Markov Dynamics on Interlacing Arrays

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Main goals:
By introducing suitable axioms, unify and deform existing nice Markov dynamics (related to Dyson Brownian motion). Get new models and examples out of this.

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1. "Schur level"
   - Dyson Brownian motion and its discrete version
   - Push-block dynamics
   - RSK dynamics
   - **Unifying axioms**
   - **New RSK correspondences**
2. "Macdonald level"
   - From Schur to Macdonald
   - q-deformed 1d particle systems: **new examples**
   - **Randomized insertion algorithm for triangular matrices over a finite field**

"Schur level"

1. Dyson Brownian motion and its discrete version
2. Push-block dynamics
3. RSK dynamics
4. Unifying axioms
5. New RSK correspondences

Dyson Brownian motion: GUE

GUE random matrix of size $N \times N$ has density with respect to the Lebesgue measures on Hermitian $N \times N$ matrices given by

$$e^{-\mathrm{Tr}(X^2)/2} = \prod_{i=1}^{N} e^{-x_{ii}^2/2} \prod_{1 \le i < j \le N} e^{-(\Re x_{ij})^2} e^{-(\Im x_{ij})^2}, \quad X = [x_{ij}]_{i,j=1}^{N}.$$

Equivalently, the N^2 quantities

$$(x_{ii}; \sqrt{2} \cdot \Re x_{ij}, \sqrt{2} \cdot \Im x_{ij} : i < j)$$

are independent identically distributed standard normal random variables.

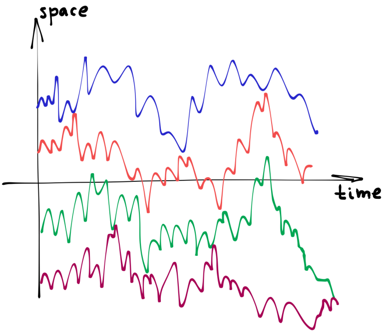
Dyson Brownian motion

Let $(x_{ii}; \sqrt{2} \cdot \Re x_{ij}, \sqrt{2} \cdot \Im x_{ij} : i < j)$ evolve as independent Brownian motions.

Then the eigenvalues $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, N$ perform a Markovian evolution — Dyson Brownian motion [Dyson '62]

Dyson BM
= independent BM's
conditioned to never collide

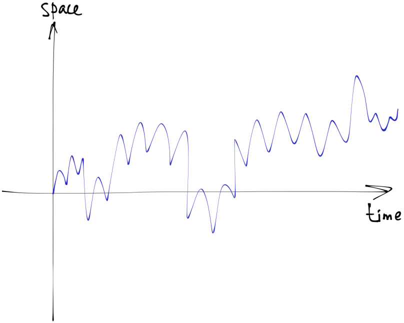
$$d\lambda_i = dB_i + \sum_{j \neq i} \frac{d\tau}{\lambda_i - \lambda_j}$$



Dyson Brownian motion

Apart from the GUE construction, there are two more multilayer (hierarchical) constructions of Dyson BM:

1. Path transformation of independent Brownian motions related to Robinson–Schensted–Knuth correspondence, e.g., [O'Connell '03]
2. Warren's construction '07 (see picture)



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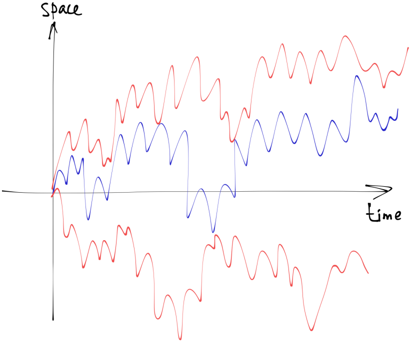

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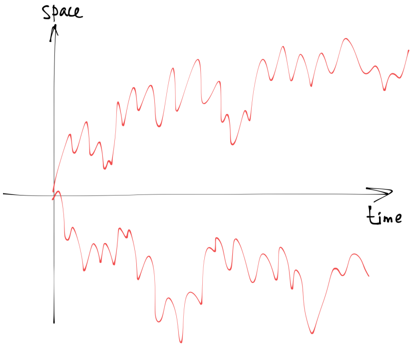

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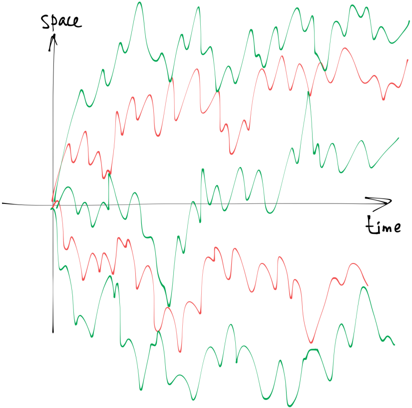

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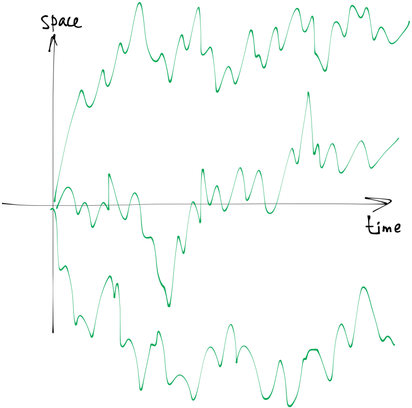
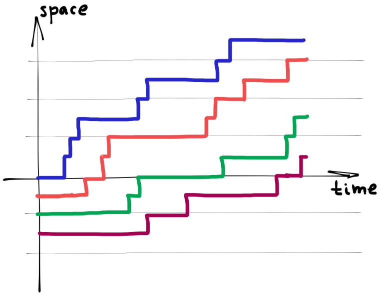


Dyson Brownian motion: discrete analogue

State space: $\{\lambda_N \le \lambda_{N-1} \le \ldots \le \lambda_1\}$, $\lambda_i \in \mathbb{Z}$.

N independent Poisson growth processes conditioned to never collide



(this picture is in *shifted* coordinates $\{\lambda_j - j\}_{j=1}^{N}$)

$$\text{jump rate}(\lambda \to \nu) = \prod_{i<j} \frac{\nu_i - i - \nu_j + j}{\lambda_i - i - \lambda_j + j} \cdot \mathbf{1}_{\nu = \lambda + \mathrm{e}_m \text{ for some } m}$$

(As the Dyson BM, this is also a “complicated” dynamics)

Discrete Dyson Brownian motion: fixed time distributions ("discrete GUE spectrum")

Start the dynamics from $\lambda_1(0) = 0, \dots, \lambda_N(0) = 0$ (shifted will be $(-N+1, \dots, -2, -1)$). Then ([Fulman], [Johansson], [Okounkov], late 1990s])

Schur polynomials

$$\mathbb{P}\left(\begin{array}{c}(0,\dots,0)\to(\lambda_N\le\dots\le\lambda_1)\\ \text{during time } \tau\end{array}\right) = \underbrace{\frac{1}{Z}\, s_\lambda(\overbrace{1,\dots,1}^{N})\, s_\lambda(\rho_\tau)}_{\text{Schur measure}}$$

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Schur polynomials

$$\mathbb{P}\left(\begin{matrix}(0,\dots,0)\to(\lambda_N\le\dots\le\lambda_1)\\ \text{during time } \tau\end{matrix}\right) = \underbrace{\frac{1}{Z}\, s_\lambda(\overbrace{1,\dots,1}^{N})\, s_\lambda(\rho_\tau)}_{\text{Schur measure}}$$

Next: Two multilayer (hierarchical) constructions of discrete Dyson Brownian motion and Schur measures. And other constructions like this.

(want to understand dynamics "simpler" than the discrete Dyson BM)

“Schur level”

1. Dyson Brownian motion and its discrete version
2. Push-block dynamics
   - Interlacing integer arrays
   - Definition of the push-block dynamics
   - Asymptotic properties: KPZ universality
   - 1d Markovian projections
3. RSK dynamics
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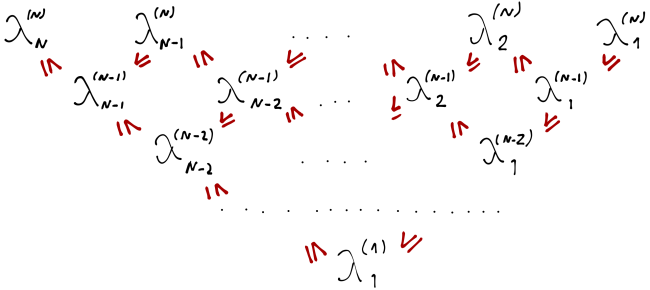
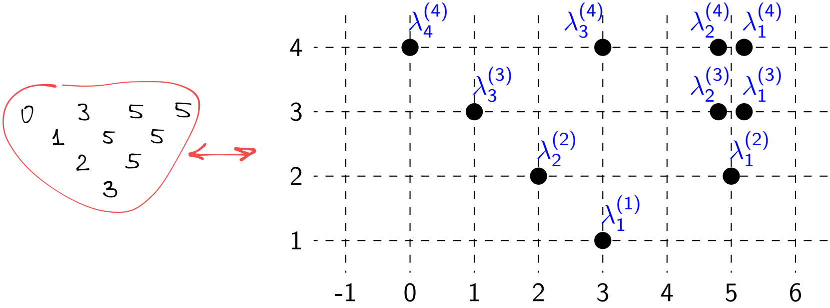
Interlacing integer arrays (= Gelfand-Tsetlin schemes)



Main object: continuous-time Markov dynamics on the space of interlacing integer arrays.

interlacing integer arrays $\longleftrightarrow$ particles in 2 dimensions



1 particle at level 1,
2 particles at level 2, etc.

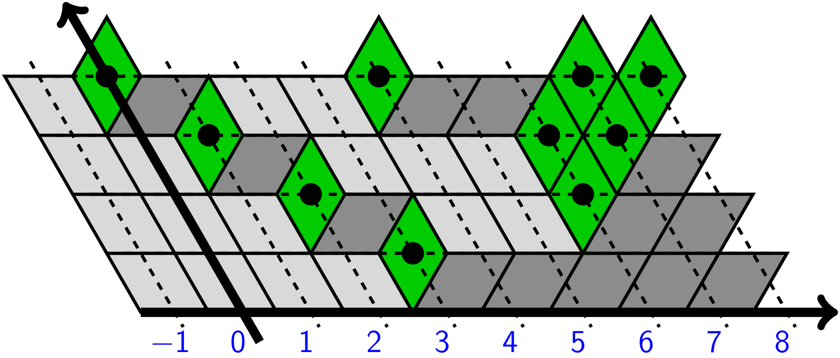
Interlacing integer arrays $\longleftrightarrow$ lozenge tilings

+0 .... +(N-2) +(N-1)

0 3 5 5
1 5 5
2 5
3

shift coordinates so that in each row they become distinct

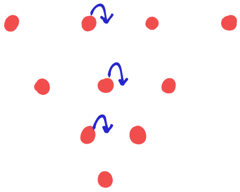
0 4 7 8
1 6 7
2 6
3



Stochastic dynamics on interlacing arrays

I will describe stochastic Markov dynamics on interlacing arrays in which (in continuous time) particles jump to the right by one.

During a small time interval, at most one particle on each level jumps.

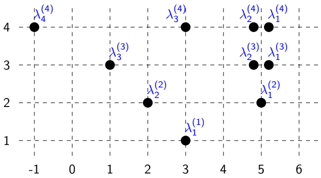


Such dynamics on interlacing arrays will be multilayer extensions of the discrete Dyson Brownian motion.

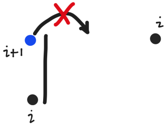
“Schur level”

Push-block dynamics [Borodin–Ferrari '08] — discrete analogue of Warren's construction

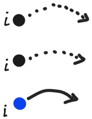
1. Each particle $\lambda_j^{(k)}$ jumps to the right by one according to an independent exponential clock of rate 1.



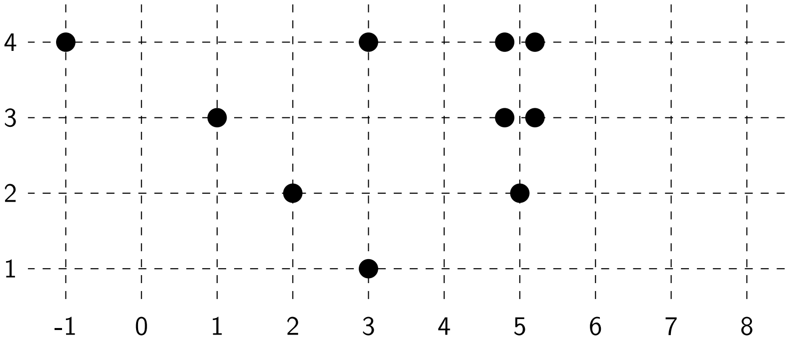
2. If it is blocked from below, there is no jump



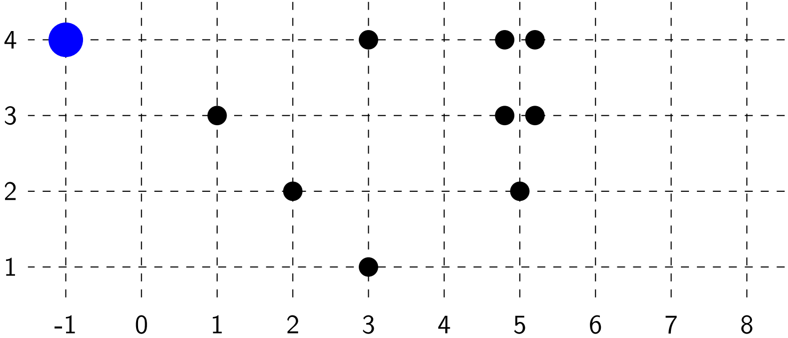
3. If violates interlacing with above, it pushes the above particles



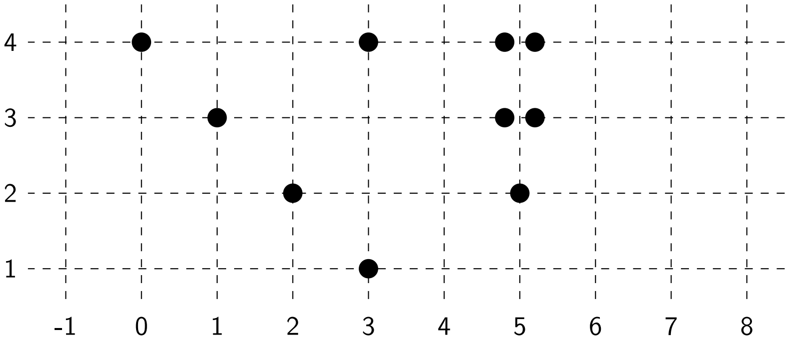
Push-block dynamics



Push-block dynamics



Push-block dynamics



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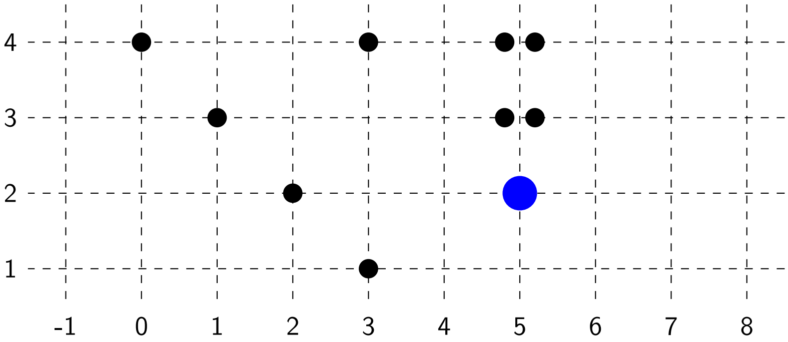

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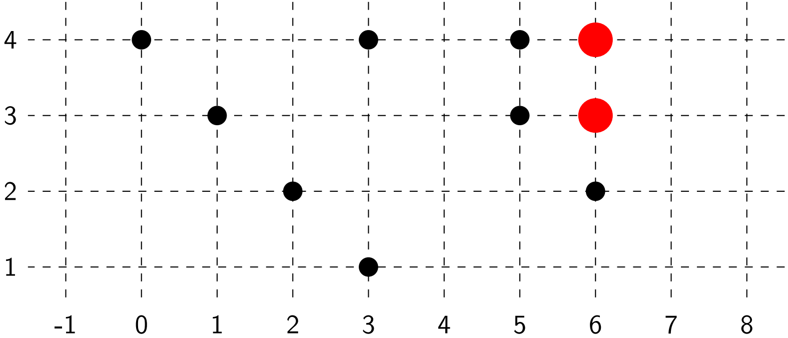

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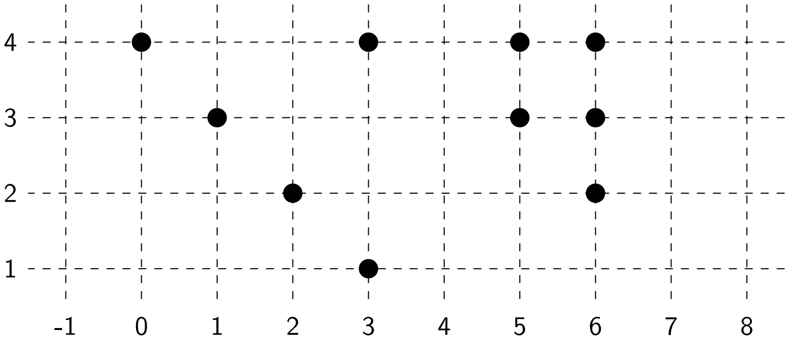

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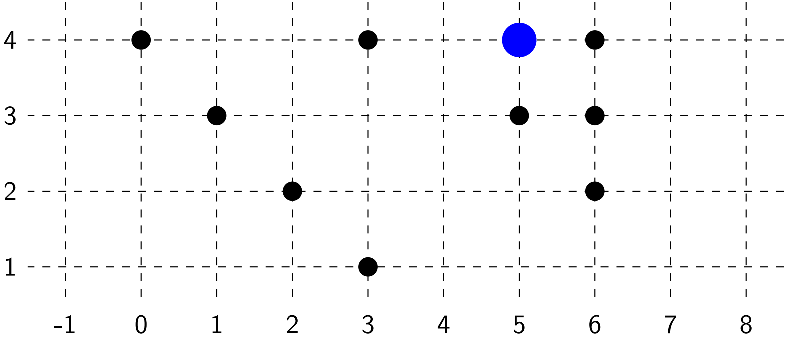

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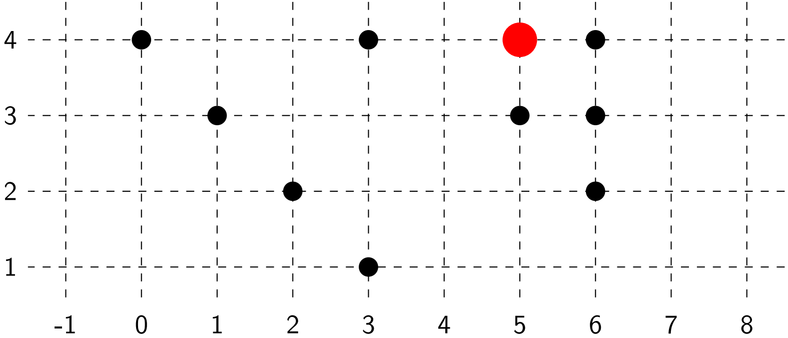

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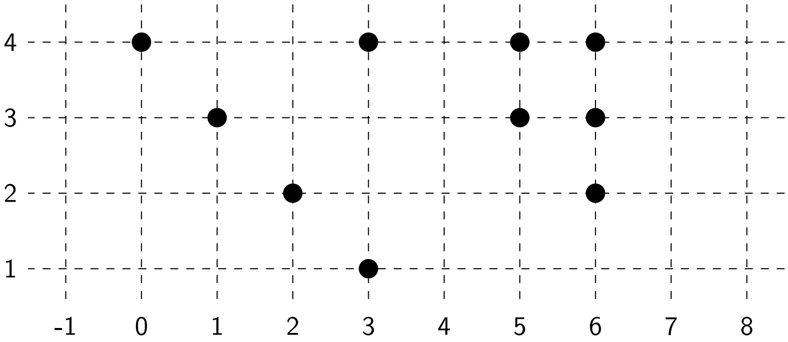

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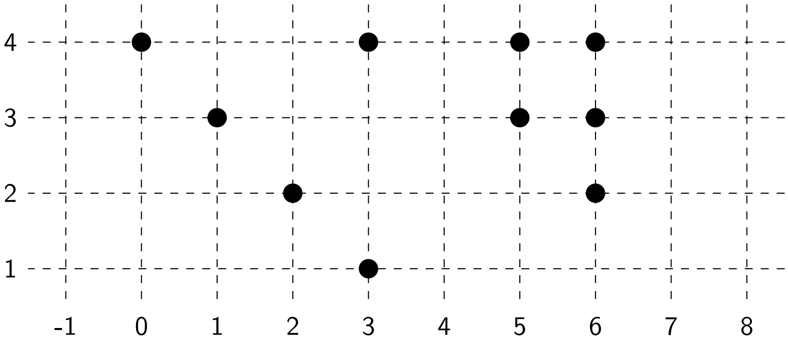

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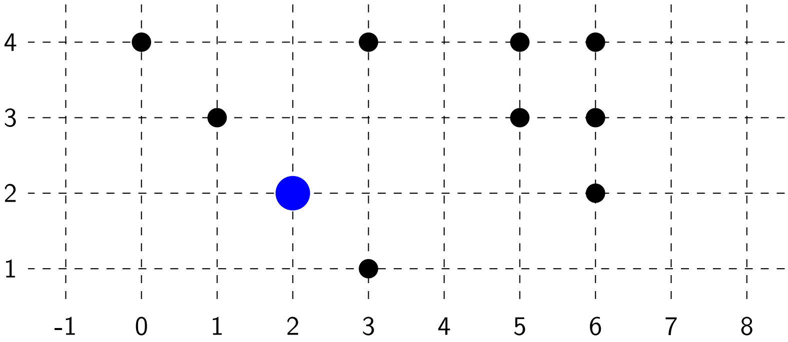

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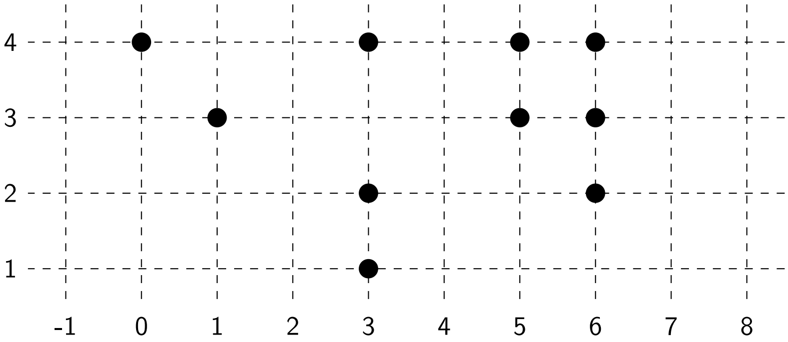

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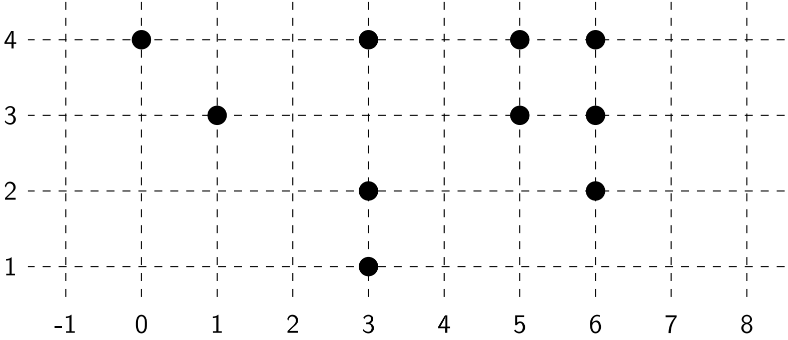

Simulation

“Schur level”

1. Dyson Brownian motion and its discrete version
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   - Interlacing integer arrays
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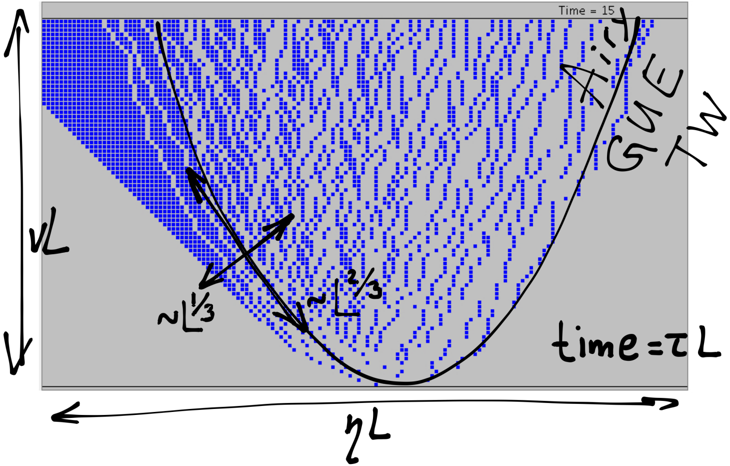
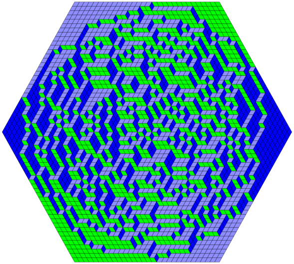
Asymptotic properties of the push-block dynamics [BF '08]: KPZ universality

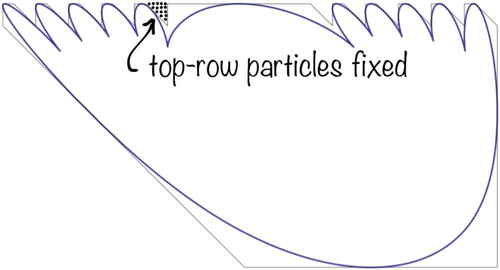


+ fluctuations $\sim L^{1/3}$ with time (L — large parameter)

Remark: Other tiling models

Scaling orders $L^{1/3}$ – $L^{2/3}$, GUE Tracy–Widom distribution and Airy process found in other models of random lozenge tilings: [Okounkov–Reshetikhin '07], [Baik–Kriecherbauer–McLaughlin–Miller '07], **[P. '12] (proved Airy edge fluctuations and Gaussian Free field fluctuations inside the shape)**





(uniformly random configuration with fixed top row)

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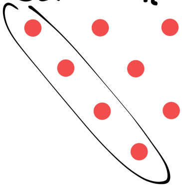
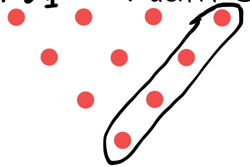
TASEP and PushTASEP

TASEP: $x_n = \lambda_n^{(n)} - n$



Markovian projection to the leftmost particles — TASEP

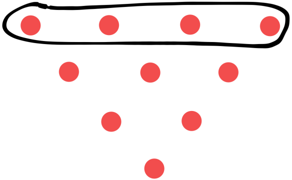
$x_n = \lambda_1^{(n)} + n$: PushTASEP



Markovian projection to the rightmost particles — PushTASEP

Discrete Dyson Brownian motion

noncolliding Poisson processes



$x_n = \lambda_n^{(N)} - n$

Started from the empty initial state $\lambda_j^{(k)} = 0$, the evolution of the particles in each Nth row is Markovian:

- Rate 1 Poisson processes conditioned never to intersect;
- Equivalently, Doob's h-transform of independent Poisson processes, with $h(x_1, \ldots, x_N) = \prod_{i<j}(x_i - x_j)$.

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1. Dyson Brownian motion and its discrete version
2. Push-block dynamics
3. RSK (Robinson–Schensted–Knuth) dynamics
    - Dynamics on interlacing arrays
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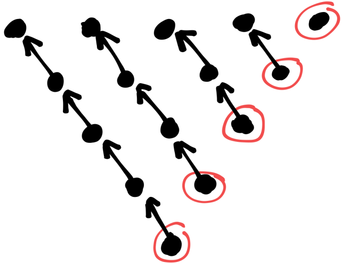
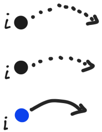
RSK dynamics [Johansson '99,'02], [O'Connell '03]

1. Each rightmost particle $\lambda_1^{(k)}$ jumps to the right by one according to an *independent exponential clock* of rate 1.

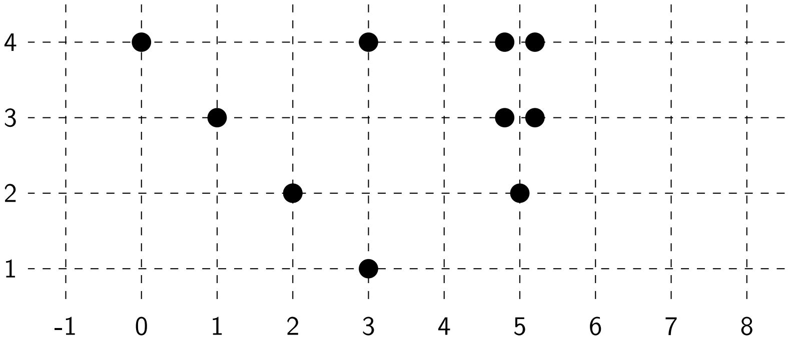
2. When any particle $\lambda_j^{(h)}$ moves, it triggers either the move $\lambda_j^{(h+1)} \mapsto \lambda_j^{(h+1)} + 1$, or $\lambda_{j+1}^{(h+1)} \mapsto \lambda_{j+1}^{(h+1)} + 1$ (exactly one of them).



The second one is chosen generically, while the first one is chosen only if $\lambda_j^{(h+1)} = \lambda_j^{(h)}$, i.e., if the move $\lambda_j^{(h)} \mapsto \lambda_j^{(h)} + 1$ violated the interlacing constraint (push rule).



RSK dynamics



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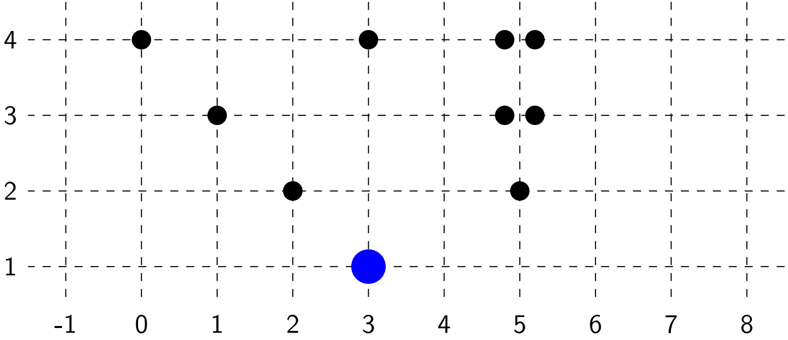

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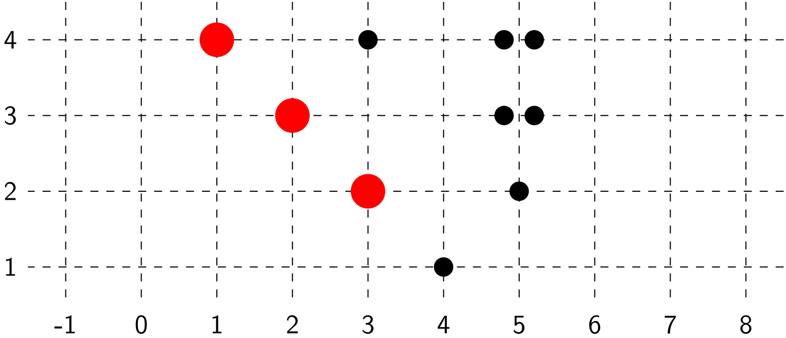
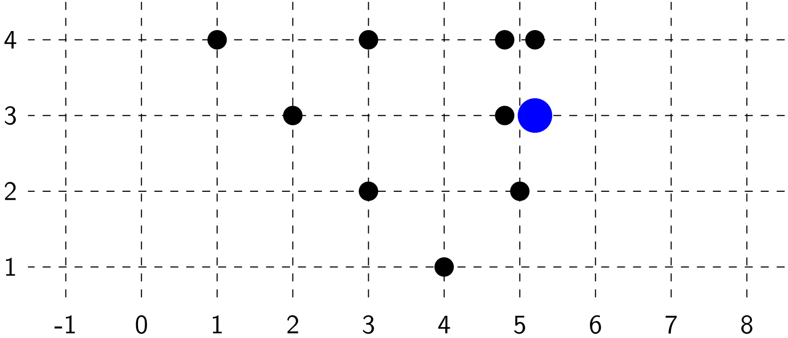
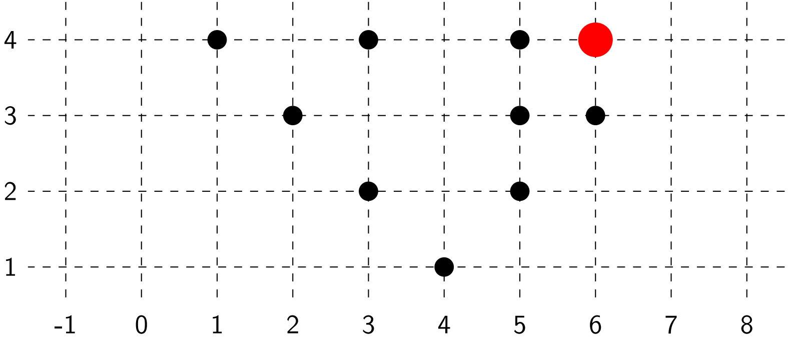


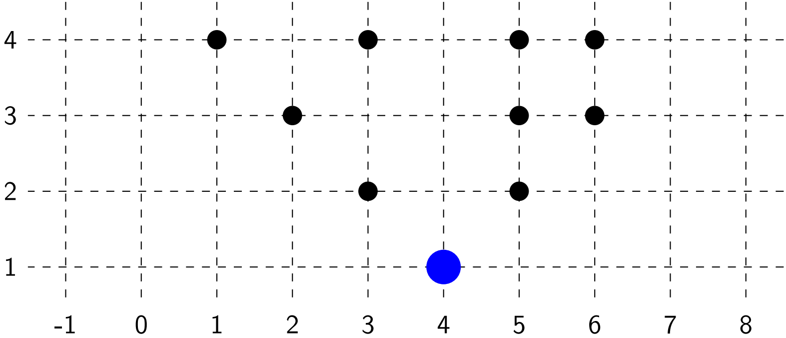
RSK dynamics



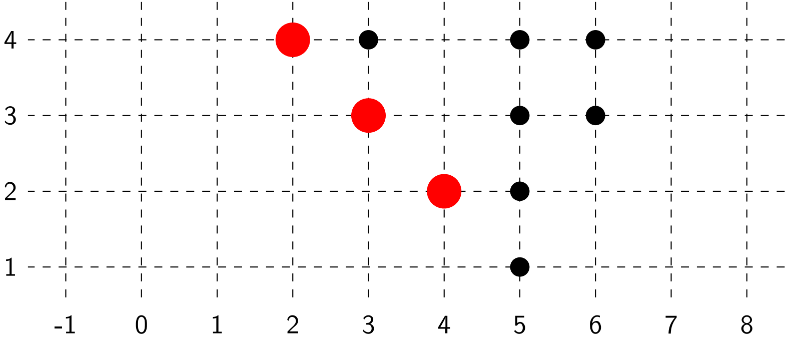
RSK dynamics



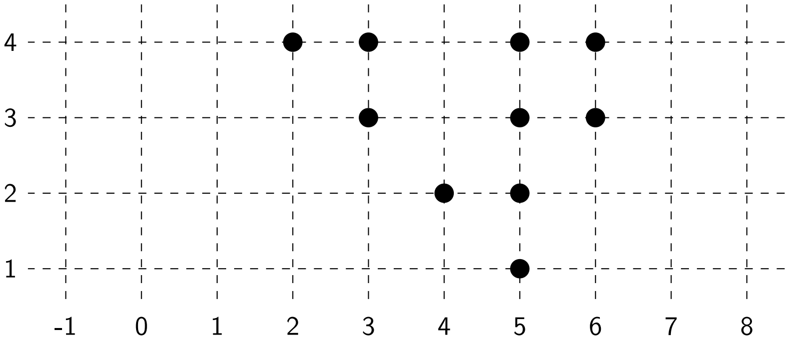
RSK dynamics



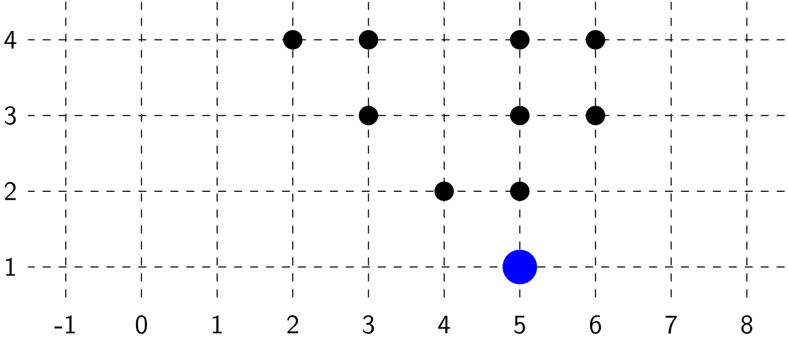
RSK dynamics



RSK dynamics



RSK dynamics



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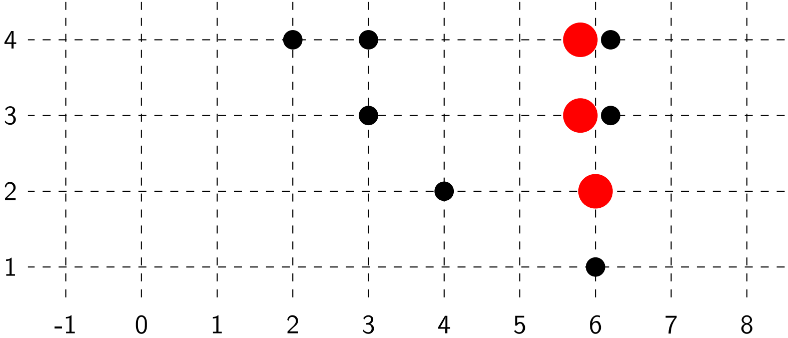

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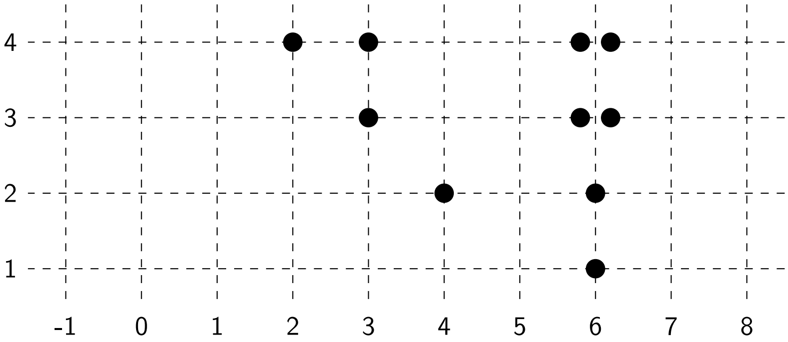

“Schur level”

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   - Dynamics on interlacing arrays
   - Relation to the classical RSK correspondence
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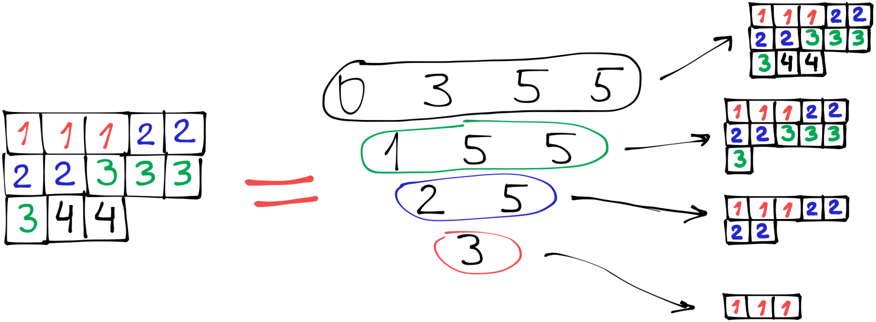
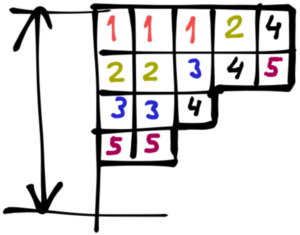
Nonnegative interlacing integer arrays $\longleftrightarrow$ semistandard Young tableaux (via row-lengths of shapes)

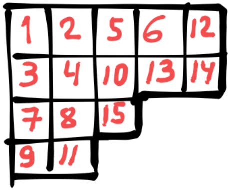


Classical RSK insertion

- *Semistandard* Young tableau P



- *Standard* Young tableau Q

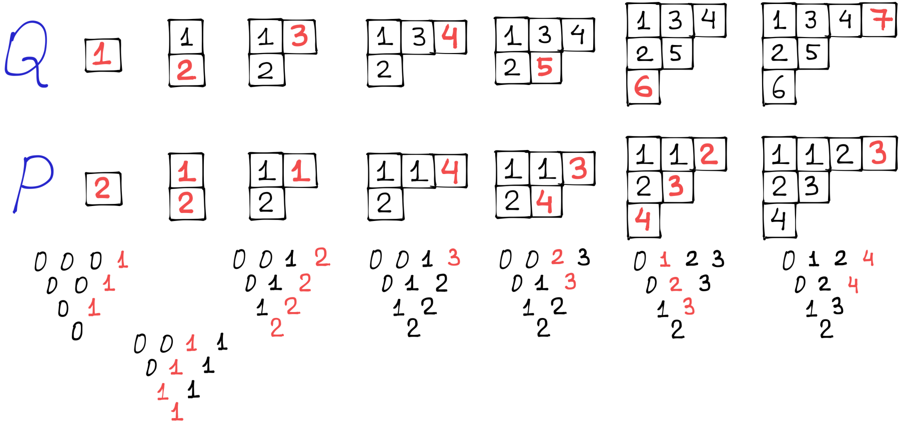


Classical RSK insertion

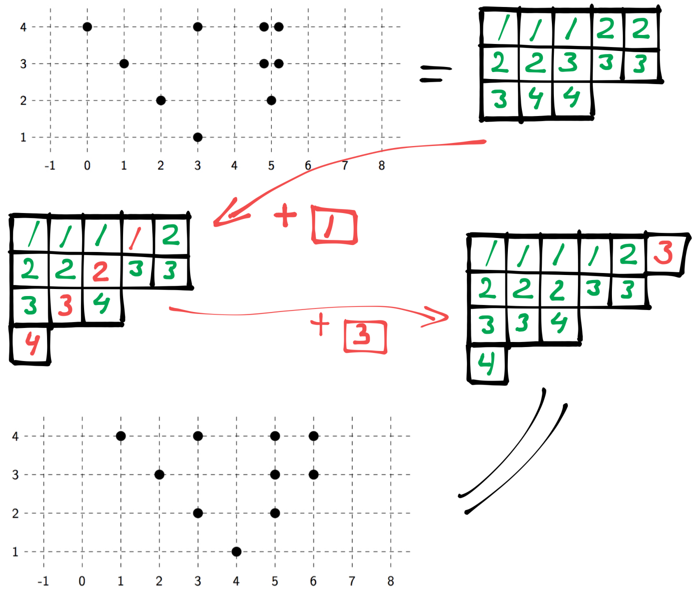
Interlacing arrays $\longleftrightarrow$ semistandard Young tableaux ("P-tableaux");

Independent jump at level h $\longleftrightarrow$ RSK-insert the letter h into the tableau P.

Example: word $= 2114323$



Classical RSK insertion (only P tableaux)

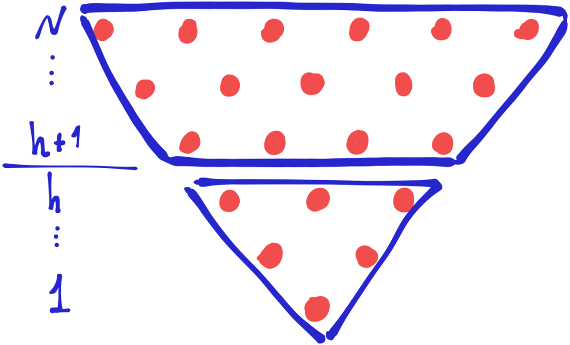


“Schur level”

1. Dyson Brownian motion and its discrete version
2. Push-block dynamics
3. RSK dynamics
4. Unifying axioms
    - Common properties of push-block and RSK dynamics
    - Nearest neighbor dynamics
5. New RSK correspondences

1. "Interaction goes up"

evolution of the lower floors $\{\lambda^{(1)}, \dots, \lambda^{(h)}\}$ is independent of the upper floors $\{\lambda^{(h+1)}, \dots, \lambda^{(N)}\}$ for any $h = 1, \dots, N$.



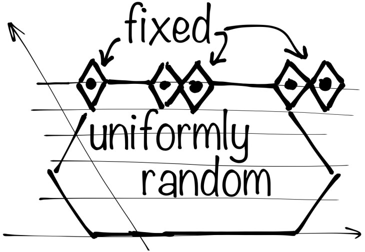
2. Preserve the class of Gibbs measures

Definition. Gibbs probability measures on interlacing arrays

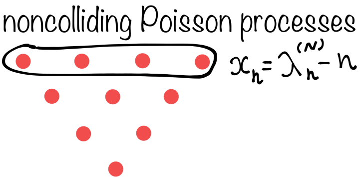
A measure M is called *Gibbs* if for each $h = 1, \dots, N$:

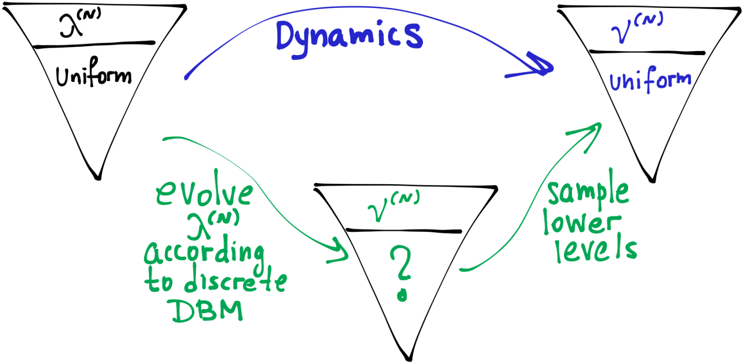
> *Given (fixed)* $\lambda_h^{(h)} \le \ldots \le \lambda_1^{(h)}$, *the distribution of all the lower levels* $\lambda^{(1)}, \dots, \lambda^{(h-1)}$ *is uniform (among configurations satisfying the interlacing constraints).*

Dynamics on arrays *preserves the class of Gibbs measures* if it maps one Gibbs measure into another.



3. On Gibbs measures, each row marginally evolves as a discrete Dyson BM





“Schur level”

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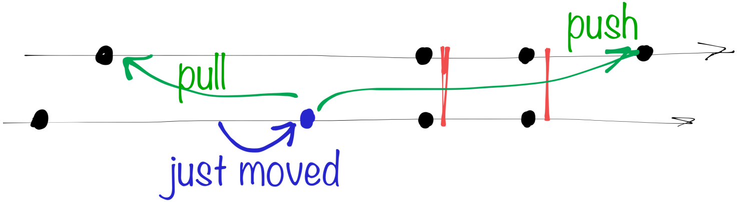
Nearest neighbor dynamics

We look for other dynamics which satisfy:

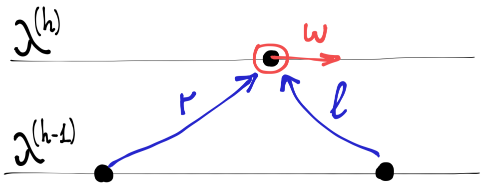
1. "Interaction goes up"
2. Preserve Gibbs measures
3. "discrete Dyson BM" on floors
4. Nearest neighbor interactions:



(push/pull with some probabilities, do nothing with the complementary probability)

[Borodin-P. '13] — introduce these axioms, and obtain complete classification of nearest neighbor dynamics.

Nearest neighbor dynamics



independent jump rate $w = w(\lambda^{(h-1)}, \lambda^{(h)})$

pushing probabilities (after lower particle jumped) $r = r(\lambda^{(h-1)}, \lambda^{(h)})$ and $\ell = \ell(\lambda^{(h-1)}, \lambda^{(h)})$

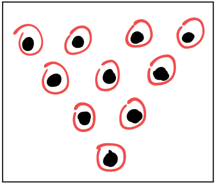
Theorem [Borodin-P. '13]. Nearest neighbor dynamics correspond to solutions of the equations

$$r(\lambda^{(h-1)}, \lambda^{(h)}) + \ell(\lambda^{(h-1)}, \lambda^{(h)}) + w(\lambda^{(h-1)}, \lambda^{(h)}) = 1$$

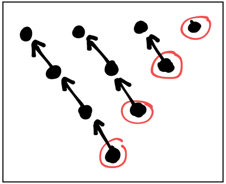
written for all states $\lambda^{(1)}, \ldots, \lambda^{(N)}$ of the array and each particle in it.

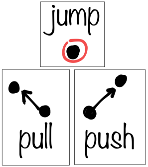
“Basis” dynamics are encoded by pictures such as:

Push-block:

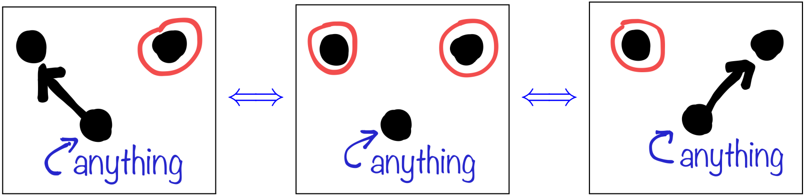


RSK:





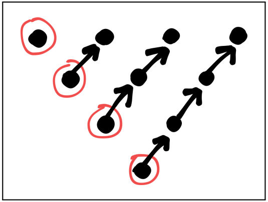
plus local flips



All other dynamics are linear combinations of “basis” ones

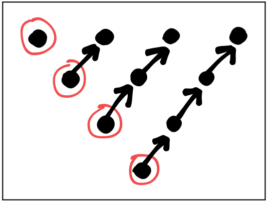
“Basis” nearest neighbor dynamics examples

Column (= dual) RSK [O’C ’03]

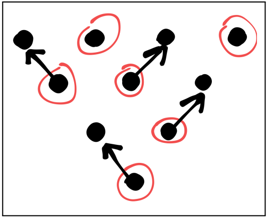


“Basis” nearest neighbor dynamics examples

Column (= dual) RSK [O’C ’03]

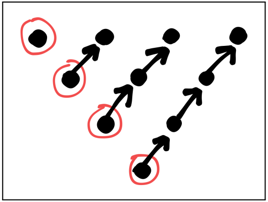


Some “basis” dynamics



“Basis” nearest neighbor dynamics examples

Column (= dual) RSK [O'C '03]



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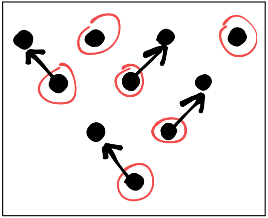
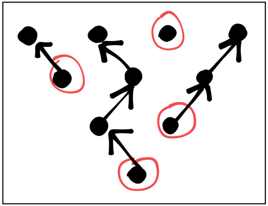


RSK-type dynamics ($\Rightarrow$ **we obtain** $N!$ **bijections between words and pairs of tableaux**)



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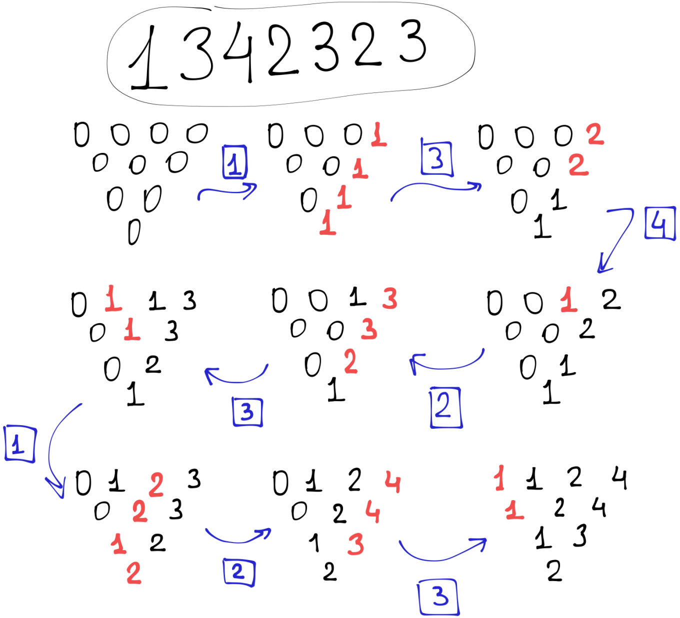
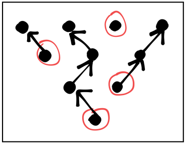
New RSK correspondences





“Macdonald level”

1. From Schur to Macdonald
   - Schur and Macdonald processes
   - Nearest neighbor dynamics
2. q-deformed 1d particle systems: new examples
3. Randomized insertion algorithm for triangular matrices over a finite field

Schur polynomials in dynamics on interlacing arrays

Distribution of the dynamics

$$\mathbb{P}_{\tau}\left[\begin{matrix} 0 \dots 0 \\ \dots \\ 0 \end{matrix} \longrightarrow \begin{matrix} \lambda_N^{(N)} \dots \lambda_1^{(N)} \\ \dots \\ \lambda_1^{(1)} \end{matrix}\right]$$

is the Schur process [Okounkov–Reshetikhin '03].
The Schur process is the Gibbs extention of the Schur measure

$$Prob_{\tau}(\lambda^{(N)}) = \frac{1}{Z} \cdot s_{\lambda^{(N)}}(1, \dots, 1) \cdot s_{\lambda^{(N)}}(\rho_{\tau})$$

to the whole interlacing array.

The Schur process is a determinantal point process, which is the source of integrability of the model.

Schur polynomials in dynamics on interlacing arrays

Schur polynomials:

$$s_\mu(x_1,\ldots,x_k) = \frac{\det\left[x_i^{\mu_j+N-j}\right]_{i,j=1}^k}{\det\left[x_i^{N-j}\right]_{i,j=1}^k}, \text{ where } \mu_1 \geq \ldots \geq \mu_k.$$

Remark. Relation to RSK through Young tableaux:

$\#SSYT(\lambda^{(N)}) = s_{\lambda^{(N)}}(\underbrace{1,\ldots,1}_{N})$, and

$$\frac{\tau^{|\lambda^{(N)}|}}{|\lambda^{(N)}|!}\cdot\#SYT(\lambda^{(N)}) = s_{\lambda^{(N)}}(\rho_\tau) = \lim_{L\to\infty} s_{\lambda^{(N)}}(\underbrace{\frac{\tau}{L},\ldots,\frac{\tau}{L}}_{L}).$$

ρ_τ — "Plancherel specialization".

Macdonald polynomials

$P_\lambda(x_1,\ldots,x_N) \in \mathbb{Q}(q,t)[x_1,\ldots,x_N]^{S(N)}$ labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\mathcal{D}_1 = \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}, \qquad (T_q f)(z) := f(zq),$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \ldots + q^{\lambda_N}) P_\lambda.$$

Macdonald polynomials have many remarkable properties (similar to those of Schur polynomials corresponding to $q = t$) including orthogonality, simple reproducing kernel (Cauchy identity), Pieri and branching rules, index/variable duality, etc. There are also simple higher order Macdonald difference operators commuting with $\mathcal{D}_1$.

From Schur to Macdonald

In short, replace all Schur polynomials by Macdonald polynomials. All previous constructions of dynamics work.

From Schur to Macdonald

In short, replace all Schur polynomials by Macdonald polynomials. All previous constructions of dynamics work.

Get Markov dynamics on interlacing arrays whose distributions are Macdonald processes

[Borodin–Corwin '11], [O'Connell–Pei '12], **[Borodin–P. '13] (complete classification of these dynamics)**.

Macdonald processes [BC ‘11], [BC–Gorin–Shakirov ‘13]

Macdonald process is the (q, t)–analogue of the Schur process. Macdonald process is obtained by a “(q, t)–Gibbs” continuation of the “(q, t)–Schur measure”

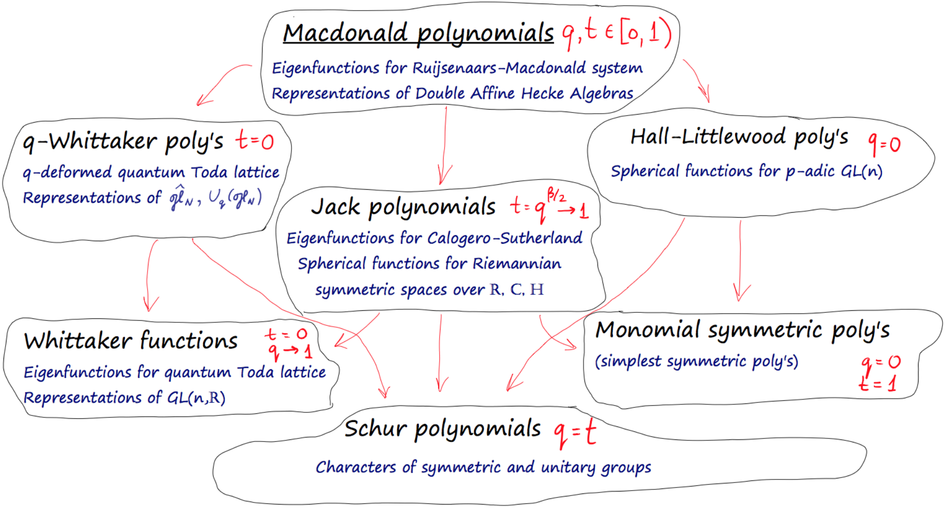
Macdonald poly   Dual basis

$$\mathrm{Prob}_{\tau}(\lambda^{(N)}) = \frac{1}{Z} P_{\lambda^{(N)}}(\underbrace{1, \ldots, 1}_{N}) \, Q_{\lambda^{(N)}}(\rho_{\tau})$$

to the whole interlacing array.

Macdonald processes turn out to be tractable as well [BC ‘11], [BCGS ‘13] (thanks to the q-difference operators $\mathcal{D}_1, \mathcal{D}_2, \ldots$)

Symmetric polynomials and related objects



([BC '11])

1. From Schur to Macdonald
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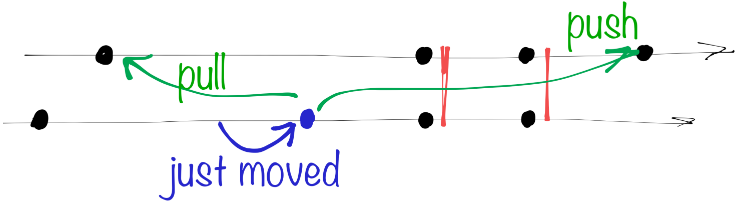
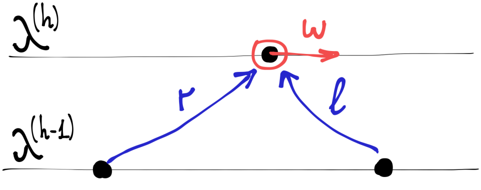
Nearest neighbor dynamics on Macdonald processes

1. "Interaction goes up"
2. Preserve "(q, t)-Gibbs" measures
3. (q, t)-"discrete Dyson BM" on floors
4. Nearest neighbor interactions:



(push/pull with some probabilities, do nothing with the complementary probability)

Nearest neighbor dynamics on Macdonald processes



independent jump rate $w = w(\lambda^{(h-1)}, \lambda^{(h)})$

pushing probabilities (after lower particle jumped) $r = r(\lambda^{(h-1)}, \lambda^{(h)})$ and $\ell = \ell(\lambda^{(h-1)}, \lambda^{(h)})$

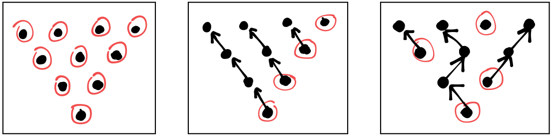
Theorem [Borodin-P. '13]. Nearest neighbor dynamics on Macdonald processes correspond to solutions of the equations

$$T \cdot r(\lambda^{(h-1)}, \lambda^{(h)}) + \tilde{T} \cdot \ell(\lambda^{(h-1)}, \lambda^{(h)}) + w(\lambda^{(h-1)}, \lambda^{(h)}) = S.$$

Here $T, \tilde{T}, S$ are certain coefficients depending on q, t, and also on $\lambda^{(h-1)}, \lambda^{(h)}$.

Nearest neighbor dynamics on Macdonald processes

The "basis" nearest neighbor dynamics are encoded by the same pictures as before.



Not all of the "Schur level" pictures lead to dynamics with nonnegative jump rates. We have to speak about *formal* Markov processes.

1. From Schur to Macdonald
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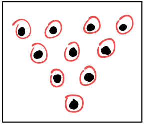
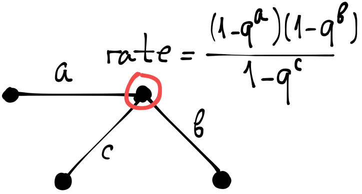
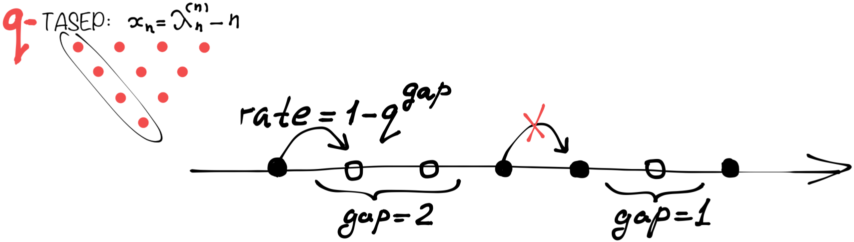
Push-block dynamics [Borodin–Corwin ‘11]

Let the second Macdonald parameter $t = 0$. The push-block dynamics gives:

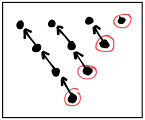




Markovian projection — q-TASEP [BC ‘11], [BC–Sasamoto ‘12], [O’Connell–Pei ‘12], **[BC–P.–Sasamoto ‘13]**, [Povolotsky ‘13]

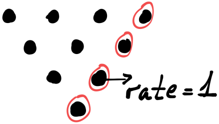


RSK-type dynamics [Borodin–P. ‘13]



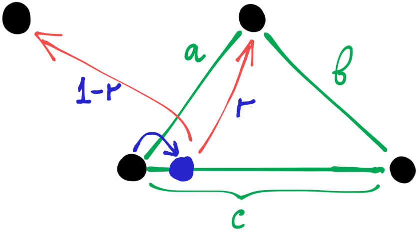
Let the second Macdonald parameter $t = 0$. Then the q-deformation of the classical RSK is:

1. Only the rightmost particles make independent jumps with rate 1



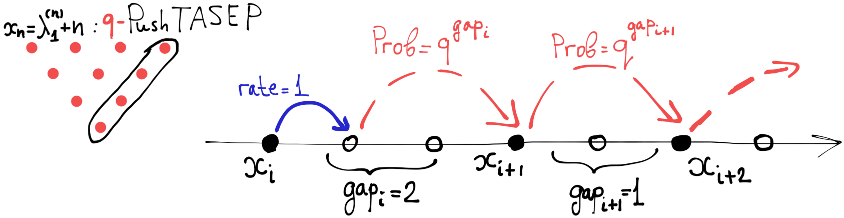
2. If a particle moves, it pushes its immediate upper neighbors with probabilities r and $1 - r$, where

$$r = q^a \frac{1 - q^b}{1 - q^c}$$

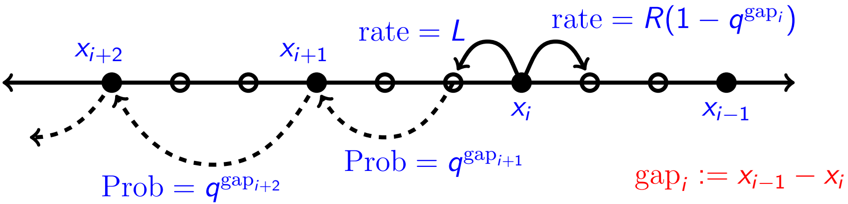


q-PushTASEP [Borodin–P. '13], [Corwin–P. '13]

Another Markovian projection:



q-PushASEP [Corwin–P. '13]



$R * (q\text{-TASEP, to the right}) + L * (q\text{-PushTASEP, to the left})$

Traffic model (relative to a time frame moving to the right)

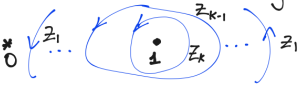
- Right jump = a car *accelerates*. Chance $1 - q^{\mathrm{gap}}$ is lower if another car is in front.
- Left jump = a car *slows down*. The car behind sees the brake lights, and may also quickly slow down, with probability q^{gap} (chance is higher if the car behind is closer).

q-PushASEP integrability

Theorem [Corwin–P. '13]. q-moment formulas for the q-PushASEP with the step initial condition $x_i(0) = -i$, $i = 1, \ldots, N$.

$$\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{N_i}(t)+N_i}\right) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} (1-z_j)^{-N_j} \frac{G_t(q z_j)}{G_t(z_j)} \cdot \frac{dz_j}{z_j}$$

$N_1 \geq N_2 \geq \ldots \geq N_k > 0$



$G_t(z) = e^{t(Rz + Lz^{-1})}$

(obtained via a quantum many body systems approach dating back to H. Bethe '31)

1. From Schur to Macdonald
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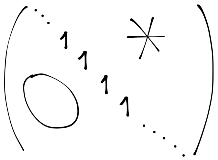
3. Randomized insertion algorithm for triangular matrices over a finite field

Random triangular matrices over a finite field

Consider the group $\mathbf{U}$ of infinite unipotent upper-triangular matrices over the finite field $F_{t^{-1}}$, where $t \in (0, 1)$, and t^{-1} is a prime power.



[Vershik–Kerov ’80s], [Kerov ‘03]: Problem of classification of probability measures μ on $\mathbf{U}$ which are

- Conjugation-invariant: $\mu(X) = \mu(gXg^{-1})$ for $X \subset \mathbf{U}$ and g a matrix over $F_{t^{-1}}$ which differ from the identity at finitely many positions.
- Ergodic ($=$ extreme as elements of the convex set of all conjugation-invariant measures).

Random triangular matrices over a finite field

Through Jordan normal form of truncations of matrices from **U**, the problem reduces to measures μ_n on Young diagrams $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell$ with fixed number n of boxes. The measures μ_n are related to Hall–Littlewood polynomials (these are Macdonald polynomials with $q = 0$; and t as in $F_{t^{-1}}$).

Conjectural classification of measures μ on **U** [Kerov '03]: measures depend on parameters

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq 0; \qquad \sum_{i=1}^{\infty}\left(\alpha_i + \frac{\beta_i}{1-t}\right) \leq 1.$$
$$\beta_1 \geq \beta_2 \geq \ldots \geq 0;$$

These measures $\mu^{\alpha;\beta}$ exist and are ergodic.
The problem is to show the completeness of classification.
See [Gorin–Kerov–Vershik '12].

Random triangular matrices over a finite field

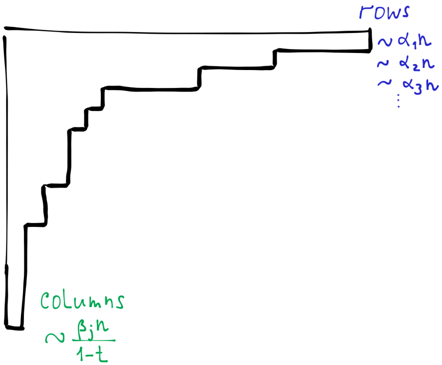
We construct a randomized RSK to sample these ergodic measures. Input of the RSK is a random Bernoulli word.

Using this RSK, we prove another conjecture of Vershik–Kerov — a law of large numbers for the measures $\mu_n^{\alpha;\beta}$ ($t = 0$ — infinite symmetric group)

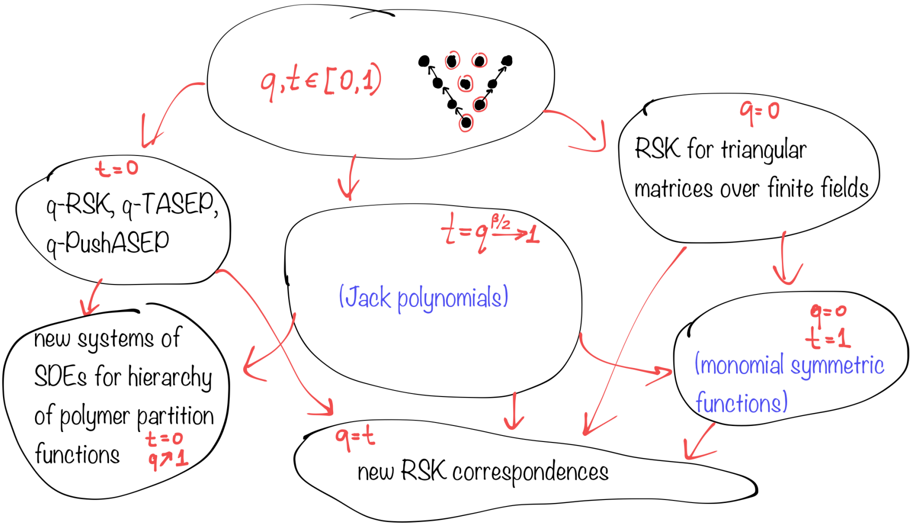
Theorem [Bufetov–P., in progress]. For random Young diagrams distributed according to $\mu_n^{\alpha;\beta}$, as $n \to \infty$:

$$\frac{\mathrm{row}(i)}{n} \to \alpha_i$$

$$\frac{\mathrm{column}(j)}{n} \to \frac{\beta_j}{1-t}$$



Conclusion



q,t ∈ [0,1)
t=0
q-RSK, q-TASEP,
q-PushASEP
t=q^{β/2} → 1
(Jack polynomials)
q=0
RSK for triangular
matrices over finite fields
new systems of
SDEs for hierarchy
of polymer partition
functions
t=0
q↗1
q=0
t=1
(monomial symmetric
functions)
q=t
new RSK correspondences