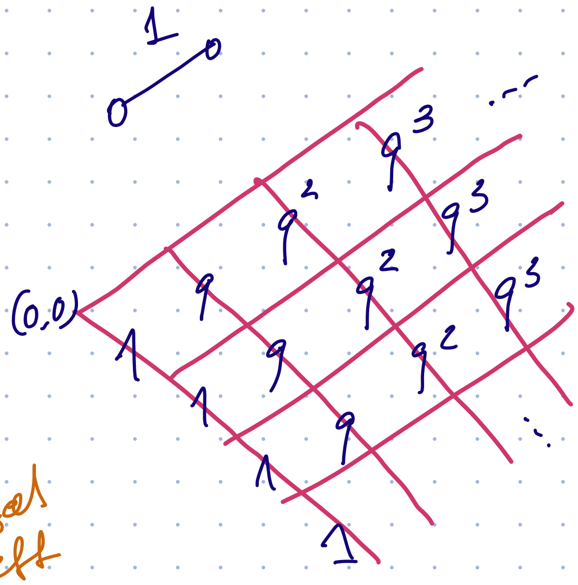
□

$$\varphi(a,b)$$

$$\mathbb{P}(\underbrace{1 \ldots 1}_{k+1}\ \underbrace{0 \ldots 0}_{n-k})$$

q-Pascal graph





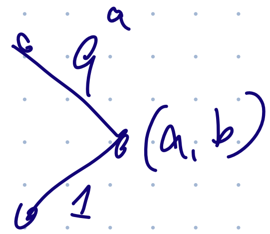
q-binomial coeff

Prop. $\dim_q \lambda = \begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$

$$(\alpha;q)_k = (1-\alpha)(1-q\alpha)\cdots(1-q^{k-1}\alpha)$$

$$\boxed{(q,q)_b = [b]!}$$

Proof



$\boxed{\dim_q \lambda \text{ satisfies the same req. as q-binomial}}$

need to check

$$\begin{bmatrix} a+b \\ b \end{bmatrix} = q^a \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix} + \begin{bmatrix} a+b-1 \\ b \end{bmatrix}$$

$$\frac{1-q^{b+a}}{[a]!\,[b]!} = q^a \frac{1}{[a]!\,[b-1]!} + \frac{1}{[a-1]!\,[b]!}$$

$$[b]! = (1-q)(1-q^2)\cdots(1-q^b)$$

$$1-q^{a+b} = q^a(1-q^b) + (1-q^a) \quad \checkmark$$

<u>Prop.</u> (similar, by shift) $\nu = (a-b, b) \in \mathbb{P}_a$

$\lambda = (n-k, k) \in \mathbb{P}_n$

$$\dim(\nu,\lambda) = \underbrace{q^{(k-b)(a-b)} \begin{bmatrix} n-a \\ k-b \end{bmatrix}}_{\text{skew dim}}$$

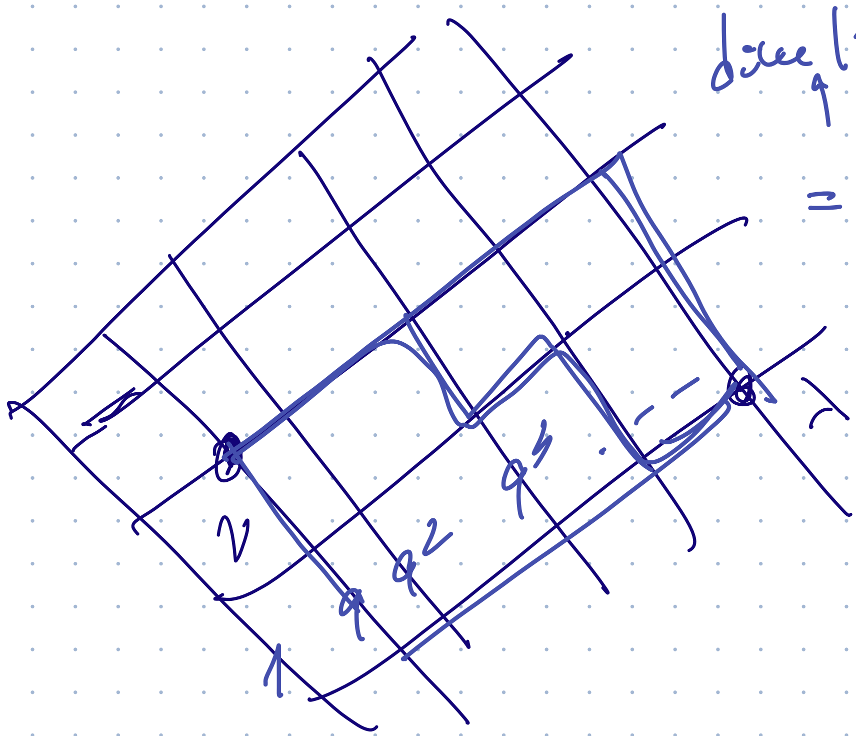
$\lambda(n) \longrightarrow$ point in the boundary

$\Updownarrow$

Asymptotics of skew dimension

$$\boxed{\lim \frac{\dim_q(\nu, \lambda(n))}{\dim_q \lambda(n)} \quad \exists}$$

$n \to \infty$
ν fixed



$\dim_q(\nu, \lambda) =$

$= \sum_{\text{paths}}$ weight of path

Gnedin - Olshanski 2009

Theorem

$\lambda = (n-k, k)$

limit converge s.t. $k = k(n)$ number of 1's

stabilizes or goes to ∞.

The boundary is $\underbrace{\{0,1,2,\dots\} \cup \{\infty\}}_{\text{closed} \Rightarrow \text{compact}}$

$\nu = (b-a, a)$

$$\frac{\dim(\nu,\lambda)}{\dim \lambda} \to \boxed{q^{(\varkappa-a)(b-a)} \frac{(q,q)_\varkappa}{(q,q)_{\varkappa-a}}}$$

if $k(n) \to \varkappa < \infty$

$$\to \mathbb{1}_{a=b} \quad \text{if} \quad k(n) \to \infty$$

Proof. $k = k(n)$ $\nu = (b-a, a)$

$$\frac{q^{(k-a)(b-a)} \begin{bmatrix} n-b \\ k-a \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix}}$$

$$\prod_{j=1}^{\infty} (1-q^j)$$

$$\frac{(q;q)_{n-b}}{(q;q)_n} \cdot \frac{(q;q)_k\,(q;q)_{n-k}}{(q;q)_{k-a}\,(q;q)_{n-b-k+a}}$$

$$\frac{(q;q)_k}{(q;q)_{k-a}} = \left(q^k; \tfrac{1}{q}\right)_a$$

$\square$

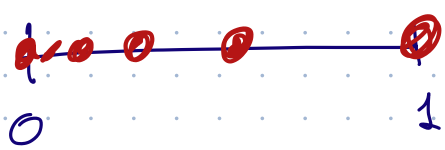
Note. (1) $$q^{(x-a)(b-a)} \frac{(q;q)_x}{(q;q)_{x-a}}$$

is a polynomial in q^x

So the boundary may be described as $\{q^x\}$

$$\Delta_q = \{0\} \cup \{1, q, q^2, \dots\} \subset [0,1]$$

$q \to 1$ limit.



$$\lim_{q\to 1} \Delta_q = [0,1]$$

(2) $$\underbrace{\Phi_{(b-a,a)}}_{\text{analogue of } p^a(1-p)^{b-a}}(q^x) = q^{(x-a)(b-a)} \frac{(q,q)_x}{(q,q)_{x-a}}$$

$p \simeq q^x$

Boundary theory $\Rightarrow$ $\forall$ q-harmonic φ
$\exists\, \mu$ on Δ_q

$$\varphi(b-a,a) = \int_{\Delta_q} \Phi_{b-a,a}(q^x)\, \mu(dx)$$

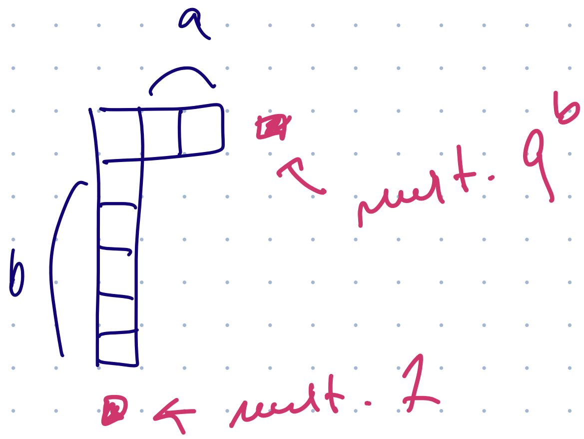
$\boxed{\int_0^1 \dots d_q x}$

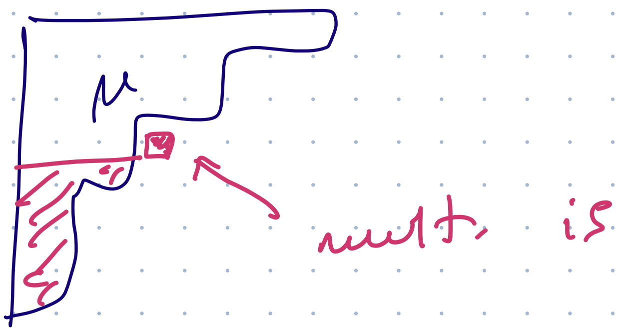
$$= \sum_{k=0}^{\infty} \Phi_{b-a,a}(q^k)\,\mu(q^k) + \mu(0)\cdot \mathbb{1}_{a=b}$$

($q\to 1$, these work as Riemann sums.)

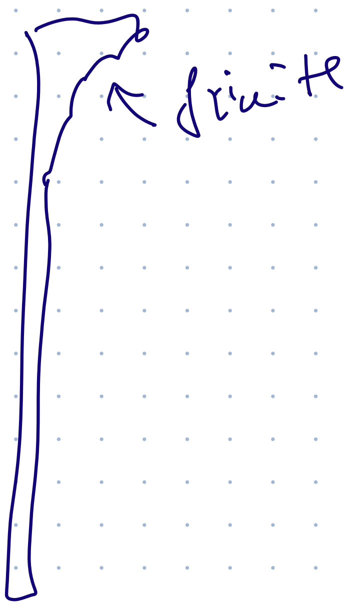
③ q-Pascal $\subset$ q-Young / q-Catalan Connection

$\kappa(\mu, \lambda)$





Conjecture: Ergodic $\lambda(n)$ grows as

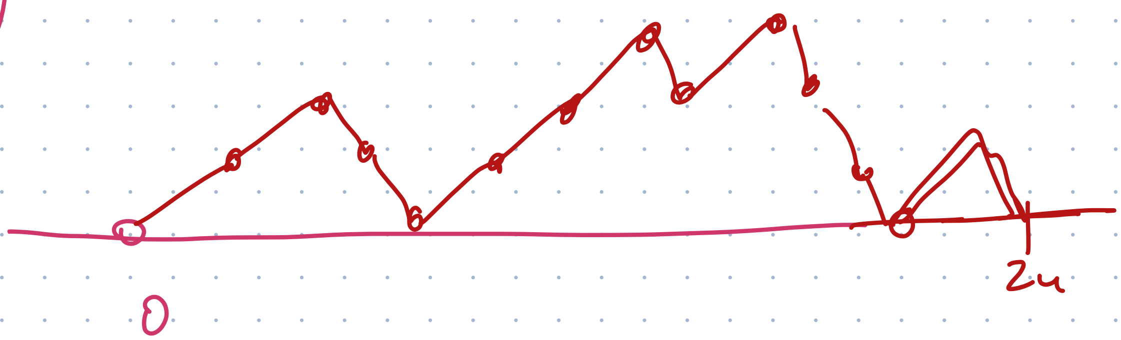


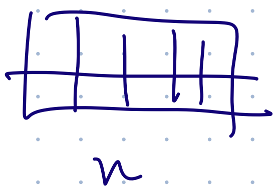
Catalan numbers

1	2	5	6	7	9		
3	4	8	10	.	–	–	

$\updownarrow$ Dyck paths (= parentheses)

# of these $= \frac{1}{n+1}\binom{2n}{n}$



Y - dim () $= \frac{1}{n+1}\binom{2n}{n}$

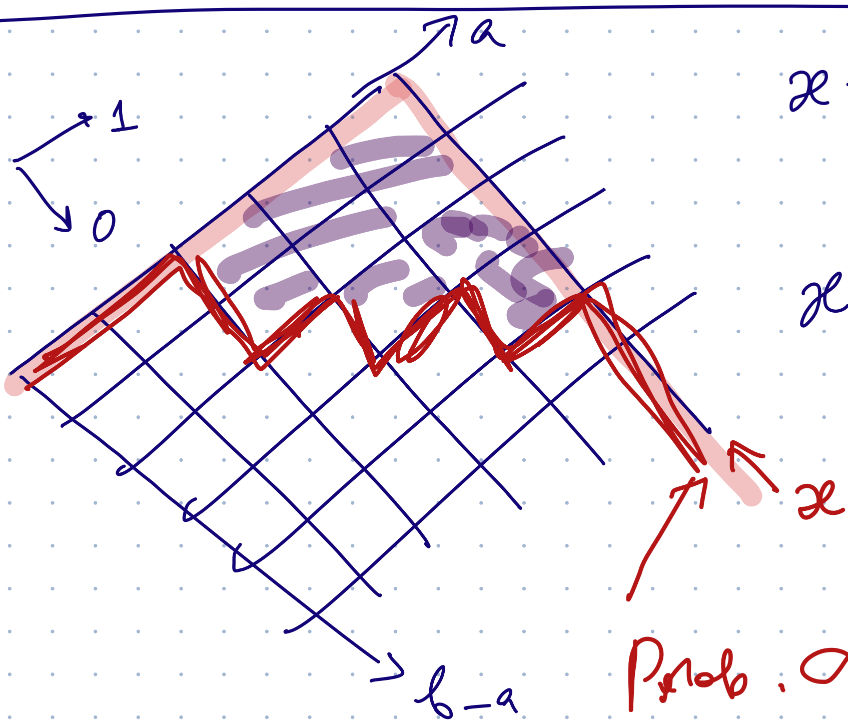
Y_q - dim (———) $= \underbrace{C_n(q)}_{q\text{-Catalan}}$

$\neq$

$\frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}$

④ Growth process with extreme measure ~ $\varkappa$

(& the q-simple random walk)



$\varkappa = \infty$: wave $(1\,1\,1\,1\,1 - - - -)$

$\varkappa < \infty$

$(\underbrace{1\,0\,0\,1\,1\,1}_{\varkappa}\ \text{"1"}\ 0\,0 - -)$

Prob. $\propto q^{\text{area above}}$

__Lemma.__

$$\mathbb{P}\Big(\underset{(b-a,a)}{\circ}\!\nearrow\Big) = 1 - q^{\varkappa - a}$$

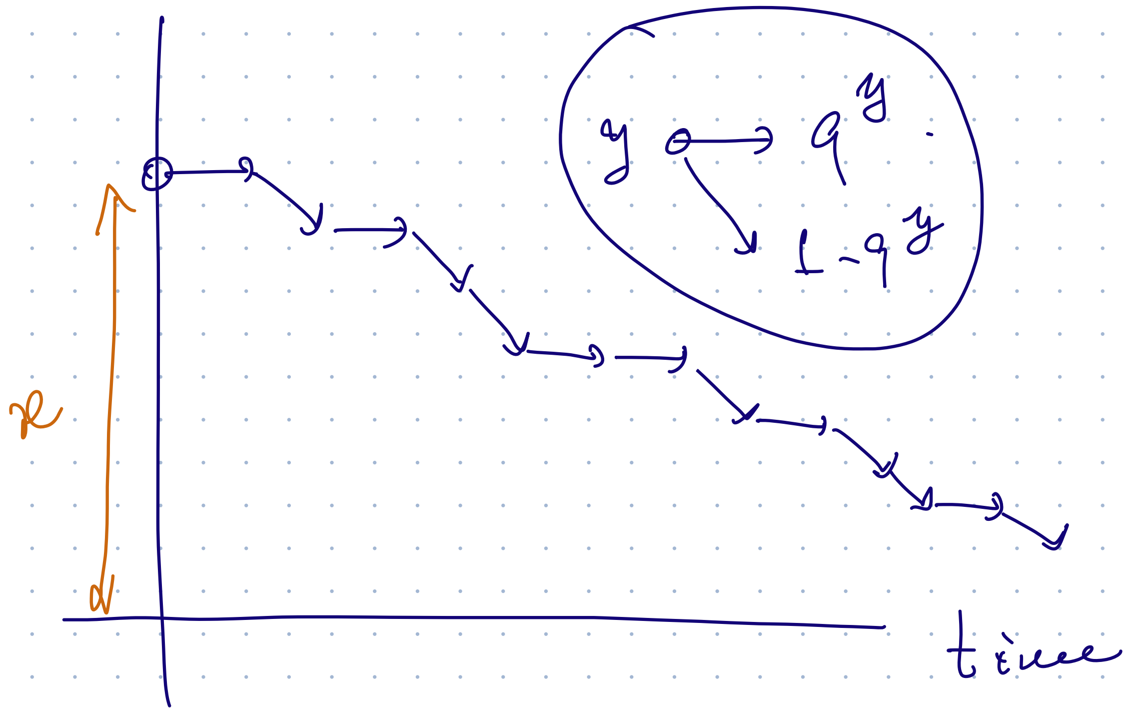
$$\mathbb{P}\Big(\ldots\searrow\Big) = q^{\varkappa - a}$$

$$\mathbb{P}\Big(\underset{(b-a,a)}{\circ}\nearrow\overset{(b-a,\,a+1)}{\circ}\searrow\Big) \Big/ \mathbb{P}\Big(\searrow\nearrow\Big) = \frac{1}{q}$$

$$\frac{(1-q^{x-a})\,q^{x-a-1}}{q^{x-a}(1-q^{x-a})}$$

---

there is <u>only one</u> q-simple r.w.
(no p)



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⑤ μ on Δ_q or $[0,1]$

$$\boxed{P(\lambda \to \nu)} = \underbrace{P(\nu \to \lambda)}_{\substack{\text{canonical} \\ \text{in graph}}} \frac{\mu_{n+1}(\nu)}{\mu_n(\lambda)}$$

$$= \frac{\cancel{\dim \lambda}}{\cancel{\dim \nu}} \cdot \kappa(\lambda,\nu) \cdot \frac{\cancel{\dim \nu} \cdot \varphi(\nu)}{\cancel{\dim \lambda} \cdot \varphi(\lambda)}$$

$\lambda = (n-k, k)$
$\nu = (n-k, k+1)$

$$= \kappa(\lambda,\nu) \frac{\varphi(\nu)}{\varphi(\lambda)}$$

$$= \kappa(\lambda,\nu) \cdot \frac{\int_0^1 p^{k+1}(1-p)^{n-k}\, d\mu(p)}{\int_0^1 p^k (1-p)^{n-k}\, d\mu(p)}$$

≤ 1

$q=1$

Grassmannian over $\mathbb{F}_q$

$q = (\text{prime})^d$

$V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots,$ complete flag

$V_n = (\mathbb{F}_q)^n$

$V_\infty = \bigcup_{n=0}^{\infty} V_n$

$Gr(V_\infty)$ Grassmannian

$\parallel$

subspaces in V_∞, $\varprojlim_n Gr(V_n)$

$$\boxed{\begin{array}{ccc} Gr(V_{n+1}) & \longrightarrow & Gr(V_n) \\ \rotatebox{90}{$\in$}\; X & \longmapsto & X \cap V_n \end{array}}$$

$$\bigcup_{n=1}^{\infty} GL(n, \mathbb{F}_q)$$

$\parallel$

$\underbrace{GL(\infty, \mathbb{F}_q)}_{\text{which matrices?}}$ acts on $Gr(V_\infty)$

$X \in Gr(V_\infty)$ random,

with distr. invar. under

$GL(\infty, \mathbb{F}_q)$

Thm [G-O] $X \leftrightarrow$ Harmonic function on q-Pascal

coherent measures, sum to 1 over k

$\lambda = (n-k, k)$

$$\varphi(\lambda) \cdot \dim_q \lambda$$

$$\text{Prob}\big(\dim (X \cap V_n) = k \big)$$

<u>Proof.</u> $X \subset V_n$ of dim k

$\downarrow$

there are $1+q^{n-k}$ subspaces

$Y \subset Gr(V_{n+1})$ s.t. $Y \cap V_n = X$

1 of dim k &

q^{n-k} of dim $n-k$

explain

Therefore, extreme $GL(\infty, \mathbb{F}_q)$-inv. subspaces $\subset V_\infty$ are parametrized by codimension*

$$\{0, 1, 2, \dots\} \cup \underbrace{\{\infty\}}$$

$X = \{0\}$
w. prob. 1

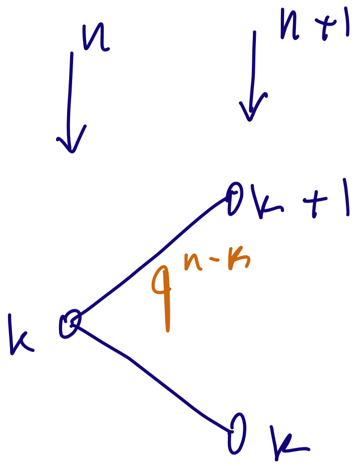
* not dimension
because $q > 1$ so
in q-Pascal, $k \to \infty$
$n-k$ stabilize

& there is a geometric explanation

Recall q-Pascal harm. f.

edge weight.

$$\varphi(n-k,k) = \varphi(n+1-k,k) + \boxed{q^{n-k}}\,\varphi(n-k,k+1)$$



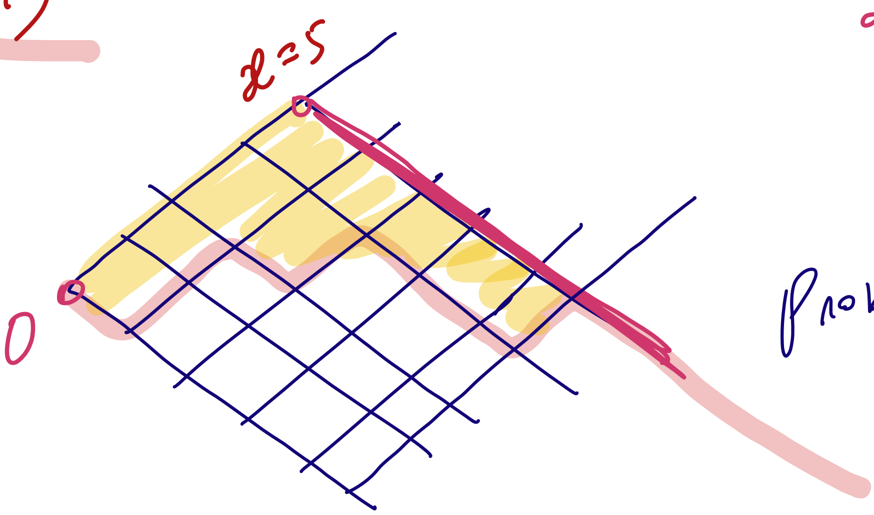
$$P(100\,10\ldots) = q \cdot P(100\,01\ldots)$$

$$\varphi(n-k,k) = P(1^k 0^{n-k})$$

Proved: (q-de Finetti)

Extreme $\varphi(\varnothing)=1$, $\varphi \geq 0$ $\longleftrightarrow$ $\Delta_q = \{1, q, q^2, \ldots\} \cup \{0\}$

$x \in \Delta_q$



$$\text{Prob} = \frac{1}{Z} q^{\text{area}}$$

random path

Grassmannian over $\mathbb{F}_q$

$q = (\text{prime})^d$

$V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots,$

$V_n = (\mathbb{F}_q)^n$

complete flag

$V_\infty = \bigcup_{n=0}^{\infty} V_n$

(which vectors $\in V_\infty$)

all but finitely many coord. are 0.

$Gr(V_\infty)$ Grassmannian

$\shortparallel$

subspaces in V_∞, $\boxed{\varprojlim_n Gr(V_n)}$

$$\boxed{\begin{array}{ccc} Gr(V_{n+1}) & \longrightarrow & Gr(V_n) \\ \rotatebox{90}{$\in$}\; X & \longmapsto & X \cap V_n \end{array}}$$

(Guess: $Gr(V_\infty)$ is uncountable)

$X \in Gr(V_\infty)$

$X = (X_1 \subset X_2 \subset X_3 \subset X_4 \subset X_5 \subset \cdots)$

$X_n \subset V_n, \quad X_{n+1} \cap V_n = X_n$

$$\bigcup_{n=1}^{\infty} GL(n, \mathbb{F}_q) = \underbrace{GL(\infty, \mathbb{F}_q)}_{\text{which matrices:}}$$

acts on V_∞, on $Gr(V_\infty)$

$$\begin{pmatrix} * & 0 & & \\ 0 & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & \ddots \end{pmatrix}$$

Classify:

$X \in Gr(V_\infty)$ random,

with distr. invar. under $GL(\infty, \mathbb{F}_q)$

$\forall A \subset Gr(V_\infty)$, A - Borel

$\forall u \in GL(\infty, \mathbb{F}_q)$,

$$P(uX \in A) = P(X \in A)$$

<u>Thm</u> [G.-D.] $X \leftrightarrow \boxed{\text{Harmonic function } \varphi \text{ on } q\text{-Pascal}}$

<u>Via</u>

$$\varphi(n-k, k) = \frac{P(\dim X \cap V_n = k)}{\begin{bmatrix} n \\ k \end{bmatrix}_q}$$

using invariance

$$= P(X \cap V_n = V_k)$$

# of k-dim subspaces of V_n

Proof. $X \subset V_n$ of dim k // $\varphi(n-k,k)$

$\downarrow$

there are $1+q^{n-k}$ subspaces

$Y \subset Gr(V_{n+1})$ s.t. $\boxed{Y \cap V_n = X}$

1 of dim k &

q^{n-k} of dim $k+1$

indeed:

Fix $X_n \in G(n,k)$. We claim that there are precisely $q^{n-k}+1$ subspaces $X_{n+1} \in \mathrm{Gr}(V_{n+1})$ such that $X_{n+1} \cap V_n = X_n$: one subspace from $G(n+1,k)$ and q^{n-k} subspaces from $G(n+1,k+1)$. Indeed, $\dim X_{n+1}$ equals either k or $k+1$. In the former case $X_{n+1} = X_n$, while in the latter case X_{n+1} is spanned by X_n and a nonzero vector from $V_{n+1} \setminus V_n$. Such a vector is defined uniquely up to a scalar multiple and addition of an arbitrary vector from X_n. Therefore, the number of options is equal to the number of lines in V_{n+1}/X_n not contained in V_n/X_n, which equals

$$\boxed{\frac{q^{n+1-k}-1}{q-1} - \frac{q^{n-k}-1}{q-1} = q^{n-k}.}$$

$\square$

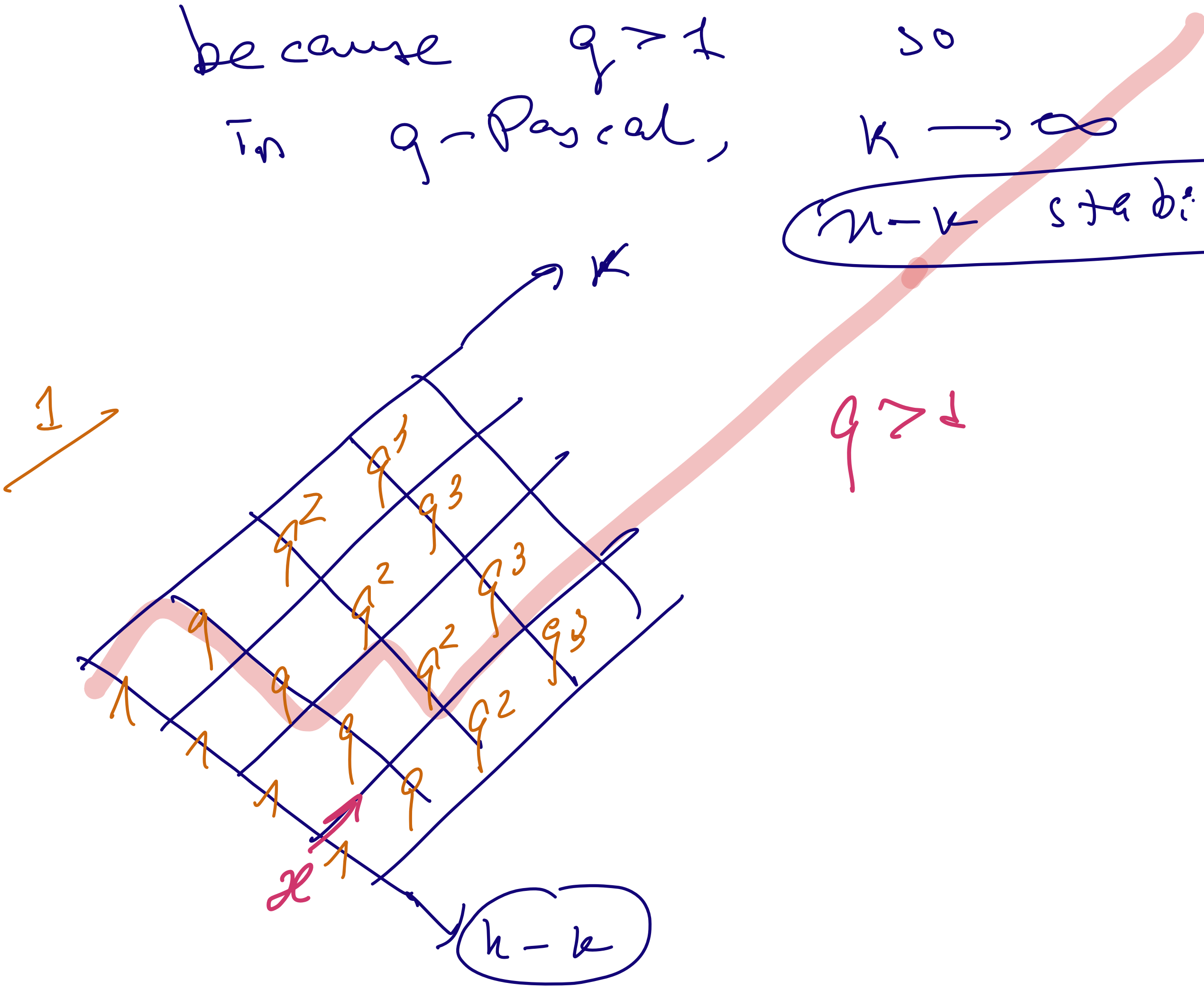
Therefore, extreme $GL(\infty, \mathbb{F}_q)$-invar. subspaces $\subset V_\infty$ are parametrized by codimension*

$$x \in \{0, 1, 2, \dots\} \cup \underbrace{\{\infty\}}$$

$$\{1, q, q^2, \dots\} \cup \{0\}$$

$X = \{0\}$ w. prob. 1

* not dimension because $q > 1$ so in q-Pascal, $k \to \infty$ $n-k$ stabilizes



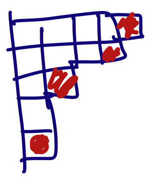
10. Young – Fibonacci

10.1. Differential posets (partially ordered set)

1) Let $U = x$, $D = \frac{\partial}{\partial x}$

$$[D,U]f = (DU-UD)f$$
$$= (xf)' - xf' = f$$

2) $\mathbb{Y}$, $\ell^2(\mathbb{Y})$ basis $\{\underline{\lambda}\}_{\lambda\in\mathbb{Y}}$

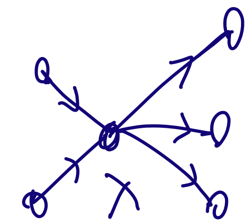


$$D\underline{\lambda} = \sum_{\mu=\lambda-\square} \underline{\mu} \qquad U\underline{\lambda} = \sum_{\nu=\lambda+\square} \underline{\nu}$$

$$[D,U] = \mathrm{Id}$$

$$\left(\Rightarrow \boxed{\sum_{|\lambda|=n} (\dim\lambda)^2 = n!}\right)$$

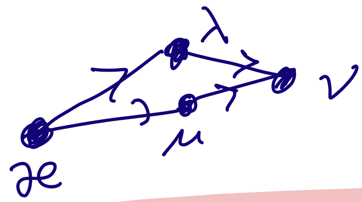
Abstract setting: Differential posets

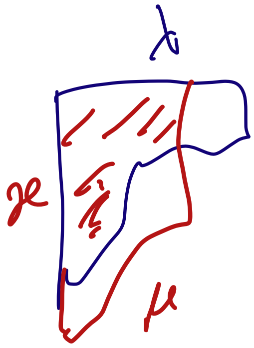


$\longrightarrow$ branching graph

$\longrightarrow$ $\lambda \neq \mu$,

$\# \nu = \# \varkappa$
$= 0$ or 1





$\longrightarrow$ $\boxed{\#\{\nu : \nu \succ \lambda\} = \#\{\mu : \mu \to \lambda\} + 1}$

$\nu = \lambda \cup \mu$

$\Rightarrow$ $\boxed{[D, U] = \mathrm{Id}}$

3) Any other posets like $\mathbb{Y}$?

Just one

— Young–Fibonacci graph $\quad \mathbb{YF}$

[Stanley '88], [Fomin '88]

— Boundary: [Kerov – Goodman '97]

Intermission: $\mathbb{Y}$ and examples of harmonic functions

$$\varphi(\lambda) = \sum_{\nu = \lambda + \square} \varphi(\nu)$$

we know $p_1 s_\lambda = \sum_{\nu = \lambda + \square} s_\nu$

$p_1 = x_1 + x_2 + \cdots$

So $\varphi(\lambda) = s_\lambda(\alpha_1 \ldots \alpha_n)$ $\quad \left(\sum \alpha_i = 1, \ \alpha_i \geq 0\right)$

is an example of a nonnegative harmonic f.

because $p_1(\alpha_1 \ldots \alpha_n) = 1$

$$s_\lambda(\alpha_1 \ldots \alpha_n) = \frac{\det\left[\alpha_i^{\lambda_j + n - j}\right]}{\prod_{i<j}(\alpha_i - \alpha_j)}$$

"Magically", these $\varphi(\lambda)$ are extreme

10.2 Young – Fibonacci graph

fibonacci words

Let $YF_n = \{$ words on $\{1,2\}$ with $|w| = n$ $\}$

$|112212| = 9$

$$YF_3 = \left\{ \begin{matrix} 21 \\ 111 \\ 12 \end{matrix} \right\}$$

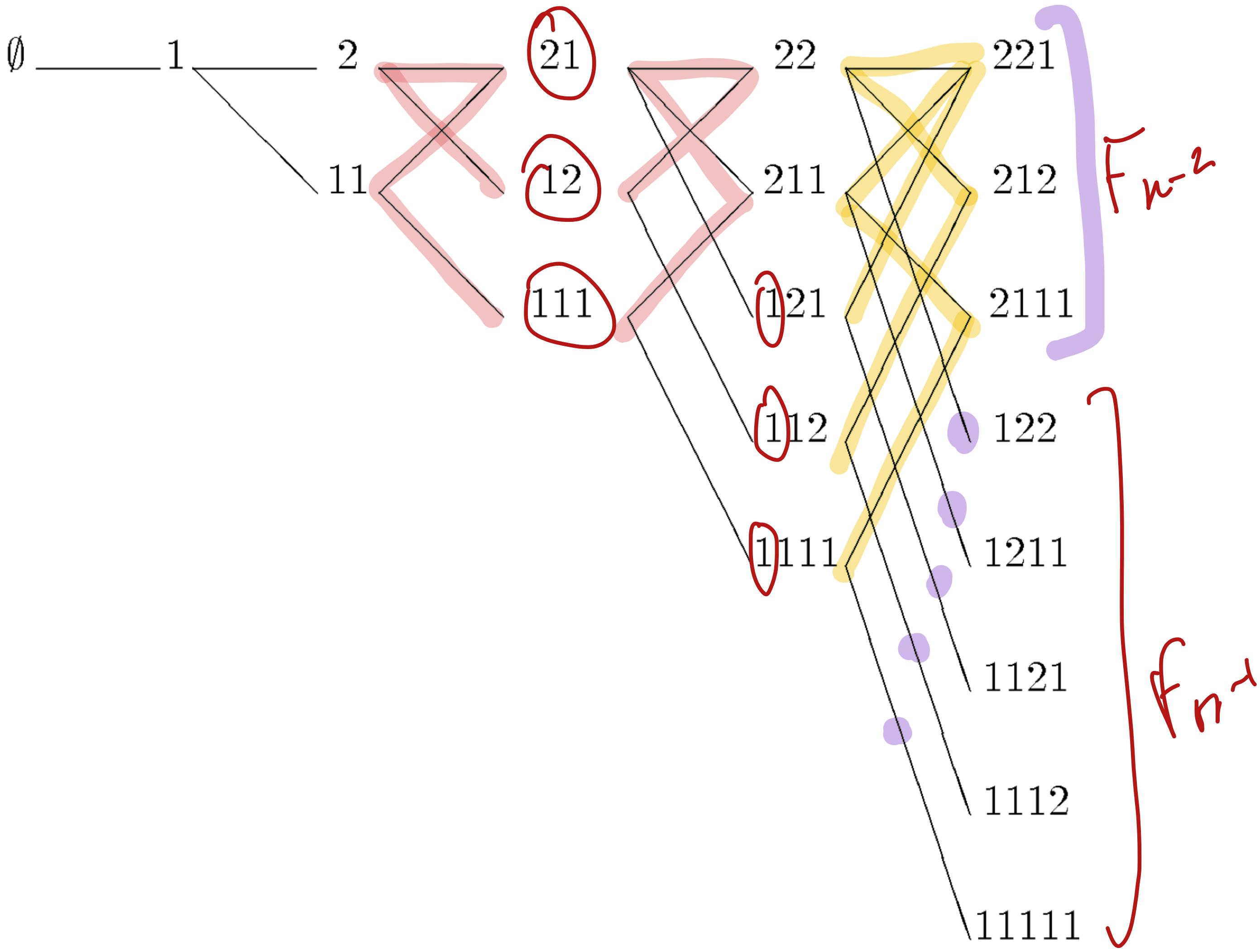
$|YF_n| = |YF_{n-1}| + |YF_{n-2}|$ $\qquad$ $|YF_0| = |YF_1| = 1$

YF_n = Fib. number (n)

$$|Y/F_n| \sim C \cdot \left(\frac{\sqrt{5}+1}{2}\right)^n \leftarrow \text{faster growing}$$

Contrast to $|Y/_n| = p(n) \sim e^{c\sqrt{n}}$

Branching : $\boxed{\text{Reflection}}$ operation on floors



(Kerov-Goodman
1997)

Given a Fibonacci word v, we first define the set $\overline{v} \subset \mathbb{YF}$ of its successors. By definition, this is exactly the set of words $w \in \mathbb{YF}$ which can be obtained from v by one of the following three operations:

(i) put an extra 1 at the left end of the word v;

(ii) replace the first 1 in the word v (reading left to right) by 2;

(iii) insert 1 anywhere in between 2's in the head of the word v, or immediately after the last 2 in the head.

Example. Take 222121112 for the word v of rank 14. Then the group of 3 leftmost 2's forms its head, and v has 5 successors, namely

$$\overline{v} = \{\mathbf{1}222121112,\ 2\mathbf{1}22121112,\ 22\mathbf{1}2121112,\ 222\mathbf{1}121112,\ 222\mathbf{2}21112\}.$$

1222

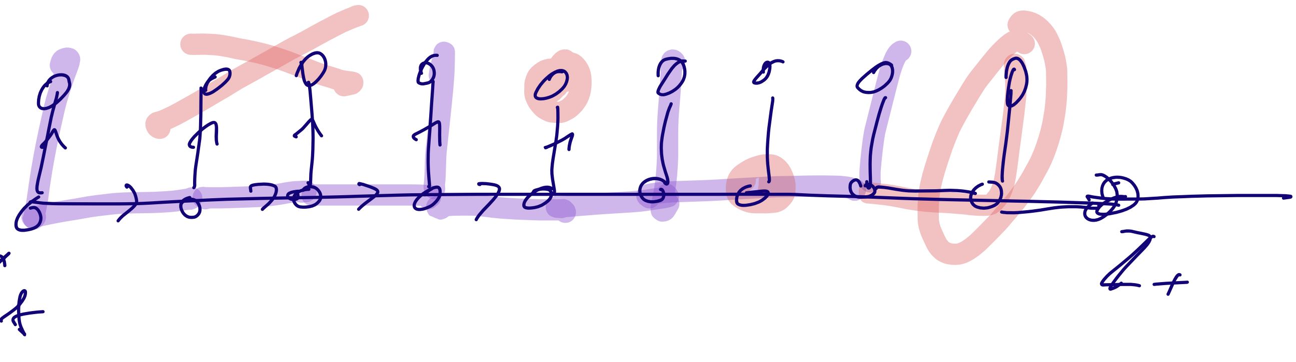
2122

2212

2221

2221... → 2222...

Fact. $\mathcal{Y}F$ = graph of branching of subtrees



2 2 2 1 2 1 1 1 2

10.3 Examples of harmonic funct. on $\mathbb{YF}$

(before talking about the bdry)

$[D, U] = 1$

1) Plancherel function,

$$\varphi(v) = \frac{\dim v}{n!}$$

$|v| = n$

$|v| = n$

Indeed:

harmonicity of φ_{Pl}.

$$\boxed{\sum_{w \searrow v} \dim w = (n+1) \dim v}$$

Proof.

Show using $[D, U] = 1$.

$$\dim w = (U^n \varnothing, w)$$

$$\sum_{w \searrow v} \dim w = (U^{n+1} \varnothing, U v) =$$

$$D = U^*.$$

$\Leftrightarrow (D u^{n+1} \emptyset, \underline{v})$

$$Du = uD + 1$$

$$Du^2 = (uD+1)u =$$

$$= uDu + u$$

$$= u(uD+1) + u$$

$$= u^2 D + 2u$$

$$Du^n = n u^{n-1} + u^n D$$

$\Leftrightarrow ((n+1) u^n \underline{\emptyset}, \underline{v}) + (u^{n+1} \boxed{D \underline{\emptyset}}, \underline{v})$ $=0$

$\square$

Next?

2) „Close Schur functions"

$$\vec{\alpha} = (\alpha_1, \alpha_2, \ldots)$$

$$A_\ell(\vec{\alpha}) = \det_{\ell\times\ell} \begin{bmatrix} 1 & \alpha_1 & & & 0 \\ 1 & 1 & \alpha_2 & & \\ & 1 & 1 & \alpha_3 & \\ 0 & & 1 & 1 & \alpha_4 \\ & & & & \ddots \end{bmatrix}$$

$$B_{\ell-1}(\vec{\alpha}) = \det_{\ell\times\ell} \begin{bmatrix} \alpha_1 & \alpha_2 & & & 0 \\ 1 & 1 & \alpha_3 & & \\ & 1 & 1 & \alpha_4 & \\ 0 & & 1 & 1 & \alpha_5 \\ & & & & \ddots \end{bmatrix}$$

Note, $B_0(\vec{\alpha}) = \alpha_1$ & $A_0(\vec{\alpha}) = 1$

<u>Define</u>

$$S^{\alpha}_{u}(\vec{\alpha}) = \begin{cases} A_k(\vec{\alpha}) & u = 1^k \\ B_k(sh_{|\vec{v}|}\vec{\alpha}) \cdot S^{\alpha}_{v}(\vec{\alpha}) & u = 1^k 2v \end{cases}$$

u - Fibonacci word

$$sh_m \vec{\alpha} = (\alpha_{m+1}, \alpha_{m+2}, \ldots)$$

<u>Ex.</u>

$$S_{221}(\alpha) = \alpha_4 \, S_{21}(\vec{\alpha}) =$$

$$= \alpha_4 \, \alpha_2 \, S_1(\alpha)$$

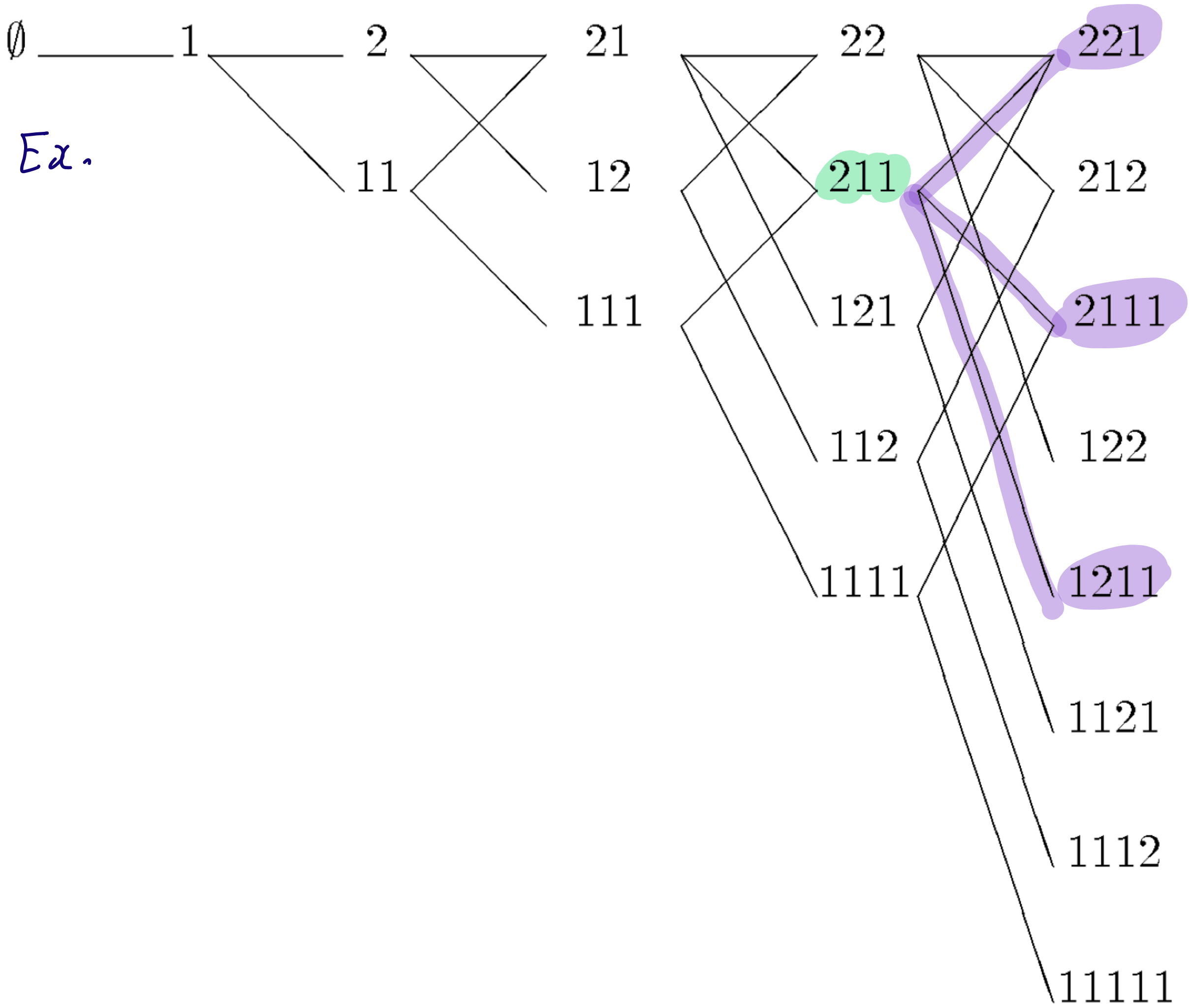
$$= \alpha_4 \, \alpha_2 \, .$$

Theorem (Okada) $\varphi(u) = s'_u(\vec{\alpha})$ is harmonic on $\mathbb{YF}$

(Normalized, $\varphi(\emptyset) = 1$)

Example, not proof

Ex.



$S_{221}(\alpha) = \alpha_4 \alpha_2$

$S_{2111}(\alpha) = \alpha_4 \circ A_3(\vec{\alpha})$

$S_{1211}(\alpha) = B_1(sh_2 \vec{\alpha}) \circ A_2(\vec{\alpha})$

$S_{211}(\vec{\alpha}) = \alpha_3 \circ A_2(\vec{\alpha})$.

```
AM[l_] := Table[If[i == j || i == j + 1, 1, 0] + If[i == j - 1, α[i], 0], {i, 1, l}, {j, 1, l}]
AM[3] // MatrixForm
```

$$\begin{pmatrix} 1 & \alpha[1] & 0 \\ 1 & 1 & \alpha[2] \\ 0 & 1 & 1 \end{pmatrix}$$

```
BM[l_, k_] :=
  Table[If[i == j || i == j + 1, 1, 0] + If[i == j - 1, α[i + k + 1], 0] +
    If[i == j == 1, α[k + 1] - 1, 0], {i, 1, l + 1}, {j, 1, l + 1}]
BM[1, 2] // MatrixForm
```

$$\begin{pmatrix} \alpha[3] & \alpha[4] \\ 1 & 1 \end{pmatrix}$$

```
α[2] × α[4] + α[4] × Det[AM[3]] + Det[AM[2]] × Det[BM[1, 2]] // Expand
```

$\alpha[3] - \alpha[1] \times \alpha[3]$

```
α[3] × Det[AM[2]] // Expand
```

$\alpha[3] - \alpha[1] \times \alpha[3]$

Caveat / interesting property:

$\mathbb{Y} \rightsquigarrow \varphi(\lambda) = s_\lambda(\vec{\alpha})$ are extreme

$\mathbb{Y}/F \rightsquigarrow \varphi(n) = S'_n(\vec{\alpha})$, not extreme

(except Plancherel)

In fact, $\varphi_{Pl.}(\lambda) = \dfrac{\dim \lambda}{n!}$ is given by $\varphi_{Pl}(\lambda) = S_n(\vec{\alpha})$,

$$\alpha_i = \frac{1}{i+1}$$

(How is it for $\mathbb{Y}$, the Young gr.)

there is a similar property

Young – Fibonacci graph & harm. functions & their positivity

$\{1, 2\}$

Today: show some examples of harmonic functions & explain how they differ from $\mathbb{Y}$ & some related challenges

---

$\mathbb{Y}$.

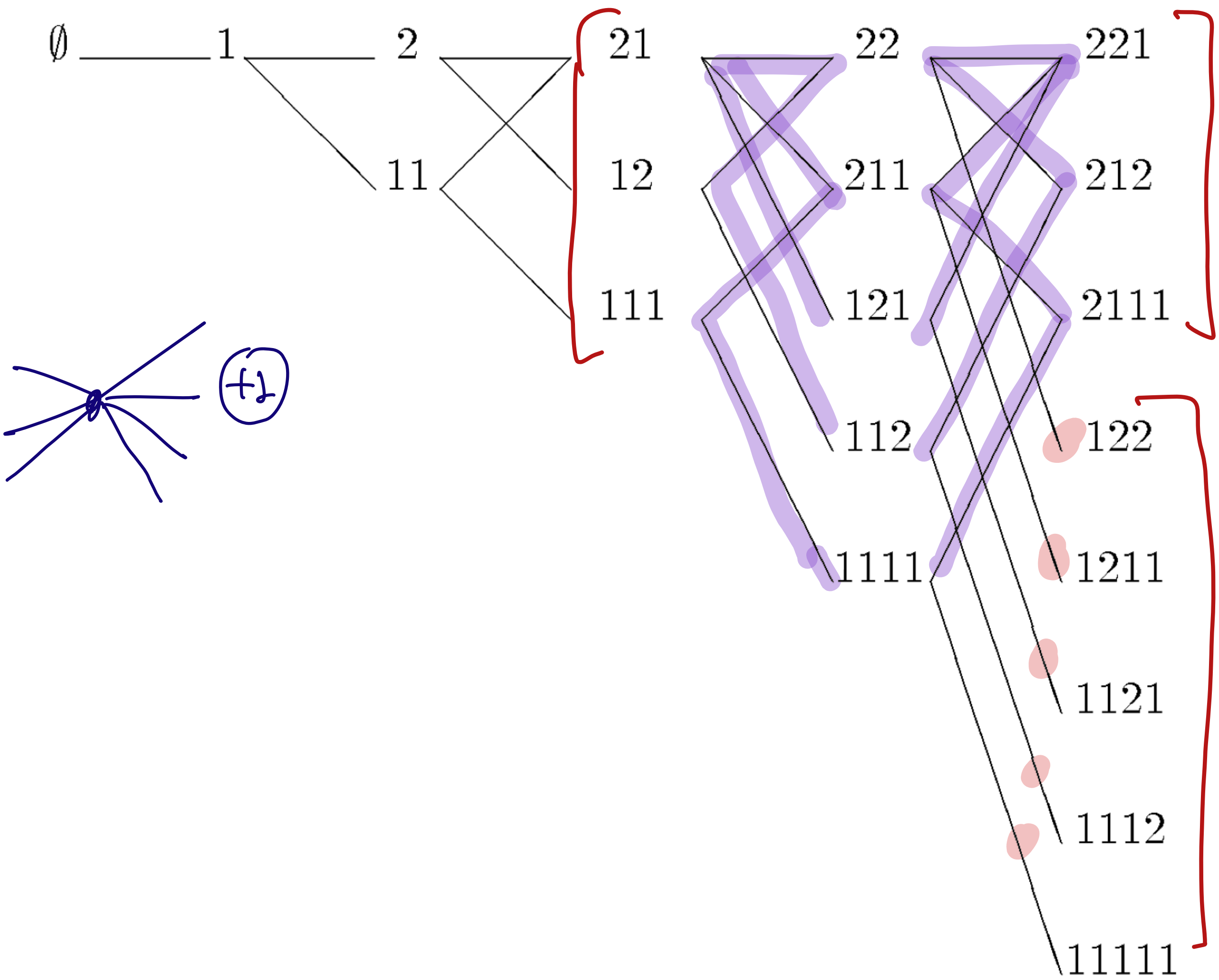
$$\varphi(\lambda) = S_\lambda(\alpha_1 \ldots \alpha_n)$$

$$\sum \alpha_i = 1$$

harmonic on $\mathbb{Y}$

& extreme

(10.4)



$$w \to 1w$$
$$w \to 2v \quad \text{if} \quad w = 1v$$
$$w \to 2^k 1 v \quad \text{if} \quad w = 2^k v$$

Goal: show some harmonic functions & do some experiments

Harmonic functions

$$\varphi(v) = \sum_{w \searrow v} \varphi(w)$$

Example 1. $\varphi_{Pl}(w) = \dfrac{\dim w}{n!}$

$\dim w = \#$ paths $\emptyset \longrightarrow w$

Jacobi – Trudi

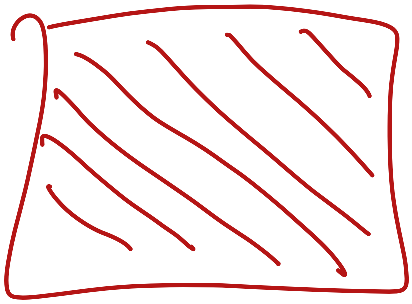
$h_{\lambda_i + j - i}$

$$S_\lambda(\vec{x}) = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} \cdots \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} \\ & \ddots & \ddots \end{bmatrix}$$

σ

$h_n(\vec{x})$ = sum of all monomials of deg $= n$

Minor of the Toeplitz matrix



$$\begin{bmatrix} 1 & h_1 & h_2 & & \ddots & \\ & 1 & h_1 & h_2 & & \ddots \\ & & 1 & h_1 & h_2 & \\ & 0 & & 1 & h_1 & \\ & & & & 1 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

want minors to be ≥ 0

Example 2.

Let x, y be two sequences

$$A_\ell(x|y) = \det_{\ell\times\ell}\begin{pmatrix} x_1 & y_1 & & & \\ 1 & x_2 & y_2 & & 0 \\ & 1 & x_3 & y_3 & \\ 0 & & & \ddots & \\ & & & & \ddots \end{pmatrix}$$

$$B_{\ell-1}(x|y) = \det_{\ell\times\ell}\begin{pmatrix} y_1 & x_1 y_2 & & 0 \\ 1 & x_3 & y_3 & \\ & 1 & x_4 & \\ 0 & & & \ddots \end{pmatrix}$$

$A_0 = 1, \quad B_0 = y_1$

Closed Schur functions

<u>Define</u> $\mathcal{S}_u(x|y)$ $\quad u$- fib. word

$$= \begin{cases} A_k(x|y), & u = 1^k \\ B_k(x+|v|\,|\,y+|v|)\circ S_v(x|y), & u = 1^k 2v \end{cases}$$

$$x+r = (x_{r+1}, x_{r+2}, x_{r+3}, \ldots)$$

$$A \quad \begin{pmatrix} x_1 & y_1 & & 0 \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & y_3 \end{pmatrix}$$

<u>Ex.</u>

$$S_{2\underbrace{21}_{v}}(x|y) = B_0(x+3|y+3) \circ S_{\underbrace{21}_{v}}(x|y)$$

$$= y_4 \circ B_0(x+1|y+1) \circ S_1(x|y)$$

$$= \boxed{y_4 \circ y_2 \circ x_1}$$

$$S_{2\underbrace{11}_{2}}(x|y) = B_0(x+2|y+2) \circ S_{11}(x|y)$$

$$= \boxed{y_3 \circ \begin{vmatrix} x_1 & y_1 \\ 1 & x_2 \end{vmatrix}}$$

Harmonicity (example & test)

$y_4 y_2 x_1$

221

211

$2\underline{111}$ $\quad y_4 \cdot A_3$

$12\overline{11}$ ✓

$y_3 \cdot \begin{vmatrix} x_1 & y_1 \\ 1 & x_2 \end{vmatrix}$

$B_1(x+2 \mid y+2) \cdot A_2$

$x_i \equiv 1$

Theorem (Okada '94)

$$\varphi(w) = S'_w(1 \mid \vec{y})$$

are harmonic on YF

(proof later)

Caveat / interesting properties

① Extremality

$$\varphi(w) = S_w(1 \mid y)$$

are usually **not** extremal

Q: how S_w decomposes into extremes?

(extremes are known)

② Plancherel as a special case (is extreme)

Fact. $\varphi_{Pl}(w) = \dfrac{\dim w}{n!}$

$= S_w(\vec{1} \mid y)$,

$\boxed{y_i = \dfrac{1}{i+1}}$

(For Young graph, we have a similar specialization)

$S_\lambda(\alpha_1, \dots, \alpha_k) = \varphi(\lambda)$

Fix λ, let $k \to \infty$, $\alpha_i = \dfrac{1}{k}$ then

$S_\lambda\left(\frac{1}{k}, \dots, \frac{1}{k}\right) \longrightarrow \dfrac{\dim \lambda}{n!}$

$|\lambda| = n$

10.5 Positivity

(we need $\boxed{\varphi \geq 0}$ & $\varphi(\emptyset) = 1$)

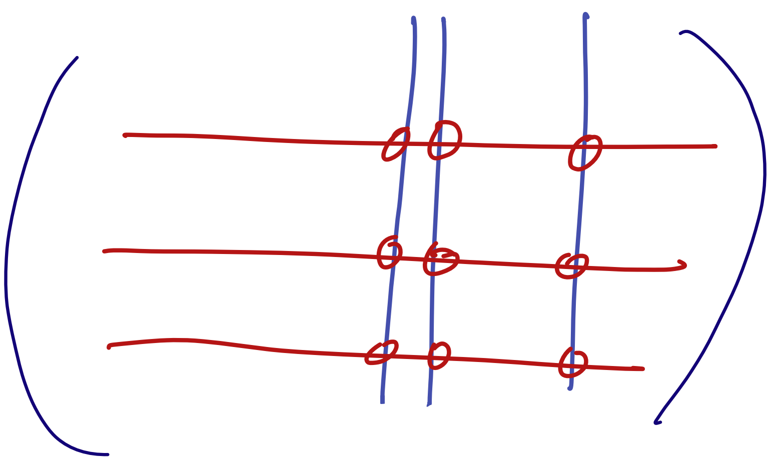
Def. $\vec{y}$ is fibonacci positive if

$$\boxed{\mathfrak{S}_w(\vec{1} \mid \vec{y}) \geq 0} \quad \forall\, w \in \mathbb{YF}$$

(obvious) $\Longleftrightarrow$ $A_\ell(\vec{1} / \vec{y}),\ B_\ell(\vec{1} / \vec{y} + r) \geq 0 \quad \forall \ell, r$

Def. Matrix T is totally positive if (totally nonnegative)

all its minors are ≥ 0



Young graph, monoms. φ

$\updownarrow$

TP Toeplitz matrices

$$\begin{pmatrix} 1 & h_1 & h_2 & \cdots \\ & 1 & h_1 & h_2 & \cdots \\ & & 1 & h_1 & h_2 \\ 0 & & & \cdots & \cdots \end{pmatrix}$$

YF $\longleftrightarrow$ TP tridiagonal matrices.

$\underline{Y}$.
$$\begin{pmatrix} 1 & h_1 & h_2 & \cdots \\ & 1 & h_1 & \cdots \\ 0 & & 1 & h_1 \cdots \end{pmatrix} \text{ is TP}$$

$$\gamma = 1 - \sum(\alpha_i + \beta_i)$$

iff

$$\boxed{\sum_{n=0}^{\infty} h_n z^n} = \left(\prod_{i=1}^{\infty} \frac{1+\beta_i z}{1-\alpha_i z}\right) e^{\gamma z}$$

$(\vec{\alpha}, \vec{\beta})$ – Ocer $\delta(\infty)$ parameters

$\boxed{\text{Fact}}$ $\vec{y}$ is fib-pos. iff

$$T(\vec{y}) = \begin{pmatrix} 1 & y_1 & & & & \\ 1 & 1 & y_2 & & 0 & \\ & 1 & 1 & y_3 & & \\ 0 & & 1 & 1 & y_4 & \\ & & & & & \ddots \end{pmatrix}$$

is totally positive

& the shifted sequence

$$\vec{y}^{(r)} = (y_r^{-1} y_{r+1}, \; y_{r+2}, \; y_{r+3}, \; \ldots)$$

is totally positive $\forall r$ (in the sense of T)

Young graph parallel.

① $S_\lambda = \det(h's)$

② Total positivity of Toeplitz matrices

10.6. Connection to continued fractions

Back to Y/F

Define
$$T(x|y) = \begin{pmatrix} x_1 & y_1 & & & 0 \\ 1 & x_2 & y_2 & & \\ & 1 & x_3 & y_3 & \\ & 0 & & & \ddots \end{pmatrix}$$

A_ℓ = det's of its principal corners

Recursion on A_ℓ (three-term)

$$\boxed{A_\ell(x|y) = x_1 A_{\ell-1}(x+1|y+1) - y_1 A_{\ell-2}(x+2|y+2)}$$

$$\begin{pmatrix} x_1 & \boxed{y_1} & & \\ 1 & x_2 & y_2 & \\ & 1 & x_3 & y_3 \\ & & & \ddots \end{pmatrix}$$

Overview

Tridiag. matrices

$\updownarrow$

Discrete versions of $\left(a(x)f(x)\right)''$

$\updownarrow$

Orthogonal poly's which are (eigenfunctions) solutions to "interesting" 2nd degree ODEs

$$Lf = \underbrace{f'' + xf'}_{} \qquad Lf_n = \lambda_n f_n$$

Hermite poly's

$$T(x|y) = \begin{pmatrix} x_1 & y_1 & & & \\ 1 & x_2 & y_2 & & 0 \\ & 1 & x_3 & y_3 & \\ 0 & & & & \ddots \end{pmatrix}$$

Let

$$J(z) = \cfrac{1}{1 - x_1 z - \cfrac{y_1 z^2}{1 - x_2 z - \cfrac{y_2 z^2}{1 - x_3 z - \cfrac{y_3 z^2}{\cdots}}}}$$

We have

$$\boxed{\frac{1}{J_{x,y}(z)} - 1 + x_1 z + y_1 z^2 J_{x+1|y+1}(z) = 0}$$

$$1 = J(0) = a_0 = \int_0^\infty 1 \, d\mu(x)$$

Theorem. $T(x|y)$ is Totally positive

(Sokal 1990s) $\iff$ $\boxed{J_{x,y}(z)} = \sum_{n=0}^{\infty} a_n z^n$,

and $a_n = \int_0^{+\infty} x^n \, d\mu(x)$

nonnegative Borel measure

probab.

Then $p_n(t) = (t - x_n) p_{n-1}(t) - y_{n-1} p_{n-2}(t)$

are orthog. poly's wrt μ.

$$\int_0^{\infty} p_n(x) \, p_m(x) \, d\mu(x) = 0$$

$\deg p_n(x) = n$

$n \neq m$

x, y params. T

via cont. fract. (indirect)

direct

μ – prob. meas. on $[0, \infty)$

(indirect also)

orthogonal poly's

$$p_n(t) = (t - x_n)\, p_{n-1}(t) - y_{n-1}\, p_{n-2}(t)$$

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= x_1 \\
a_2 &= x_1^2 + y_1 \\
a_3 &= x_1^3 + 2x_1y_1 + x_2y_1 \\
a_4 &= x_1^4 + 3x_1^2y_1 + y_1^2 + 2x_1x_2y_1 + x_2^2y_1 + y_1y_2
\end{aligned}
$$

$T \leftrightarrow \mu$ (indirect)

$$a_n = \int_0^\infty x^n \, d\mu$$

10.7. Continued fractions. Continued

Ex. 1. Let
$$\boxed{\begin{aligned} x_k &= k+\rho-1 \\ y_k &= k\rho \end{aligned}}$$

$$\Rightarrow \underbrace{S_{1n}(\rho)}\; = \rho^n \qquad \& \qquad \boxed{\rho = 1 \text{ is Planch}}$$

Check:

$$\begin{pmatrix} \rho & \rho & & 0 \\ 1 & \rho+1 & 2\rho & \\ & 1 & \rho+2 & 3\rho \\ 0 & & & \ddots \end{pmatrix}$$

column operations

$$\begin{pmatrix} \rho & 0 & 0 & \\ 1 & \rho & 0 & \\ 0 & 1 & \rho & \ddots \\ \vdots & \vdots & 1 & \rho \\ & & & \ddots \end{pmatrix} \longrightarrow \rho^n$$

$$\boxed{\dim(1 \dots 1) = 1}$$

(we know these are fib.-nonnegative)

$\Rightarrow$ $a_0 = 1$ $\quad$ ξ

$a_1 = \rho$ $\quad$ $E\xi$

$a_2 = \rho^2 + \rho$ $\quad$ $E(\xi^2)$

$a_3 = \rho^3 + 3\rho^2 + \rho$ $\quad$ $E(\xi^3)$ $\vdots$

$a_4 = \rho^4 + 6\rho^3 + 7\rho^2 + \rho$

etc.

$$\boxed{P(\xi = k) = e^{-\rho} \frac{\rho^k}{k!}}$$

Poisson (ρ)

$$\boxed{E\xi = \rho = \operatorname{Var}\xi}$$

$$p_n(t) = (t - n - \rho + 1)\, p_{n-1}(t) - (n-1)\rho\, p_{n-2}(t)$$

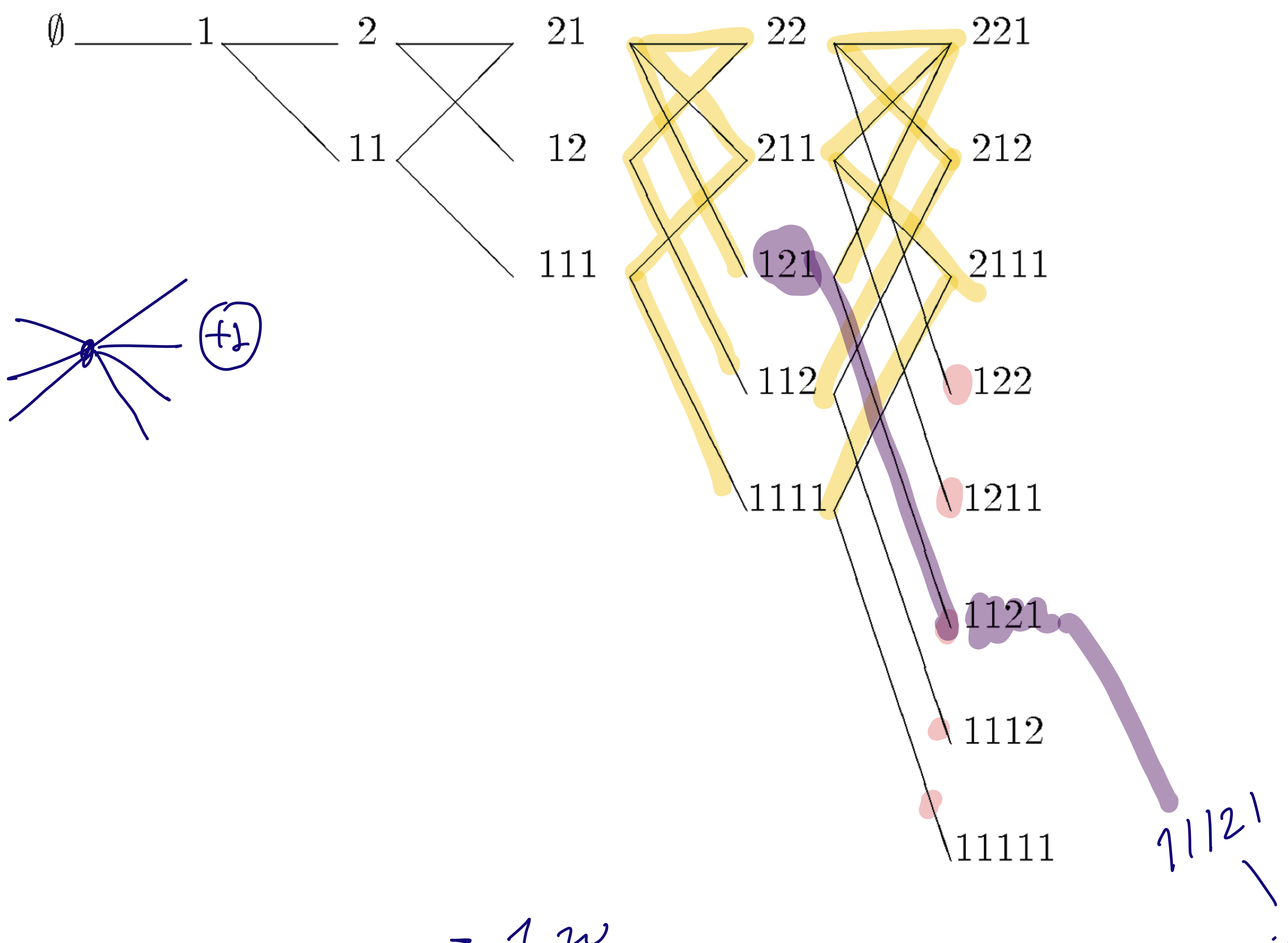
$$\boxed{\begin{aligned} x_k &= k + \rho - 1 \\ y_k &= k\rho \end{aligned}}$$

μ ?

$p_n(t)$

from the hand book

Young – Fibonacci graph (recall 10.4–10.5)



$$w \to 1w$$

$$w \to 2v \quad \text{if} \quad w = 1v$$

$$w \to 2^k 1 v \quad \text{if} \quad w = 2^k v$$

→ differential poset

(Stanley . Enumerative Combinatorics I)

$\rightarrow$ $\boxed{\varphi_{P_l}(W) = \frac{\dim W}{n!}}$ is harmonic

$\rightarrow$ connection to tridiagonal matrices

$$A_l(x|y) = \det_{l\times l}\begin{pmatrix} x_1 & y_1 & & 0 \\ 1 & x_2 & y_2 & \\ & 1 & x_3 & y_3 \\ 0 & & & \ddots \end{pmatrix}$$

$$B_{l-1}(x|y) = \det_{l\times l}\begin{pmatrix} y_1 & x_1 y_2 & & 0 \\ 1 & x_3 & y_3 & \\ 0 & 1 & x_4 & \\ & & & \ddots \end{pmatrix}$$

$A_0 = 1, \quad B_0 = y_1$

$$\mathfrak{S}_u(x|y) \qquad u\text{ - fib. word}$$

$$= \begin{cases} A_k(x|y), & u = 1^k \\ B_k(x+|v| \,|\, y+|v|) \circ \mathfrak{S}_v(x|y), & u = 1^k 2 v \end{cases}$$

$$x+r = (x_{r+1}, x_{r+2}, x_{r+3}, \ldots)$$

Clone Schur functions

<u>Claim</u>, (Okada 1994)

$$\varphi(w) = \mathfrak{S}_w(\vec{1} \,|\, \vec{y}) \text{ is}$$

harmonic on $\mathbb{YF}$

Positive harmonic f. (correction)

Def. $\vec{y}$ is Fib positive if $\forall w$,

$$\boxed{\mathcal{S}_w(\vec{1}\,|\,\vec{y}) \geq 0.}$$

$$\Downarrow$$

$A_\ell(\vec{1}/\vec{y})\,,\quad B_\ell(\vec{1}/\vec{y}+r) \geq 0 \quad \forall \ell, r.$

$$\boxed{\text{Fib.Pos} \subset \text{Tot.Pos. } \vec{y}}$$

$x_i \equiv 1$, $\quad T(\vec{y}) \equiv \begin{pmatrix} 1 & y_1 & & 0 \\ 1 & 1 & y_2 & \\ & 1 & 1 & y_3 \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}$

Def $\vec{y}$ is tot. pos. (tot. nonneg.) if principal minors of $T(\vec{y})$ are ≥ 0

<u>Claim</u> $\vec{y}$ TP & $\forall r$,

$$\vec{y}^{(r)} := (y_r^{-1}, y_{r+1}, y_{r+2}, y_{r+3}, \ldots)$$

is TP

$\Updownarrow$

$\vec{y}$ is Fibonacci positive

($\Leftarrow$ clear, $\Rightarrow$ follows from properties of TP)

Role of y_r^{-1} is to get $B_{\ell-1}$

Note

Not all of TP seq, $\vec{y}$ are FP.

Example. $y_n = \dfrac{n^2}{(2n-1)(2n+1)}$

then $A_\ell = \dfrac{\ell!}{(2\ell-1)!!} \geq 0$

(exercise)

$$A_\ell = \det_{\ell\times\ell} \begin{pmatrix} 1 & y_1 & & & 0 \\ 1 & 1 & y_2 & & \\ & 1 & 1 & y_3 & \\ 0 & & 1 & 1 & y_4 \\ & & & & \ddots \end{pmatrix}$$

But: $\vec{y}^{(1)}$ is not TP

$$A_3(\vec{y}^{(1)}) = \det \begin{pmatrix} 1 & \frac{4}{5} & 0 \\ 1 & 1 & \frac{27}{35} \\ 0 & 1 & 1 \end{pmatrix} < 0$$

$\Rightarrow$ $Sw(\vec{1}|\vec{y})$ may be negative.

TP $\vec{y}$ $\longleftrightarrow$ prob. meas. on $[0,\infty)$

(nice, related to ϕpl) $\longleftarrow$ Poisson distr. on $[0,\infty)$

Which μ on $(0,\infty)$ are Fib. positive?
(except Poisson)

11. Boundary of $\mathbb{V}/F$ (Goodman–Kerov)

11.1 Answer & non-uniqueness

Recall basic definitions about boundary:

$$\left\{ \varphi : \varphi(w) = \sum_{v \searrow w} \varphi(v), \quad \varphi(\varnothing) = 1, \ \varphi \geq 0 \right\}$$

$$\boxed{\gamma(\mathbb{Y}/F)}, \text{ convex set}$$

Extreme pts $\boxed{\mathrm{Ex}\,\gamma(\mathbb{Y}/F) = \,?}$

Note: why harmonic, again

$$\boxed{(\Delta\varphi)(w) = -\varphi(w) + \sum_{v \searrow w} \varphi(v)}$$

$\varphi_w(v) = \dim(v, w)$ — Green f.

$$-(\Delta \psi_w)(v) = 1_{v=w}$$ (exercise)

We're after limits of

$$K(v,w) = \frac{\dim(v,w)}{\dim w}, \quad |w| \to \infty$$

(martin kernel)

Martin boundary of graph G

$\mathrm{Fun}(G)$, pointwise convergence

$\tilde{E} \subset \mathrm{Fun}(G)$ — closure of

$$\{v \mapsto K(v,w)\}_{w \in G}$$

$$K(v,w) = \frac{\dim(v,w)}{\dim w}$$

$\tilde{E}$ is compact because $0 \leq K \leq 1$

$w \in G_{op} \to K(\cdot, w) \in \tilde{E}$

$$\boxed{E = \tilde{E} \setminus G}$$ $\leftarrow$ Martin boundary, definition

$K(v,w)$ extends to $K(v, \alpha)$,

$\alpha \in E$

&

$\varphi(v) = K(v, \alpha)$ belongs to $\mathcal{H}$,

$\alpha \in E$

<u>Thm.</u> (Choquet) $\forall\, \varphi \in \mathcal{H}$,

$\exists$ probab. measure μ on E s.t.

$$\varphi(v) = \int_E K(v, \alpha)\, \mu(d\alpha)$$

Note. μ might be non-unique

(For $\mathbb{Y}/\mathbb{F}$, uniqueness announced in 2020)

$E_{min} \subset E$

$\alpha \in E_{min} \iff \varphi(v) = K(v,\alpha)$ is extreme

then choose μ supported by E_{min}, and it is unique.

Goodman-Kerov (1997) — described E

Def. $w \in \{1,2\}^\infty$ inf word.

d_i = positions of 2's

$w = 1112111112112\ 2221\ 2 \ldots$

$\uparrow_{d_1} \qquad \uparrow_{d_2}$

Called summable iff

$$\sum_i \frac{1}{d_i} < \infty \iff \pi(w) := \prod_i \left(1 - \frac{1}{d_i}\right) > 0$$

converges

<u>Thm.</u> $E = E(\mathbb{Y}/\mathbb{F})$ consists of

① Plancherel $\varphi_{Pl}(w) = \dfrac{\dim w}{n!}$

② (β, w) s.t. $0 < \beta \leq 1$,

$w \in \{1, 2\}^{\infty}$ summable

Topology on Ω. 1) $(\beta^n, w^n) \longrightarrow Pl$ iff

$\beta^n \to 0$ or $\pi(w^n) \to 0$

2) $(\beta^n, w^n) \longrightarrow (\beta, w)$ iff

$\varphi_{Pl}(w) = \dfrac{\dim w}{n!}$

$w^n \to w$ digitwise, &

$\beta^n \pi(w^n) \to \beta \pi(w)$

$\pi(w) := \prod_i \left(1 - \frac{1}{d_i}\right)$

Examples of convergence.

1. $\pi(\omega) = 0$, what it means?

Many 2's

2. $(\beta^n, \omega^n) \longrightarrow (\beta, \omega)$

11.2 Plancherel & $\boxed{\text{Type I functions}}$

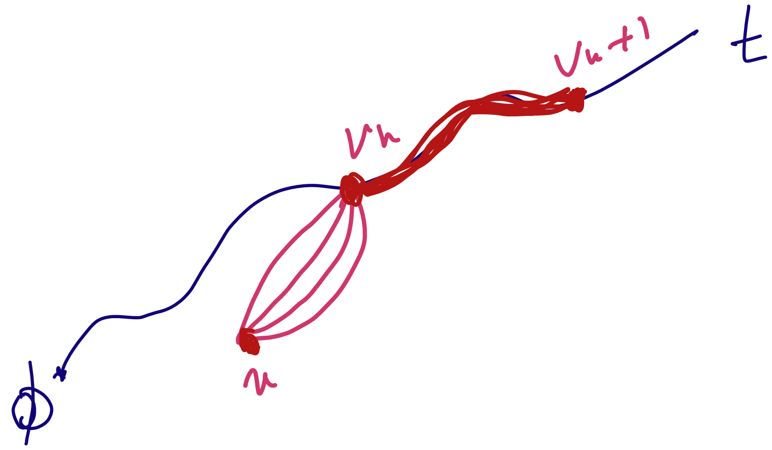
we know $\varphi_{Pl}(w) = \dfrac{\dim w}{n!}$, $\varphi_{Pl} \in \mathcal{X}$

Type I. Let $t = (v_0 \nearrow v_1 \nearrow \dots)$ inf path.

$\forall w$ $\dim(u, v_n)$ increases,

$$\dim(u,t) := \lim_n \dim(u, v_n)$$

(can be infinite)

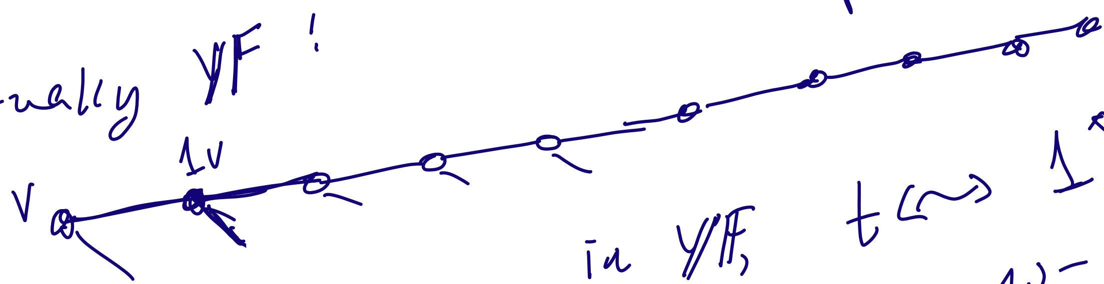


Lemma. t — inf. path. The foll. are equiv.

1. $\dim(\emptyset, t) < \infty$
2. $\dim(u, t) < \infty \quad \forall u$
3. For almost all n,

   $v_{n-1} \to v_n$ is the only edge predecessor

eventually V/F!

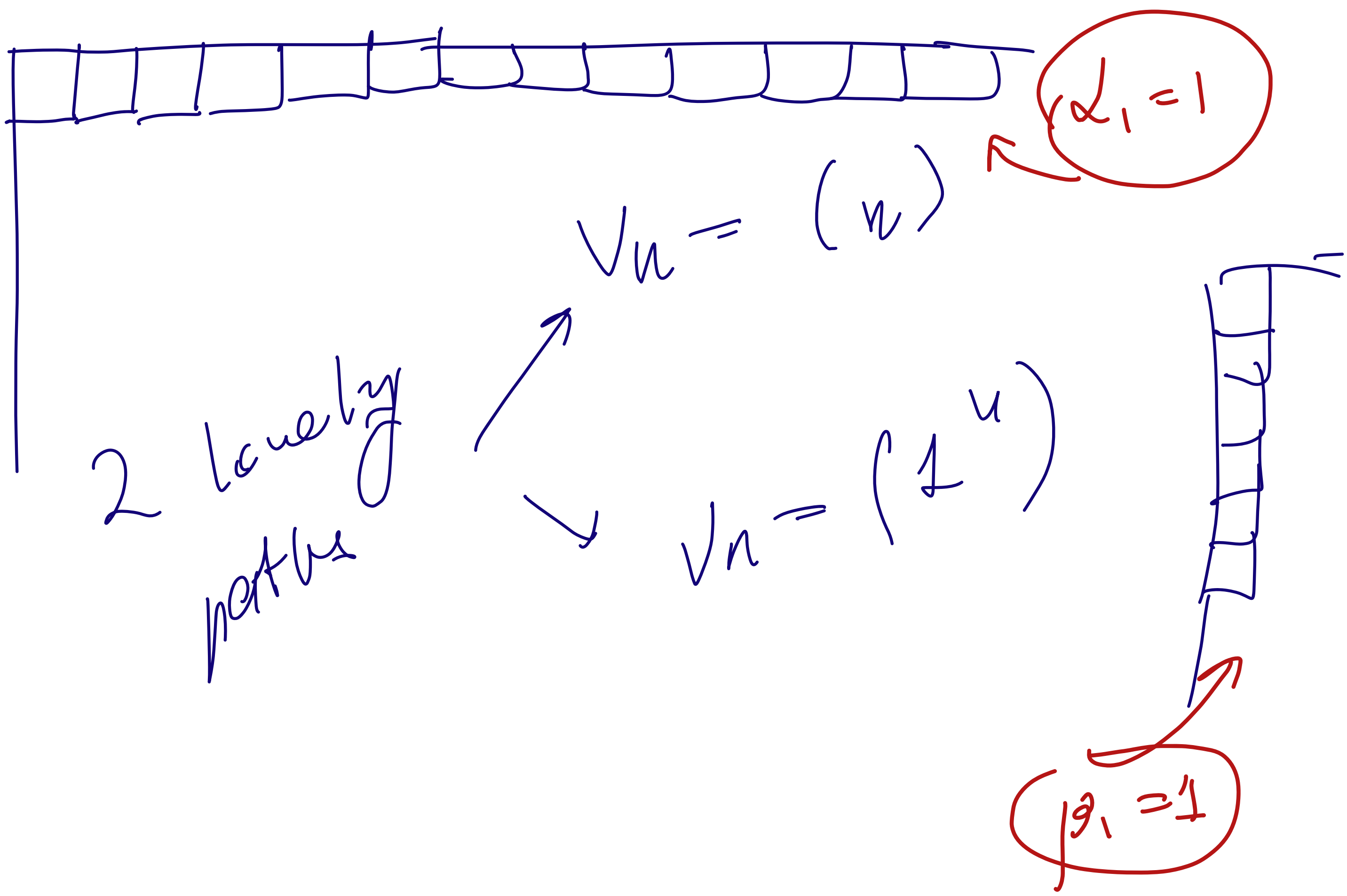


in V/F, $t \rightsquigarrow 1^{\infty} w$

w - fixed

4. There are finitely many paths which eventually coincide with t

For ψ, which are the paths s.t. $\dim(\phi, t) < \infty$?



Lemma. $\varphi_t(v) = \dim(v,t) / \dim(\varnothing,t)$,

$\varphi_t \in \mathcal{H}$

Def. φ_t are called type I harmonic functions

& $\mathbb{YF}$ has many of these.

Also, φ_t's are extremal.

(because on $\mathbb{YF}_n$, $n \gg 1$, φ_t is a delta function)

(what words $w \in E$ do they correspond to ?)

w with finitely many 2's.

& summable w
with ∞ many 2's
are limits of type I φ's

$\boxed{E}$ Pl ✓

(β, w) $\beta = 1$, w - finitely many 2

type I ✓

$\beta < 1$ — ?

11.3. Contraction of harmonic function to Plancherel

$\varphi \in \mathcal{H}$ $\Longrightarrow$ Random growth process on $\mathbb{G}$

$|u| = n-1, \quad |v| = n \quad , \quad u \to v$

$$\varphi(u) = \sum_{v \searrow u} \varphi(v)$$

$$\boxed{p^{\uparrow}_{\varphi}(u,v) = \frac{\varphi(v)}{\varphi(u)}}, \quad \Sigma \text{ to } 1 \text{ over } v.$$

random growth process

Not every random growth is harmonic

Need the "exchangeability" ("centrality") condition

— $\mathbb{P}_{\varphi}(\emptyset \to v_1 \to v_2 \to \cdots \to v_n)$

depends only on v_n.

φ_1, φ_2 – 2 harm. funct., fix $\tau \in [0,1]$

<u>Define</u> $\boxed{\varphi_1 *_\tau \varphi_2}$

growth process s.t.

$$\mathbb{P}_{\varphi_1 *_\tau \varphi_2}(v) = \mathbb{E}\, \mathbb{P}_{\varphi_1}(u)\, \mathbb{P}_{\varphi_2}(u \to v)$$

$\boxed{|v| = n}$

$|u| \sim \mathrm{Bin}(n, \tau)$

$$= \sum_{k=0}^{n} \binom{n}{k} \tau^k (1-\tau)^{n-k} \sum_{|u|=k} \varphi_1(u)\, \dim u \cdot \dim(u,v) \frac{\varphi_2(v)}{\varphi_2(u)}$$

Q: Is this harmonic?

Kerov - Goodman $\varphi_2 = \varphi_{pl}$.

$$\varphi_2(\lambda) = \frac{\dim \lambda}{k!}$$

$$\sum_{k=0}^{n} \frac{1}{(n-k)!} \tau^k (1-\tau)^{n-k} \sum_{|\lambda|=k} \varphi_1(\lambda) \dim \lambda \cdot \dim(\lambda, \nu)$$

$$\| $$

$$\underbrace{\dim \nu \cdot (\varphi_1 *_\tau \varphi_{pl})(\nu)}_{\text{probab.}}$$

$$\Rightarrow \quad \underset{\tau \in [0,1]}{C_\tau(\varphi)(v)} = \sum_{k=0}^{n} \frac{\tau^k (1-\tau)^{n-k}}{(n-k)!} \sum_{|u|=k} \varphi(u)\, d_{\mathrm{in}}(u,v)$$

$$C_\tau(\varphi) = \varphi *_\tau \varphi_{pl}$$

Properties

$C_\tau(\varphi) \in \mathcal{P}$

$C_\tau(C_\sigma(\varphi)) = C_{\tau\sigma}(\varphi)$

$C_0(\varphi) = \varphi_{pl}$

$C_\tau(\varphi_{pl}) = \varphi_{pl}$.

$$C_\tau(\underbrace{\varphi_{\beta,w}}_{}) = \varphi_{\tau\beta,w}$$

harmonic f. $\in E$

Exercise $\forall$, $\varphi = \varphi_w$, $w = (\alpha;\beta)$

$$\Rightarrow \boxed{C_\tau(\varphi)} \sim (\tau\alpha;\tau\beta)$$

and $\tau = 0$ is Plancherel.

a way to mix in some Plancherel

Next: use C_τ to show that φ_w are not extreme

Today: boundary of $\mathbb{Y}$ via
close functions (as much as we have time)

Next: Back to $\mathbb{Y}$ & Plancherel measures $\longrightarrow$ $\boxed{\text{reg. rep. of } S(\infty)}$

$\to$ limit shape

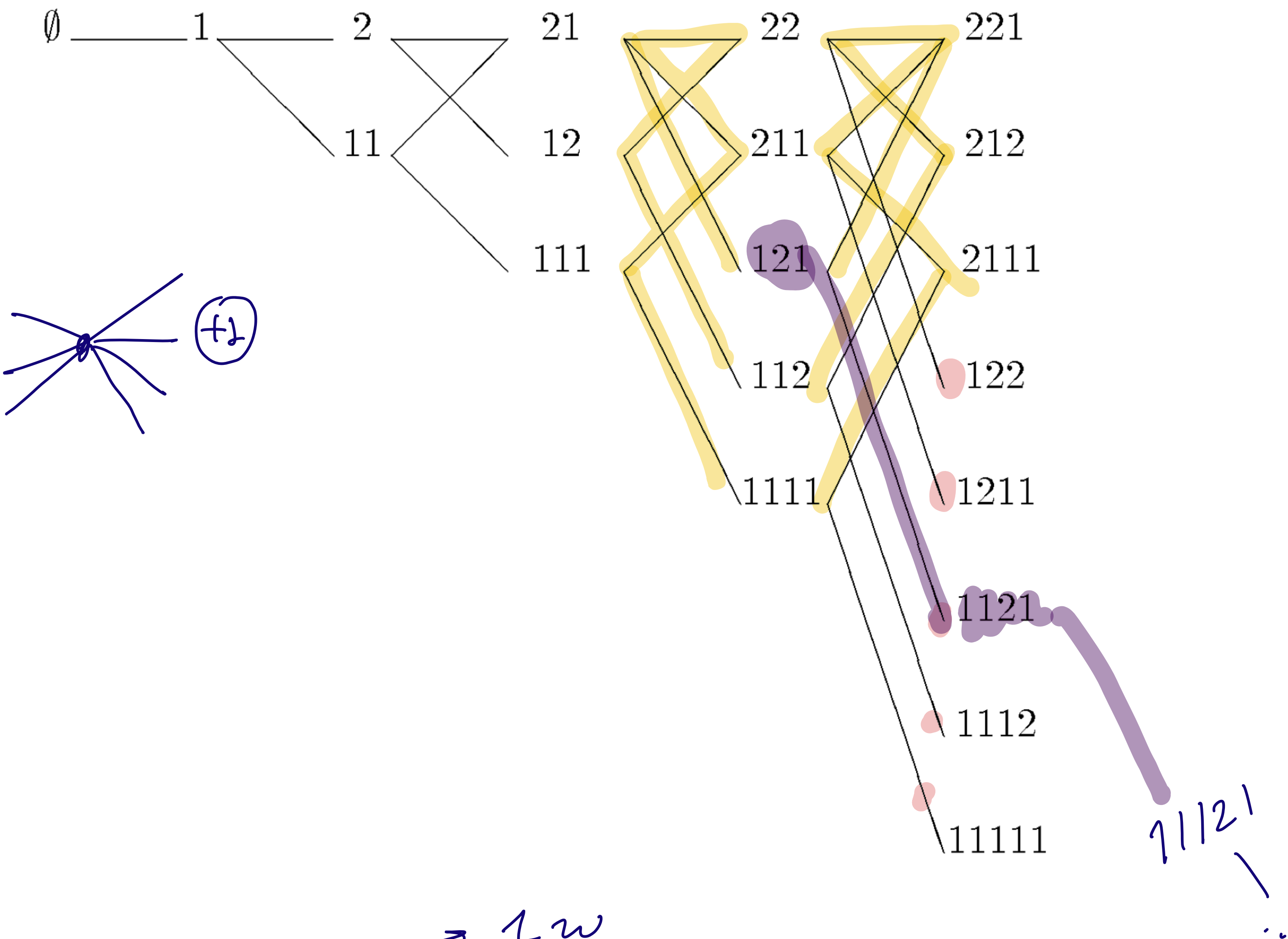
$\to$ Plancherel growth process & its hydrodynamics

$\to$ Inequalities (including some on rep-th coefficients like Littlewood-Richardson & Kronecker)

$\lambda \in \mathbb{Y}_n$

$$\max_{\lambda \in \mathbb{Y}_n} \dim \lambda \sim ? \quad n \to \infty$$

Young–Fibonacci graph



$$w \to 1w$$
$$w \to 2v \quad \text{if} \quad w = 1v$$
$$w \to 2^k 1 v \quad \text{if} \quad w = 2^k v$$

Thm. Martin boundary is $\left. \dfrac{\dim(v,w)}{\dim w} \right|_{w \to \infty}$

$\to$ Plancherel

$\to (\beta, \alpha)$, $\quad 0 < \beta \le 1$,

$\varphi(w) = \frac{d_{\mathrm{Pa}}(w)}{n!}$

$\alpha \in \{1,2\}^\infty$, s.t.

$$\pi(\alpha) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{d_i}\right) > 0$$

d_i – positions of 2's in α

($\beta = 1$, α – finitely many 2's $\swarrow$ $1^\infty v$

come from „lonely paths" &

type 1 φ's)

$\dim(w, 1^\infty v)$

Flow $\quad \varphi \longmapsto C_\tau(\varphi) \qquad \tau \in [0,1]$

Extremes: $\varphi_{\beta,\alpha} \longmapsto \varphi_{\tau\beta,\alpha}$

Def. $\varphi_1 *_\tau \varphi_2(v) = \sum_{k=0}^{n} \binom{n}{k} \tau^k (1-\tau)^{n-k} \qquad |v| = n$

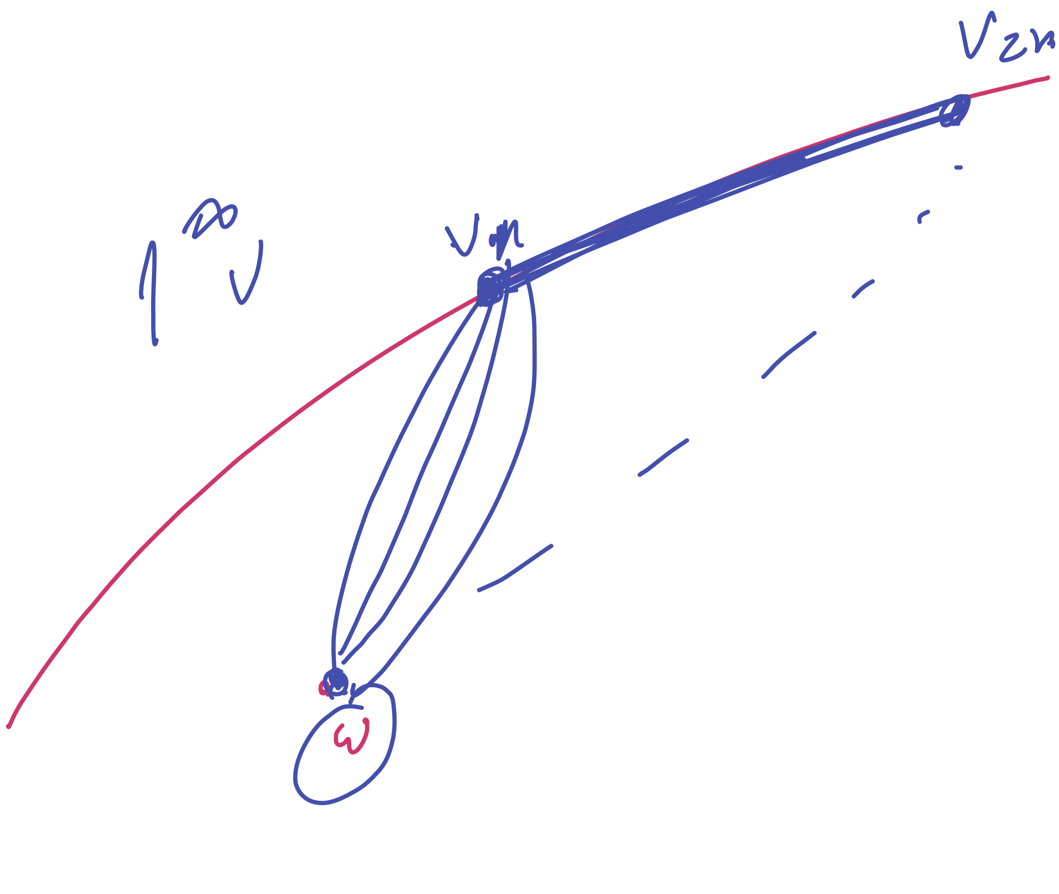
$\boxed{0 \le \tau \le 1}$

$$\cdot \sum_{|u|=k} \frac{\varphi_1(u)\varphi_2(v)}{\varphi_2(u)} \cdot \frac{\dim u \dim(u,v)}{\dim v}$$

$$C_\tau(\varphi) = \boxed{\varphi_1 *_\tau \varphi_{Pl}}$$

$$\sum_{k=0}^{n} \cancel{\binom{n}{k}} \tau^k (1-\tau)^{n-k} \cancel{} \ (n-k)!$$

$$\cdot \sum_{|u|=k} \frac{\varphi_1(u)\,\cancel{\varphi_2(v)}}{\cancel{\varphi_2(u)}} \cdot \frac{\cancel{\dim u}\ \dim(u,v)}{\cancel{\dim v}}$$



Prop. ① $C_\tau(\varphi)(v)$ is harmonic
(in general, $\varphi_1 *_\tau \varphi_2$ — not harm.)

② $C_\sigma \circ C_\tau = C_{\sigma\tau}$

③ $C_0(\varphi) = \varphi_{pc}$

④ $C_\tau(\varphi_{pc}) = \varphi_{pl}$

Proof. ② n. — fix

Let $\xi \sim Bin(n, \tau)$
$\eta \sim Bin(\xi, \sigma)$
$\Rightarrow \eta \sim Bin(n, \tau\sigma)$ $\square$

(Note: q-analogue
$$P(\xi = k) = \tau^k (\tau; q)_{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

①
$$\sum_{w \searrow v} \underbrace{C_\tau(\varphi)(w)} = C_\tau(\varphi)(v)$$

$$\sum_{w \geq v} \sum_{k=0}^{n+1} \tau^k (1-\tau)^{n-k+1} \cancel{(n+1-k)!}$$

$$\cdot \sum_{|u|=k} \varphi_1(u)\, \dim(u, w)$$

$|u| = k$

$|w| = n+1$

$\dim(u, v) =$

$= \langle U^{n+1-k} \underline{u}, \underline{w} \rangle$

$$\sum_{w \geq v} \langle U^{n+1-k} \underline{u}, \underline{w} \rangle$$

$$= \langle U^{n+1-k} \underline{u}, U \underline{v} \rangle$$

$[D, U] = 1$

$$= \langle D U^{n+1-k} \underline{u}, \underline{v} \rangle \stackrel{?}{=}$$

$$\boxed{D U^{\ell} = U^{\ell-1} \cdot \ell + U^{\ell} D}$$

$DU = 1 + UD$

$DUU = (1 + UD) U = U + UDU$

$= U + U + UUD$

$$\stackrel{?}{=} \underbrace{(n+1-k) \dim(u, v)} + \underbrace{\langle U^{n+1-k} D \underline{u}, \underline{v} \rangle}$$

$$\underbrace{\sum_{k=0}^{n+1} \frac{\tau^{k}(1-\tau)^{n-k+1}}{(n-k)!} \cdot \sum_{|u|=k} \varphi_1(u)\dim(u,v)}_{1)\ \boxed{(1-\tau)\,C_\tau(\varphi)(v).}}$$

$$\sum_{k=0}^{n+1} \frac{\tau^{k}(1-\tau)^{n+1-k}}{(n+1-k)!} \sum_{|u|=k}\ \sum_{\rho\nearrow u} \varphi_1(u)\dim(\rho,v)$$

$$\sum_{|\rho|=k-1} \dim(\rho,v)\cdot \overbrace{\boxed{\sum_{u:\,u\searrow\rho} \varphi_1(u)}}^{\varphi_1(\rho)}$$

$$\Rightarrow \sum_{\substack{k=0\\ \boxed{k>0}}}^{n+1} \frac{\tau^{k-1}(1-\tau)^{n-(k-1)}}{(n-(k-1))!} \cdot \tau \cdot \sum_{|\rho|=k-1} \varphi_1(\rho)\dim(\rho,v)$$

$$= \boxed{\tau\, C_\tau(\varphi)}$$

$\square$

The princ. implies that $\boxed{\varphi(\nu) = S_\nu(\vec{1} \mid \vec{y})}$ ←
are not extreme
(the class of S_ν's is not the same as $\varphi_{\beta,\alpha}$)

$$\boxed{C_\tau : \varphi_{\beta,\alpha} \to \varphi_{\tau\beta,\alpha}}$$

Indeed,

$$C_\tau(S_\nu) = \sum_{k=0}^{n} \binom{n}{k} \tau^k (1-\tau)^{n-k} \cdot \sum_{|\mu|=k} \frac{S_\mu \, \varphi PL(\nu)}{\varphi PL(\mu)} \cdot \frac{\dim \mu \, \dim(\mu,\nu)}{\dim \nu}$$

$$= \underbrace{\boxed{\sum_{k=0}^{n} \frac{\tau^k (1-\tau)^{n-k}}{(n-k)!} \cdot \sum_{|\mu|=k} S'_\mu \cdot \dim(\mu,\nu)}}_{}$$

has a simplification
which prevents this from
being of the form $S_{\nu}(\vec{1} \mid \vec{y})$
for another ν

$$\sum_{k=0}^{n} \frac{t^{k}(1-t)^{n-k}}{\cancel{(n-k)!}} \cdot \underbrace{\sum_{|\lambda|=k} s_{\lambda}(\alpha_1 \dots \alpha_N)\, s_{\mu/\lambda}(PL)}_{s_\mu(PL \cup \alpha_1 \dots \alpha_N)} \cdot \cancel{(n-k)!}$$

Y

Y|F

q-Y|F

?

Clone funct. ring

(like symm. funct. but for $\mathcal{Y}/\mathbb{F}$)

R = noncomm. poly's in X, Y

$$w = 1^{k_t}\, 2\, 1^{k_{t-1}}\, 2 \cdots 1^{k_1}\, 2\, 1^{k_0}$$

$$h_w = \underbrace{X^{k_0}\, \cancel{Y}\, X^{k_1} \cdots X^{k_t}}$$ (Reverse!)

R_n — deg n

R_∞ = inductive limit

$$R_n \longmapsto \underline{R_n}\boxed{X} \subset R_{n+1}$$

$$f \longmapsto \boxed{fX^\infty \in R_\infty}$$

$$\boxed{R_\infty = R/_{X-1}}$$

φ on R_∞,

$$\varphi(f) = \varphi(fX)$$

deg 7

$XXYYXXY \rightarrow (x_1 x_2 y_3 y_4 x_5 x_6 y_7)$

A_ℓ

Let $P_n = \det \begin{vmatrix} X & Y & & & 0 \\ 1 & X & Y & & \\ & 1 & X & Y & \\ & & & \ddots & Y \\ 0 & & & 1 & X \end{vmatrix}$

$B_{\ell-1}$

$Q_{n-1} = \det \begin{vmatrix} Y & Y & & & & 0 \\ X & X & Y & & & \\ & 1 & X & Y & & \\ & & 1 & X & Y & \\ & & & & \ddots & Y \\ 0 & & & & 1 & X \end{vmatrix}$ $n \times n$

$$\boxed{\sum_{\sigma} (-1)^{\sigma} a_{\sigma(1),1} \, a_{\sigma(2),2} \cdots a_{\sigma(n),n}}$$

$P_{n+1} = P_n X - P_{n-1} Y$

$Q_0 = Y$

$Q_{n+1} = Q_n X - Q_{n-1} Y$

$Q_1 = \begin{vmatrix} Y & Y \\ X & X \end{vmatrix} \neq 0$

$= YX - XY$

$Q_0 X = X Q_0 + Q_1$

$Q_0 = Y$

$YX = XY + YX - XY$ ✓

$$\boxed{\begin{aligned} P_n X &= P_{n+1} + P_{n-1} Q_0 \\ Q_n X &= Q_{n+1} + Q_{n-1} Q_0 \end{aligned}} \quad (*)$$

Schur poly's (clone version)

$$S_v = P_{k_0} Q_{k_1} \cdots Q_{k_t} \,, \quad \boxed{v = 1^{k_t} 2 \ldots 2 1^{k_1} 2 1^{k_0}}$$

(same def as before, but non commutative)

Commut. rel. (*) $\Rightarrow$

$$\boxed{S_w X = \sum_{v \searrow w} S_v}$$ $\leftarrow$ $\mathbb{YF}$ branching

$\mathbb{Y}$. $p_1 S_\lambda = \sum_{\nu = \lambda + \square} S_\nu$

$\Downarrow$

$\varphi(w)$ on $\mathbb{YF}$, harm. $\varphi \geq 0$, $\varphi(\emptyset) = 1$

$\longleftrightarrow$

φ on R s.t.

1) $\varphi(fX) = \varphi(f)$

2) $\varphi(1) = 1$

3) $\varphi(S_v) \geq 0$

$$\varphi(v) = \varphi(S_v')$$

p - functions. $\qquad v = 1^{k_t} 2\, 1^{k_{t-1}} 2 \dots 2 1^{k_0}$

$$p_v \doteq \left(X^{k_0+2} - (k_0+2) X^{k_0} y \right)$$

$$\dots \left(X^{k_{t-1}+2} - (k_{t-1}+2) X^{k_{t-1}} y \right) X^{k_t}$$

$$p_v X = p_{1v}$$

$$D(p_{2v}) = 0$$

$$uf \doteq fX$$

$$u\, p_v = p_{1v}$$

$$Df = \text{"}\frac{\partial}{\partial x}\text{"} f$$

i.e. $\langle f, ug \rangle = \langle Df, g \rangle$
where $\langle S_u, S_v \rangle = \delta_{uv}$

$$[D, u] = Id$$

$\forall\, v \in Y/F$, we have $\boxed{p_v = p_{1^\infty v}}$

$\mathcal{Y}$: $\quad s_\lambda \longleftrightarrow p_\rho = p_{\rho_1} p_{\rho_2} \cdots$

$$p_\rho = \sum_\lambda s_\lambda \cdot \underbrace{\chi^\lambda_\rho}_{}$$

λ-character of S_n on ρ

Def.

$$p_\mu = \sum_\nu \boxed{\chi^\nu_\mu}\, s_\nu$$

have an explicit product formula (skip)

Recall $\mathcal{YL}$, $\varphi_{\alpha\beta}(p_\beta)$ = product form & very explicit

same for $\mathcal{YF}$.

$$\varphi_{00}(p_1) = 1$$
$$\varphi_{00}(p_k) = 0, \; k \geq 2$$

1. $\varphi_{pl}(p_u) = 0$ if u contains 2

Proof

$$\varphi_{pl}(\underline{Df}) = n\,\varphi_{pl}(f)$$

f - deg n

$$\varphi_{pl}(S_v) = \frac{\dim v}{n!}$$

$$\sum_{w \nearrow v} \dim w = \dim v \cdot (n+1)$$

$|v| = n$

$$\varphi_{pl}(p_{2v}) = 0$$

because $\boxed{D\, p_{2r} = 0}$

2 Type 1 herm. f. $\sim$ path $1^\infty w$

$\mathbb{Y}/\mathbb{F} \ni w = 1 \ldots 1\, 2\, 1 \ldots 1\, 2 \ldots 2 \ldots 1$

$\uparrow d_1$ $\qquad \uparrow d_2$

$u = 1^\infty 1 . 1212 \ldots 122 \ldots \in 1^\infty \mathbb{Y}/\mathbb{F}$

$\uparrow \delta_1 \quad \uparrow \delta_2 \quad \uparrow \delta_3$

$\Downarrow$

$$\boxed{\varphi_w(p_u) = \prod_{i=1}^{u} \prod_{j:\, \delta_i \le d_j < \delta_{i+1}} \left(1 - \frac{\delta_{i+1}}{d_j}\right)}$$

(follows from explicit formula for characters χ^u_v, skip)

So, here the summability is natural
for α – summable word,
define $\varphi_\alpha(p_n)$ by same

$$\varphi_\alpha(p_n) = \prod_{i=1}^{n} \prod_{j:\ \delta_i \le d_j < \delta_{i+1}} \left(1 - \frac{\delta_i + 1}{d_j}\right)$$

(& we have $\varphi_\alpha(p_{1n}) = \varphi_\alpha(p_n)$)

Next. $\underbrace{\varphi_{\beta,\alpha}(p_u)}_{\text{defined as } C_\beta(\varphi_\alpha)} = \underbrace{\beta}_{}^{\|u\|}\varphi_\alpha(p_u)$

$\|u\| = (\text{position of leftmost 2 in } u) + 1$

defined as $C_\beta(\varphi_\alpha)$

<u>Proof</u>.

Defined : $\varphi_{\beta,\alpha}(p_n)$ $\forall$ $0<\beta\leq 1$

$\alpha \in \{1,2\}^{\infty}$

$\boxed{\text{summable}}$

Remaining steps

→ All φ_{pl} & $\varphi_{\beta,\alpha}$ are distinct

→ $\varphi_{\beta,\alpha}$ = lim of type 1 herm. f

→ regularity conditions :

$\frac{\dim(V,W)}{\dim W}$

a sequence of type I φ's converges to φ_{PL} or $\varphi_{\beta\alpha}$ iff ---

(1) $\lim \pi(V^n) = 0$

(2) $V^n \to \alpha$ summable, and

$\pi(\alpha)^{-1} \lim \pi(V^n) \to \beta > 0$

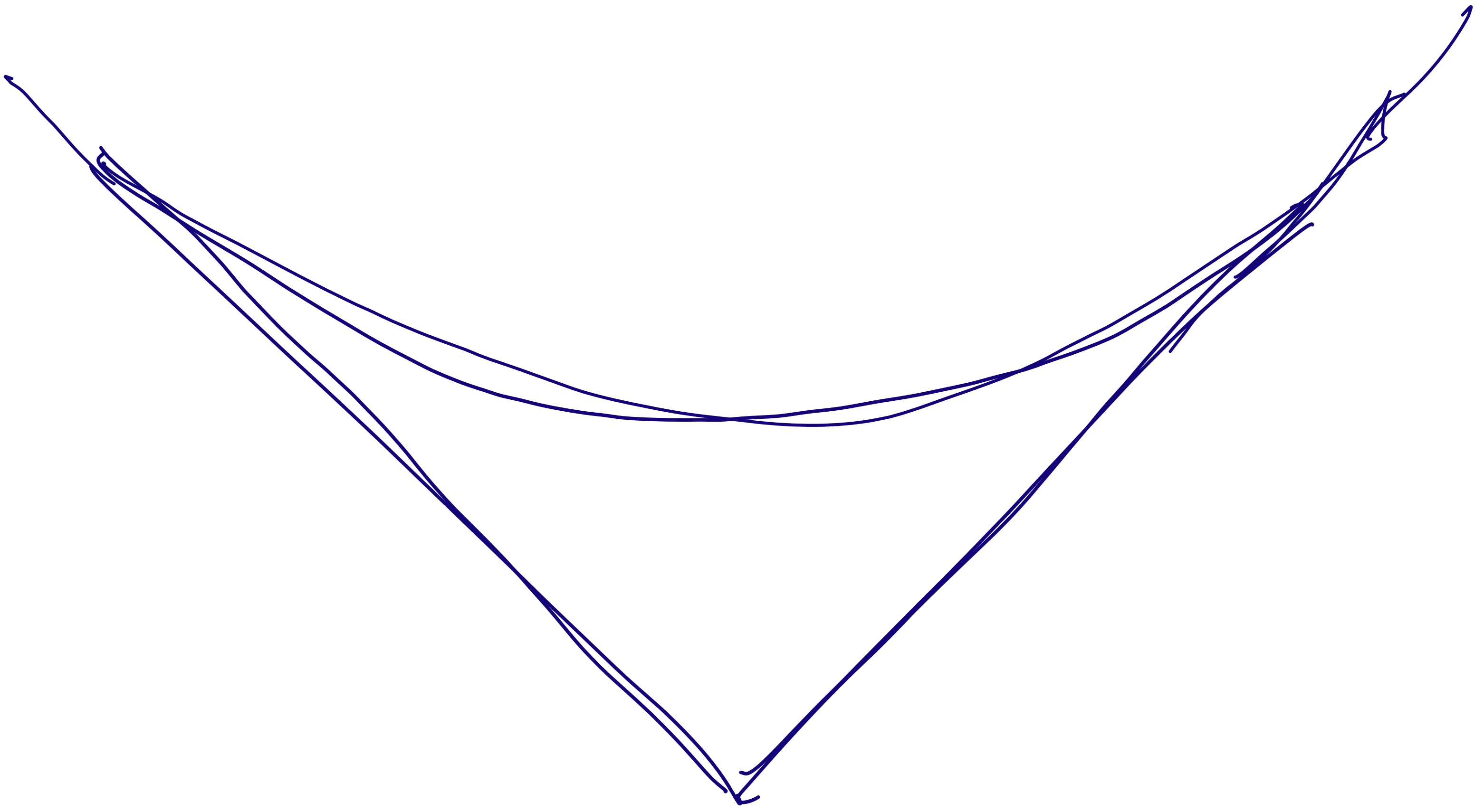
<u>Theorem</u> If V^n is regular then

(1) $\varphi_{V^n}(f X^{n-m}) \to \varphi_{PL}(f)$

(2) $\varphi_{V^n}(f X^{n-m}) \to \varphi_{\beta\alpha}(f)$

& these are all possible limits of harm. f's

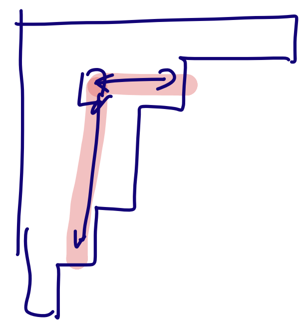
(so, Martin boundary)



12 Plancherel measure on Young diagrams

12.1 Recall what we know

$$\boxed{M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}}$$



→ Biregular representation of $S(\infty)$

→ Sum to 1

→ Hook formula

$$\boxed{\dim \lambda = \frac{n!}{\prod_{\square \in \lambda} h(\square)}}$$

→ up recursion for $\dim \lambda$

→ Plancherel growth process

→ Transition distribution:

where do we add a box?

RT $(G,K) = (S(\infty)\times S(\infty),\ \operatorname{diag} S(\infty))$

acts on $\boxed{\ell^2(K)}$ $\quad f(g)\mapsto f(h_1^{-1} g h_2)$

$(h_1, h_2)\in G$

bi-reg. rep.

K-inv. vector

$$\xi(g) = \begin{cases} 1, & g=e \\ 0, & \text{else} \end{cases}$$

$(T(h,e)\xi, \xi) = \varphi(h) =$ Plancherel character

$= \boxed{1_{h=e}}$ funct on $S(\infty)$

$$\boxed{\varphi\Big|_{S(n)} = \sum_\lambda c_\lambda \cdot \frac{\chi^\lambda}{\dim\lambda}}$$

Plancherel theorem

irrep. of $S(n)$

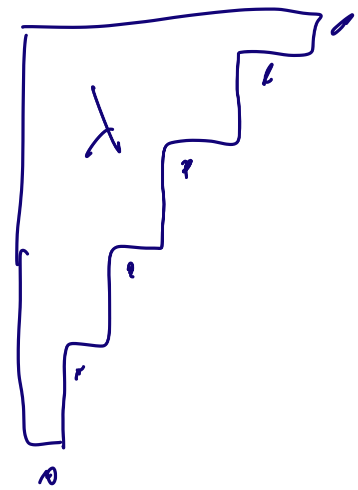
Then $c_\lambda = \dfrac{(\dim\lambda)^2}{n!}$

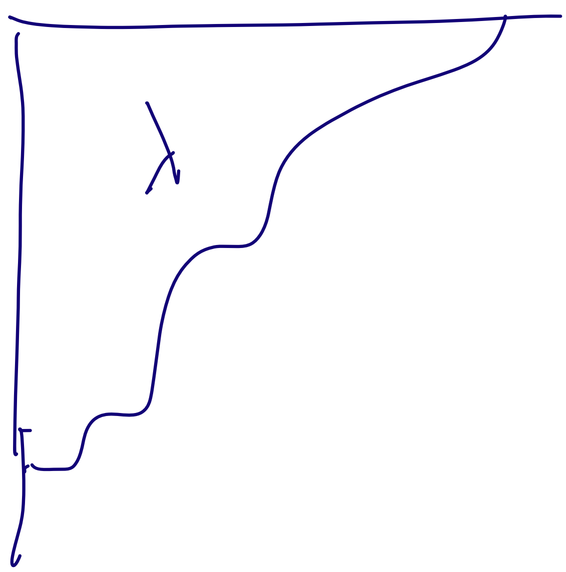
plach. growth.

$\mu = \lambda + \square$

$$\boxed{p(\lambda \to \mu) = \frac{\dim \mu}{\dim \lambda \cdot (n+1)}}$$

$$\boxed{\varphi_{pl}(\lambda) = \frac{\dim \lambda}{n!}}$$





$p(\lambda \to \lambda + \square) \leftarrow$ where?

(12.2) Plancherel measure & Longest increasing subseq's (history & motivation for Limit shape) → RSK ←

(a) $LIS(n)$

(= time to board the airplane)

(b) $\lambda_1 \sim \text{Planch}(n)$

σ_i

$\sigma \in S_n$

$$\sigma = (\sigma_1 \, \sigma_2 \, \ldots \, \sigma_n)$$

$$3 \; \underline{2} \; \underline{5} \; \underline{6} \; 4 \; \underline{7} \; 1$$

$LIS(\sigma)$ = length of longest incr. subseq.

$LIS = 4.$

$\exists$ Dynamical programming to find LIS

$\boxed{LIS(n)}$ = random var., $= LIS(\sigma)$, $\sigma \in S_n$ uniform

(Ulam) 1960's

$LIS(n) \sim ?$ $n \to \infty$

$c\sqrt{n}$ $c = ?$

1970's

$\exists c$

$c = 2$?

RSK → 1977

$c = 2$

Vershik-Kerov

Logan-Shepp

1999

Baik
Deift
Johansson

$LIS(n) \simeq 2\sqrt{n} + \boxed{?}$

$\xi \cdot n^{1/6}$

random

$\boxed{\text{RSK}}$

1. Bijection $\boxed{S(n) \longleftrightarrow \,?}$

2. LIS matching

<u>Prop</u> (w/o proof)
LIS(σ) algorithm in $O(n \cdot \log^{\gamma} n)$ time

$\sigma = 3\ 2\ 5\ 6\ 9\ 4\ 7\ 1\ 8 \in S_9$

3	2	2 5	2 5 6	2 ~~5~~ 4 6 9

2 4 6 ~~9~~ 7	~~2~~ 4 6 7	1 4 6 7	1 4 6 7 8

bumping process

Claim. this has correct LIS length.

RSK bijection (Robinson – Schensted – Knuth)

RS

$$\boxed{S_n} \longleftrightarrow \boxed{\text{pairs of Std. Young Tableaux of same shape with } n \text{ boxes}}$$

$\Downarrow$

$$n! = \sum_{|\lambda|=n} (\dim \lambda)^2$$

$\Rightarrow$ Plancherel random λ = Shape (RS – Image) of uniform $\sigma \in S_n$

Def. (RS)

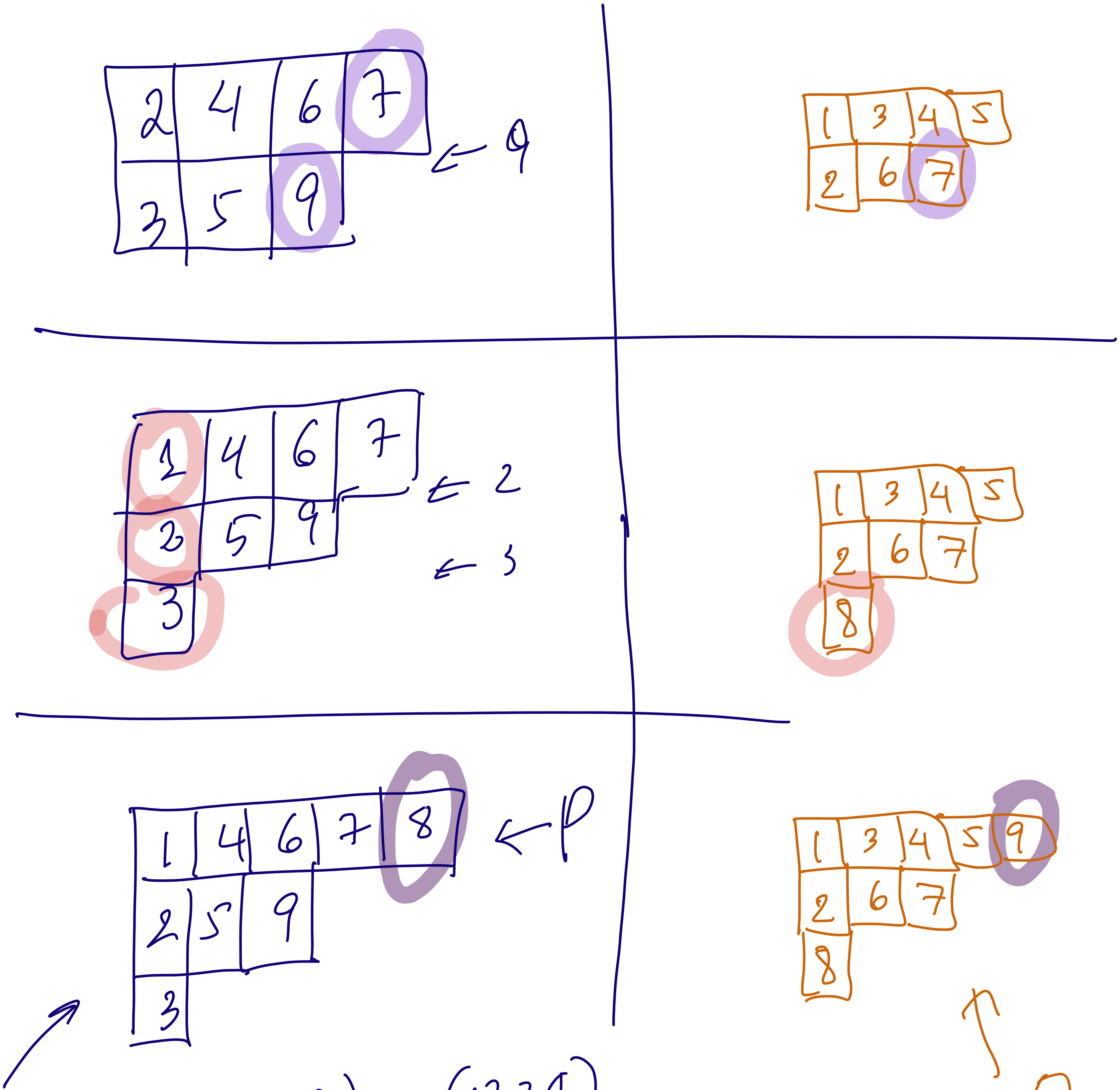
$$\sigma \longmapsto (P, Q)$$

3 2 5 6 9 4 7 1 8

recording tableau

insertion tableau

P	Q
$\begin{array}{\|c\|}\hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$
$\begin{array}{\|c\|}\hline 2 \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|c\|}\hline 2 & 5 & 6 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|c\|c\|}\hline 2 & 5 & 6 & 9 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|c\|c\|}\hline 2 & 4 & 6 & 9 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \leftarrow 5$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2}\end{array}$

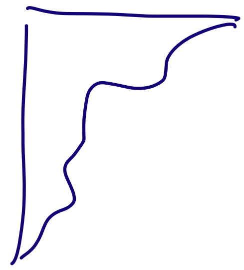


$\lambda = \text{shape}(P) = (5\,3\,1)$

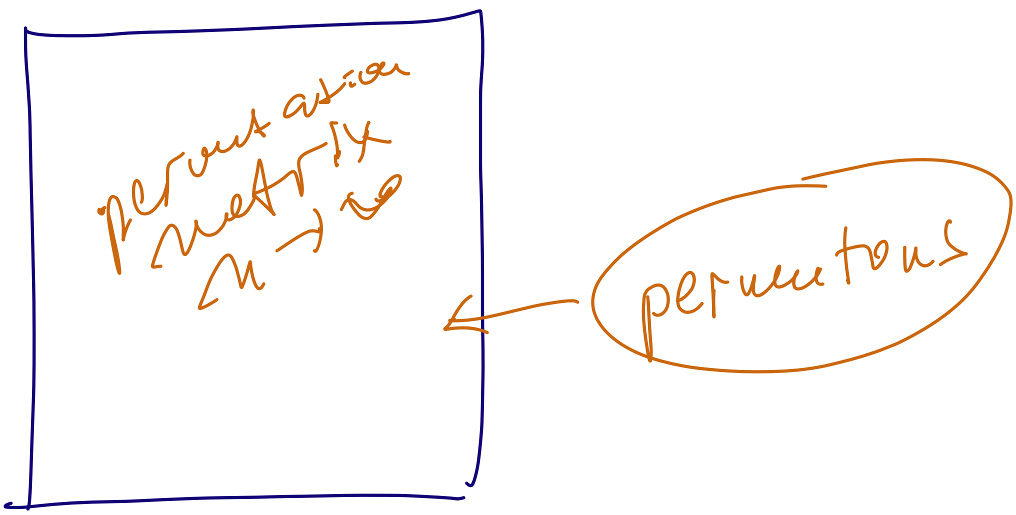
<u>Why bijection?</u>

Can invert each step

Problem

n – large, $\lambda =$ 

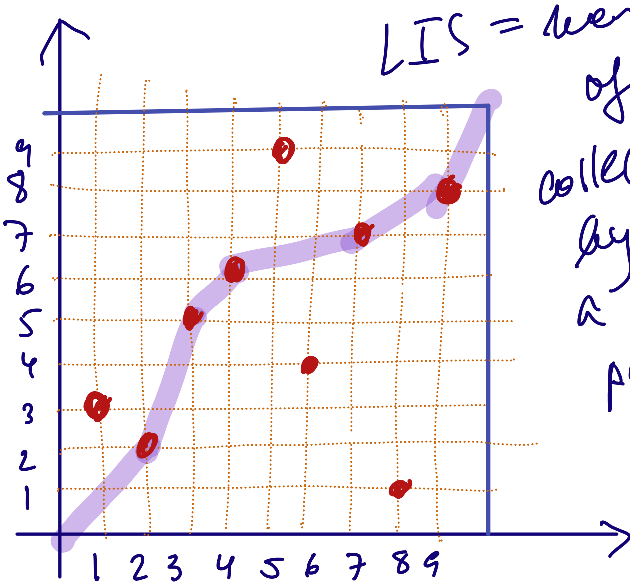
how does uniform $\sigma \in S_n$ look like, given $\mathrm{shape}(RS(\sigma)) = \lambda$



σ

$$\sigma \xleftrightarrow{\text{bij}} (P, Q)$$

$\boxed{\mathrm{LIS}(\sigma)} = \text{function}(P, Q)$ (only depends on λ) $= \boxed{\lambda_1}$



LIS = max # of pts collected by a path

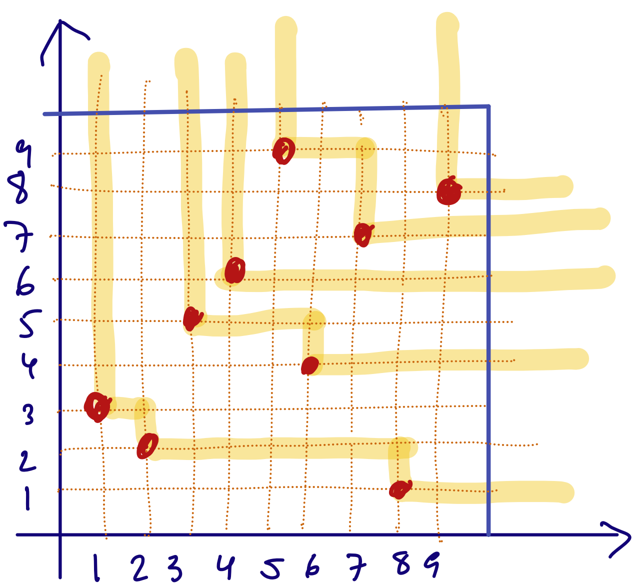
$LIS(\sigma) = \lambda_1$

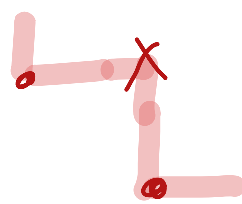
(Greene's theorem)

shadow lines (Viennot)

$\sigma = 3\ 2\ 5\ 6\ 9\ 4\ 7\ 1\ 8$

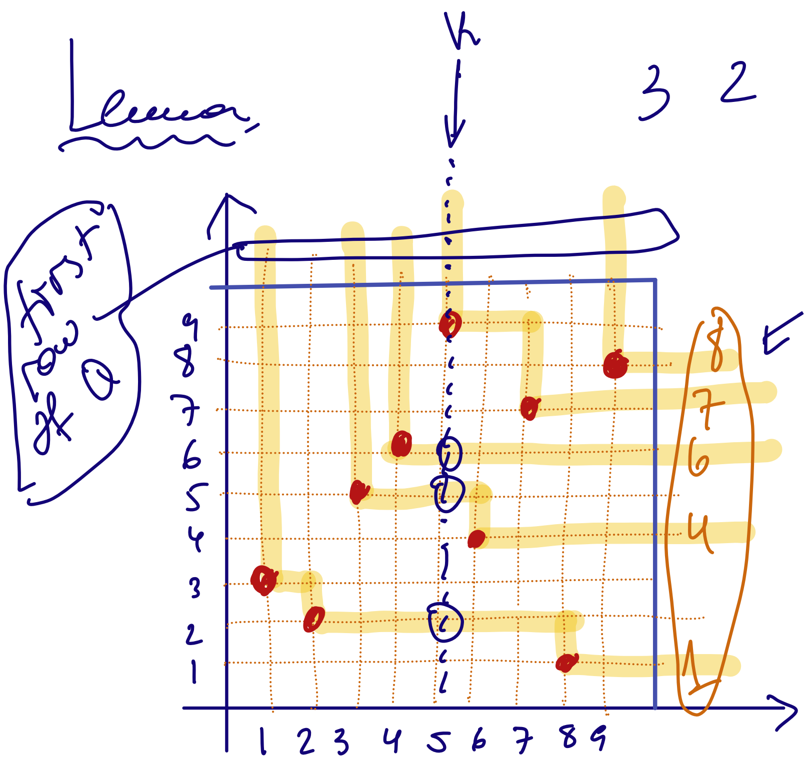
Shadow lines (Sagan 2000)





Lemma

3 2 5 6 9 4 7 1 8



$\forall k$, intersection

(2 5 6 9)

Then this is the first row of P after inserting $b_1, b_2, \dots b_k$

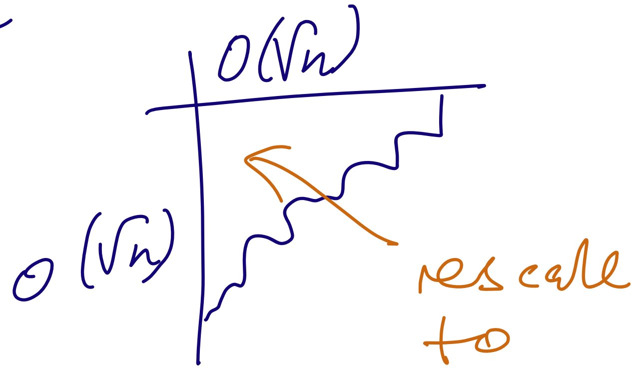
Proof.. Induction on k., easy exercise

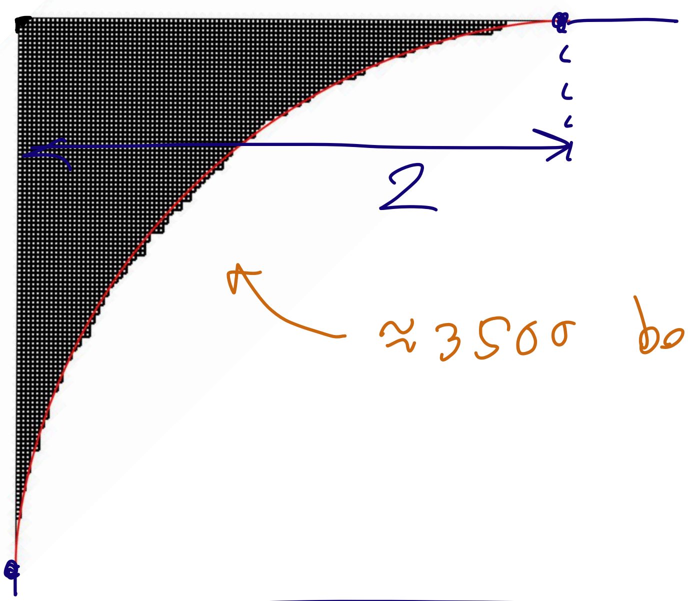
$\boxed{\text{evolution of intersection}}$ equivalent to the first bumping lemma (w/o proof)

$\square$

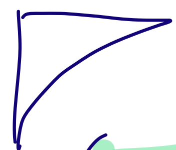
Limit shape problem

$n \to \infty$, $\boxed{M_n(\lambda)}$





$LIS(n) \sim \boxed{2\sqrt{n}}$

Heuristics: the shape  should have $\max_{\lambda \in \mathbb{Y}_n} \left(\boxed{\dim \lambda}\right)$

known formula

Recall

Plancherel measure on partitions

$$\boxed{M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}}$$

RSK: $\lambda_1 \overset{d}{=}$ length of LIS of unif. $\sigma \in S_n$

$$n! = \sum_{\lambda} (\dim \lambda)^2$$

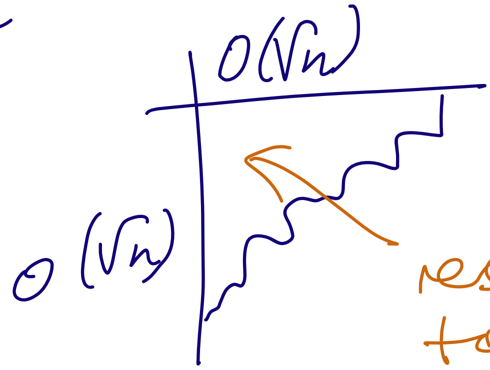
RSK $\prod_{i,j=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{\text{all } \lambda} s_\lambda(x_1, \ldots, x_N)\, s_\lambda(y_1, \ldots, y_N)$

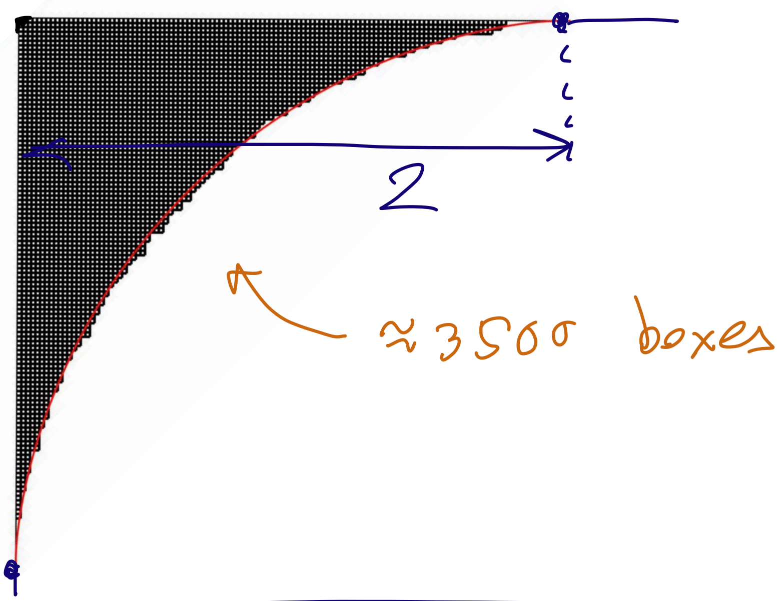
Cauchy identity

$$\boxed{\frac{\lambda_1}{\sqrt{n}} \to 2, \quad n \to \infty}$$

Limit shape problem

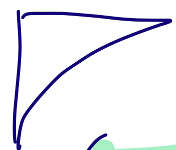
$n \to \infty$, $\boxed{M_n(\lambda)}$





$LIS(n) \sim \boxed{2\sqrt{n}}$

Heuristics: the shape



should have $\max_{\lambda \in \mathbb{Y}_n} (\boxed{\dim \lambda})$

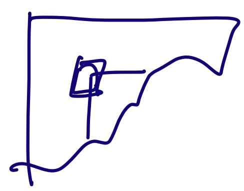
known formula

we will look for $\dim\lambda \to \max$

$$\dim\lambda = \frac{n!}{\prod_{\square} h(\square)}$$

so $\prod_{\square} h(\square) \longrightarrow \min$

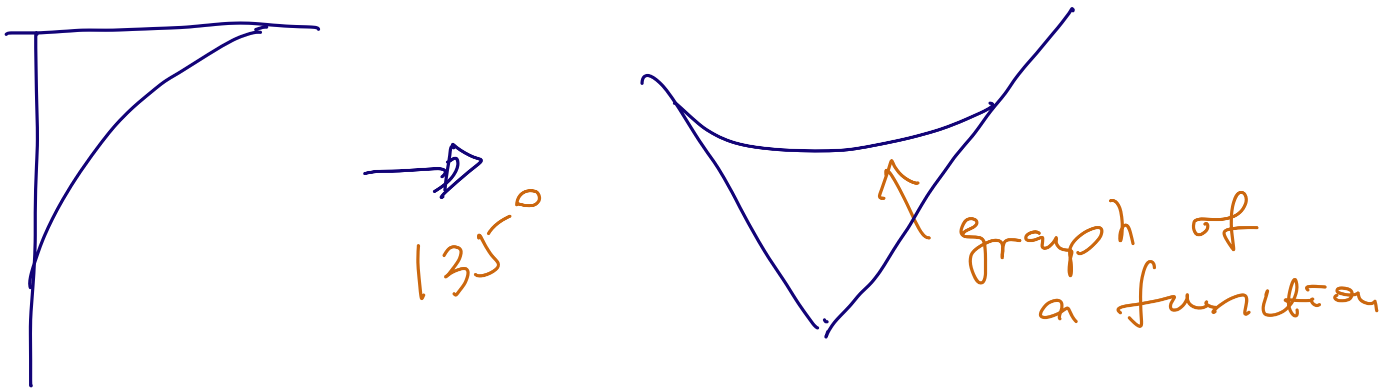
$$\Longleftrightarrow \boxed{\sum_{\square\in\lambda} \log h(\square) \to \min}$$



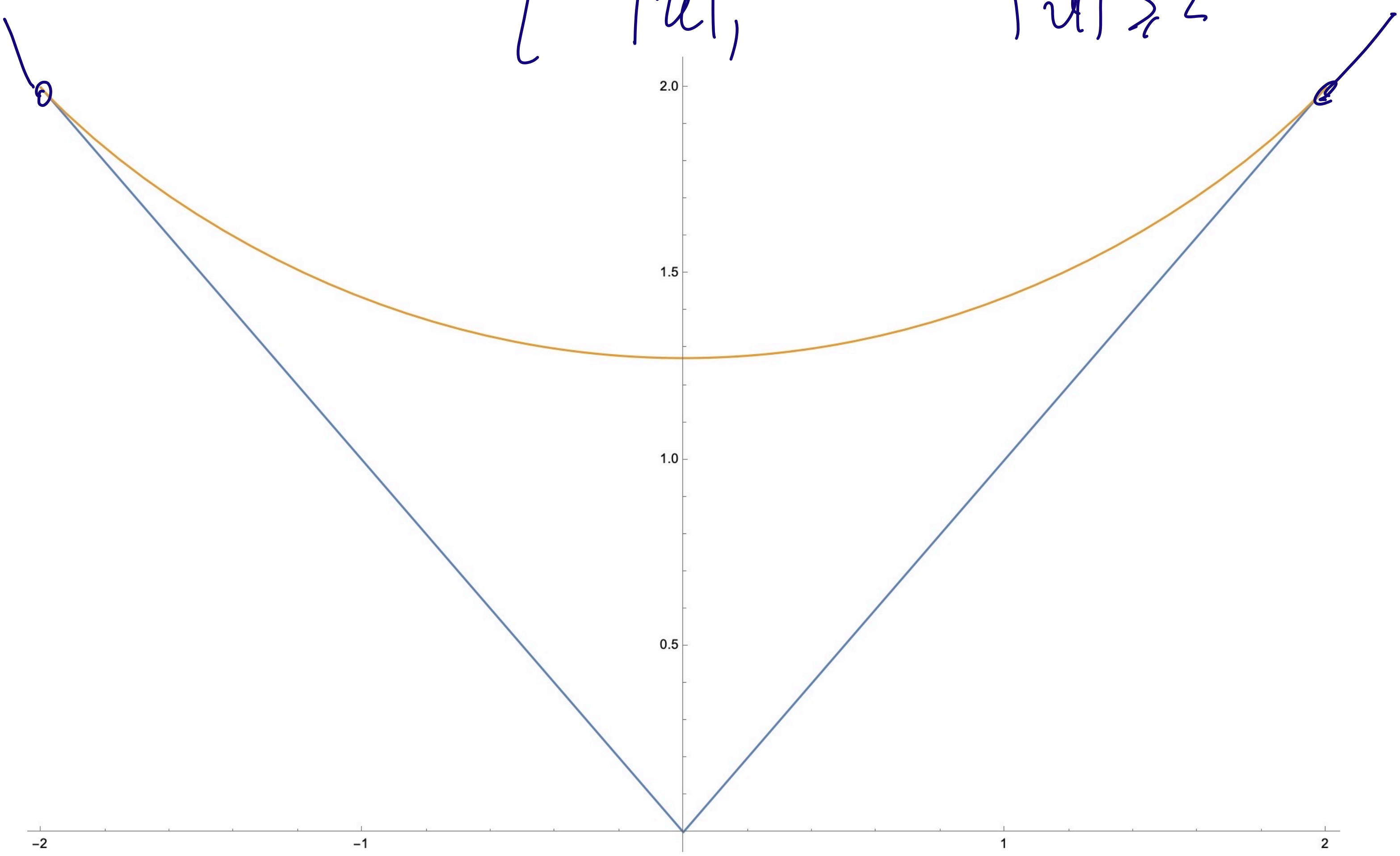
idea, $\sum_{\square\in\lambda} \approx \iint_{\text{inside }\lambda} dx\,dy$

12.3 hook functional & minimizer

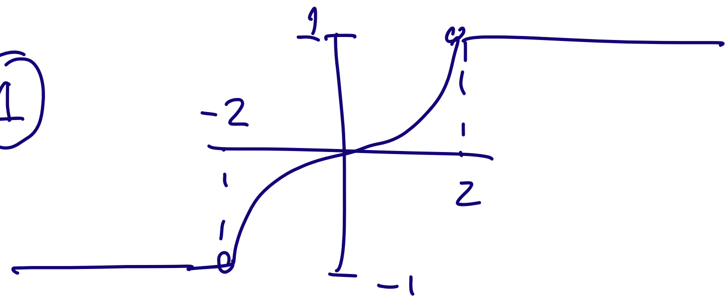
→ VKLS shape



Def. $\Omega(u) = \begin{cases} \frac{2}{\pi}\left(u \arcsin \frac{u}{2} + \sqrt{4-u^2}\right), & |u| \le 2 \\ |u|, & |u| \geq 2 \end{cases}$



Notes. $\Omega'(u) = \frac{2}{\pi} \arcsin\left(\frac{u}{2}\right)$ slope

① 

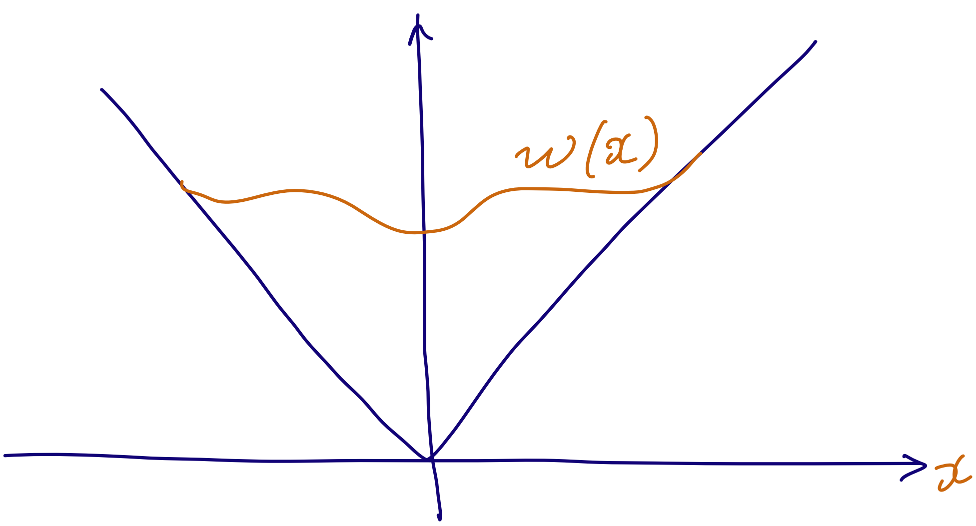
② $\int_{-2}^{2} \big(\Omega(u) - |u|\big)\, du = \boxed{2}$

(exercise, area)

Def. $A(w) := \frac{1}{2} \iint_{v<u} d\big(u - w(u)\big)\, d\big(v + w(v)\big)$

„continual young diagram"

$= 1$ for Ω

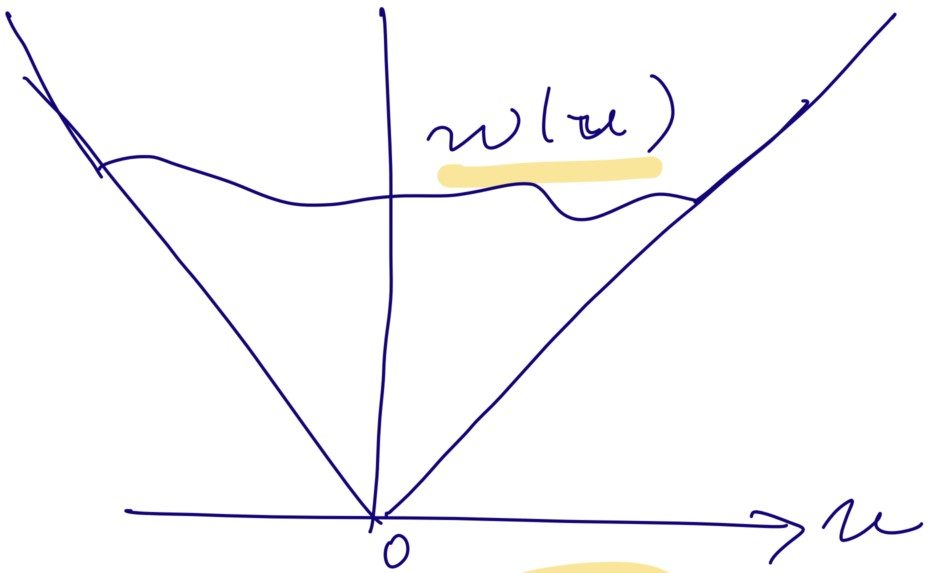


Exercise (HW)

$A(w)$ is the same as

$$\frac{1}{2}\int \left(w(u) - |u|\right) du$$

Def. $w(u)$ — continual Y.D.

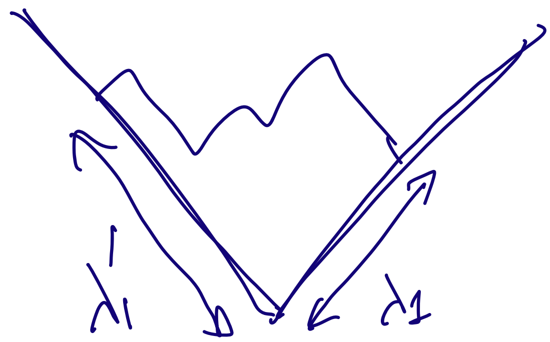


Def. $w(u)$ — c.Y.d. if

→ $|w(u_1) - w(u_2)| \leq |u_1 - u_2|$ $\forall u_1, u_2$

→ $|w(u)| = |u|$ for large u

Note: Y.d. $\lambda \to w_\lambda(x)$



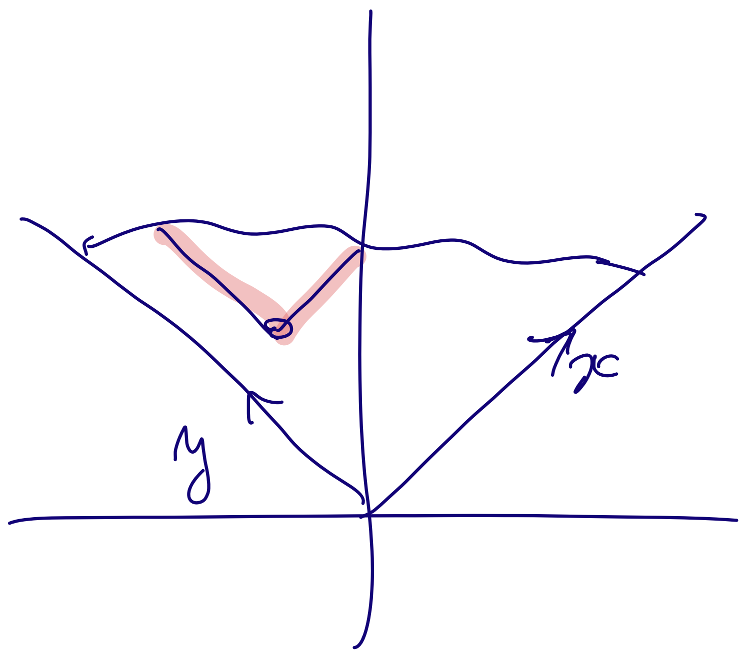
$$w_\lambda'(x) = \pm 1$$

and cont. Y-d. are uniform limits of rescaled w_λ s.t. the area is 1.

Goal,

$$\sum_{\square \in \lambda} \log h(\square) =$$

$$= \iint_{\substack{\text{below} \\ \omega(u)}} \text{length}\left(\bigvee_{x,y} \right) dx\, dy$$

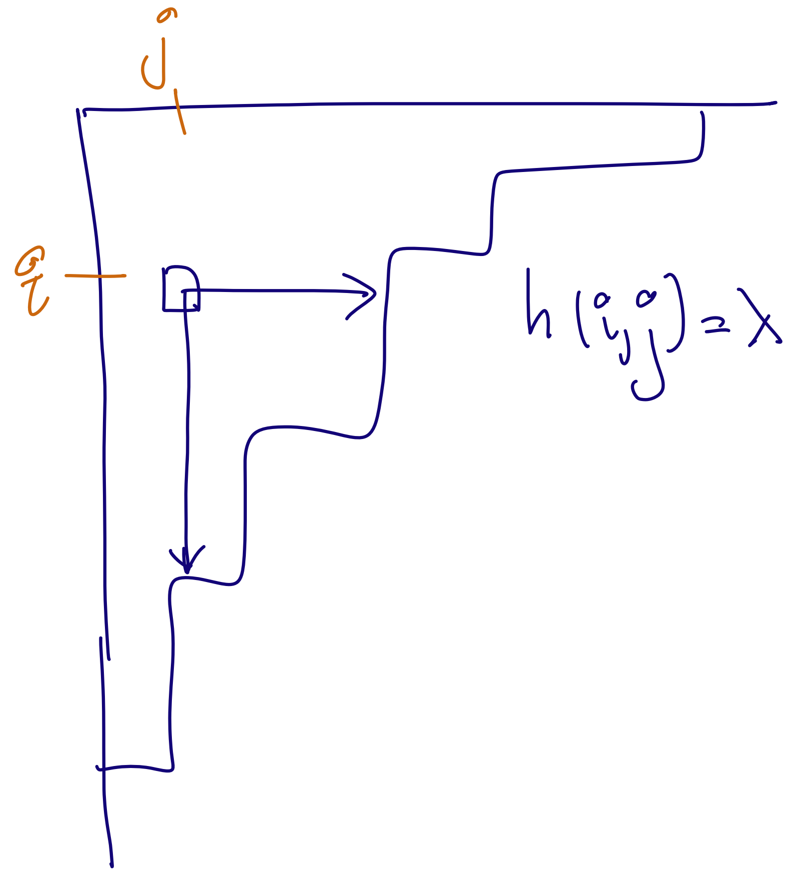


Hook integral

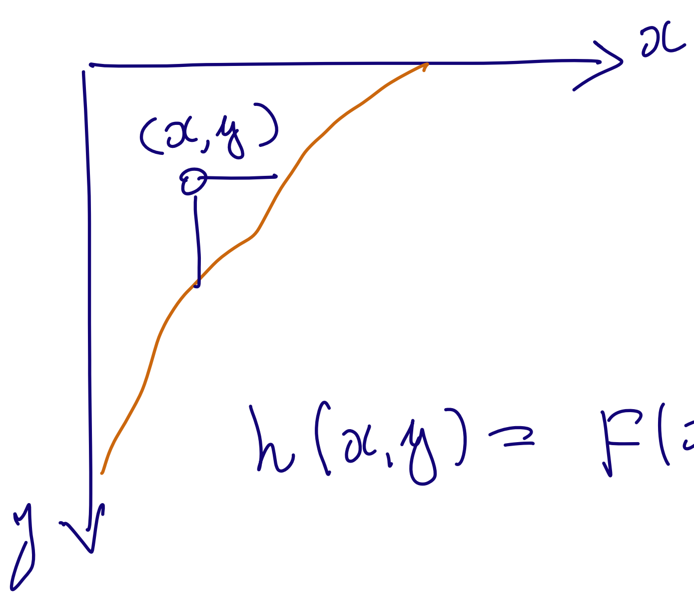
minimize

$$\lambda \longmapsto \prod_{\square \in \lambda} h(\square)$$

(but in a continuous setting)



$$h(i,j) = \lambda_i - j + \lambda'_j - i + 1$$

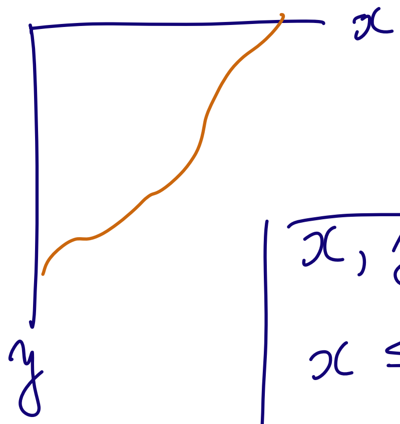


$$y = F(x)$$

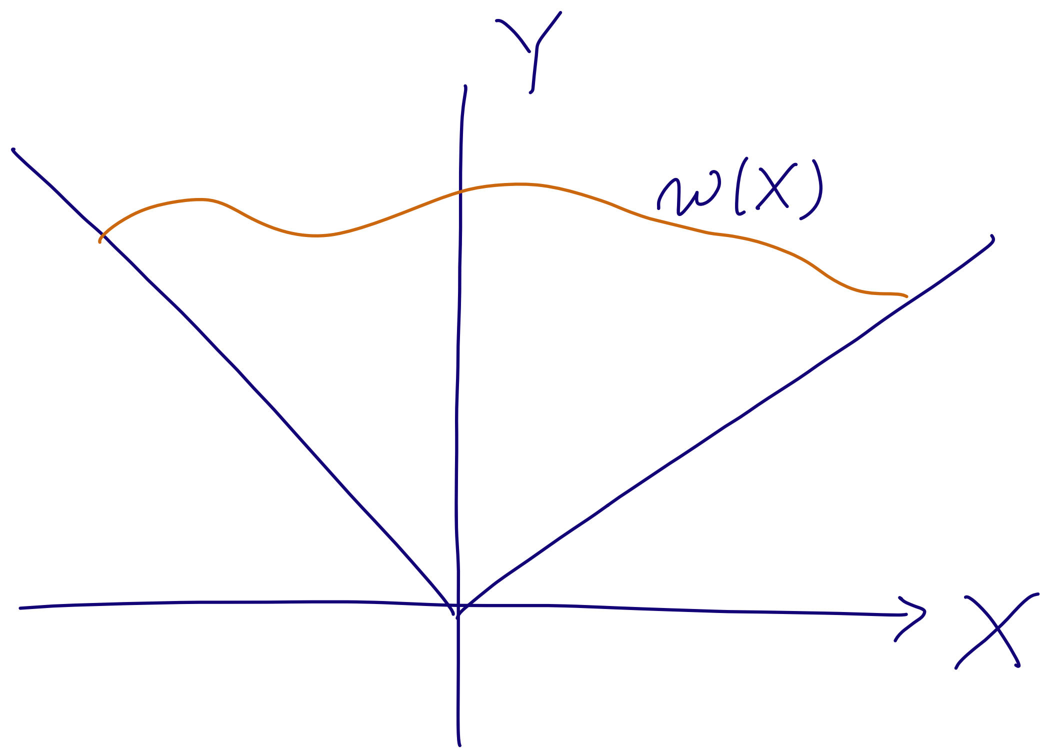
$$h(x,y) = F(x) + F^{-1}(y) - x - y$$

$$\prod h(\square) \longrightarrow \sum \log h(\square)$$

135° rotation



$$\boxed{\begin{array}{l} x, y \geq 0 \\ x \leq F^{-1}(y) \\ y \leq F(x) \end{array}}$$

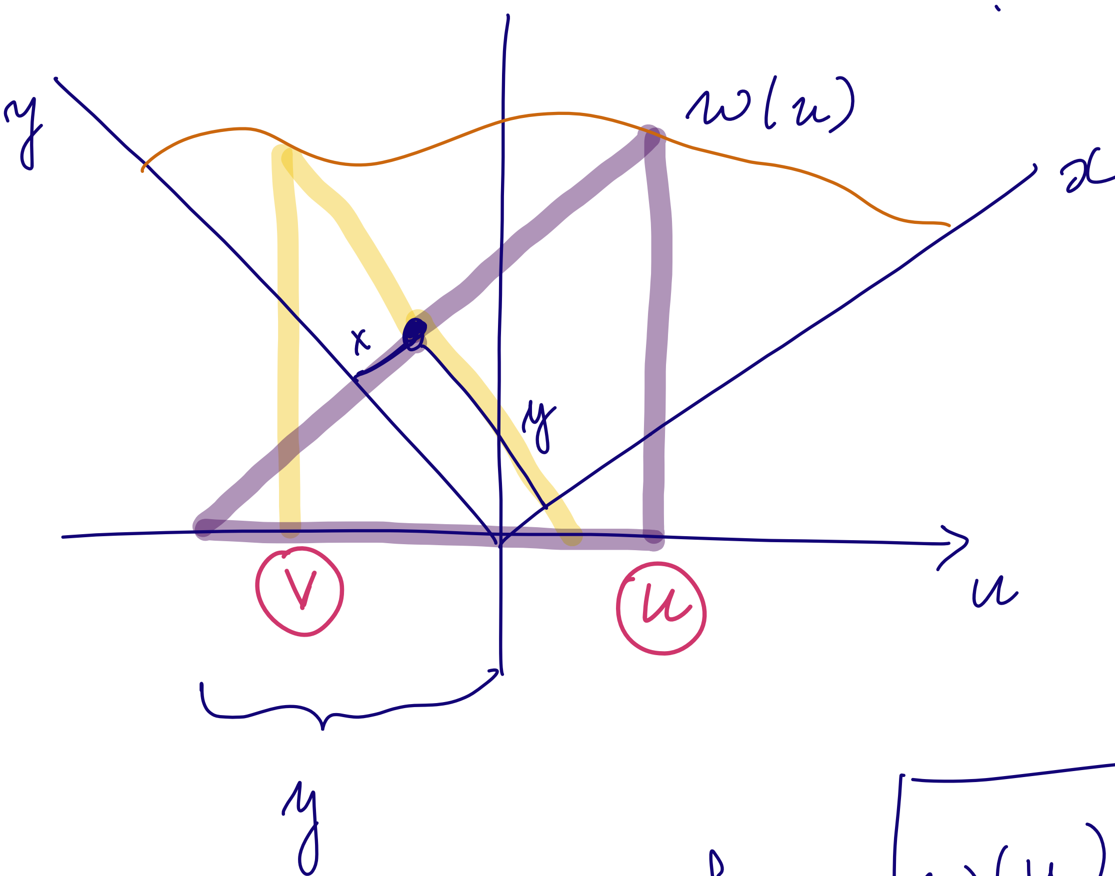


$$\boxed{|X| \leq Y \leq w(X)}$$

Coordinates

$u > v$

$h(x, y) =$ ✓

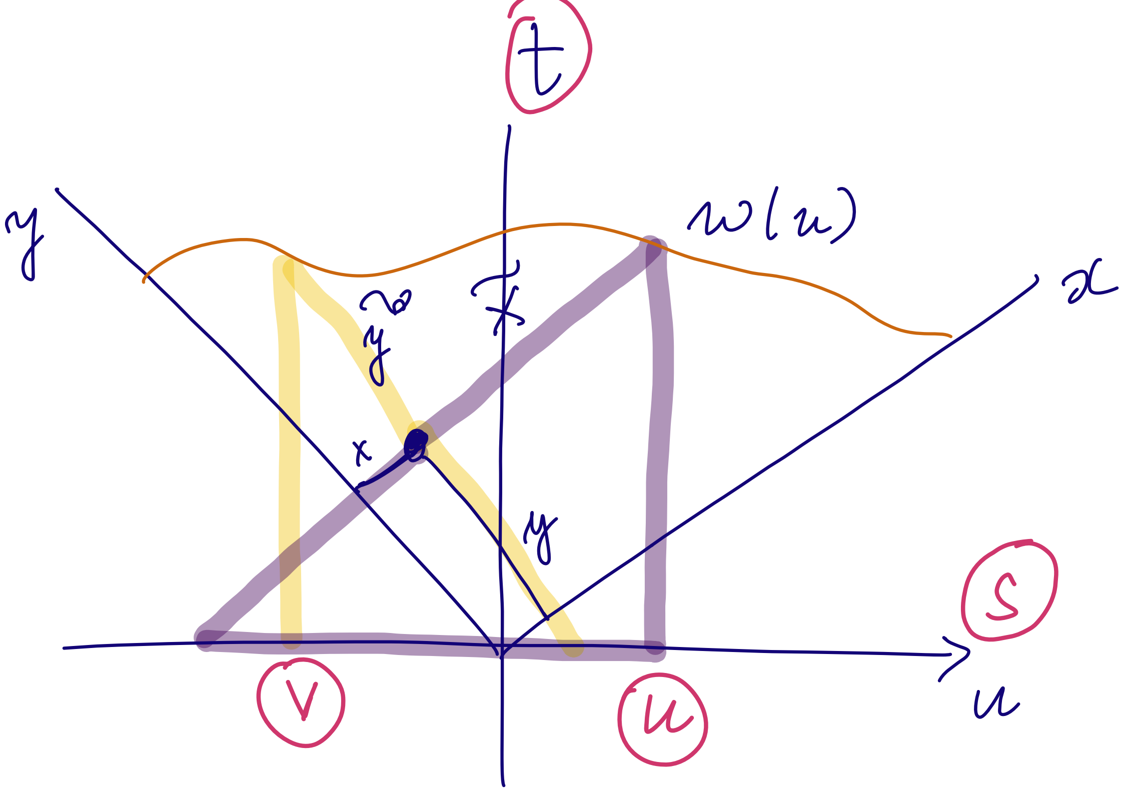


change of coord

$$\begin{cases} y = \boxed{w(u) - u} \\ x = \boxed{w(v) + v} \end{cases}$$ ✓ ✓

$\frac{\partial y}{\partial u} < 0$

$$\boxed{dx\,dy = d(u - w(u)) \cdot d(v + w(v))}$$



$$t = s + \big(w(u) - u\big) = s + y$$

$$t = -s + \big(v + w(v)\big) = -s + x$$

$$s + y = -s + x, \qquad s = \frac{x-y}{2}$$

$$t = \frac{x+y}{2}$$

$$h(x,y) = -2t + w(u) + w(v)$$

$$= w(u) + w(v) - x - y = \boxed{u - v}$$

$\Downarrow$

<u>To minimize</u> :

$$\boxed{\theta(\omega) = 1 + \frac{1}{2} \iint_{v<u} \overbrace{\log(u-v)}^{\text{log of hook length}} \, d(\overbrace{u-\omega(u)}^{-y}) \, d(\overbrace{v+\omega(v)}^{x})}$$

under the <u>area</u> constraint

$$\boxed{A(\omega) = \frac{1}{2} \iint_{v<u} d(v+\omega(v)) \, d(u-\omega(u)) = 1}$$

[Logan – Shepp 1977]

[Vershik – Kerov 1977]

$\Omega(u)$ – VKLS limit shape

12.4 VKLS shape as unique minimizer

Let $\boxed{f(u) = \omega(u) - \Omega(u)}$

$$\Omega(u) = \frac{2}{\pi}\left(u \arcsin\frac{u}{2} + \sqrt{4-u^2}\right)$$

$|u| \leq 2$

Then :

Prop.

$$\theta(\omega) = -\frac{1}{2}\iint_{u,v\ \text{any}} \log|u-v|\, f'(u) f'(v)\, du\, dv$$

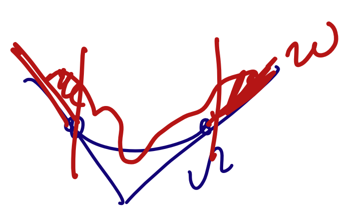
≥ 1

$\parallel$

$\Omega + f$

$$+ 2\int_{|u|>2} f(u) \operatorname{arccosh}\left|\frac{u}{2}\right| du$$

$(\Rightarrow \theta(\Omega) = 0)$



Proof

$$\operatorname{arccosh} x = \log\left(x + \sqrt{x^2-1}\right)$$

$$e^x + e^{-x} = 2t$$

$$\theta(\Omega + f) =$$

$$= 1 + \frac{1}{2} \iint_{v<u} \log(u-v)\, d\left(u - f(u) - \Omega(u)\right) \cdot d\left(v + \Omega(v) + f(v)\right)$$

$$= 1 - \frac{1}{2} \iint_{v<u} \log|u-v|\, f'(u) f'(v)\, du\, dv$$

$$+ \text{rest}$$

$$\underline{\text{Rest}} = 1 + \frac{1}{2} \iint_{v<u} \log(u-v) \cdot$$

$$\cdot \Big[1 - \Omega'(u) - f'(u) + \Omega'(v) + f'(v)$$

$$- \Omega'(u)\,\Omega'(v) +$$

$$- \Omega'(u) f'(v) - \Omega'(v) f'(u) \Big]$$

„Calculus" gives -1

Then, $\frac{1}{2} \iint_{v<u} \log(u-v) \Big\{ - f'(u) + f'(v)$

$$- \Omega'(u) f'(v)$$

$$- \Omega'(v) f'(u) \Big\}$$

$\|$ more „Calc."

$$2 \int_{|u|>2} f(u) \operatorname{arccosh}\left|\frac{u}{2}\right| du$$

$\square$

Sobolev Norm

$$\| f \|_\theta^2 = \iint \left(\frac{f(s) - f(t)}{s - t} \right)^2 ds\, dt \quad = \quad -c \iint \log|u-v|\, f'(u)\, f'(v)\, du\, dv$$

$f \longmapsto \hat{f}$ Fourier

$$\| f \|_\theta^2 = \frac{1}{2} \int_{\mathbb{R}} |\xi|\, |\hat{f}(\xi)|^2\, d\xi$$

Hilbert transform

$$(Hf)(s) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)\, dt}{s - t}$$

$$= -\frac{1}{\pi} \int_{\mathbb{R}} f'(t) \log|s-t|\, dt$$

$$\widehat{Hf}(\xi) = -i \cdot \operatorname{sign} \xi \cdot \hat{f}(\xi)$$

$$\int \Big(\underbrace{\int \log|u-v|\, f'(u)\, du}_{Hf} \Big) \cdot f'(v)\, dv$$

$$\shortparallel$$

$$\left\langle \widehat{Hf}, \widehat{f'} \right\rangle$$

$$\shortparallel$$

$$\left\langle i \operatorname{sign} \xi \cdot \hat{f}(\xi), i \xi \hat{f}(\xi) \right\rangle$$

$$= \boxed{\int_{\mathbb{R}} |\xi| \cdot \left|\hat{f}(\xi)\right|^2 d\xi}$$

Norm
(Sobolev)

we only need

$$-\iint \log f' f' > 0 \quad \text{for } f \neq 0$$

$$\Downarrow$$

$$\Theta(w = f + \Omega) = \underbrace{\|f\|_\Theta^2 + 2\int_{|u|>2} f(u) \operatorname{arctanh}\left|\frac{u}{2}\right| du}_{\substack{>0 \text{ if } f \neq 0 \\ =0 \text{ if } f = 0}}$$

$\Rightarrow$ $w = \Omega$ is the unique minimizer

$\Rightarrow$

$\max(\dim \lambda) \simeq \lambda \simeq$ VKLS shape

let $\omega_\mu = \Omega + \underbrace{\varepsilon f}_{\text{correction}}$

μ large

$\boxed{\theta(\omega_\mu)} \simeq \varepsilon$

$$\frac{(\dim \mu)^2}{n!} \ll \text{maximal value}$$

<u>Want</u> : $P(\mu \text{ has shape } \simeq \Omega + \varepsilon f) < e^{-n^\alpha \varepsilon}$

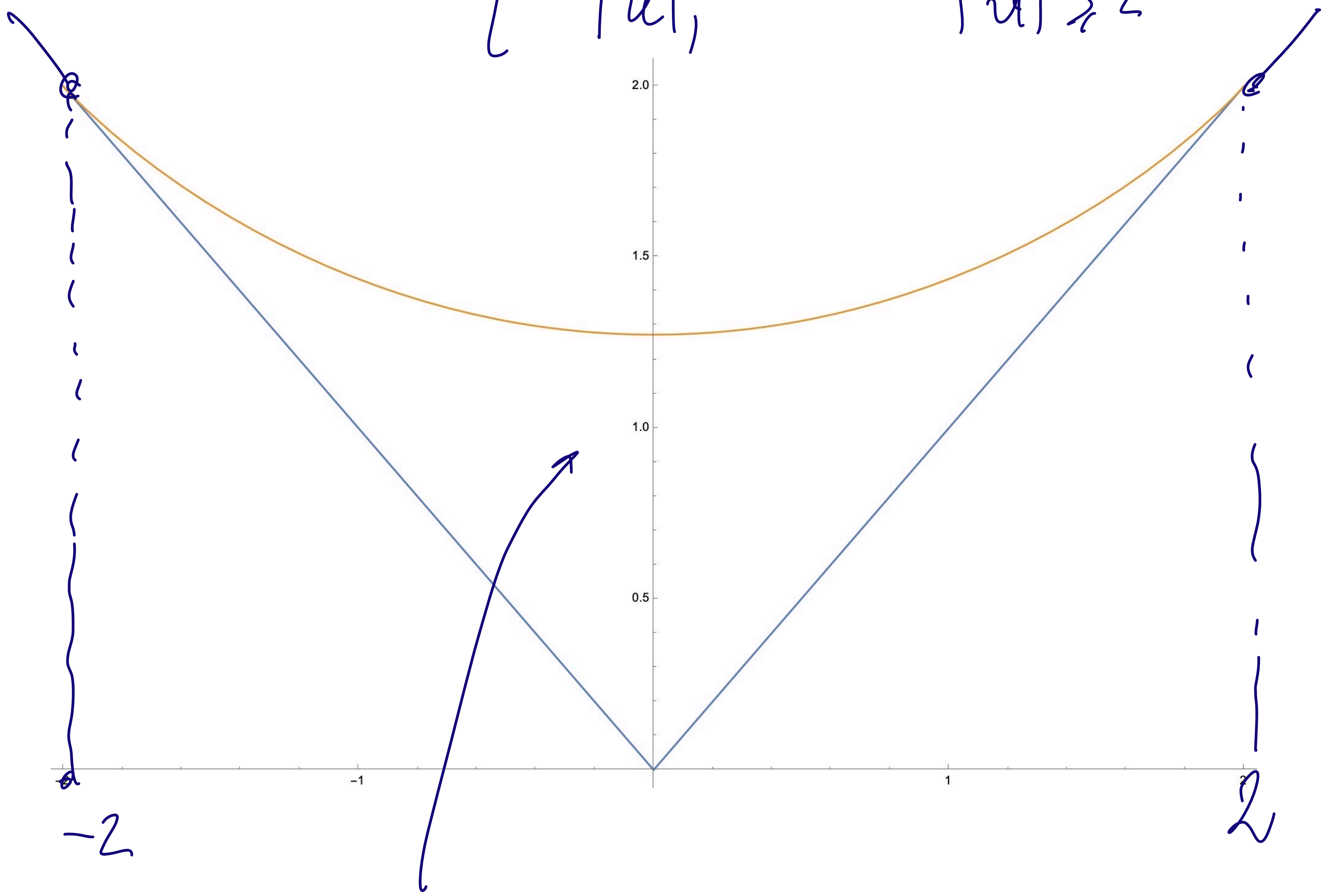
$\Rightarrow$ convergence

$\boxed{\text{HW deadlines}}$

$$M_n(\lambda) = \frac{(d(\lambda))^2}{n!}$$

Plancherel measure

Def. $$\Omega(u) = \begin{cases} \frac{2}{\pi}\left(u \arcsin \frac{u}{2} + \sqrt{4-u^2}\right), & |u| \le 2 \\ |u|, & |u| \ge 2 \end{cases}$$



VKLS shape

Proved: Ω — unique minimizer of the hook functional

$$\theta(w) = 1 + \frac{1}{2} \iint_{v<u} \log(u-v)\, d(u-w(u))\, d(v+w(v))$$

under the area constraint

$$A(w) = \frac{1}{2} \iint_{v<u} d(v+w(v))\, d(u-w(u)) = 1$$

So, Plancherel probability is maximized on Ω

[VK 1977,
VK 1985]

Also showed

$$\theta(w) = \| w - \Omega \|_S^2 + $$

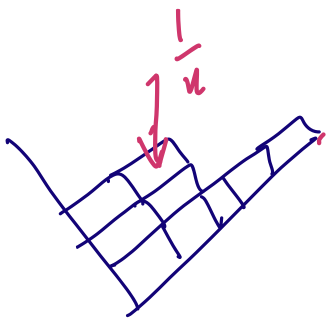
$$+ 2 \int_{|u|>2} f(u) \operatorname{arccosh} \left| \frac{u}{2} \right| du$$

Fact: $\| f \|_S^2 = \iint \frac{f(s) - f(t)}{s-t} \, ds \, dt$

$$\geq \text{Const.} \| f \|_{\text{unif}}$$

12.5. Limit shape

Next, we show:



$$\lambda^{(n)} \sim \mathrm{Plancherel}(n)$$

$$\downarrow$$

$$\omega_{\lambda^{(n)}}(u), \qquad A(\omega_{\lambda^{(n)}}) = 1$$

has $\boxed{\omega_{\lambda^{(n)}} \longrightarrow \Omega}$, $n \to \infty$

in probab, uniform

Precisely:

Plancherel measure

$$M_n\left(\lambda : \sup_{u\in\mathbb{R}} \left|\omega_\lambda(u) - \Omega(u)\right| > \varepsilon\, \boxed{n^{-1/6}}\right) \longrightarrow 0,$$

$n \to \infty$

(conv. in probab.)

Prof.

$$\sum_{\lambda} (\dim \lambda)^2 = n!$$

$\left(\frac{n}{e}\right)^n \cdot \mathrm{Poly}(n) \sim e^{n \log n}$

$\# \lambda,\ |\lambda| = n$

is $\frac{1}{4n\sqrt{3}} e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}} \sim e^{c\sqrt{n}}$

$\Downarrow$

max dim is $\sim \boxed{e^{\frac{1}{2} n \log n - c_1 n - c_2 \sqrt{n} + \dots}}$

$$-\log\left[M_n(\lambda) \big/ \sqrt{n}\,! \right]$$

$$= 2n \underbrace{\theta(w_\lambda)}_{\text{hook functional}} + \overbrace{\sqrt{n}\,\eta(w_\lambda)}^{\text{some functional}} - \frac{1}{2}\log n + O(1)$$

$$\|w_\lambda - \Omega\|_C > \varepsilon$$

$$\Downarrow$$

$$\theta(w_\lambda) > \varepsilon \cdot \text{const} \quad (\text{recall } \theta(\Omega) = 0)$$

$$M_n\left(\|w_\lambda - \Omega\|_c > \varepsilon \right) \longrightarrow 0$$

because if

$$\|w_\lambda - \Omega\|_c > \varepsilon \implies \theta(w_\lambda) > \varepsilon_1$$

then $\quad (\dim \lambda)^2 \leq e^{n \log n - 2\varepsilon_1 n}$

& $\quad \mathbb{P}(\text{such } \lambda) \leq e^{-2\varepsilon_1 n} \longrightarrow 0$

$\square$

& Also can prove that

$$w_{\lambda^{(n)}} \longrightarrow \Omega \quad \text{almost surely}$$

Corollary (LIS)

$\sigma \in S_n$ uniformly random

Then $\dfrac{LIS(\sigma)}{\sqrt{n}} \xrightarrow{P,\, a.s.} 2, \quad n \to \infty$

(1977)

1999 $LIS(\sigma) = 2\sqrt{n} + \xi\, n^{-1/6} + \dots$

Baik-Deift-Johansson

$\boxed{\text{random Tracy-Widom}}$

Matrix / $\mathbb{C}$, Hermitian, $N \times N$

$N \to \infty$

iid random Gaussian

$$\lambda_{max} \sim 2\sqrt{N} + N^{-1/6} \cdot \xi_{TW}$$

More precise statement about $\dim \lambda$:

[VK 1985]

$$\boxed{\max_{\lambda} \dim \lambda \asymp \sqrt{n!}\, e^{-\frac{c}{2}\sqrt{n}}}$$

up to constants, $\exists\, c_1, c_2, \ldots$

$$\frac{1}{\sqrt{n}} \log\left(\dim \lambda / \sqrt{n!}\right) \longrightarrow ?$$

<u>Conjecture</u> [VK 1985], open

if $\lambda^{(n)} \sim$ Plancherel (n), then

$$\frac{2}{\sqrt{n}} \log \frac{\dim \lambda^{(n)}}{\sqrt{n!}} \to c \quad \text{exists}, \quad n \to \infty$$

(c — should be between

0.3 and 2.5)

$$\sum_{\Lambda} \dim \Lambda = t_N, \text{ где } t_N \sim \mathrm{const} \cdot \left(\frac{N}{e}\right)^{N/2} \cdot e^{\sqrt{N}}$$

$$t_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{k!\,2^k}$$

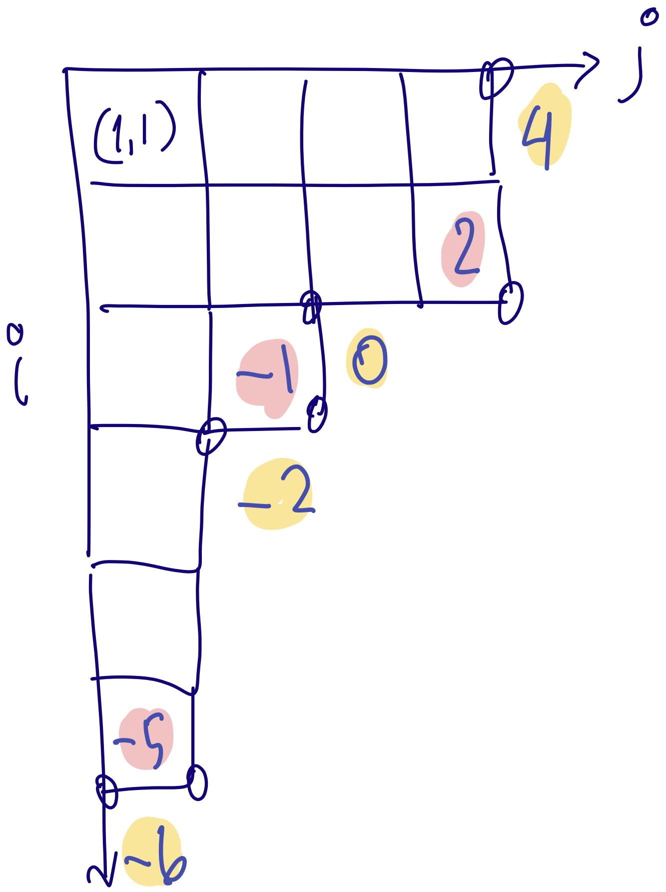
$$\sum_{\lambda} (\dim \lambda)^{\beta} = F_{n,\beta}$$

$\beta = 0$, $F_n \simeq e^{c\sqrt{n}}$

$\beta = 2$ $\quad F_{n,2} = n!$

13. Hydrodynamics of Plancherel growth

13.1 Kerov interlacing coordinates



$x_k = j_k - i_k$

$y_k = j_k - i_k$

$c(\square) = j - i$

Lemma. $x_1 > y_1 > x_2 > y_2 > \dots > y_{d-1} > x_d$ $\square$

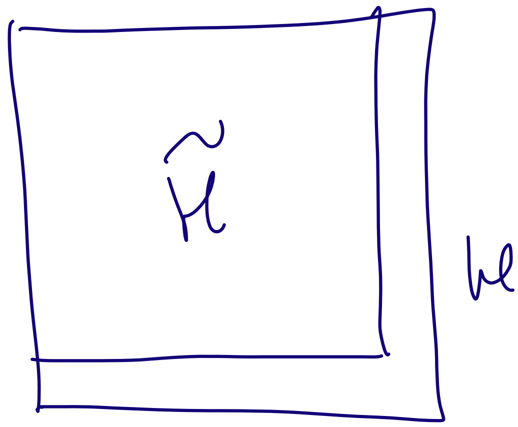
(Connection to [random] matrices & orthogonal polynomials)

Fact ① H — $N \times N$ Hermitian matrix

$$H^T = H^*$$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ — e.v.

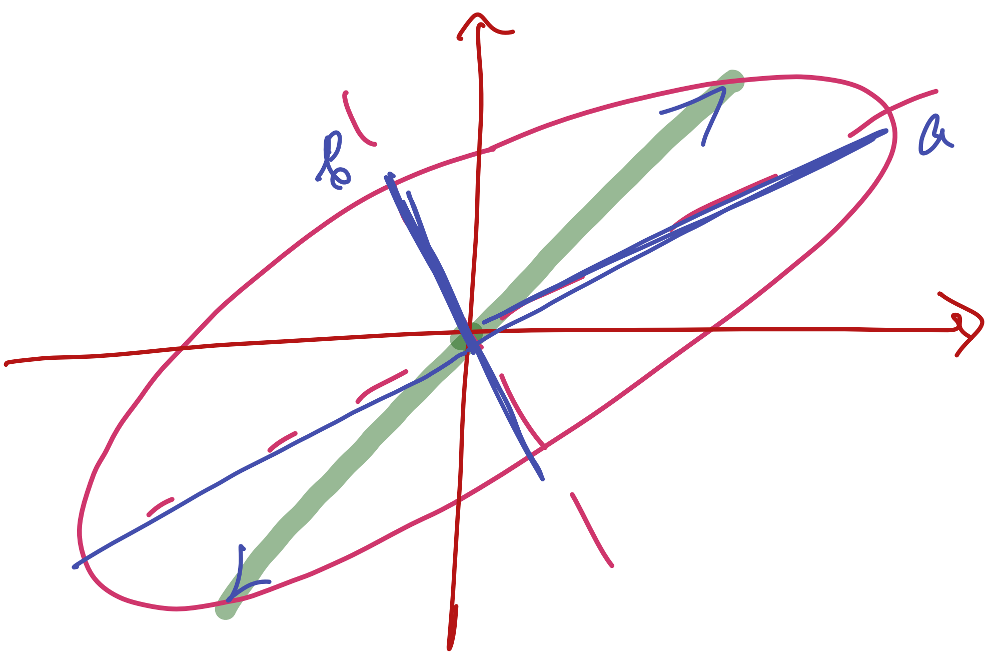
$\lambda_i \in \mathbb{R}$



$\tilde{H} \longrightarrow \mu_1 \geq \mu_2 \geq \dots \geq \mu_{N-1}$

(exercise)

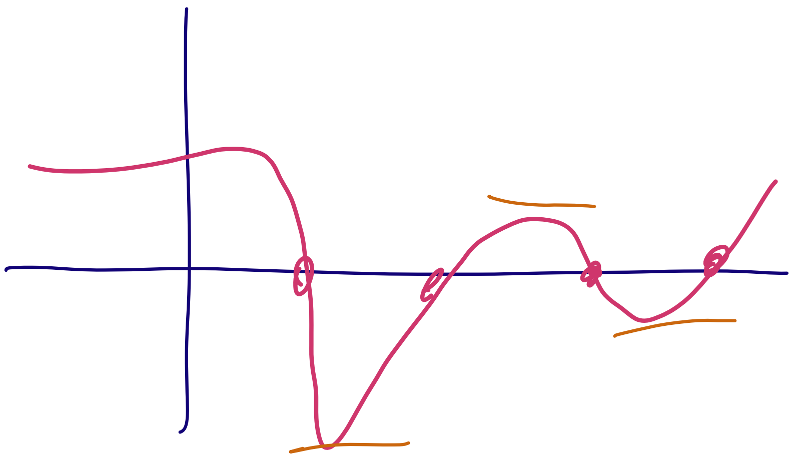
Then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{N-1} \geq \lambda_N$



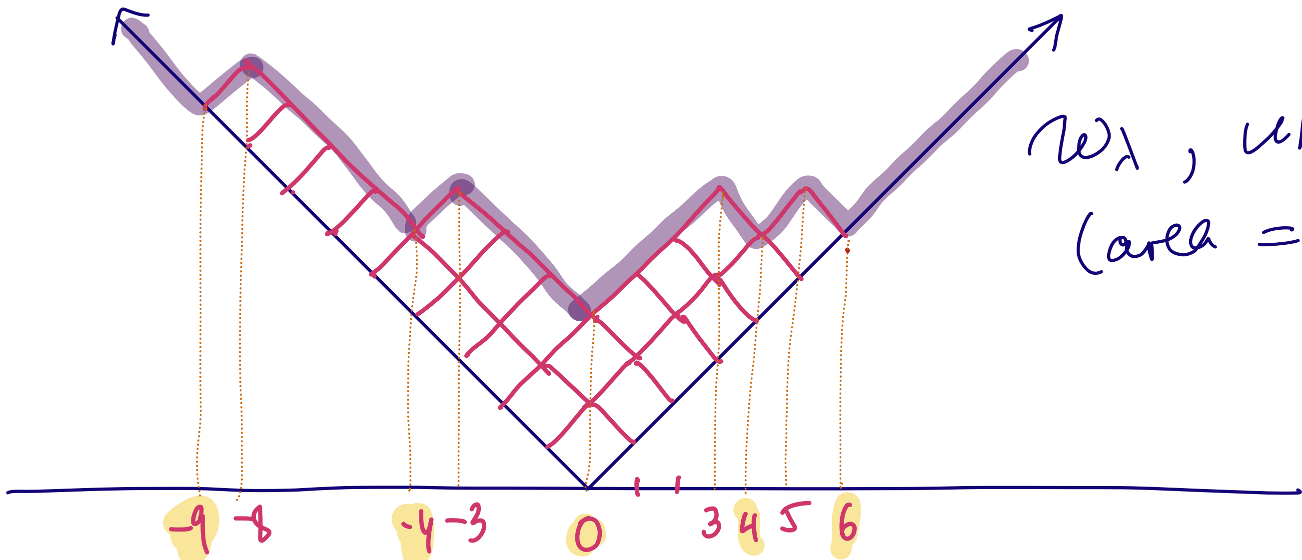
(2) $p(z) = \prod_1^N (z - \lambda_i)$ (real roots)

$p'(z) = \prod_1^{N-1} (z - \mu_i)$

Then μ, λ interlace



Facts about $\vec{x} / \vec{y}$.

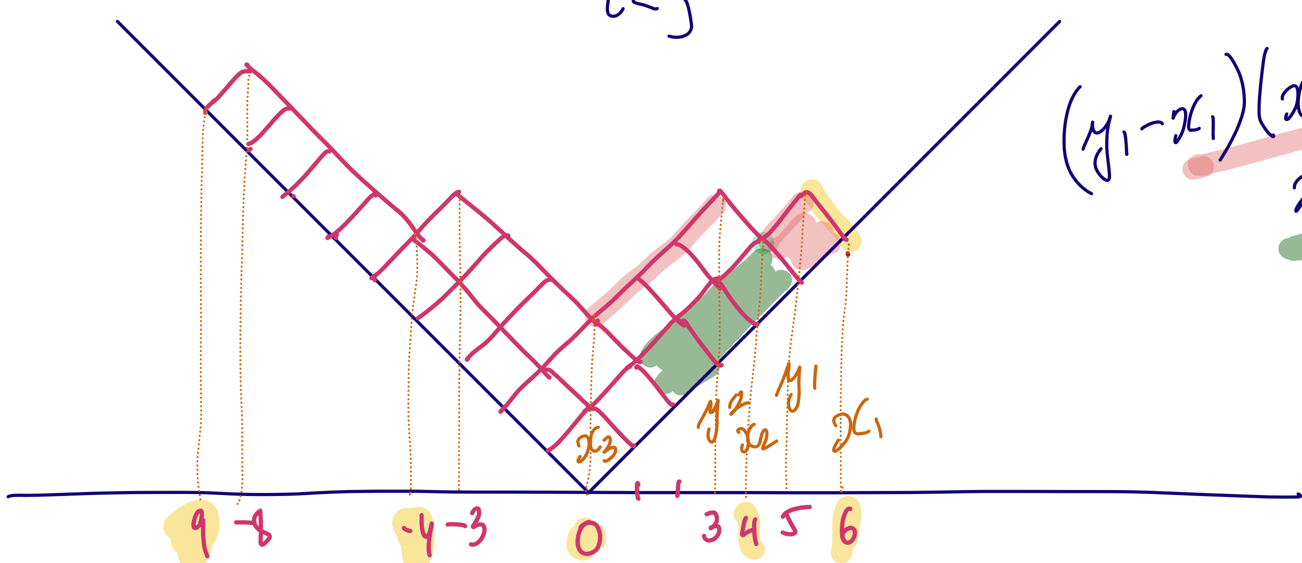


w_λ, unscaled (area $= n$)

1) (local) minima / maxima

2) $\sum_1^d x_i = \sum_1^{d-1} y_j$ (induction)

3) $|\lambda| = \text{area} = \sum_{i<j} (y_i - x_i)(x_j - y_{j-1})$



$(y_1 - x_1)(x_2 - y_1)$

$x_3 - y_2$

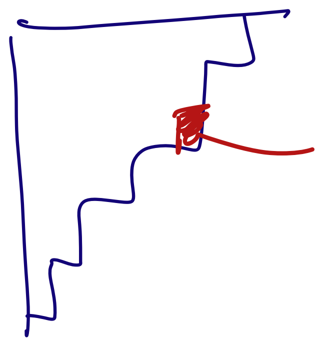
13.2 Plancherel growth process

(Young graph story)

Recall $\{M_n\}$ – coherent

$$\sum_{\lambda=\mu+\square} M_n(\lambda)\,\frac{\dim\mu}{\dim\lambda} = M_{n-1}(\mu)$$

$$p^{\downarrow}(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}$$



Joint distr. of λ, μ $\quad |\lambda| = n$, $|\mu| = n-1$

$$P(\mu,\lambda) = M_n(\lambda)\,p^{\downarrow}(\lambda,\mu) \qquad \mathbb{Y}_n \times \mathbb{Y}_{n-1}$$

Then $\mu \sim M_{n-1}$

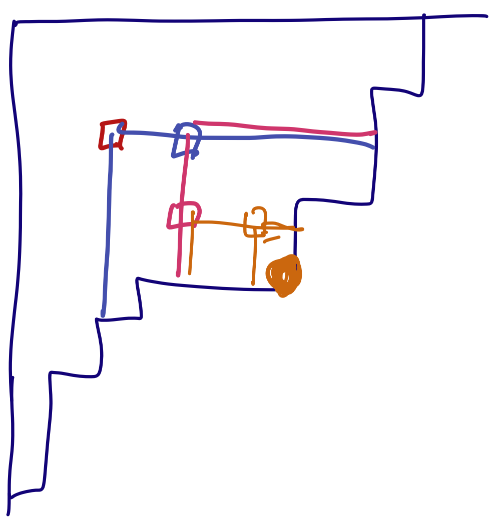
Define $p^{\uparrow}(\mu,\lambda) = P(\lambda \mid \mu)$ under this cond. distr.

$$= \frac{M_n(\lambda)\, p^{\downarrow}(\lambda,\mu)}{M_{n-1}(\mu)}$$

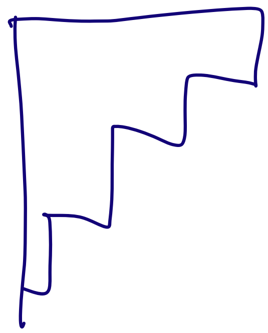
$p^{\uparrow}$ depends on M_n.

$$\boxed{p^{\uparrow}_{\text{Plancherel}}(\lambda,\nu) = \frac{\dim \nu}{(|\lambda|+1)\dim \lambda}}$$

$p^{\downarrow} \longleftrightarrow$ hook walk algo



$p^{\uparrow}_{Pl.} \longleftrightarrow$ RSK

λ ,  $\leftarrow$ $k+\frac{1}{2}$

pick a unif random SYT of shape λ (hook walk)

RSK - insert "letter" $k+\frac{1}{2}$ into tableau, where $k \in \{1, 2, \ldots, n\}$.

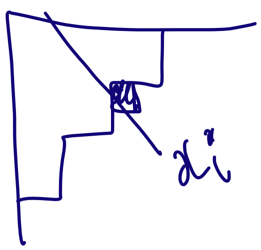
Proposition.

$$\frac{\prod_{j=1}^{d-1}(u-y_j)}{\prod_{i=1}^{d}(u-x_i)} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{u-x_i}$$

$$\frac{\prod_{i=1}^{d}(u-x_i)}{\prod_{j=1}^{d-1}(u-y_j)} = u - \sum_{j=1}^{d-1} \frac{\pi_j^{\downarrow}}{u-y_j}.$$

Then:

$$\boxed{p^{\uparrow}(\lambda,\nu) = \pi_i^{\uparrow}}\,, \qquad p^{\downarrow}(\lambda,\mu) = \pi_j^{\downarrow}/|\lambda|$$



Proof. (formula for $\pi_i^{\uparrow}$)

$$\frac{\prod_{j=1}^{d-1}(u-y_j)}{\prod_{i=1}^{d}(u-x_i)} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{u-x_i}$$

$$\prod_j (u-y_j) = \sum_{i=1}^{d} \pi_i^{\uparrow} \prod_{j\neq i} (u-x_j)$$

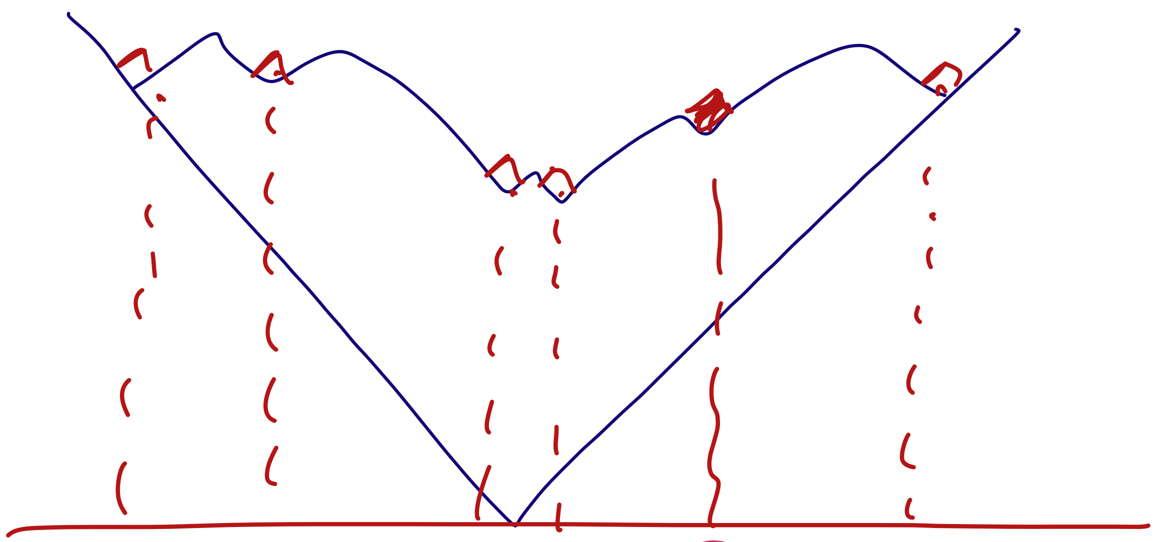
$u = x_i$:

$$\boxed{\pi_i^T = \frac{\prod_j (x_i - y_j)}{\prod_{j \neq i} (x_i - x_j)}} \overset{?}{=} \frac{\dim}{\dim(n+1)}$$

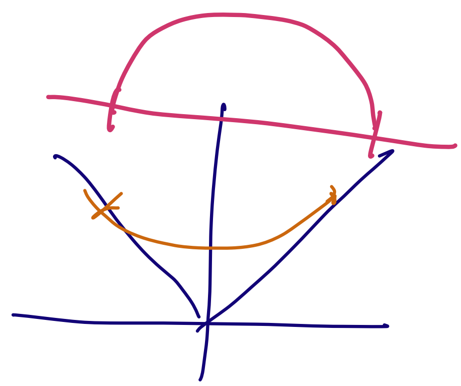
w/o proof

Def. Transition distribution $\pi^\uparrow$ of λ

(= probab. on $\mathbb{R}$)



$\pi^\uparrow = \sum_i \delta_{x_i} \pi_i^\uparrow$



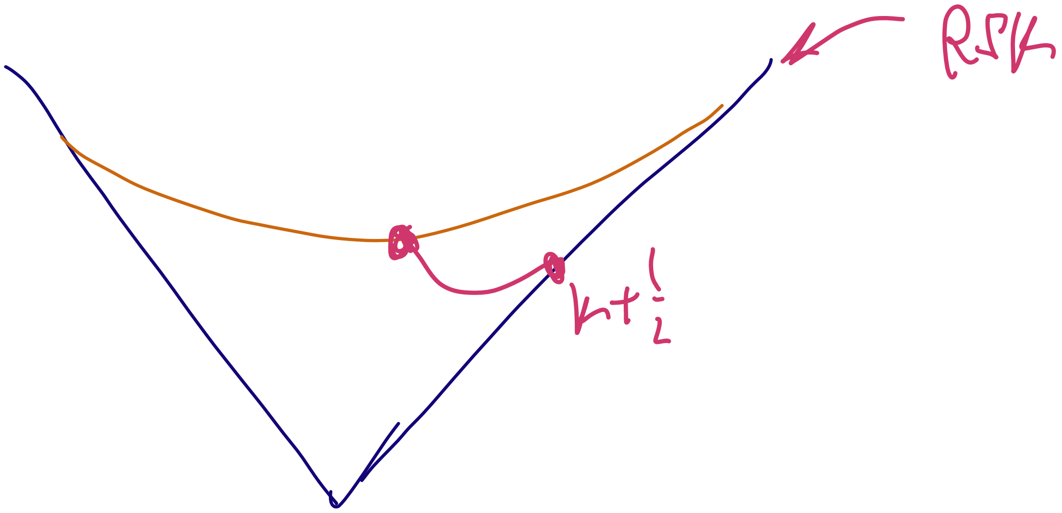
Natural questions:

$\lambda \sim$ Plancherel, n large

① $\pi^\uparrow(\lambda) \sim$?

$\pi^\uparrow(\Omega) \sim$ semi-circle

② RSK – insertion path ?



$\exists\ \pi^{\uparrow}$ for cont. curve?

like Ω

Next

13.3 Transition probabilities of continual Y.d.

w : $|w(x)-w(y)| \le |x-y|$

$w(x) = |x-x_0| \ \forall$ large x

(x_0 — center)

$w(u) \rightsquigarrow \sigma(u) = \frac{1}{2}\left(w(u)-|u|\right)$

(1) $\sigma' \ \exists$ a.e.

$|\sigma'| \le 1$. σ' comp. supp.

(2) w is determined by σ' or σ''

(σ'' — discrete measure)

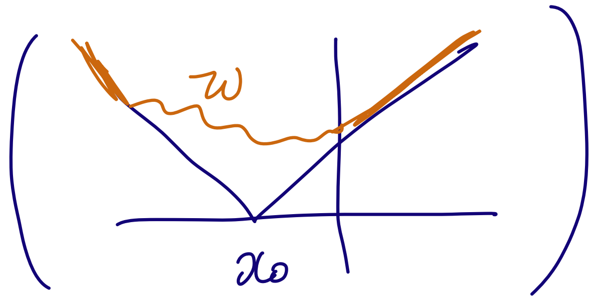
Example

Def. $$\tilde{p}_k = \int_{-\infty}^{+\infty} x^k \, \delta''(x)\, dx,$$

$$= -k \int_{-\infty}^{+\infty} x^{k-1} \, \delta'(x)\, dx$$

$$k = 1, 2, \dots$$

Facts ① $x_0 = \tilde{p}_1$



② area $= \frac{1}{2}\left(\tilde{p}_2 - \tilde{p}_1^{\,2}\right)$

For discrete y. d.

$$\tilde{p}_k(w) =$$

Def.

$$S(z) = \overbrace{\sum_{n=1}^{\infty} \frac{\tilde{p}_n(w)}{n} z^{-n}}^{\text{object from symm.-f.}}$$

$$= \underbrace{\int_{\mathbb{R}} \frac{\sigma'(x)\,dx}{z - x}}_{\text{Stieltjes transform}}$$

Fact Ω has

$$\tilde{p}_{2m-1}(\Omega) = 0, \quad \tilde{p}_{2m}(\Omega) = \binom{2m}{m}$$

$$\tilde{p}_{2m} = -2\int_{\mathbb{R}+} \sigma'(u)\,du^{2m} = \int_0^2 \left(1 - \frac{2}{\pi}\arcsin\frac{u}{2}\right) du^{2m}.$$

The substitution $u = 2\sin\varphi$ and integration by parts imply

$$\tilde{p}_{2m} = 2^{2m}\int_0^{\pi/2} (1 - 2\varphi/\pi)\, d\sin^{2m}\varphi = 2^{2m-1}\pi \int_0^{\pi/2} \sin^{2m}\varphi\, d\varphi =$$

$$= \frac{2^{2m}(2m-1)!!}{(2m)!!} = \frac{(2m)!}{m!\,m!}, \qquad \square$$

$$\Rightarrow \quad S(z) = \log\frac{z}{2} + \log\left(z - \sqrt{z^2-4}\right)$$

$$(|z| > 2)$$

Def. Transition distribution

$$\underbrace{\frac{\prod_{j=1}^{d-1}(u-y_j)}{\prod_{i=1}^{d}(u-x_i)}} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{u-x_i}$$

$\{\pi_i^{\uparrow}\}$ (for discrete)

$$\parallel$$

$$\exp\left\{ \sum_j \log(u-y_j) - \sum_i \log(u-x_i) \right\}$$

$$= \exp S(u)$$

Def. Transition probability of w is $d\pi^{\uparrow}(u)$, where

$$\exp S(z) = \int_{\mathbb{R}} \frac{d\pi^{\uparrow}(x)}{1 - x/z},$$

for large enough $|z|$

Recall symm. f -

$$\exp \sum_{k\geq 1} \frac{p_k}{k} t^k = \sum_{n\geq 0} h_n t^n$$

$$\Downarrow$$

$\tilde{h}_n$ = moments of $\pi^\uparrow$

<u>Prop.</u> $\Omega \rightsquigarrow \pi^\uparrow$ has density

$$\frac{1}{2\pi}\sqrt{4-z^2}, \quad |z|\leq 2$$

Proof.

the moments have the form

$$h_{2m+1} = 0, \quad h_{2m} = \frac{1}{m+1}\binom{2m}{m}; \quad m = 0, 1, 2, \ldots . \tag{3.4.7}$$

The moment generating function of the semicircle distribution equals

$$H(x) = \frac{x}{2}\left(1 - \sqrt{1 - (2/x)^2}\right), \quad x > 2. \tag{3.4.8}$$

Proof. Clearly, all odd moments vanish. The substitution $u = 2\sin\varphi$ implies

$$h_{2m} = \frac{1}{2\pi}\int_{-2}^{2} u^{2m}\sqrt{4-u^2}\,du = \frac{2^{2m+2}}{\pi}\int_{0}^{\pi/2}(\sin^{2m}\varphi - \sin^{2m+2}\varphi)\,d\varphi =$$
$$= \frac{2^{2m+2}}{\pi}\cdot\frac{\pi}{2}\left(\frac{(2m-1)!!}{(2m)!!} - \frac{(2m+1)!!}{(2m+2)!!}\right) = \frac{1}{m+1}\binom{2m}{m}.$$

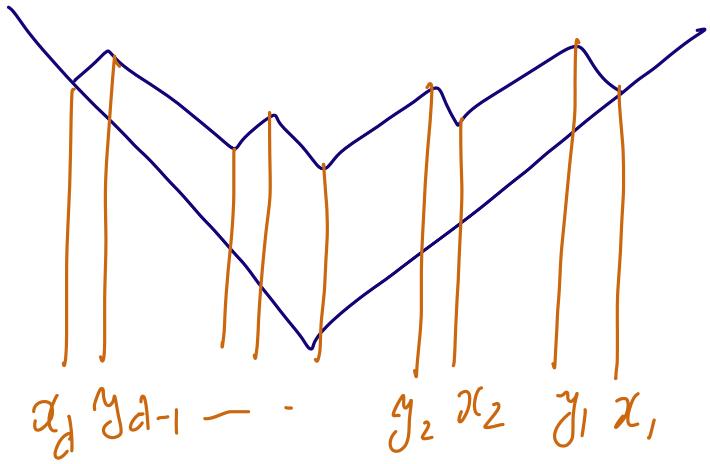
It follows from the binomial identity that

$$\frac{1}{s}(1 - \sqrt{1-s^2}) = \frac{s}{2}\sum_{m=0}^{\infty}\frac{(2m-1)!!}{(2m)!!}\frac{s^{2m}}{m+1}.$$

Using the substitution $s = 2/x$ and the formula (3.4.7) which had been proved above, we derive

$$H(x) = \sum_{m=0}^{\infty} h_{2m}\,x^{-2m} = \frac{x^2}{2}\left(1 - \sqrt{1-(2/x)^2}\right). \quad \square$$

Recall



$$p^{\uparrow}(\lambda,\nu) = \frac{\dim \nu}{(|\lambda|+1)\dim \lambda}, \quad \text{Plancherel growth}$$

$$p^{\uparrow}(\lambda, \lambda + \square_{x_i}) = \pi_i^{\uparrow}, \quad \text{where}$$

$$(*) \quad \boxed{\frac{\prod_{i=1}^{d-1}(u-y_i)}{\prod_{i=1}^{d}(u-x_i)} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{u-x_i}}$$

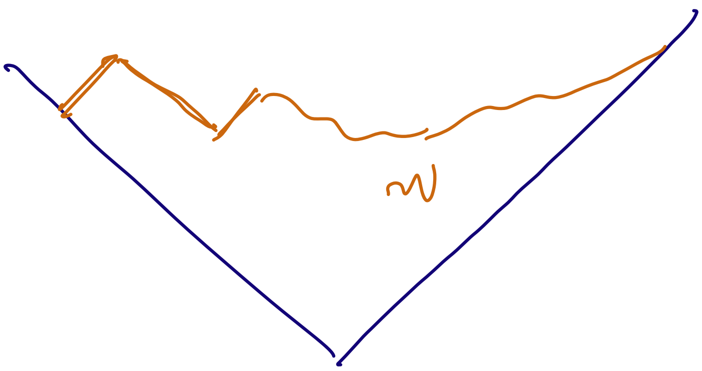
Note: $(*)$ works for any interlacing $\vec{x}/\vec{y}$, not necess. $\mathbb{Z}$

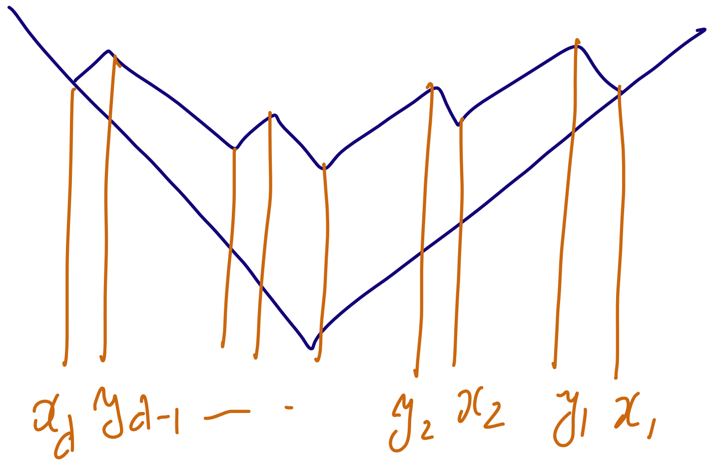
$$\exp(\Sigma \log - \Sigma \log) \qquad \int_{\mathbb{R}} \frac{d\pi^{\uparrow}(x)}{u-x}$$

Next: $\pi^{\uparrow}(w)$ as a distribution

$\longrightarrow$ continuous version of $(*)$

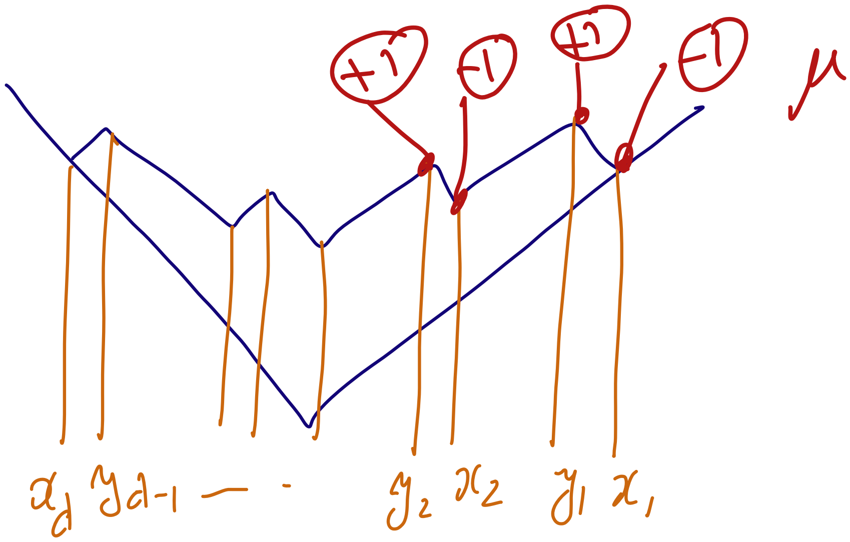
(ideas?)





$$\sum \log(u-y_j) - \sum \log(u-x_i)$$

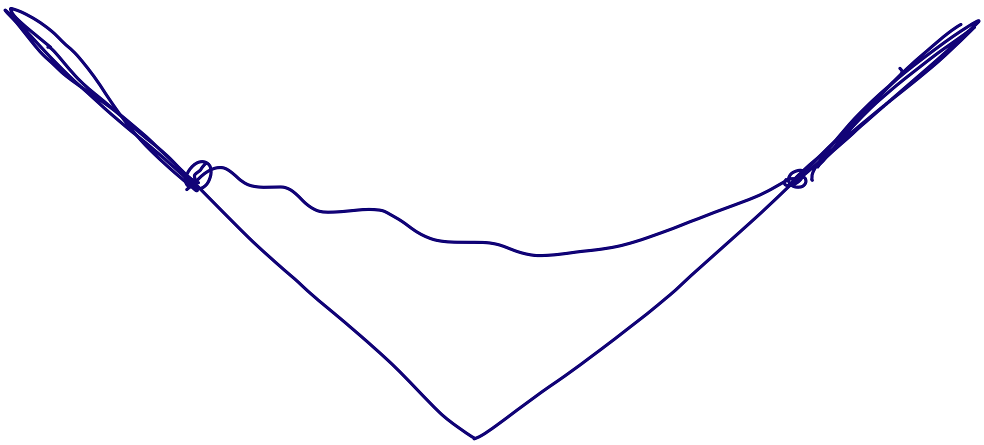
$$= \int \log(u-x) \cdot \mu(dx)$$



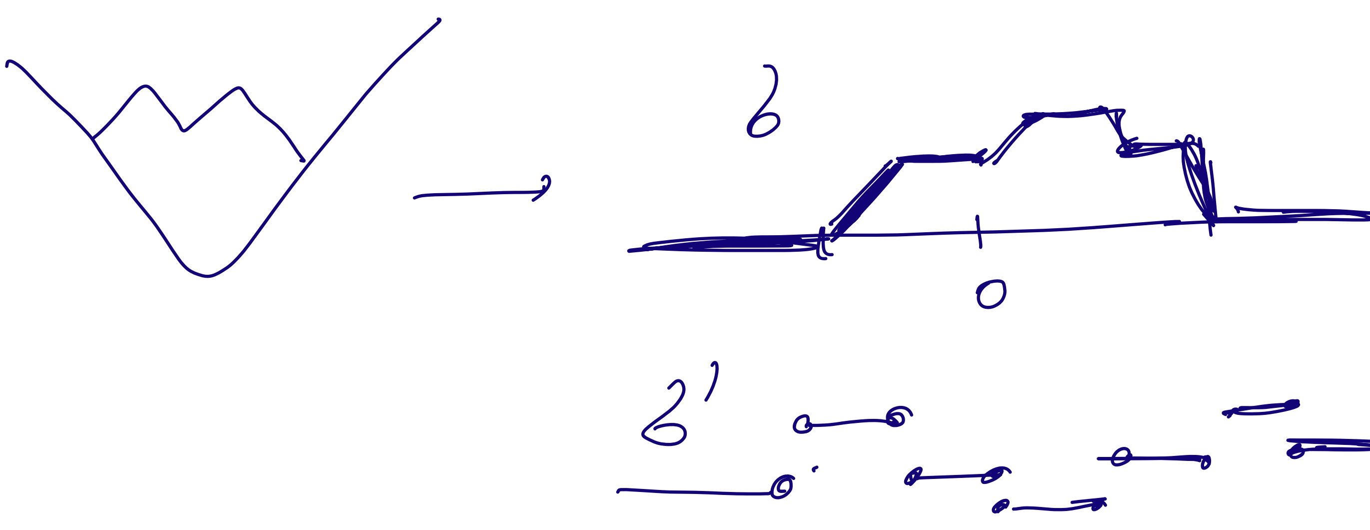
13.3 Transition probabilities of continual Y.d.

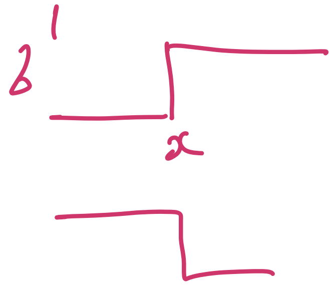
$w:$ $\quad |w(x)-w(y)| \le |x-y|$

$w(x) = |x| \quad \forall$ large x



$$w(u) \rightsquigarrow \sigma(u) = \frac{1}{2}\left(w(u) - |u|\right)$$





$\longrightarrow \sigma''$ $\quad +\delta_x$

σ'' $\quad -\delta_x$

① σ' $\exists$ a.e.

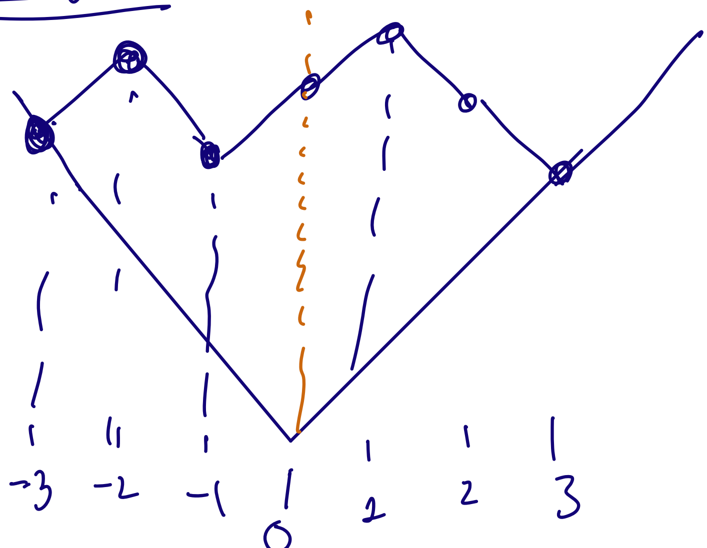
$|\sigma'| \le 1$. $\quad \sigma'$ comp. supp.

② ω is determined by σ'

or σ''

(σ'' — discrete measure)

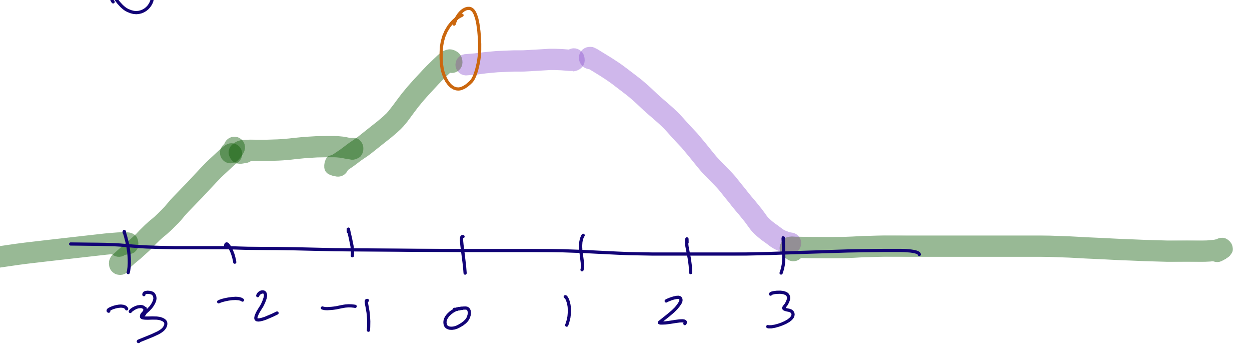
<u>Example</u> (rectangular)



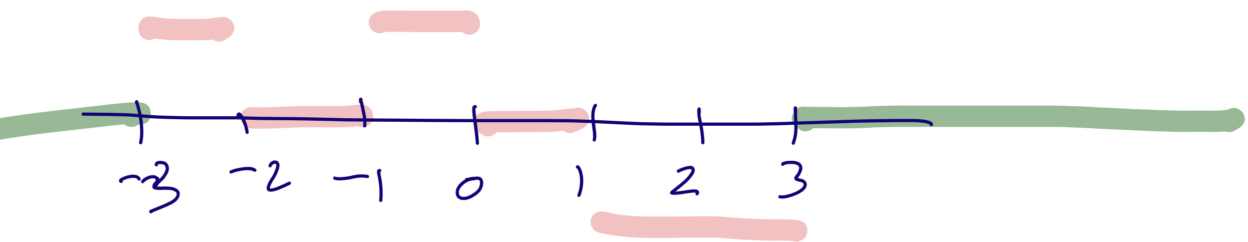
$x = (3, -1, -3)$

$y = (1, -2)$

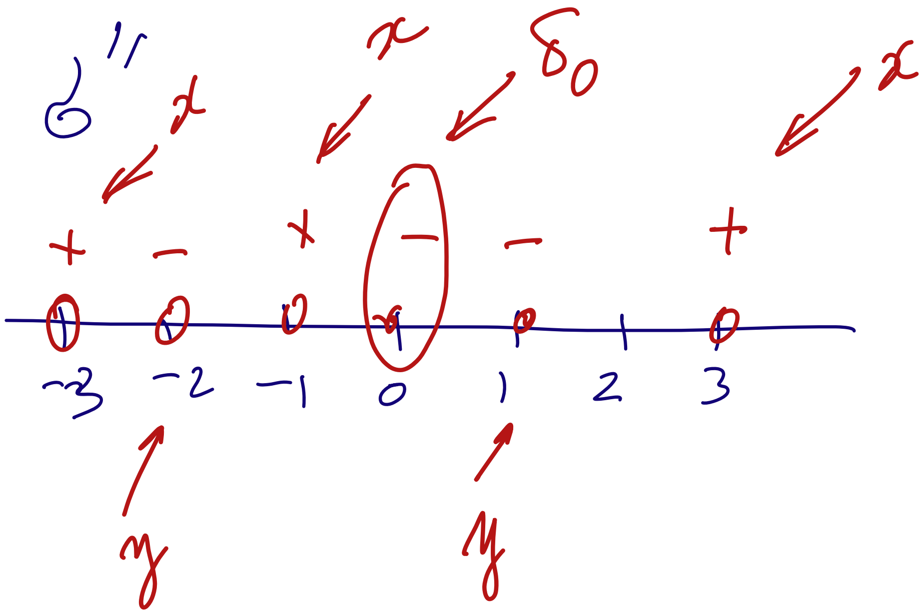
σ:



σ'



σ''



$$\sum \log(u-y_j) - \sum \log(u-x_j)$$

$$= -\int \log(u-x)\, d\sigma'(x) - \log u$$

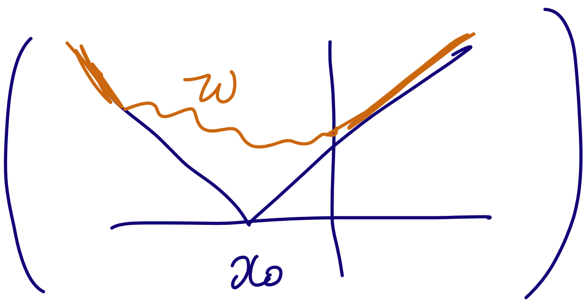
Def. $\tilde{p}_k = \int_{-\infty}^{+\infty} x^k \, d\delta'(x)$,

$= -k \int_{-\infty}^{+\infty} x^{k-1} \delta'(x)\,dx$ ← function

$k = 1, 2, \dots$

$\boxed{\sum x_i - \sum y_j = 0}$

Facts (1) $\boxed{\tilde{p}_1 = 0}$



(2) area $= \frac{1}{2}\left(\tilde{p}_2 - \tilde{p}_1^2\right)$

$\frac{1}{2}\left(\sum x_i^2 - \sum y_j^2\right)$

$\parallel$ ✓

$\tilde{p}_k$ – supersymm. power sums

$\sum x_i^k - \sum y_j^k = p_k(\vec{\alpha} = \vec{x}, \vec{\beta} = -\vec{y})$

$\sum_{i<j} (y_i - x_i)(x_j - y_{j-1})$

$\parallel$

$\frac{\tilde{p}_2 - \tilde{p}_1^2}{2}$

$$\boxed{p_k(\alpha,\beta) = \sum \alpha_i^k - (-1)^k \sum \beta_i^k}$$

$\chi_{\alpha\beta}$ (k - cycle) of $S(\infty)$

For rectangular Y.d.,

$$\tilde{p}_k(w) = \sum_{i=1}^{d} x_i^k - \sum_{j=1}^{d-1} y_j^k$$

Def.

$$S(z) := \overbrace{\sum_{n=1}^{\infty} \frac{\tilde{p}_n(w)}{n} z^{-n}}^{\text{object from symm.-f.}} = \underbrace{\int_{\mathbb{R}} \frac{\sigma'(x)\,dx}{z-x}}_{\text{Stiltjes transform}}$$

Rect. y.d. $\vec{x} / \vec{y}$

$$\int_{\mathbb{R}} \frac{\sigma'(x)\,dx}{z-x} = \frac{1}{z}\int_{\mathbb{R}} \sum \left(\frac{x}{z}\right)^k \sigma'(x)\,dx$$

$$-k x^{k-1} dx = -dx^k$$

$$\tilde{p}_k = -k \int_{-\infty}^{+\infty} x^{k-1} \sigma'(x)\,dx$$

$$\sum_{n=1}^{\infty} \frac{\tilde{p}_n(\omega)}{n} z^{-n} = \sum_{n=1}^{\infty} \frac{\sum x_i^n - \sum y_j^n}{n} z^{-n}$$

$$= -\sum_1^d \log(1 - x_i/z) + \sum_1^{d-1} \log\left(1 - \frac{y_j}{z}\right)$$

$$= -\sum_1^d \log(z - x_i) + \sum_1^{d-1} \log(z - y_j)$$

$$+ \log z$$

$$= \text{as before} \quad \boxed{\int_{\mathbb{R}} \log(z-x)\, d\sigma'(x)}$$

$$\int_{\mathbb{R}} \frac{\sigma'(x)\,dx}{z-x}$$

Ω - VKLS

$\Omega'(u) = \frac{2}{\pi} \arcsin\left(\frac{u}{2}\right), \quad |u| \le 2$

Fact Ω has

$\tilde{p}_{2m-1}(\Omega) = 0, \quad \tilde{p}_{2m}(\Omega) = \binom{2m}{m}$

$\sigma = \frac{1}{2}(\Omega - |u|)$

$$\tilde{p}_{2m} = -2\int_{\mathbb{R}+} \sigma'(u)\, du^{2m} = \int_0^2 \left(1 - \frac{2}{\pi}\arcsin\frac{u}{2}\right) du^{2m}.$$

The substitution $u = 2\sin\varphi$ and integration by parts imply

$$\tilde{p}_{2m} = 2^{2m}\int_0^{\pi/2} (1 - 2\varphi/\pi)\, d\sin^{2m}\varphi = 2^{2m-1}\pi \int_0^{\pi/2} \sin^{2m}\varphi\, d\varphi =$$

$$= \frac{2^{2m}(2m-1)!!}{(2m)!!} = \frac{(2m)!}{m!\, m!}, \qquad \square$$

$$\Rightarrow \boxed{S(z) = \log\frac{z}{2} + \log\left(z - \sqrt{z^2-4}\right) \quad (|z|>2)}$$

$$= \sum_{n=1}^{\infty} \frac{z^{-2n}}{2n}\binom{2n}{n}$$

<u>Def</u>. Transition distribution

$$\boxed{\frac{\prod_{j=1}^{d-1}(u-y_j)}{\prod_{i=1}^{d}(u-x_i)}} = \boxed{\sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{u-x_i}}$$

$\{\pi_i^{\uparrow}\}$ (for discrete)

$$= \boxed{\exp S(u)} = \exp\Big(\sum \log(u-y_j) - \sum \log(u-x_i) + \underline{\log u} \Big)$$

Def. Transition probability of ω is $d\pi^T(x)$, where

$$\boxed{\exp\left(S(z)\right) = \int_{\mathbb{R}} \frac{d\pi^T(x)}{1 - x/z}},$$

for large enough $|z|$

$$\frac{1}{1-x/z} = \sum_{n=0}^{\infty} \left(\frac{x}{z}\right)^n$$

$$\text{RHS} = \sum_{n=0}^{\infty} z^{-n} \overbrace{\int x^n \, d\pi^T}^{n\text{-th moment}}$$

$\Rightarrow$ correspondence $\tilde{P}_n \longleftrightarrow \pi^T$

$$\tilde{P}_n \longleftrightarrow w$$

Recall symm. f -

$$\exp\left(\sum_{k\geq 1} \frac{p_k}{k} t^k\right) = \overbrace{\sum_{n\geq 0} h_n t^n}^{\prod \frac{1}{1-tx_i}}$$

$$\Downarrow$$

$\tilde{h}_n$ = moments of π^T

$$W \longrightarrow \tilde{p}_n = \text{moments of "}\sigma\text{"}$$

$$\swarrow$$

$$\tilde{h}_n = \text{moments of } \pi^{\uparrow}(w)$$

Prop. $\Omega \rightsquigarrow \pi^{\uparrow}$ has density

$$\frac{1}{2\pi}\sqrt{4-z^2}\,, \quad |z|\le 2$$

(semicircle law)

$\Omega \longrightarrow$

$$S(z) = \log\frac{z}{2} + \log\left(z-\sqrt{z^2-4}\right)$$

$$e^{S(z)} = \frac{z}{2}\cdot\left(z-\sqrt{z^2-4}\right) = z^2 C\left(\frac{1}{z}\right)$$

$$C(z) = \frac{z-\sqrt{4-z^2}}{2z}$$

Catalan numbers g.f.

2m-th moment of π^T

Proof.

the moments have the form

$$\tilde{h}_{2m+1} = 0, \quad \tilde{h}_{2m} = \frac{1}{m+1}\binom{2m}{m}; \quad m = 0, 1, 2, \ldots. \tag{3.4.7}$$

The moment generating function of the semicircle distribution equals

$$H(x) = \frac{x}{2}\left(1 - \sqrt{1-(2/x)^2}\right), \quad x > 2. \tag{3.4.8}$$

Proof. Clearly, all odd moments vanish. The substitution $u = 2\sin\varphi$ implies

$$\tilde{h}_{2m} = \frac{1}{2\pi}\int_{-2}^{2} u^{2m}\sqrt{4-u^2}\,du = \frac{2^{2m+2}}{\pi}\int_0^{\pi/2}(\sin^{2m}\varphi - \sin^{2m+2}\varphi)\,d\varphi =$$
$$= \frac{2^{2m+2}}{\pi}\cdot\frac{\pi}{2}\left(\frac{(2m-1)!!}{(2m)!!} - \frac{(2m+1)!!}{(2m+2)!!}\right) = \frac{1}{m+1}\binom{2m}{m}.$$

It follows from the binomial identity that

$$\frac{1}{s}(1-\sqrt{1-s^2}) = \frac{s}{2}\sum_{m=0}^{\infty}\frac{(2m-1)!!}{(2m)!!}\,\frac{s^{2m}}{m+1}.$$

Using the substitution $s = 2/x$ and the formula (3.4.7) which had been proved above, we derive

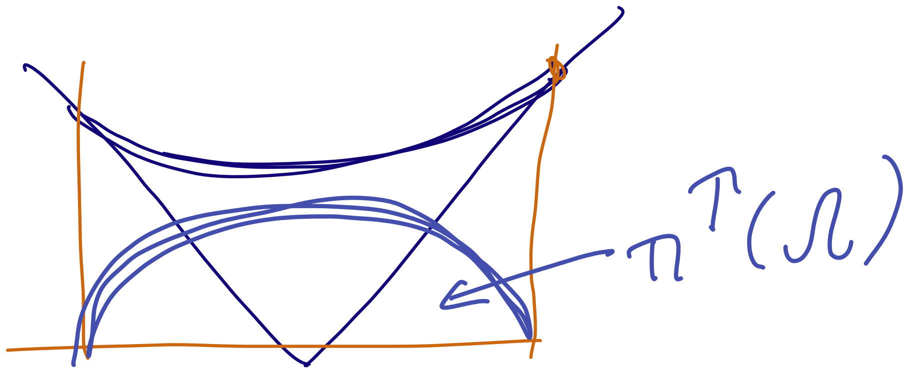
$$H(x) = \sum_{m=0}^{\infty}\tilde{h}_{2m}\,x^{-2m} = \frac{x^2}{2}\left(1-\sqrt{1-(2/x)^2}\right). \quad \square$$

$\Rightarrow$ π^T density is

$$\frac{1}{2\pi}\sqrt{4-x^2}, \quad |x|<2$$

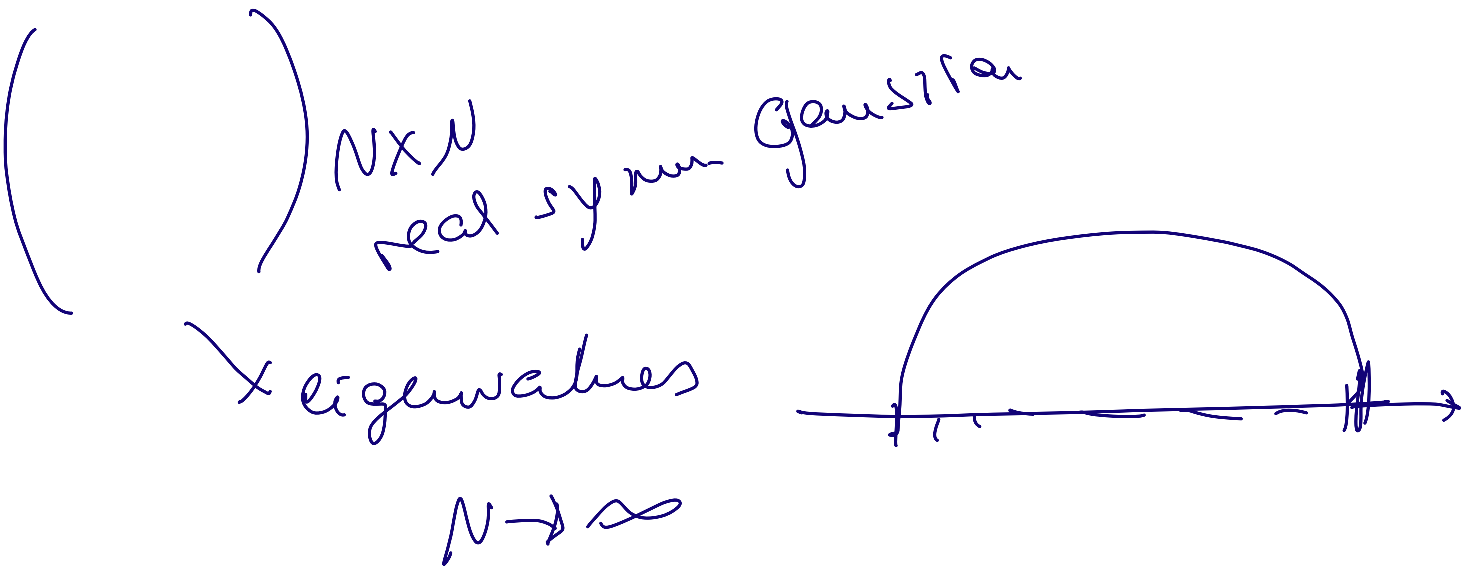
because $$\frac{1}{m+1}\binom{2m}{m} = \int_{-2}^{2}\frac{1}{2\pi}x^{2m}\sqrt{4-x^2}\,dx$$

So, in a Plancherel random partition, add a box at semicircle distr.



Appearance of semicircle law $\boxed{\frac{1}{2\pi}\sqrt{4-x^2}\,dx}$

(Wigner, 1950s)

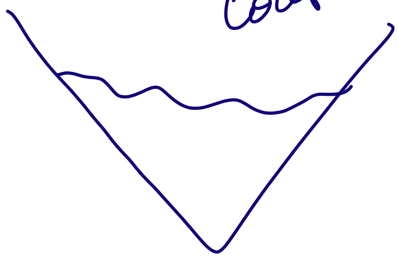


$\boxed{\text{VKLS} \longleftrightarrow \text{SC}}$ in random matrices

Rect. y.d. $\longrightarrow$ $\pi^{\uparrow}_i$

$\downarrow$

continuous



$\longrightarrow$ $\pi^{\uparrow}(w)$

via moments

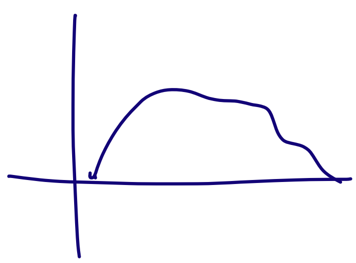
$$\boxed{e^{S(z)} = \int_{\mathbb{R}} \frac{d\pi^{\uparrow}(x)}{1 - x/z}}$$

X $\quad \boxed{\mathcal{N}(0,1)}$ αN

N

$XX^* \longrightarrow$ eigenvalues

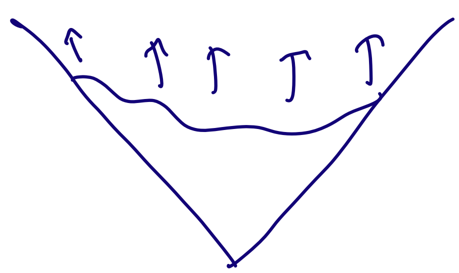
random y.d. $\longleftarrow$



$\pi^{\uparrow}$

13.4 Differential model of growth ("hydrodynamics")

Large scale behavior of Plancherel growth.



Area $= t$ (time)

$$\| $$

$$\int_{\mathbb{R}} \sigma_t(z)\,dz, \qquad \boxed{\sigma_t(z) = \frac{w_t(z)-|z|}{2}}$$

Let $\quad T_t(z) = \frac{\partial}{\partial t} \sigma_t(z),$

So $\quad \int_{\mathbb{R}} T_t(z)\, dz = 1$

probab. distribution

At each time t,

want that w_t "grows by"

$\pi^T(w_t)$, i.e.

density variable

$$d\pi^{\pi}(w_t)(z) = T_t(z)\, dt$$

(equality of 2 probab. distr.)

(∞-dim. ODE in space

of cont. diagrams)

Forms of diff. eq. for ν_t

(1) $\exp \int_{\mathbb{R}} \frac{\sigma_t'(x)\,dx}{z-x} =$

$$= \int_{\mathbb{R}} \frac{\partial/\partial t\ \sigma_t(x)\,dx}{1-u/x}$$

for large $|x|$

(2) via moments:

$$\frac{\frac{\partial}{\partial t}\widetilde{p}_{n+2}(t)}{n+2} = (n+1)\,\widetilde{h}_n(t)$$

$n = 0, 1, 2, \ldots$

where $\tilde{p}_n = \int_{-\infty}^{+\infty} x^n \, \delta''(x) \, dx$

$\tilde{h}_n$ = moments of $\Omega^{\uparrow}$

③ Define $R_t(x) = \sum_{n=0}^{\infty} \tilde{h}_n x^{-(n+1)}$

large $|x|$

Then : $$\boxed{\frac{\partial}{\partial t} R + R \frac{\partial}{\partial x} R = 0}$$

Proof $S = \sum \frac{\tilde{\rho}_{n+2}}{n+2} x^{-(n+2)}$

$= \log(xR)$

$(\forall w, \text{ as } \tilde{\rho}_1 = 0)$

$S = \log(xR) = \log x + \log R$

$\frac{\partial}{\partial t} \Rightarrow$

$$\frac{\partial}{\partial t} S = \frac{\frac{\partial}{\partial t} R}{R}.$$

& note $\frac{\partial}{\partial x} R = \sum_{n=0}^{\infty} -(n+1)\,\tilde{h}_n\, x^{-(n+2)}$

$\frac{\partial}{\partial t} S = \ ---$

(using moments relation in the evolution eq'n)

Semicircle / VKLS

$$S(x) = \log(xR)$$

$$\Rightarrow R = \frac{1}{2}\left(x - \sqrt{x^2-4}\right)$$

(essentially, gener. function of Catalans at $\frac{1}{x}$)

Def

$$r(x) = \frac{1}{2}\left(x - \sqrt{x^2-4}\right)$$

$$R_t(x) = \frac{r(x/\sqrt{t})}{\sqrt{t}}$$

R_t

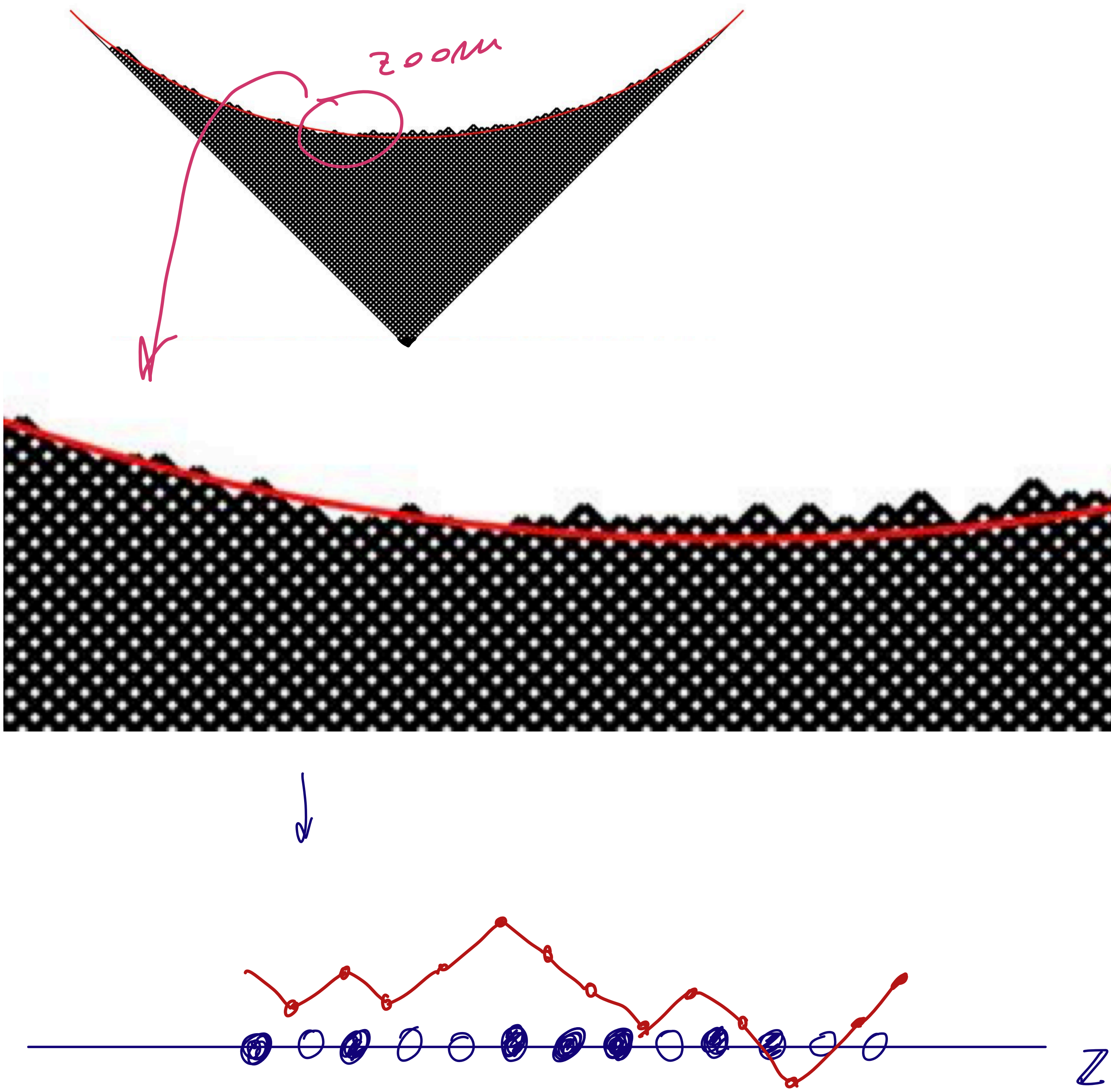
① Satisfies: $\boxed{\frac{\partial}{\partial t} R + R \frac{\partial}{\partial x} R = 0}$

② $\frac{r(x/\sqrt{t})}{\sqrt{t}}$ is the unique automodel solution

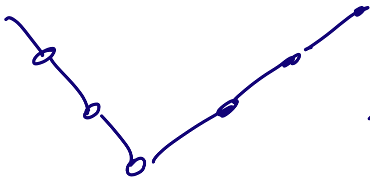
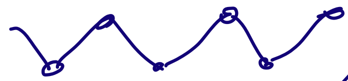
③ can prove (by moments) that Planchel growth converges to this R started from any $R_{t=1}$, i.e. our

automodel solution is
an attractor

14. Plancherel limit shape via determinantal formulas.



$\longrightarrow$ Random walk (locally)?

$\mathbb{P}\Big($  $\Big)$ vs $\mathbb{P}\Big($  $\Big)$

$\longrightarrow$ Something else?

14.1 Infinite wedge space

(1) $$\lambda \in \mathbb{Y} \longrightarrow \left\{ \lambda_i - i + \tfrac{1}{2} \right\}_{i \geq 1}$$

$$v_\lambda = v_{\lambda_1 - 1 + \frac{1}{2}} \wedge v_{\lambda_2 - 2 + \frac{1}{2}} \wedge \cdots$$

$$\lambda = (4, 3, 1) \longrightarrow v_\lambda = \cdots$$

$$v_\emptyset = v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge \cdots$$

(„vacuum“)

(2) ψ_i, ψ_i^* $i \in \mathbb{Z}$

create conjugate

(3) Let $U v_\lambda = \sum_{\mu = \lambda + \square} v_\mu$

$$D v_\lambda = \sum_{\mu = \lambda - \square} v_\mu$$

Lemma.

$$U = \sum_k \psi_k \psi_{k-1}^*$$

$$D = \sum_k \psi_k \psi_{k+1}^*$$

(4) dim λ via $\mathcal{U}, \mathcal{D}$

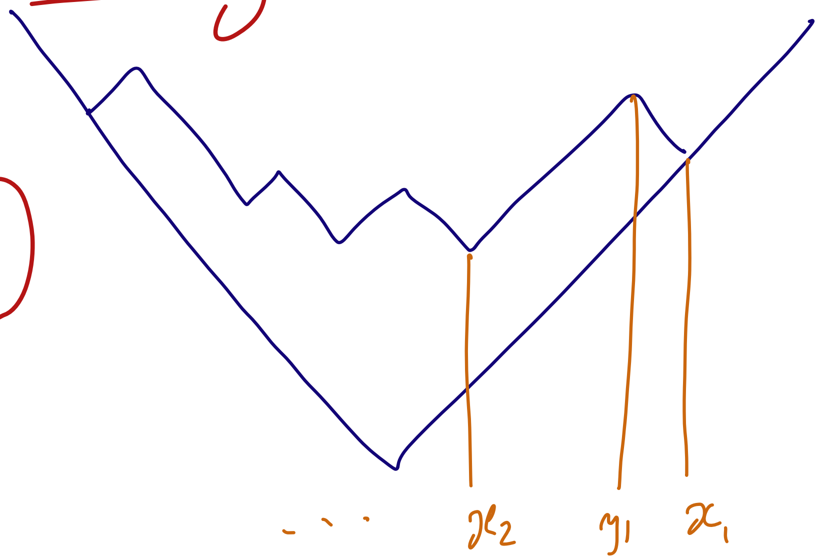
Plancherel measure via $\mathcal{U}, \mathcal{D}$

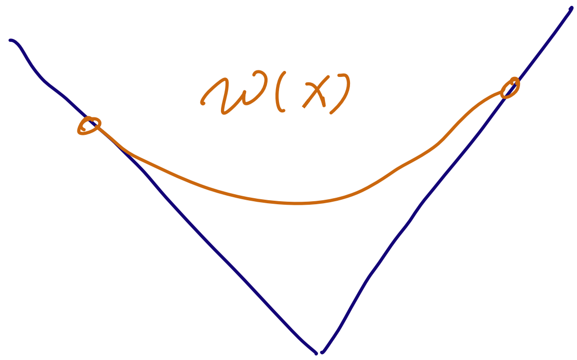
(5) Define $\Gamma_+(\theta) = e^{\theta U}$

$$\Gamma_-(\theta) = e^{\theta D}.$$

Summary.

(1)





$$p^{\uparrow}(\lambda, \nu) = \frac{\dim \nu}{(|\lambda|+1)\dim \lambda} = \pi_i^{\uparrow},$$

$$\frac{\prod_{i=1}^{d-1}(z-y_i)}{\prod_{i=1}^{d}(z-x_i)} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{z-x_i}$$

$$\sigma(x) = \frac{1}{2}\left(w(x) - |x|\right)$$

$$S(z) = \int_{\mathbb{R}} \frac{\sigma'(x)\,dx}{z-x}$$

Def.

$\pi^{\uparrow}(x) =$ $\lvert z\rvert$ large	$e^{S(z)} = \int_{\mathbb{R}} \frac{d\pi^{\uparrow}(x)}{1 - x/z}$

2

Moments: $\tilde{p}_k = \int_{\mathbb{R}} x^k \, d\sigma'(x)$

$= -k \int_{\mathbb{R}} x^{k-1} \sigma'(x)\, dx$

$= - \int_{\mathbb{R}} \sigma'(x)\, d(x^k)$

(for rect.) $= \sum x_i^k - \sum y_j^k$

$$S(z) = \sum_{n=1}^{\infty} \frac{\tilde{p}_n}{n} z^{-n}$$

$$\exp\left(S(z)\right) = \sum_{n=0}^{\infty} \tilde{h}_n z^{-n} = \int_{\mathbb{R}} \frac{d\pi^T(x)}{1 - x/z}$$

$\Rightarrow$ $\tilde{h}_n$ are moments of π^T:

$$\tilde{h}_n = \int_{\mathbb{R}} x^n \, d\pi^T(x)$$

3 Symm. f.

$p_k = \sum x_i^k$

h_n = complete homog.

$$\boxed{e^{\sum_1^\infty p_n/n\, t^n} = \sum_0^\infty h_n t^n}$$

Also $\exists$ s_λ (continual y.d.)

Ex. $s_\lambda(\Omega) = \det\left[\underbrace{h_{\lambda_i+j-i}(\Omega)}_{\substack{0 \text{ or Catalan}}}\right]$

$C_n = \frac{1}{n+1}\binom{2n}{n}$

$\det\left(C_{j+i}\right) = \det\begin{bmatrix} C_1 & C_2 & C_3 & \cdots \\ C_2 & C_3 & \cdots & \\ C_3 & \cdots & & \\ \cdots & & & \end{bmatrix} = 1$

④ VKLS Ω

$$\tilde{p}_{2m-1}(\Omega) = 0, \quad \tilde{p}_{2m}(\Omega) = \binom{2m}{m}$$

$$S(z) = \sum_{n=1}^{\infty} \frac{\tilde{p}_n}{n} z^{-n}$$

arcsin(2z)

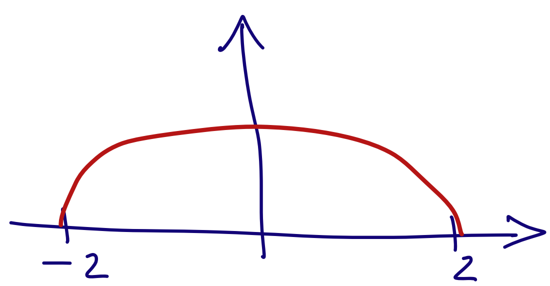
$$= \log \frac{z}{2} + \boxed{\log\left(z - \sqrt{z^2-4}\right)}$$

(Taylor series for arcsine)

$$\tilde{h}_{2m-1} = 0, \quad \tilde{h}_{2m} = \underbrace{\frac{1}{m+1}\binom{2m}{m}}_{\text{Catalan}}$$

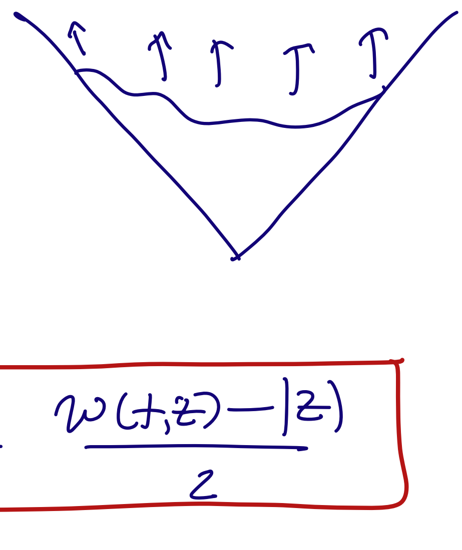
$$\Rightarrow d\pi^{\Omega}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$

semicircle density



13.4. Growth model

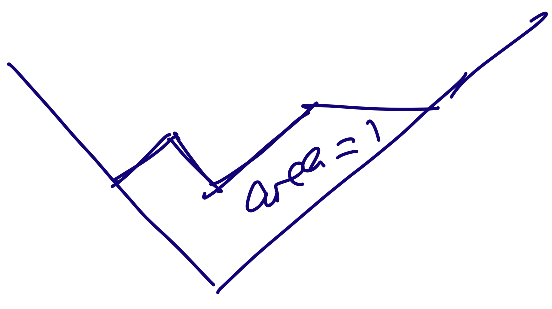
Large scale behavior of Plancherel growth.



Area $= t$ (time)

$$\| \quad \int_{\mathbb{R}} \sigma(t,z)\,dz, \qquad \boxed{\sigma(t,z) = \frac{w(t,z) - |z|}{2}}$$

Start at $t=1$



π^0

Let $T(t,z) = \frac{\partial}{\partial t} \delta(t,z)$,

So $\int_{\mathbb{R}} T(t,z)\, dz = 1$

probab. distribution

Def. Planned growth w_t:

$$\frac{\partial}{\partial t} \delta(t,x)\, dx = d\, \pi^T(w(t,\cdot))\,(x)$$

equality of 2 probab. densities

(∞-dim ODE in space of cont. r.d.

$$\boxed{\frac{\partial}{\partial t} \delta = F(\delta)}$$)

Rewrite equation

(1) $\int_{\mathbb{R}} x^n T(t,x)\, dx =$

$$= \frac{\tilde{p}_{n+2}(t)}{(n+1)(n+2)}$$

because

$$\int_{\mathbb{R}} x^n T(t,x)\, dx = \frac{\partial}{\partial t} \int_{\mathbb{R}} x^n \sigma(t,x)\, dx$$

$\int_{\mathbb{R}} x^n \sigma(t,x)\, dx =$

= twice by parts

$$= \int \frac{x^{n+2}}{(n+1)(n+2)} \sigma''(t,x)\, dx$$

$$= \frac{p_{n+2}(t)}{(n+1)(n+2)}. \qquad \square$$

② $T(t,x)\,dx = d\pi^T(\omega(t,\cdot))(x)$

$\Rightarrow$ moments :

$$\frac{\tilde{p}'_{n+2}(t)}{(n+1)(n+2)} = \overbrace{\tilde{h}_n(t)}^{\text{moments of } \pi^T}$$

③ via $S(t,z)$:

know

$$\exp\big(S(t,z)\big) = \int \frac{\overbrace{d\pi^T(w(t,0))(x)}^{\frac{\partial}{\partial t} b}}{1 - z/x}$$

$$\Rightarrow \boxed{\exp \int \frac{b'_x(t,x)\,dx}{z-x} = \int \frac{b'_t(t,x)\,dx}{1 - z/x}}$$

④ Define

"R-transform"

$$R(t, z) = \sum_{n=0}^{\infty} \tilde{h}_n(t)\, z^{-n-1}$$

$$\left(\sum_{n\geq 1}^{\infty} \frac{\tilde{p}_n(t)}{n} z^{-n}\right) = S(t,z) = \log\left(z R(t,z)\right)$$

$$\frac{\partial}{\partial t} \Longrightarrow \boxed{S'_t = \frac{R'_t}{R}}$$

& know

specific to Planar growth

$$\boxed{\frac{\tilde{p}'_{n+2}(t)}{(n+1)(n+2)} = \tilde{h}_n(t)}$$

$\Rightarrow$ R satisfies $\boxed{R'_t + R R'_z = 0}$

Indeed, $S'_t = \sum_{n=0}^{\infty} \frac{\tilde{\rho}'_{n+2}(t)}{n+2} z^{-n-2}$ $(\tilde{\rho}_1 = 0)$

$$= \sum_{n=0}^{\infty} \tilde{h}_n(t)\cdot(n+1)\, z^{-n-2}$$

$$= \boxed{-R'_z}$$

$\Rightarrow$ $-R'_z = \frac{R'_t}{R}$, $\boxed{R'_t + R R'_x = 0}$

$\left(\text{Burgers}: \quad \rho_t + (\rho(1-\rho))_x = 0 \right)$

$\boxed{\text{Appl. to VKLS}}$

For Ω at $t=1$

Let $r(x) = \frac{1}{2}\left(x - \sqrt{x^2-4}\right)$

Check :

$R(t,x) = \dfrac{r(x/\sqrt{t})}{\sqrt{t}}$ satisfies

$$\boxed{R_t' + R R_z' = 0}$$

Facts. $\dfrac{r(x/\sqrt{t})}{\sqrt{t}}$ is

(1) the unique automodel solution to

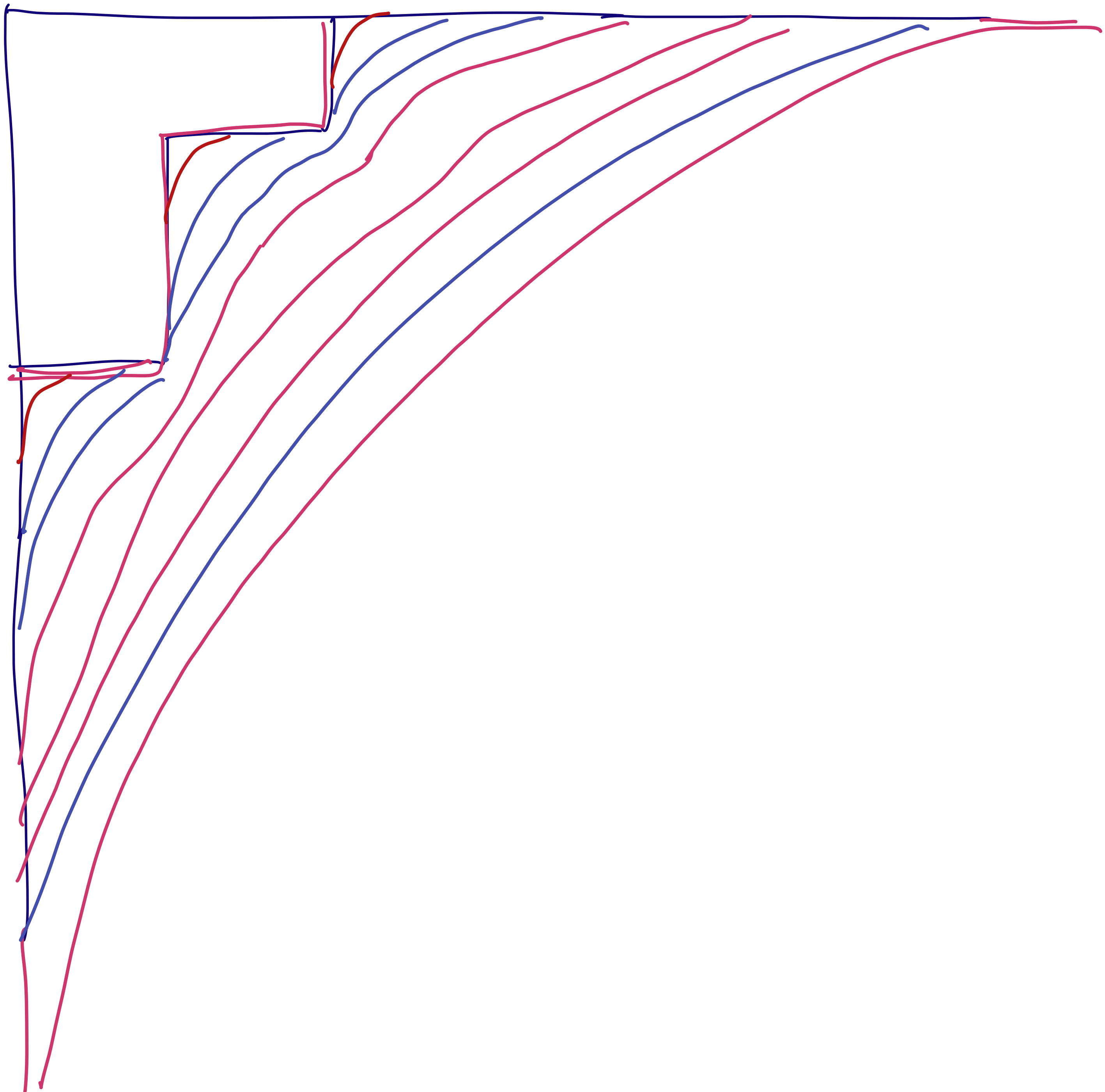
$$\boxed{R_t' + R R_z' = 0}$$

(2) Started from any continual Young diagram

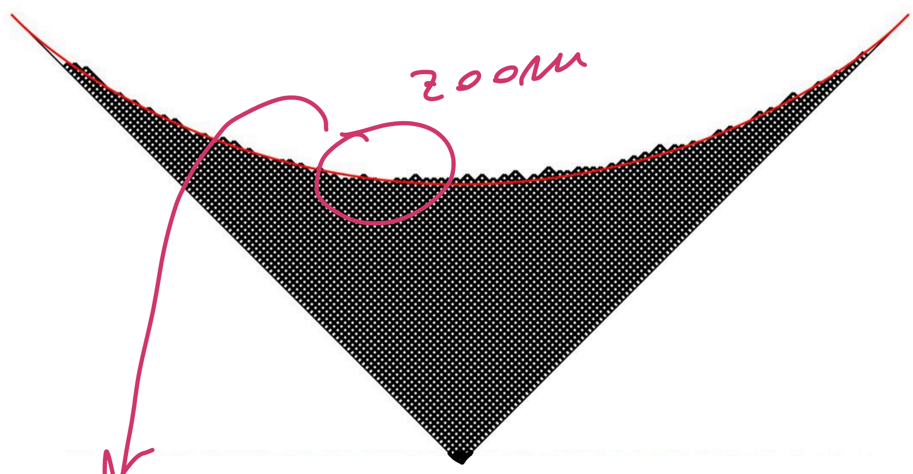
$R(t=1, x) = R_1(x)$,

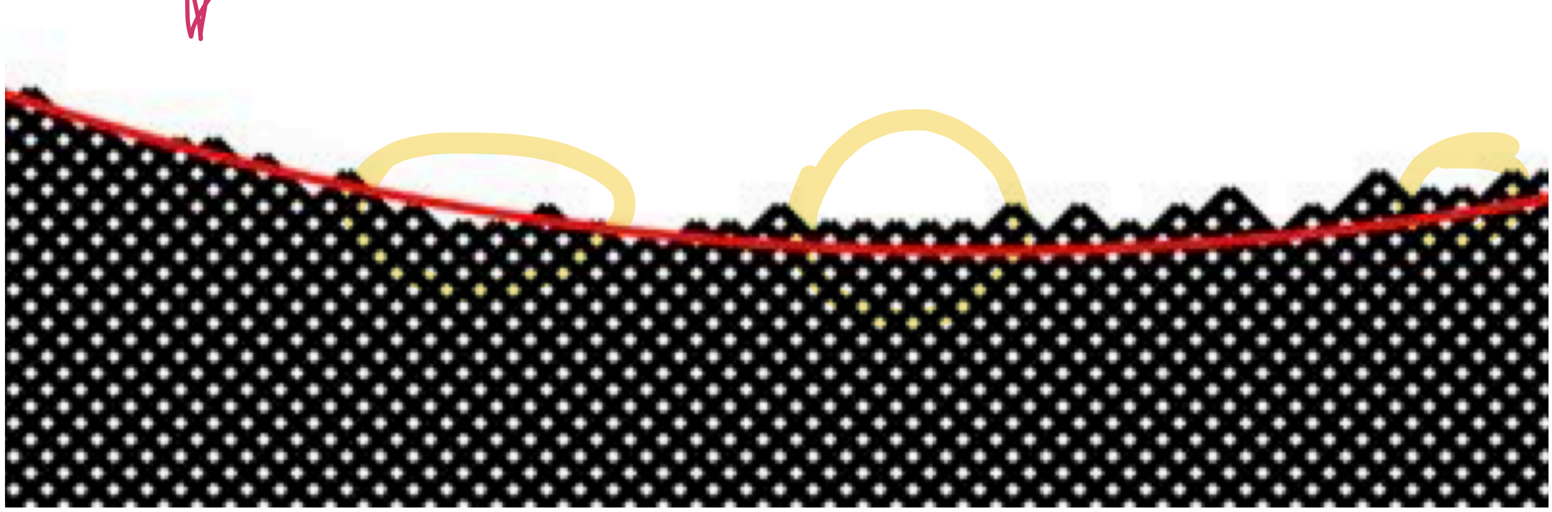
The equation's solution converges to the VKLS solution.

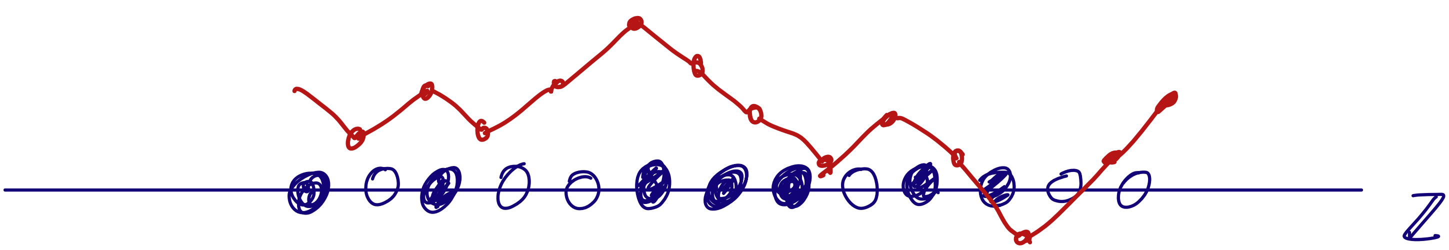
(So, $\forall$ initial y.d., the Plancherel growth produces VKLS)



14. Local correlations of Plancherel

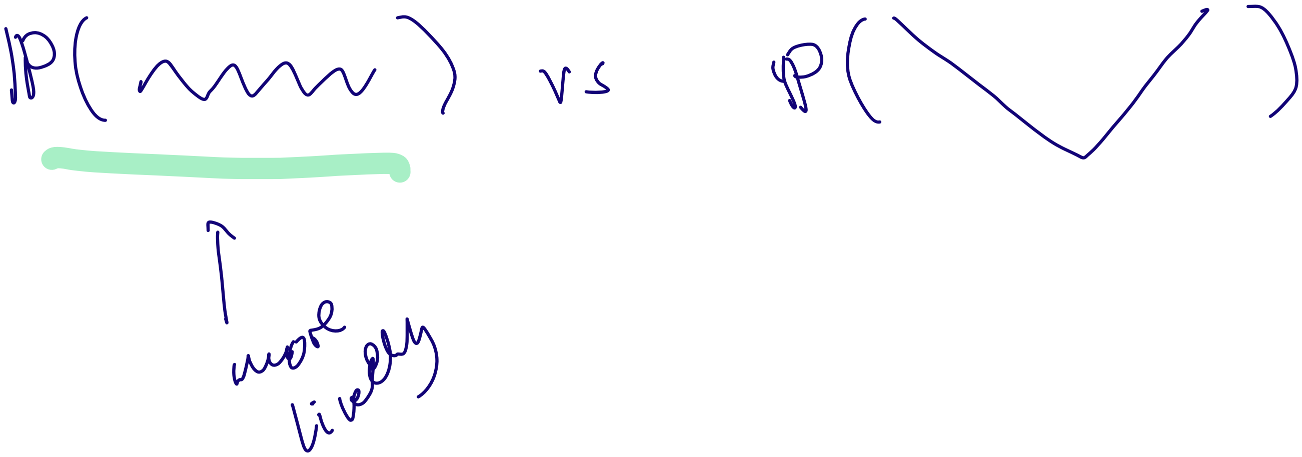






Locally Bernoulli? – $\boxed{\text{No}}$

Some other law?



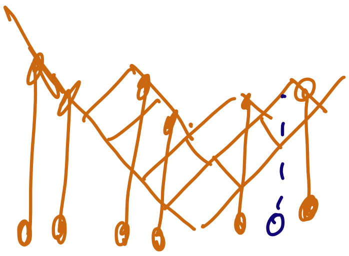
14.1 Infinite wedge space (Fock space)

[Kac ∞ dim Lie alg.]

[Okounkov 1999], ...

we only take a particular case

1. $\lambda \in \mathbb{Y} \longrightarrow \{\lambda_i - i + \frac{1}{2}\}_{i \geq 1}$



$$v_\lambda = v_{\lambda_1 - 1 + \frac{1}{2}} \wedge v_{\lambda_2 - 2 + \frac{1}{2}} \wedge \cdots$$

$$\lambda = (4,3,1) \longrightarrow \boxed{v_\lambda = v_{3+\frac{1}{2}} \wedge v_{1+\frac{1}{2}} \wedge v_{-2+\frac{1}{2}} \wedge v_{-4+\frac{1}{2}} \wedge v_{-5+\frac{1}{2}} \cdots}$$

$$v_\emptyset = v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge \cdots$$

(„vacuum")

$$\langle v_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$$ (Hilbert space)

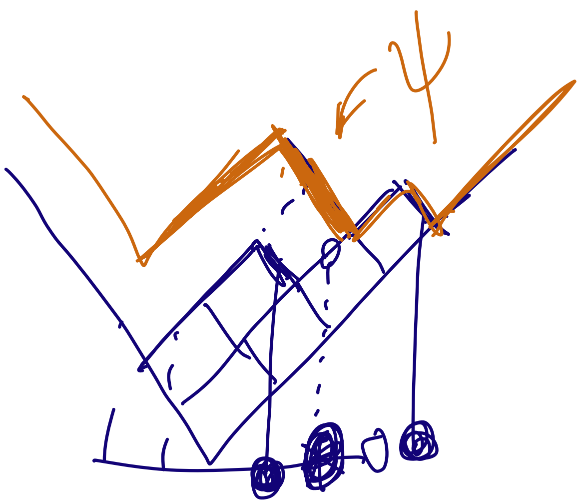
$\boxed{\ell^2(\mathbb{Y})}$

(2) $\psi_i, \ \psi_i^*$ $\qquad \boxed{i \in \mathbb{Z} + \frac{1}{2}}$

create $\qquad$ conjugate (annihilate)

$$\psi_i v_\lambda = v_i \wedge v_\lambda$$

anticommute to the place



$$v_i \wedge v_i = 0$$

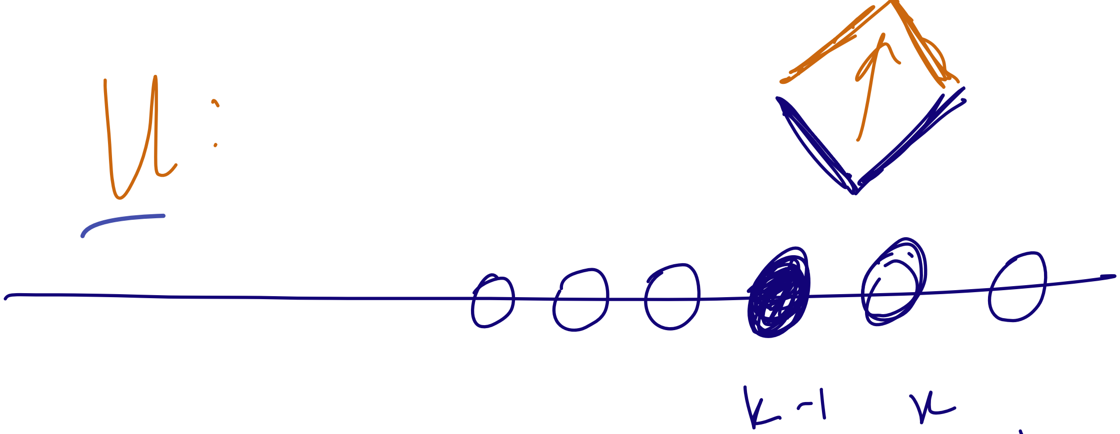
ψ_i^* - conjugate, removes v_i if it can

③ Let
$$\boxed{\begin{aligned} U v_\lambda &= \sum_{\mu = \lambda + \square} v_\mu \\ D v_\lambda &= \sum_{\mu = \lambda - \square} v_\mu \end{aligned}}$$

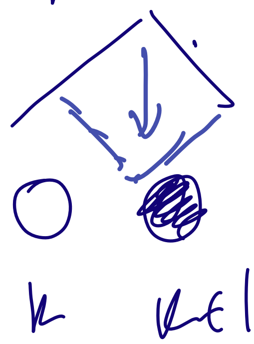
<u>Lemma</u>.
$$\boxed{\begin{aligned} U &= \sum_k \psi_k \psi^*_{k-1} \\ D &= \sum_k \psi_k \psi^*_{k+1} = U^* \end{aligned}}$$

<u>Proof</u>.

$\underline{\mathcal{U}}$:



$\underline{D}$:



(4) dim λ via $\boxed{\mathcal{U}, D}$

$$\dim \lambda = \langle \mathcal{U}^n v_\varnothing, v_\lambda \rangle, \quad |\lambda| = n$$
$$= \langle D^n v_\lambda, v_\varnothing \rangle$$

Plancherel measure via $\mathcal{U}, D$

$$M_n(\lambda) = \frac{1}{n!} \underline{\langle \mathcal{U}^n v_\varnothing, v_\lambda \rangle} \underline{\langle D^n v_\lambda, v_\varnothing \rangle}$$

vs $\langle B^? v_\varnothing, v_\varnothing \rangle$

Poissonized Plancherel (θ^2 - Parameter)

n - Poisson random $\sim \theta^2$

$$M_{\theta^2}(\lambda) = \text{Prob}(\pi_{\theta^2} = n) \cdot M_n(\lambda)$$

$$= e^{-\theta^2} \theta^{2n} \left(\frac{\dim \lambda}{n!}\right)^2 \quad \boxed{n = |\lambda|}$$

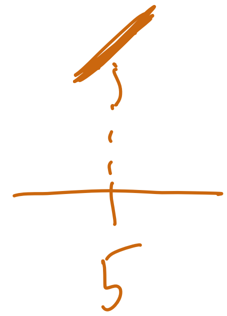
$$\# \text{boxes} \approx \theta^2 \pm c \cdot \theta$$

$$\langle e^{\theta U} v_\emptyset, v_\lambda \rangle = \sum_{n \geq 0} \frac{\theta^n}{n!} \langle U^n \emptyset, v_\lambda \rangle$$

$$M_{\theta^2}(\lambda) = e^{-\theta^2} \langle e^{\theta D} \mathbb{1}_\lambda e^{\theta U} v_\emptyset, v_\emptyset \rangle$$

$\mathbb{1}_\lambda$ operator, $\mathbb{1}_\lambda v_\mu = \begin{cases} v_\lambda, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}$

Ex. $\mathrm{Prob}\left(\exists\, i : \lambda_i - i + \frac{1}{2} = 5\right)$



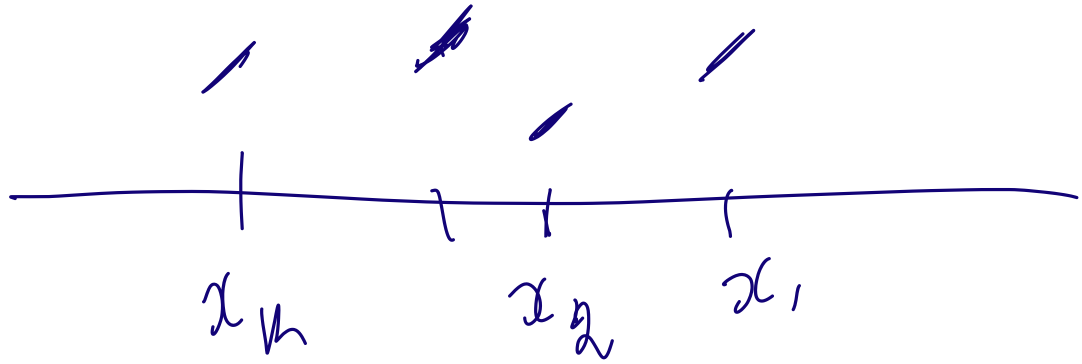
$$e^{-\theta^2} \left\langle e^{\theta D} \left(\text{indicator that } \bullet \text{ at } 5 \right) e^{\theta U} v_\emptyset, v_\emptyset \right\rangle$$

⑤ Correlation function of Poissonized Plancherel & its expression via ∞ wedge space

$\forall k, \quad x_1 \ldots x_k \in \mathbb{Z} + \frac{1}{2}$ distinct

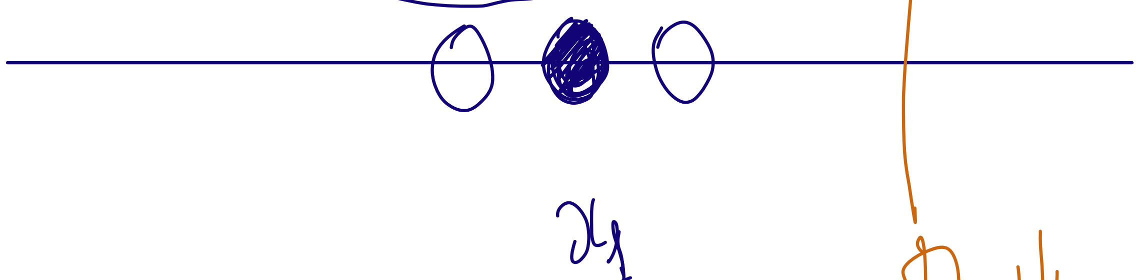
$X = \{x_1 \ldots x_k\}$

$\rho_k(X)$, $\rho(X) = \text{Prob.}\left(\begin{array}{l} \text{config } \{\lambda_i - i + \tfrac{1}{2}\}_{i=1,2,\ldots} \\ \text{contains each} \\ \text{of } x_1, x_2, \ldots, x_k \end{array} \right)$



$$= e^{-\theta^2} \left\langle e^{\theta D} \, \text{Indicator}(x_1 \ldots x_k) \, e^{\theta U} v_\varnothing, v_\varnothing \right\rangle$$

$\boxed{\psi_{x_1} \psi^*_{x_1}}$



$\text{Indicator}(x_1 \ldots x_k) = \prod_i \psi_{x_i} \psi^*_{x_i}$

$$e^{-\theta^2}\left\langle e^{\theta D}\prod_{i=1}^{k}\psi_{x_i}\psi^*_{x_i}\, e^{\theta U}\, v_\emptyset, v_\emptyset\right\rangle$$

$$\parallel$$

$$\rho_k(X)$$

Want to show:

$$\rho_k(X) = \det\left[\, K(x_i, x_j)\,\right]_{i,j=1}^{k}$$

Wick theorem

because

① $\psi_i\psi_j + \psi_j\psi_i = 0$

$$\psi_i^{0*}\psi_j^{0*} + \psi_j^{0*}\psi_i^{0*} = 0$$

$$\psi_i^{0}\psi_j^{0*} + \psi_j^{0*}\psi_i^{0} = 1_{i=j}$$

(2)

$$\left\langle e^{\theta D} \prod_i \psi_{x_i}\psi_{x_i}^{*}\, e^{\theta U} v_\emptyset, v_\emptyset \right\rangle$$

$$= \left\langle \prod_i \underbrace{\bar\Psi_{x_i}}\ \underbrace{\bar\Psi^{*}_{x_i}}\, v_\emptyset, v_\emptyset \right\rangle$$

linear combin. of ψ_j, ψ_j^* resp.

$$\left\langle \prod_i \bar\Psi_{x_i}\bar\Psi^{*}_{x_i}\, v_\emptyset, v_\emptyset \right\rangle$$

14.2 Correlations & density — formulas

14.1 Infinite wedge & random partitions

$$M_\theta(\lambda) = e^{-\theta^2}\, \theta^{2|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2 ,$$

$|\lambda| \sim$ Poisson (θ^2)

measure on whole $\mathbb{Y}$

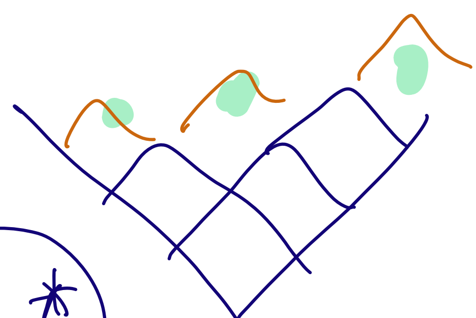
$\mathbb{E}\,|\lambda| = \theta^2$

$$|\lambda| \to \infty \iff \theta \to \infty$$

$$M_\theta(\lambda) = e^{-\theta^2} \left\langle e^{\theta D}\, \mathbb{1}_\lambda\, e^{\theta U} v_\emptyset ,\ v_\emptyset \right\rangle$$

v_λ $\ell^2(\mathbb{Y})$

$v_\emptyset \leftarrow \emptyset$ empty



$$U v_\lambda = \sum_{\mu = \lambda + \square} v_\mu$$

$D = U^*$

$$D v_\lambda = \sum_{\mu=\lambda-\square} v_\mu$$

$$1\!\!1_\lambda v_\mu = \begin{cases} v_\lambda, & \mu=\lambda \\ 0, & \text{else} \end{cases}$$

$\ell^2(\mathbb{Y})$

$$\psi_j v_\lambda = \underbrace{v_j \wedge v_\lambda}_{\begin{cases} =0 & \text{if } j \in \{\lambda_i - i + \frac{1}{2}\} \\ = \pm v_{\lambda \cup j} & \end{cases}}$$

create at j

$j \in \mathbb{Z} + \frac{1}{2}$

ψ_j^* — adjoint

<u>Anticomm rel.</u>

$$\psi_i \psi_j + \psi_j \psi_i = 0$$

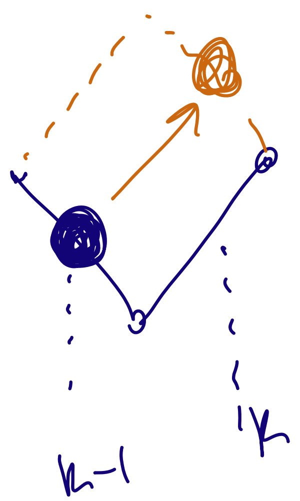
$$\psi_i^{0*} \psi_j^{0*} + \psi_j^{0*} \psi_i^{0*} = 0$$

$$\psi_i^0 \psi_j^{0*} + \psi_j^{0*} \psi_i^0 = 1_{i=j}^0$$

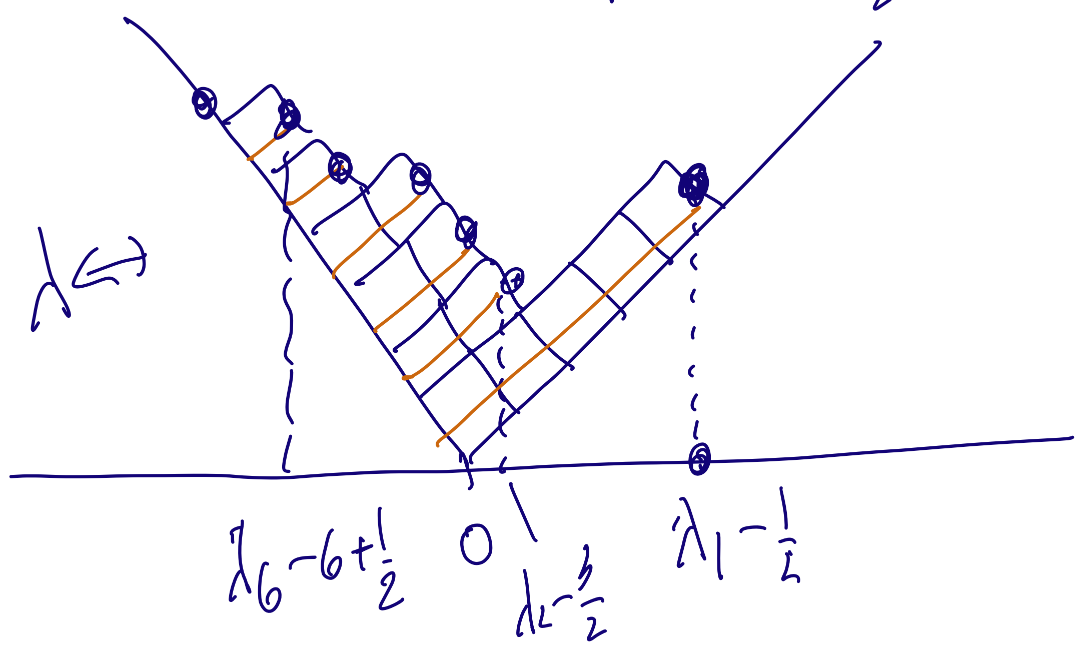
$$\boxed{DU - UD = 1}$$

$$u = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k \psi^*_{k-1}, \qquad D = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k \psi^*_{k+1}$$

(α_{-1}) $\qquad$ (α_{+1})



$$v_\lambda = v_{\lambda_1-\frac{1}{2}} \wedge v_{\lambda_2-\frac{3}{2}} \wedge \cdots$$



$X = \{x_1, \dots x_k\} \subset \mathbb{Z} + \frac{1}{2}$

$$\rho(X) \overset{\text{def}}{=} M_\theta \left(\begin{matrix} \text{configuration} \\ \{\lambda_i - i + \tfrac{1}{2}\}_{i=1,2,3,\dots} \\ \text{contains all} \\ x_1, x_2, \dots x_k \end{matrix} \right)$$

correlations

$X(\lambda) \subset \mathbb{Z} + \frac{1}{2}$

Prop (proved)

$e^{-\theta \mathcal{U}}$ $e^{-\theta D}$

$$\rho(X) = e^{-\theta^2} \left\langle e^{\theta D} \left(\prod_{i=1}^{k} \psi_{x_i} \psi_{x_i}^* \right) e^{\theta \mathcal{U}} v_\varnothing, v_\varnothing \right\rangle$$

$$\psi_x \psi_x^* v_\lambda = \begin{cases} v_\lambda, & x \in X(\lambda) \\ 0, & \text{else} \end{cases}$$

Goal:

$$\boxed{\rho(X) = \det{}_{k \times k}}$$

14.2 Determ. formula for $\rho(x)$

1) $e^{\theta D} v_\phi = v_\phi = (e^{\theta u})^* v_\phi$

2) $\boxed{e^{\alpha D} e^{\beta u} = e^{\alpha\beta} e^{\beta u} e^{\alpha D}}$ $\Leftarrow$ $[D, u] = 1$

<u>Proof of 2)</u>

(a) differential proset

(b) $\exists$ skew Cauchy identity proof

$Du = 1 + uD$

$$D^n u = D^{n-1} D u$$
$$= D^{n-1} (uD + 1)$$
$$= D^{n-1} + D^{n-1} u D$$
$$= \underline{n D^{n-1} + u D^n}$$

(b) we have skew Cauchy id. & specialization.

$$\begin{tikzcd} \lambda & \nu \\ \varkappa \arrow[u, dash, "x_1, x_2 \dots"] \arrow[r, dash, "y_1, y_2 \dots"'] & \mu \end{tikzcd}$$

$$\sum_{\nu} s_{\nu/\mu}(\vec{x})\, s_{\nu/\lambda}(\vec{y}) = \frac{1}{\prod(1-x_i y_j)} \sum_{\varkappa} s_{\lambda/\varkappa}(\vec{x})\, s_{\mu/\varkappa}(\vec{y})$$

3) $$e^{\alpha D} \underbrace{e^{\beta u}}_{u^n} e^{-\alpha D} = e^{\alpha\beta} e^{\beta u}$$

$$e^{\alpha D}\, u^n\, e^{-\alpha D}$$

$$= e^{\alpha \cdot \mathrm{ad}\, D}\, u^n$$

$$= \sum_k \frac{\alpha^k}{k!} \underbrace{(\mathrm{ad}\, D)^k u^n}$$

$$\underbrace{\cdots\, [D,[D,}_{k \text{ commutators}}[D,u^n]]]\,\cdots$$

$$e^{-\theta u} \qquad e^{-\theta D}$$

$$\rho(X) = e^{-\theta^2}\left\langle e^{\theta D}\left(\prod_{i=1}^{k}\psi_{x_i}\psi^*_{x_i}\right)e^{\theta U}v_\emptyset, v_\emptyset\right\rangle$$

Define $\quad G = e^{\theta D}e^{-\theta U}, \quad G^{-1} = e^{\theta U}e^{-\theta D}$

$$\Psi_k = G\psi_k G^{-1}$$

$$\Psi^*_k = G\psi^*_k G^{-1}$$

Prop. $\rho(X) = \langle \Psi_{x_1}\Psi^*_{x_1}\cdots\Psi_{x_n}\Psi^*_{x_n}v_\emptyset, v_\emptyset\rangle$

Proof. $\square$

$$\psi_x\psi^*_y + \psi^*_y\psi_x = 1_{x=y}$$

$$\boxed{\begin{array}{l}\text{Note:}\\ \rho(\emptyset) = 1\\ \quad \parallel\\ \langle v_\emptyset, v_\emptyset\rangle\end{array}}$$

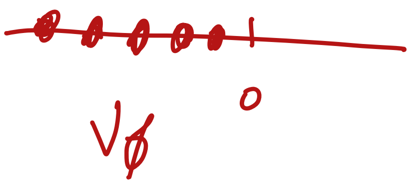
Prop. Wick theorem

$$\rho(X) = \det\left[K(x_i, x_j)\right]_{i,j=1}^{n}$$

$$K(x,y) = \langle \psi_x \psi_y^* v_\emptyset, v_\emptyset \rangle$$

$$= \| \psi_{x_n}^* \cdots \psi_{x_1}^* v_\emptyset \|^2$$

Proof. $$\rho(X) = \langle \underbrace{\psi_{x_1} \psi_{x_2} \cdots \psi_{x_n}}_{\text{put } n \text{ back}} \underbrace{\psi_{x_n}^* \cdots \psi_{x_1}^*}_{\text{remove } n} v_\emptyset, v_\emptyset \rangle$$



$n!$ orders, get sign

$$\underbrace{\langle\ \rangle \langle\ \rangle \cdots \langle\ \rangle}_{n}$$

$\square$

Crucial (hidden) algebraic property

$$\boxed{\Psi_k = \sum_m c_m \psi_k} \, , \quad \boxed{\Psi_k^* = \sum_m c_m^* \psi_k^*}$$

(Ψ_k, Ψ_k^* belong to the same algebra as generated by small $\psi_i, \psi_i^*, \ldots$)

Def. $\boxed{\psi(z) = \sum z^k \psi_k}$, $\boxed{\psi^*(w) = \sum w^{-k} \psi_k^*}$

Prop. $[D, \psi(z)] = z\psi(z)$

$[D, \psi^*(w)] = -w\psi^*(w)$

$[U, \psi(z)] = z^{-1}\psi(z)$

$[U, \psi^*(w)] = -w^{-1}\psi^*(w)$

Indeed :

$$\sum_{i,k}\big(\psi_i \underbrace{\psi^*_{i+1}\ \psi_k}_{k=i+1}\, z^k + z^k\, \psi_i\, \psi_k\, \psi^*_{i+1}\big)$$

$k = i+1$ (otherwise 0)

$$\Rightarrow \boxed{\sum_i \psi_i\, z^{i+1} = z\,\psi(z)}$$

$\Downarrow$

$$\Psi_k = e^{\theta D} e^{-\theta U} \psi_k e^{\theta U} e^{-\theta D}$$

$$\Psi(z) = \underbrace{e^{\theta D} \underbrace{e^{-\theta U} \psi(z) e^{\theta U}} e^{-\theta D}}$$

(2 adjoint actions)

$$e^{-\theta U} \psi(z) e^{\theta U} = e^{-\theta\, \mathrm{ad}_U} \psi(z)$$

$$= \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!} \boxed{(\mathrm{ad}\, U)^n \psi(z)}$$

$[\ldots[U, \psi(z)]]$

z^{-n}

$$= \boxed{e^{-\theta z^{-1}}}\, \psi(z)$$

= inf. linear combination
of the ψ_k's.

$$\boxed{\tilde{\psi}^{+}(z) = \psi(z) \cdot e^{\theta(z - z^{-1})}}$$

$\square$

14.3 Double contour integrals

Remains to compute

$$K(x,y) = \langle \psi_x \psi_y^* v_\emptyset, v_\emptyset \rangle$$

$$\tilde{K}(z,w) = \sum_{x,y} K(x,y)\, z^x w^{-y}$$

$$= \langle \psi(z) \psi^*(w) v_\emptyset, v_\emptyset \rangle$$

$$\boxed{J(z) = e^{\theta(z - z^{-1})}}$$

$$\Gamma \psi^*(z) = \psi^*(z)\, J(z)^{-1}$$

$$\Rightarrow \tilde{K}(z,w) = \frac{J(z)}{J(w)} \langle \psi(z)\psi^*(w) v_\emptyset, v_\emptyset \rangle$$

$$\psi_{-j} z^{-j} w^j \psi^*_{-j}$$

$$j > 0$$

$$= \frac{J(z)}{J(w)} \sum_{j=\frac{1}{2},\frac{3}{2},\dots} \frac{w^j}{z^j}$$

$$= \frac{J(z)}{J(w)} \frac{\sqrt{zw}}{z-w}, \quad |w|<|z|$$

$$\Rightarrow \quad K(x,y) = [z^x w^{-y}] \, \tilde{K}(z,w)$$

$$= \frac{1}{(2\pi i)^2} \oint\oint_{\substack{|w|<|z| \\ \text{around } 0}} \frac{z^{1/2-x-1} w^{1/2+y-1}}{z-w} \cdot \frac{e^{\theta(z-z^{-1})}}{e^{\theta(w-w^{-1})}}$$

(complete info on Plancherel measure!)

well, Poissonized

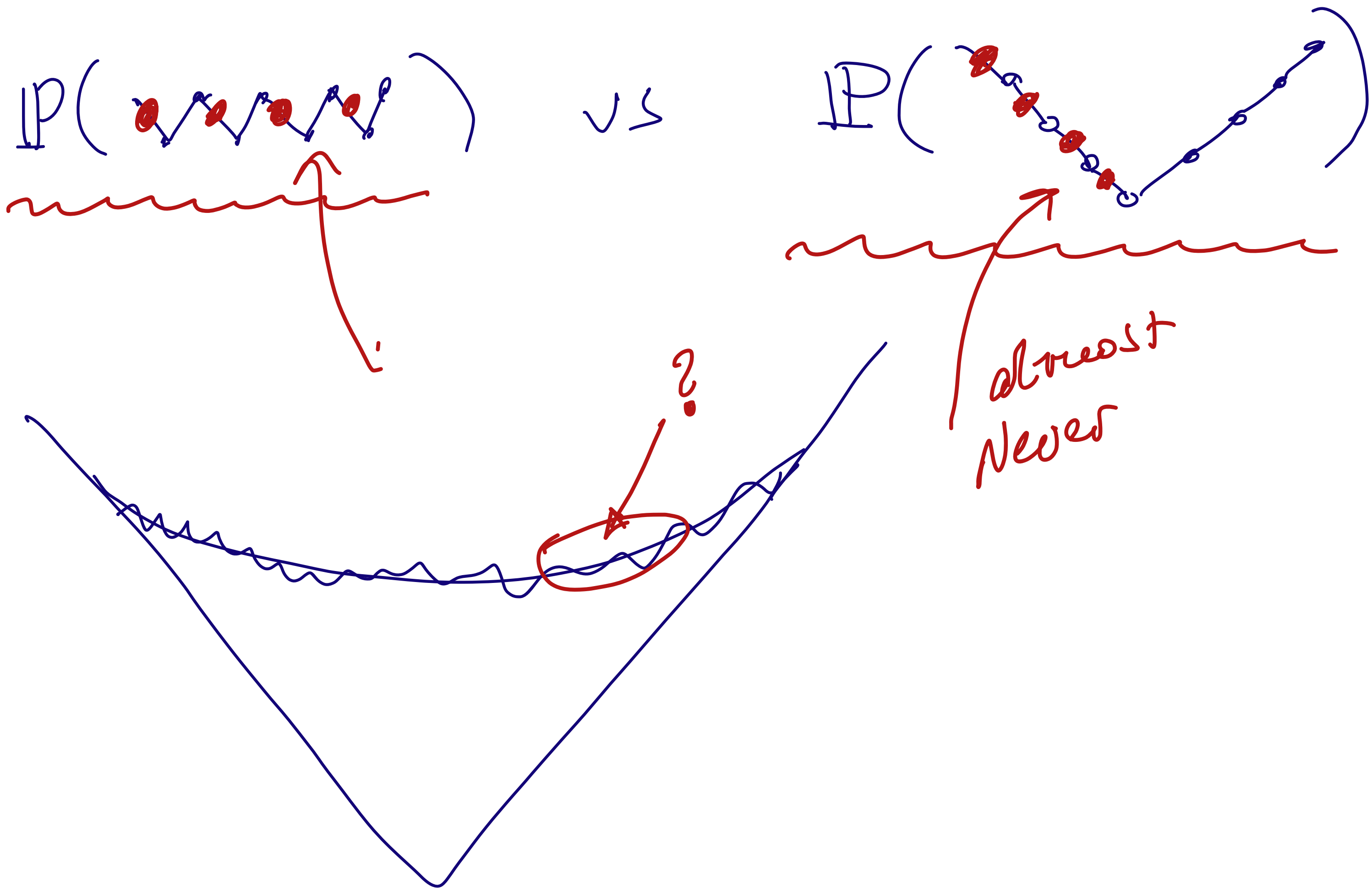
On correlations & repulsion

$$\rho\Big(\underset{\frac{1}{2}}{\bullet}\ \underset{\frac{3}{2}}{\bullet}\Big) = \det\begin{bmatrix} K\left(\frac{1}{2},\frac{1}{2}\right) & K\left(\frac{1}{2},\frac{3}{2}\right) \\ K\left(\frac{3}{2},\frac{1}{2}\right) & K\left(\frac{3}{2},\frac{3}{2}\right) \end{bmatrix}$$

$$K\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \rho\Big(\underset{\frac{1}{2}}{\bullet}\Big) \qquad K\left(\tfrac{3}{2},\tfrac{3}{2}\right) = \rho\Big(\underset{\frac{3}{2}}{\bullet}\Big)$$

$$= \underbrace{\rho\Big(\underset{\frac{1}{2}}{\bullet}\Big)\rho\Big(\underset{\frac{3}{2}}{\bullet}\Big) - \underbrace{K\left(\tfrac{1}{2},\tfrac{3}{2}\right)K\left(\tfrac{3}{2},\tfrac{1}{2}\right)}_{\text{usually } \geq 0}}_{\text{repulsion}}$$

$$\boxed{\rho(\bullet\ \bullet \mid \bullet\ *) < \rho(*\ \bullet)}$$

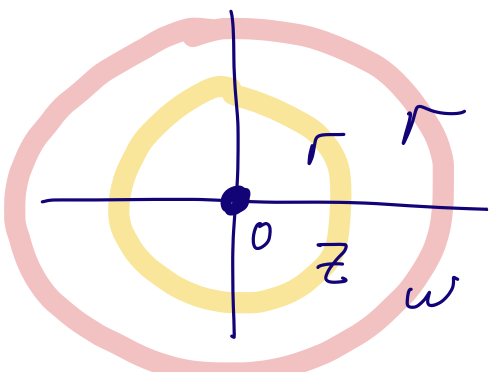

P( )
vs
P( )
!
?
almost
Never

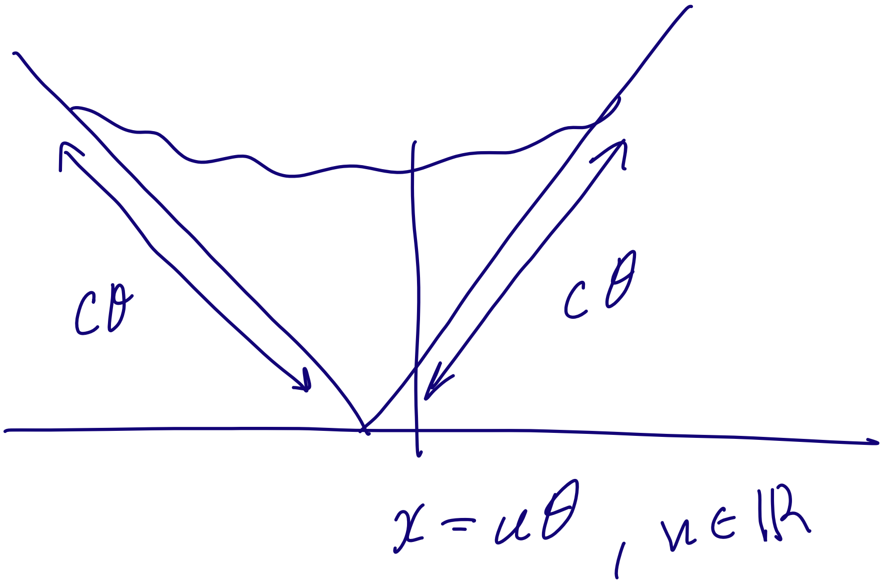
14.4. Asymptotics of density via $\iint$

$$\rho(x) = \mathrm{Prob}(\text{at } x \text{ we see } \bullet)$$

$x \in \mathbb{Z} + \frac{1}{2}$

$$K(x,x) = \frac{1}{(2\pi i)^2} \iint_{\substack{|w|<|z| \\ \text{around } 0}} \frac{z^{-x-1/2}\, w^{x-1/2}}{z-w} \, \frac{e^{\theta(z-z^{-1})}}{e^{\theta(w-w^{-1})}}$$



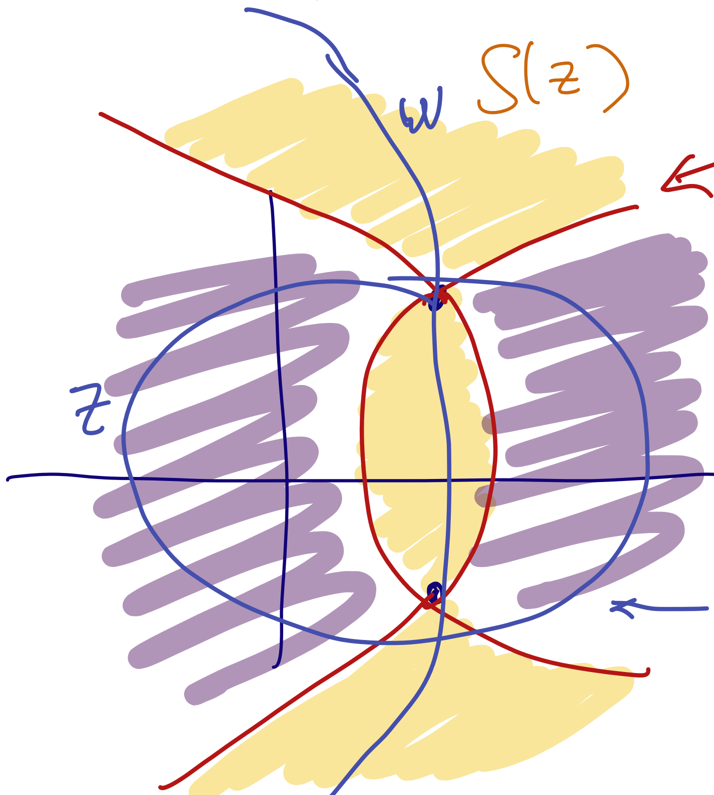


$$e^{\theta(S(z)-S(w))}$$

$$\boxed{K(u\theta, u\theta)}$$

Let $x = \theta u$ & $\theta \to \infty$

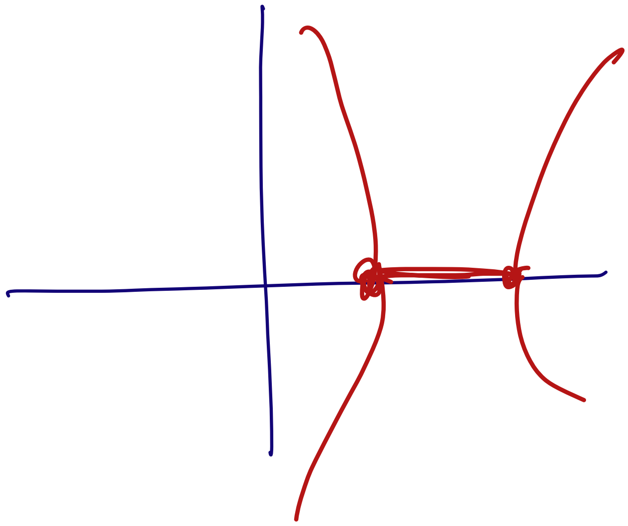
$$\exp\Big[\theta\,\underbrace{\left(z - z^{-1} - u\cdot\log z\right)}_{S(z)}\Big]$$



$\mathrm{Re}\, S(z) = \mathrm{Re}\, S(z_c)$

$|u| < 2$

$e^{\theta\cdot(\text{negative})} \longrightarrow 0$



$|u| \geq 2$

Look for critical points of $S(z)$

$$S'(z) = 1 + \frac{1}{z^2} - \frac{u}{z} = 0$$

$z^2 - uz + 1 = 0$, $\boxed{z_c = \dfrac{\sqrt{u^2-4}+u}{2}}$

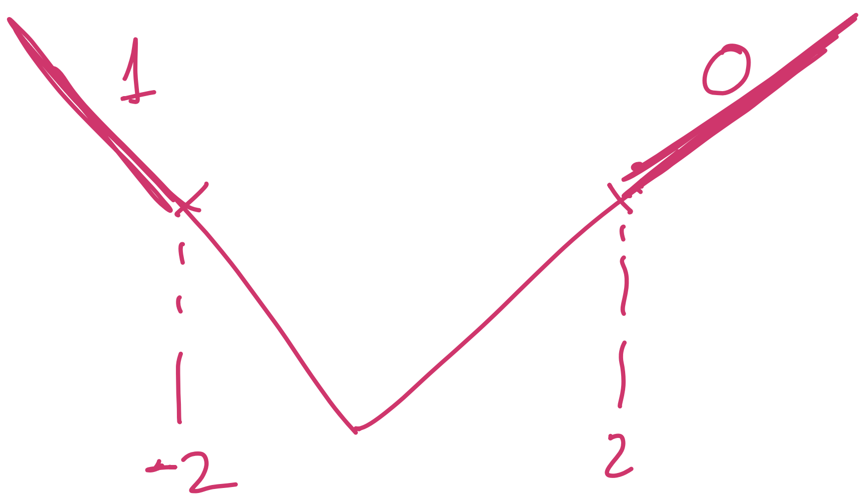
Cases: $|u| \geq 2$, 2 real or 1 real

$|u| < 2$, 2 complex

Real: $|u| \geq 2$

$K(u\theta, u\theta) \longrightarrow 0$ or 1

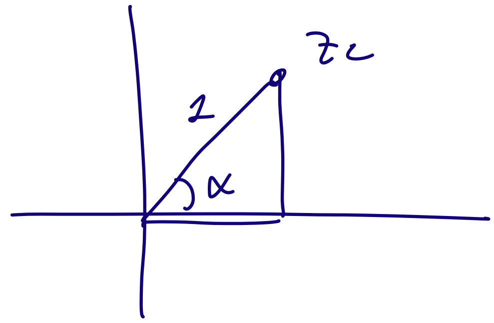
expon. fast



Complex : $k(x, x) \longrightarrow$ single $\int_{-z_c}^{z_c}$

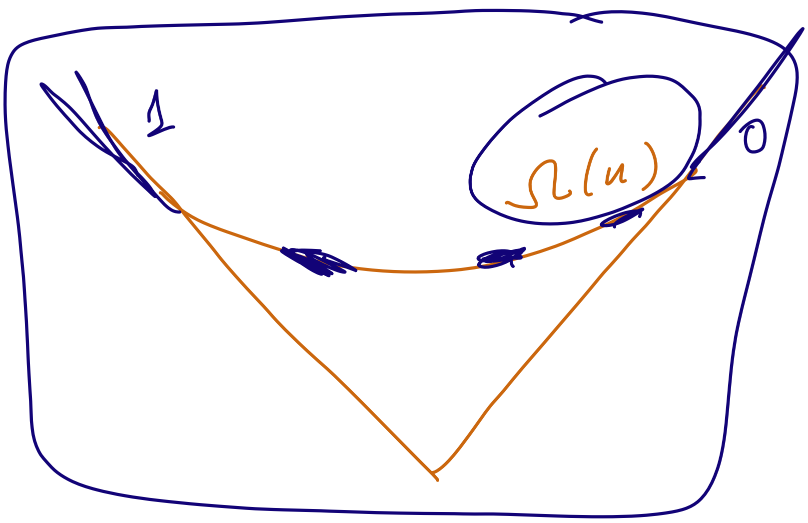
$$\frac{1}{2\pi i}\int_{\bar{z}_c}^{z_c}\frac{dw}{w} = \boxed{\frac{\arg z_c}{\pi}}$$

$$z_c = \frac{\sqrt{u^2-4}+u}{2},$$



$$|z_c| = \frac{1}{4}\left[4-u^2+u^2\right] = 1$$

$$\underline{\arg z_c = \arccos\left(\frac{u}{2}\right)}, \qquad |u|<2$$



$$\underline{K(u\theta, u\theta)}$$

$$\swarrow \quad \theta\to\infty$$

$$\boxed{\frac{1}{\pi}\arccos\frac{u}{2}} \leftarrow$$

$\Omega'(u) \in [-1,1)$ $\leftarrow$

ef

$$\boxed{\frac{1-\Omega'(u)}{2}} = \boxed{\text{density at } u}$$

$$= \boxed{\left(\arccos \tfrac{u}{2}\right) / \pi}$$

$\Rightarrow$ $\Omega(u)$ = VKLS

$$= \begin{cases} \dfrac{2}{\pi}\left(u \arcsin \dfrac{u}{2} + \sqrt{4-u^2}\right), & |u| \le 2 \\ |u|, & |u| \geq 2 \end{cases}$$

$\square$