From infinite random matrices over finite fields to square ice

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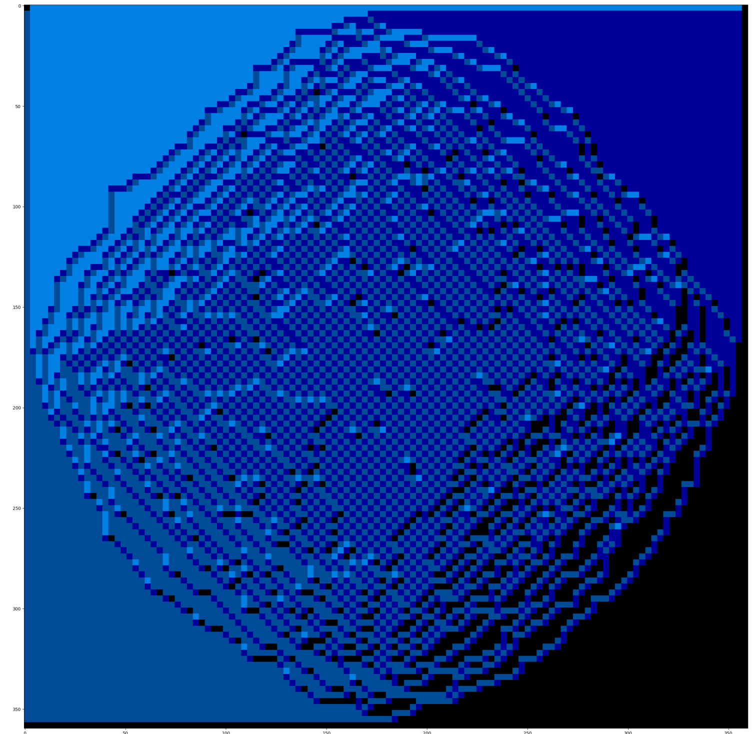


- **Infinite binary sequences**
- **Infinite triangular random matrices over a finite field**

$$11011111110111010001010010110011110011110010001010\ldots$$

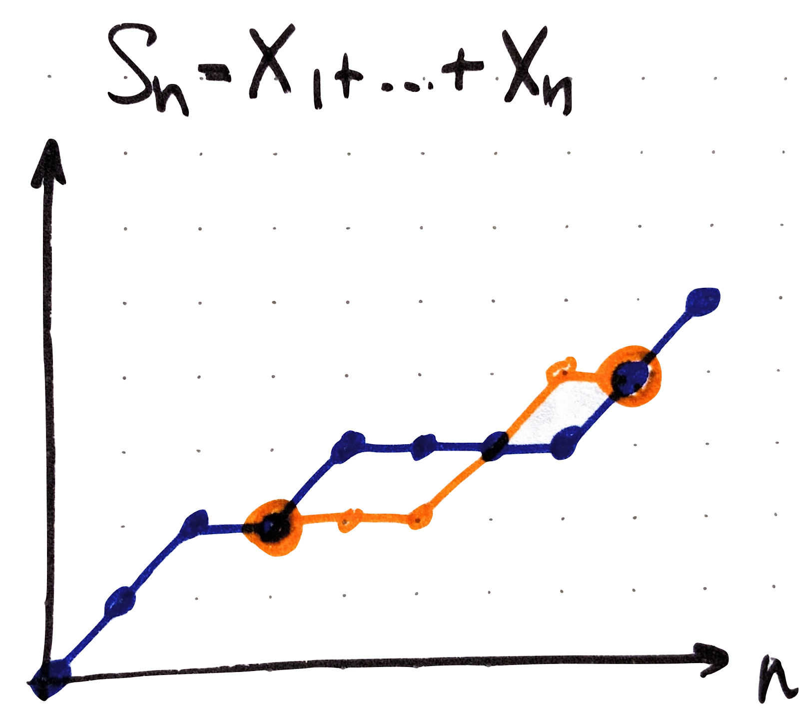
Exchangeable random binary sequences

(toy example)

Exchangeability

A random sequence $X_1, X_2, \ldots$, where $X_i \in \{0,1\}$, is called **exchangeable** if its distribution does not change under (finitary) permutations of indices.

Invariance under uniform resampling given the boundary conditions



Exchangeable distributions form a **convex set**

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2, \qquad \alpha \in [0,1]$$

Extreme exchangeable distributions are the μ's which cannot be decomposed as above with $\mu_{1,2} \neq \mu$ and $\alpha \neq 0, 1$

Classification of extreme exchangeable distributions

[de Finetti 1930s, Hewitt-Savage 1955]

Extreme exchangeable distributions are precisely the Bernoulli product measures μ_p indexed by $p \in [0,1]$

Under μ_p, the X_i's are independent with $\mathbb{P}(X_i = 1) = p$.

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How to sample

To sample an exchangeable sequence, first pick random p from a *mixing distribution* ν on $[0,1]$

Then, given p, sample independent X_i's according to μ_p

Example

The uniform mixing distribution ν on $[0,1]$ corresponds to the "Polya urn":

for each n, $X_1, \ldots, X_n$ has a uniformly random number $k \in \{0, \ldots, n\}$ of zeroes and ones (**homework problem**: how to pass from n to $n+1$?)

Parameter recovery: Law of Large Numbers

$$\lim_{n \to +\infty} \frac{X_1 + \ldots + X_n}{n} = \nu \qquad \text{in distribution and a.s.}$$

clear for extreme measures
and in the uniform example;
holds in general, too

Ergodic approach for describing "boundaries"

1. Want to classify probability distributions with certain *symmetry* and *sequential structure*

2. Distributions form a *convex set*

3. Classify *extreme* distributions using the sequential structure (each infinite-level extreme is a limit of finite-level ones)

4. Each distribution is a *convex combination* of extremes

5. Law of Large Numbers for parameter recovery ⟵ (the first focus of the talk)

6. Select non-extreme distributions are very interesting (won't discuss this in the talk)

The ergodic approach was employed by Vershik and Kerov in 1970-80s to apply to representation theory of "big" groups:

- the infinite symmetric group $S(\infty)$ **[Edrei 1950s, Thoma 1964]**
- the infinite-dimensional unitary group $U(\infty)$ **[Edrei 1950s, Voiculescu 1976], Vadim's talk on Monday**

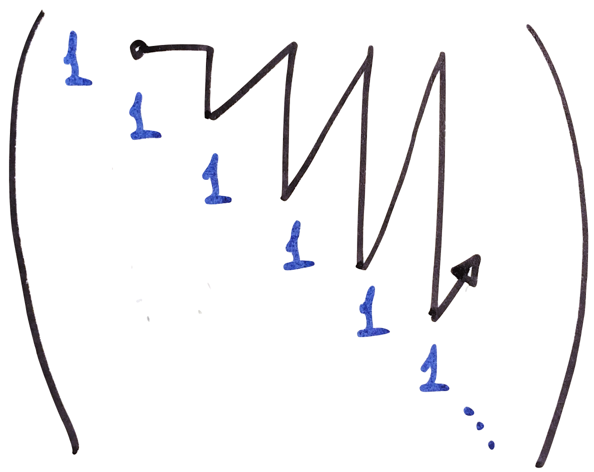
Related applications include the study of ergodic central measures on matrices, both Hermitian/$\mathbb{C}$, and over **finite fields**

- Infinite binary sequences

(different "q-analogue" of the $U(\infty)/S(\infty)$ rep theory)

- **Infinite triangular random matrices over a finite field**

1101111111011101000101001011001111001111001000101 0...



$$(X, Y) \overset{d}{=} (X+Y, Y) \mod 2$$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 & & & & & \vdots & & & & &
\end{pmatrix}$$

Another symmetry of the 1/2 i.i.d. coin flip sequence!

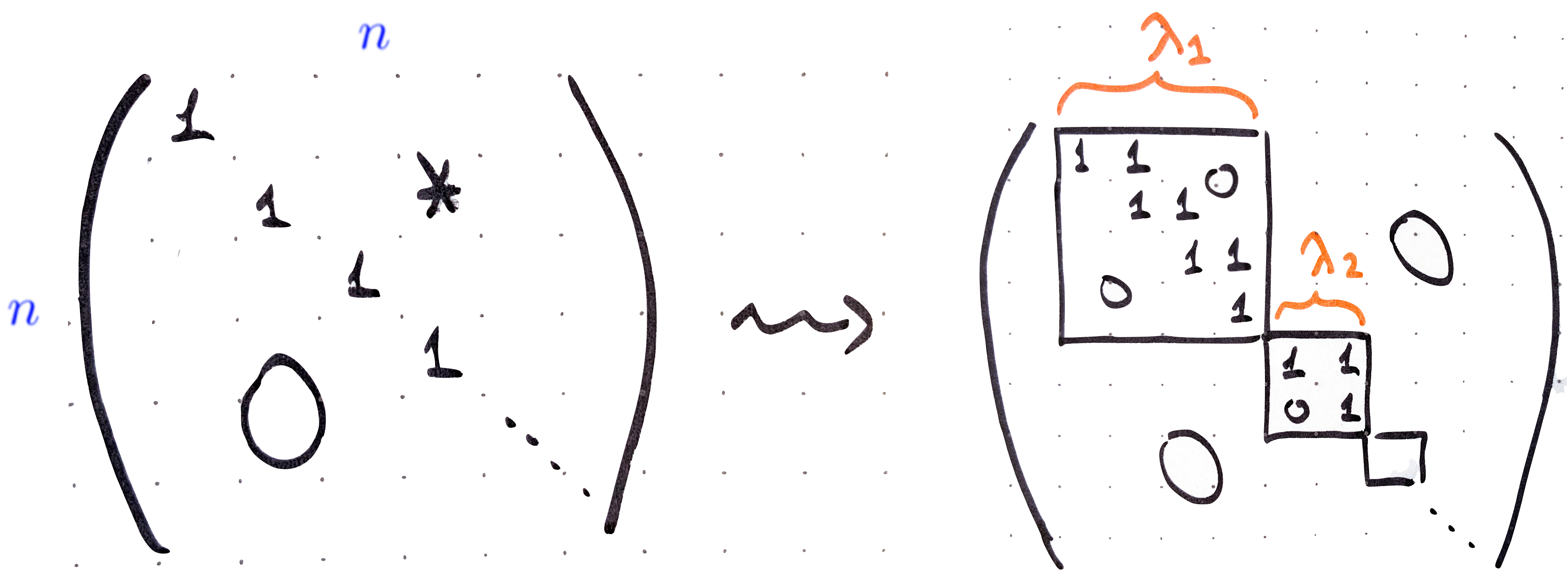
Random triangular matrices over finite fields

$\mathbb{F}_q$ — finite field ($q = 2$ when $\mathbb{F}_2 = \{0, 1\}$ suffices)

$\mathbb{U}$ — group of infinite uni upper triangular matrices over $\mathbb{F}_q$

Each $n \times n$ triangular matrix is conjugate to a Jordan form by an element of $GL_n(\mathbb{F}_q)$

Jordan forms are encoded by Young diagrams $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq 0)$, $\lambda_i \in \mathbb{Z}$



$|\lambda| = n$

$GL_\infty(\mathbb{F}_q)$ — group of infinite matrices which finitely differ from the identity

Exchangeability analogue (symmetry of measures)

A probability Borel measure μ on $\mathbb{U}$ is called **central** if $\mu(M) = \mu(gMg^{-1})$ for all measurable $M \subset \mathbb{U}$ and $g \in GL_\infty(\mathbb{F}_q)$ such that $gMg^{-1} \subset \mathbb{U}$

Exchangeability analogue

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Example: uniform product measure on $\mathbb{U}$ for which X_{ij}, $i < j$, are independent $\in \mathbb{F}_q$

$g \in GL_\infty(\mathbb{F}_q)$

(informally)

$$g \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ & & & & & \vdots & & & & & \end{pmatrix} g^{-1} \stackrel{d}{=} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ & & & & & \vdots & & & & & \end{pmatrix}$$

Exercise: for X, Y iid from $\mathbb{F}_2$, we have $(X, Y) \stackrel{d}{=} (X + Y, Y)$

Central measures form a convex set. **Goal:** classify extreme central measures

- Related to representation theory of $GL_n(\mathbb{F}_{\mathfrak{q}})$ as $n \to \infty$ [Vershik-Kerov 90s+] [Gorin-Kerov-Vershik 2012]
- At a level of (some) tools, is a one-parameter deformation of the representation theory of $S(\infty)$ (the latter corresponds to $\mathfrak{q} \to \infty$)
- The answer was conjectured by Kerov in 1992 and proven by Matveev in 2017 (together with a Macdonald generalization which adds yet one more parameter)

Theorem

Extreme central measures are in one to one correspondence with tuples

$$\alpha_1 \ge \alpha_2 \ge \ldots \ge 0, \qquad \beta_1 \ge \beta_2 \ge \ldots \ge 0, \qquad \gamma \ge 0$$

such that

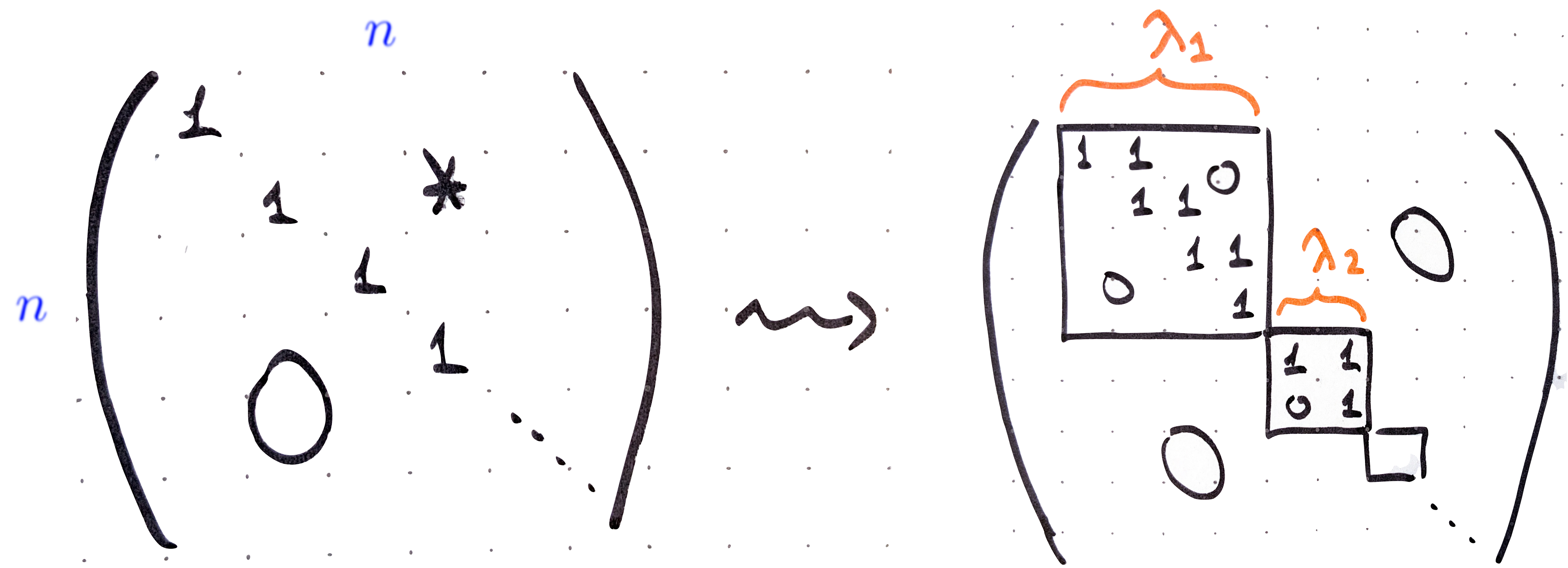
$$\frac{\gamma}{1-t} + \sum_{i \ge 1} \left(\alpha_i + \frac{\beta_i}{1-t} \right) = 1$$

$t := 1/\mathfrak{q}$

(t=0 - infinite symmetric group)

Realization of extreme central measures

"coin flips"



Central measures are determined by a sequence of random Jordan block structures $\lambda(n)$ of $n \times n$ corners, with $|\lambda(n)| = \lambda_1(n) + \lambda_2(n) + \ldots + \lambda_n(n)$, $n = 1, 2, \ldots$

Let $\omega = (\alpha; \beta; \gamma)$ where $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0)$, $\gamma \geq 0$, and $\dfrac{\gamma}{1-t} + \sum_{i \geq 0} \left(\alpha_i + \dfrac{\beta_i}{1-t} \right) = 1$

ω_0 be $\alpha_i = \beta_i = 0$, $\gamma = 1 - t$

$t := 1/\mathfrak{q}$

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Realization of extreme central measures

$$\mathrm{Prob}\,(\lambda(n) = \nu) = \frac{1}{Z} P_\nu(\omega_0) Q_\nu(\omega)$$

P_ν, Q_ν — **Hall-Littlewood symmetric polynomials**

Uniform measure is extreme and corresponds to $\alpha_i = (1 - \mathfrak{q}^{-1})\mathfrak{q}^{1-i}$, $i = 1, 2, \ldots$; $\beta_j = \gamma = 0$

Example of a Hall-Littlewood polynomial

$$P_{(4,0)}(x_1, x_2) = x_1^4 + x_2^4 + (1-t)(x_1^3 x_2 + x_1 x_2^3) + (1-t) x_1^2 x_2^2$$

$$Q_\nu = b_\nu(t) P_\nu$$

$$\text{Prob}\,(\lambda(n) = \nu) = \frac{1}{Z} P_\nu(\omega_0) Q_\nu(\omega)$$

Couple of useful facts about Hall-Littlewood polynomials

$$\sum_\nu P_\nu(x_1, \ldots, x_N) Q_\nu(y_1, \ldots, y_M) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 - t x_i y_j}{1 - x_i y_j}$$

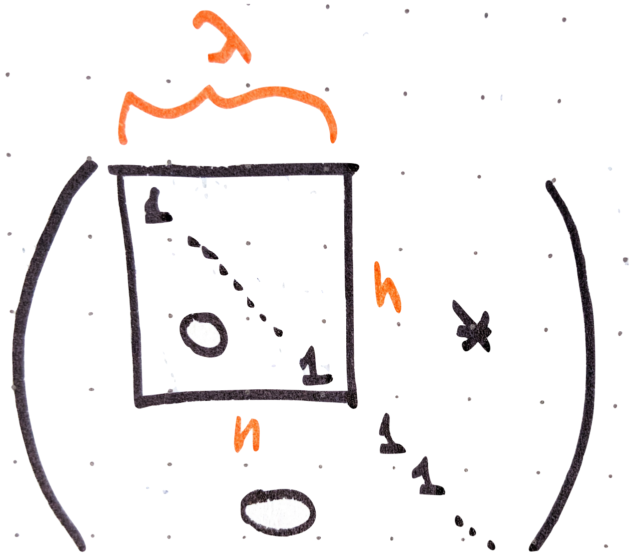
$t = 0$ — Schur polynomials $s_\lambda(x_1, \ldots, x_N) = \dfrac{\det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\prod_{1 \le i < j \le N} (x_i - x_j)}$

De Finetti	Matrices over finite fields
Random infinite binary sequences	Random infinite uni-uppertriangular matrices
Exchangeability (invariance under permutations)	Centrality (invariance under conjugations)
Extremes are parametrized by 1-d space $p \in [0,1]$	Extremes are parametrized by ∞-d space $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0),\ \beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0),\ \gamma \geq 0$ such that $\frac{\gamma}{1-t} + \sum_{i \geq 0} \left(\alpha_i + \frac{\beta_i}{1-t} \right) = 1$
Realization of extremes: iid Bernoulli (coin tossing)	Realization of extremes through Hall-Littlewood polynomials (example: uniform Bernoulli product measure on uni-uppertriangular matrices)
Reconstruction of parameters: classical Law of Large Numbers for Bernoulli trials	Reconstruction of parameters: **Law of Large Numbers** **[Bufetov-P. 2014]**

Law of Large Numbers

(Alexey Bufetov)

$t := 1/q$

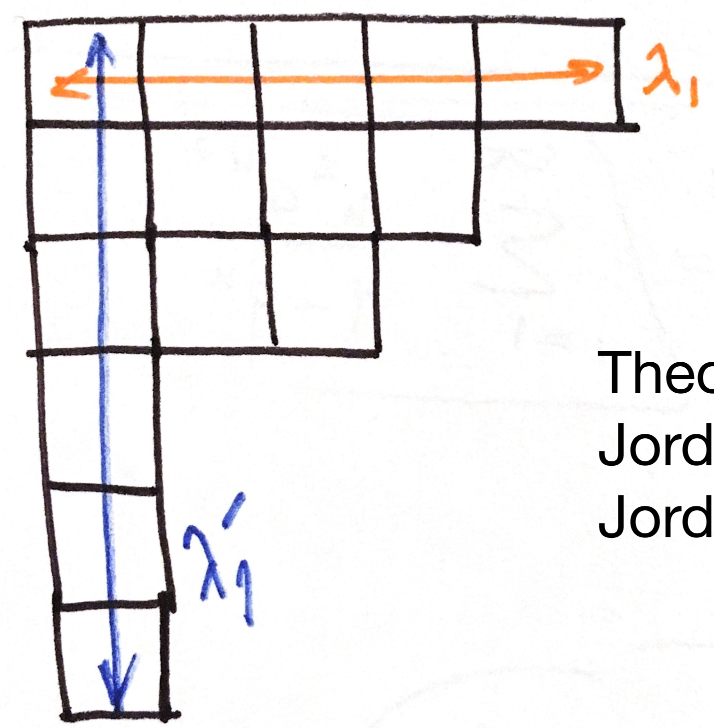


Theorem **[Bufetov-P. 2014]**

Take an extreme measure μ corresponding to $\omega = (\alpha; \beta; \gamma)$. Let $\lambda(n)$ be the Jordan block structure of the $n \times n$ corner. Then as $n \to +\infty$,

$$\frac{\lambda_i(n)}{n} \to \alpha_i, \qquad \frac{\lambda'_i(n)}{n} \to \frac{\beta_i}{1-t}$$

rows   columns



Theorem describes asymptotic sizes of large Jordan blocks & asymptotic frequencies of small Jordan blocks for matrices from extreme measures

Earlier results

- Uniform upper triangular matrices **[Borodin 1995]**, answering a question of A.A.Kirillov
- t = 0, asymptotic character theory of S(∞) **[Vershik-Kerov 1980s]**

Law of Large Numbers: idea of proof

a randomization of the Robinson-Schensted-Knuth **[O'Connell-Pei 2012], [Borodin-P. 2013]**

1. Construct a (randomized) algorithm for exact sampling of $\lambda(n)$ coming from the extreme measure μ_ω
2. Analyze the algorithm probabilistically to get limiting frequencies of rows and columns

The Young diagrams $\lambda(n)$ are sampled by constructing random Young tableaux

$$T(k+1) = T(k) \leftarrow a$$

Insertions are **randomized**; for $t=0$ reduce to the classical Robinson-Schensted-Knuth ones (with Vershik–Kerov modifications)

New letters appear independently using $\alpha_j, \beta_j, \gamma$ parameters

2	2	3	5	0.1
3	5	5	$\hat{7}$	0.34
5	$\hat{5}$			
$\hat{2}$	$\hat{5}$			
$\hat{2}$				

$\alpha\beta\gamma$-tableaux (generalize semistandard Young tableaux)

[Vershik-Kerov 1986]

Take random words with independent letters (the sum of probabilities is 1):

$$\mathbb{P}(k) = \alpha_k, \qquad \mathbb{P}(\hat{k}) = \frac{\beta_k}{1-t}, \qquad \frac{\gamma}{1-t} \text{ — continuous part}$$

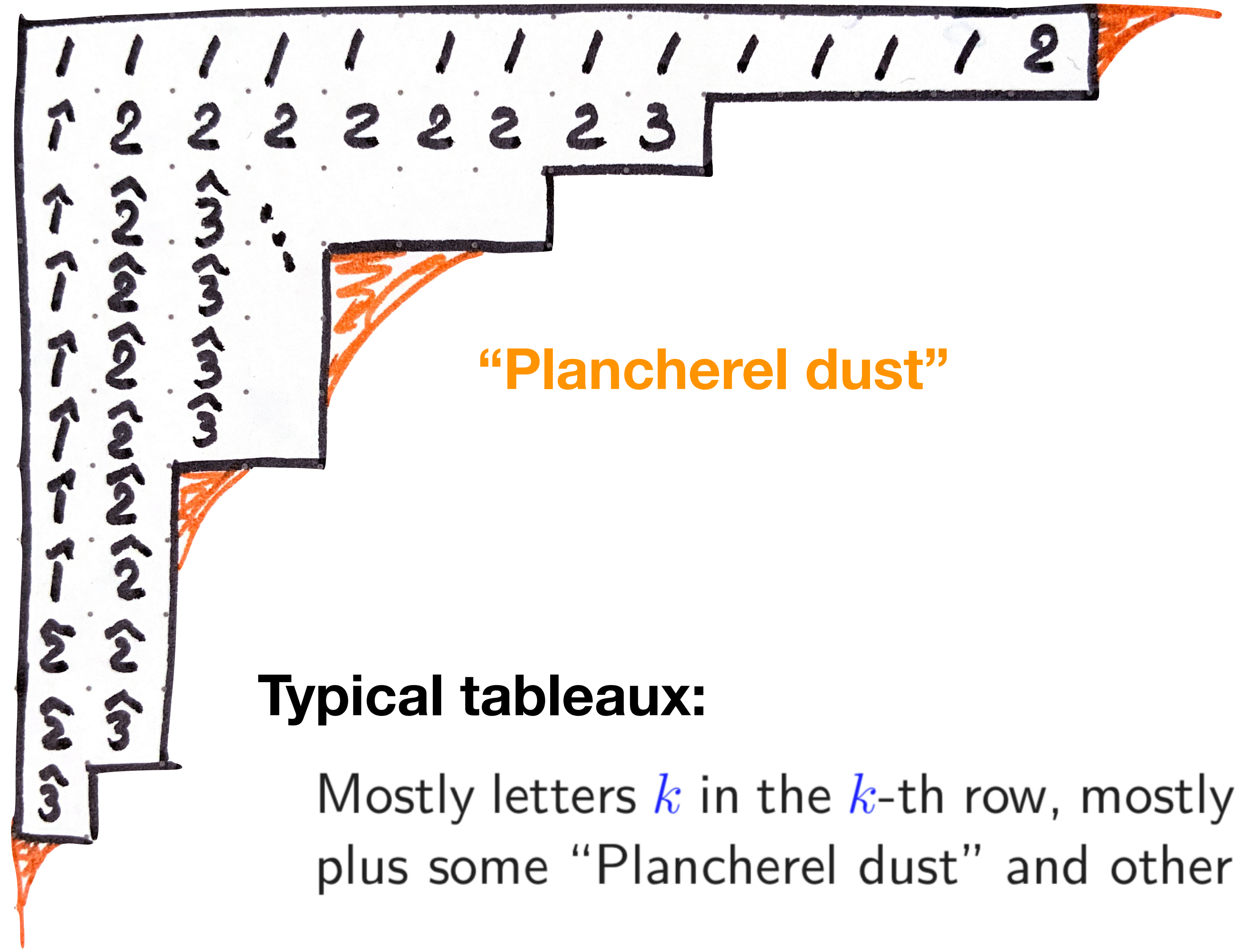
After n steps, the Hall-Littlewood RSK sampling produces a random Young diagram $\lambda(n)$, the shape of the random tableau

Theorem [Borodin-P. 2013], [Bufetov-P. 2014]

The distribution of $\lambda(n)$ coincides with the Jordan block structure of the $n \times n$ corner of the random matrix coming from the extreme measure μ_ω.

2	2	3	5	0.1
3	5	5	$\hat{7}$	0.34
5	$\hat{5}$			
$\hat{2}$	$\hat{5}$			
$\hat{2}$				

Probabilistic consequences



Typical tableaux:

Mostly letters k in the k-th row, mostly letters $\hat{k}$ in the k-th column, plus some “Plancherel dust” and other lower order errors

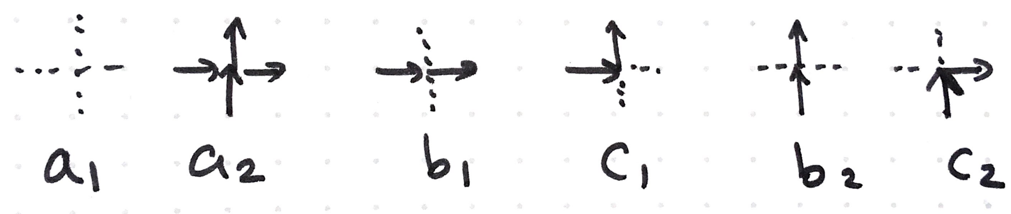
This leads to the **Law of Large Numbers** for the Jordan blocks structure of extreme central measures

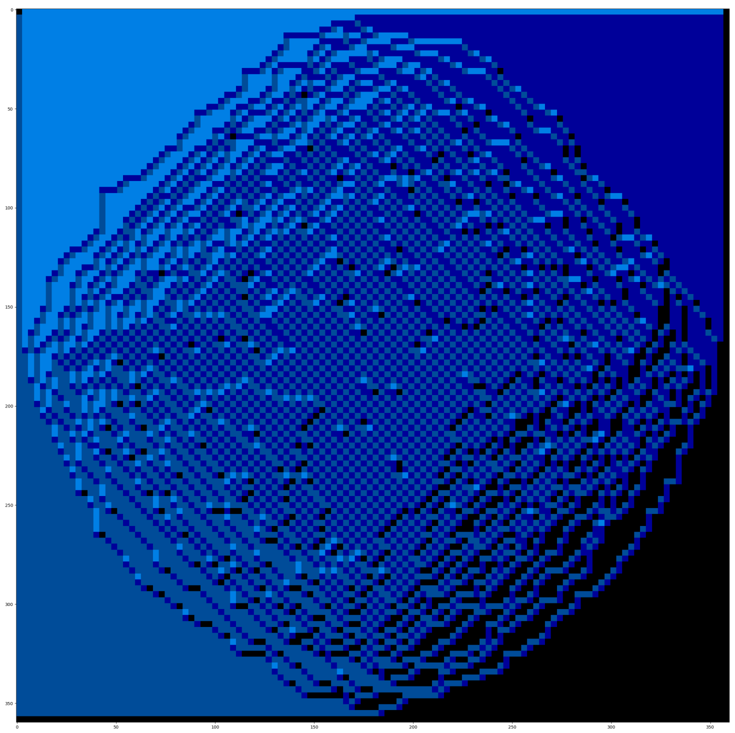
Remark

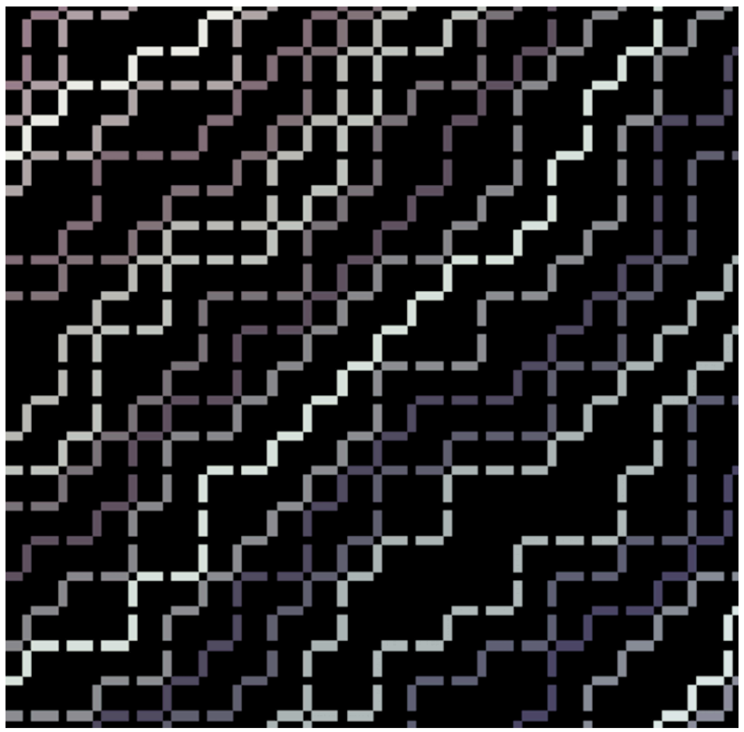
Under additional restrictions (all parameters are distinct and $\gamma = 0$), one should also get a **Central Limit Theorem** with Gaussian fluctuations of order $\sqrt{n}$

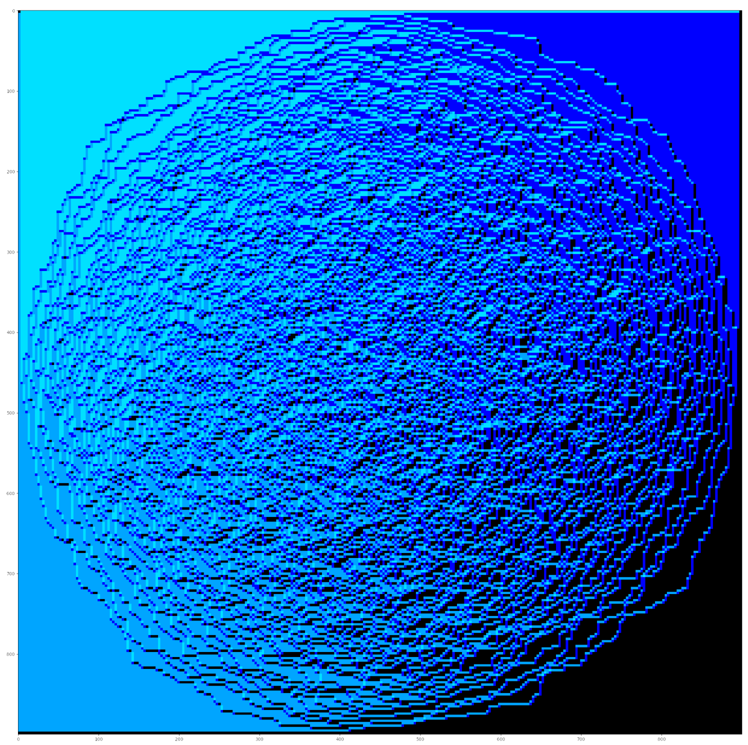
The fluctuations of each row and column are almost independent, modulo that the number of boxes is fixed

Towards the six vertex model



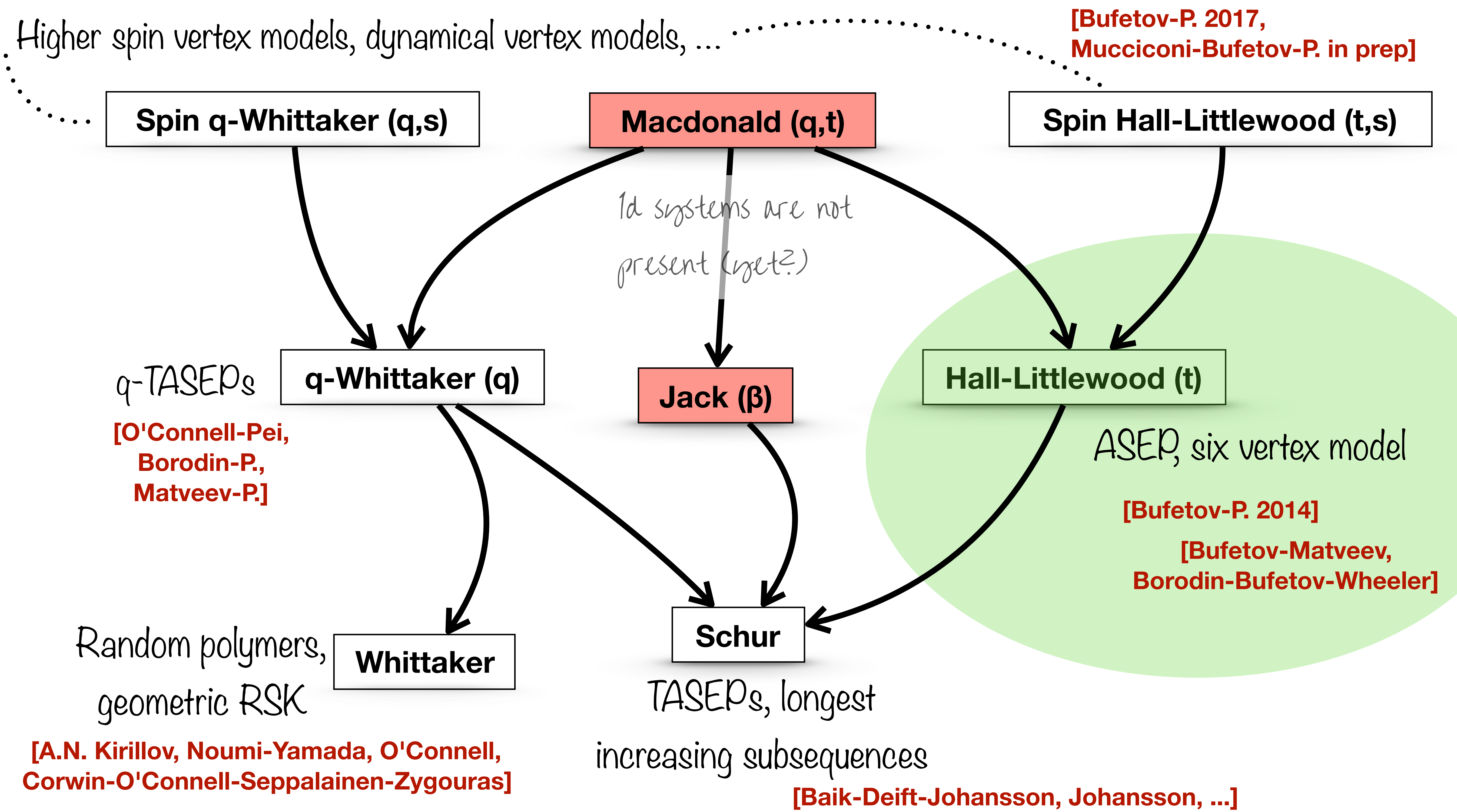






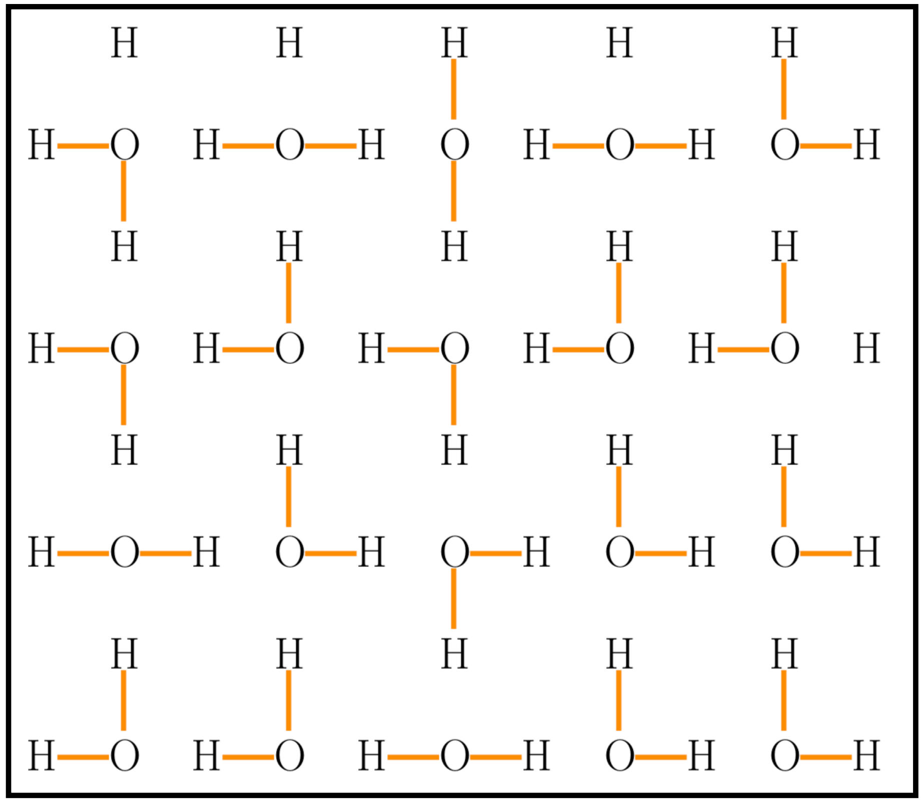
For about 20 years now, **Robinson-Schensted-Knuth** type combinatorial algorithms are providing exact observables of *1-dimensional interacting particle systems* and related models through Schur functions

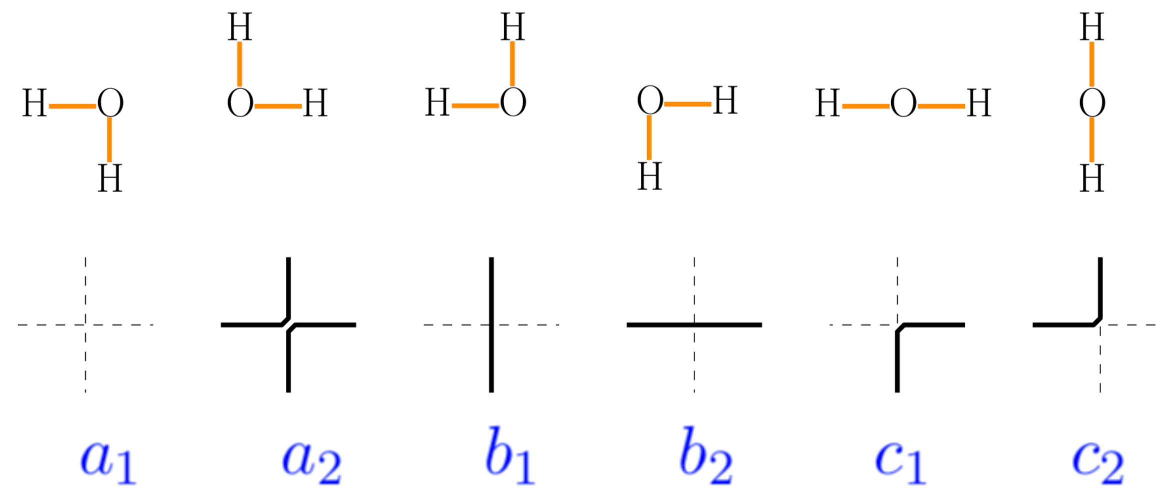
Less than a decade ago, a new wave has started involving deformations of Schur functions. The Robinson-Schensted-Knuth type constructions are also deformed (*randomized*)



Six vertex (square ice) model

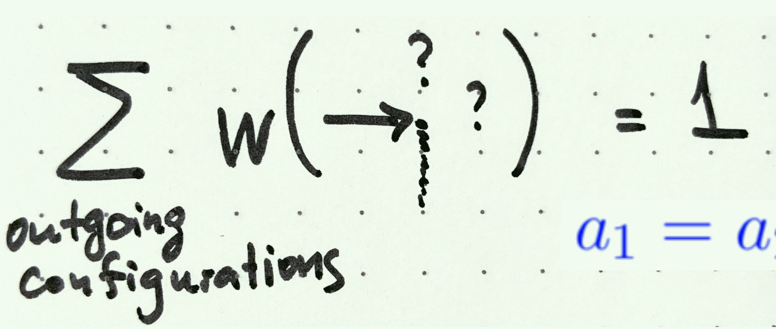
[Pauling, 1935], [Lieb,1967]





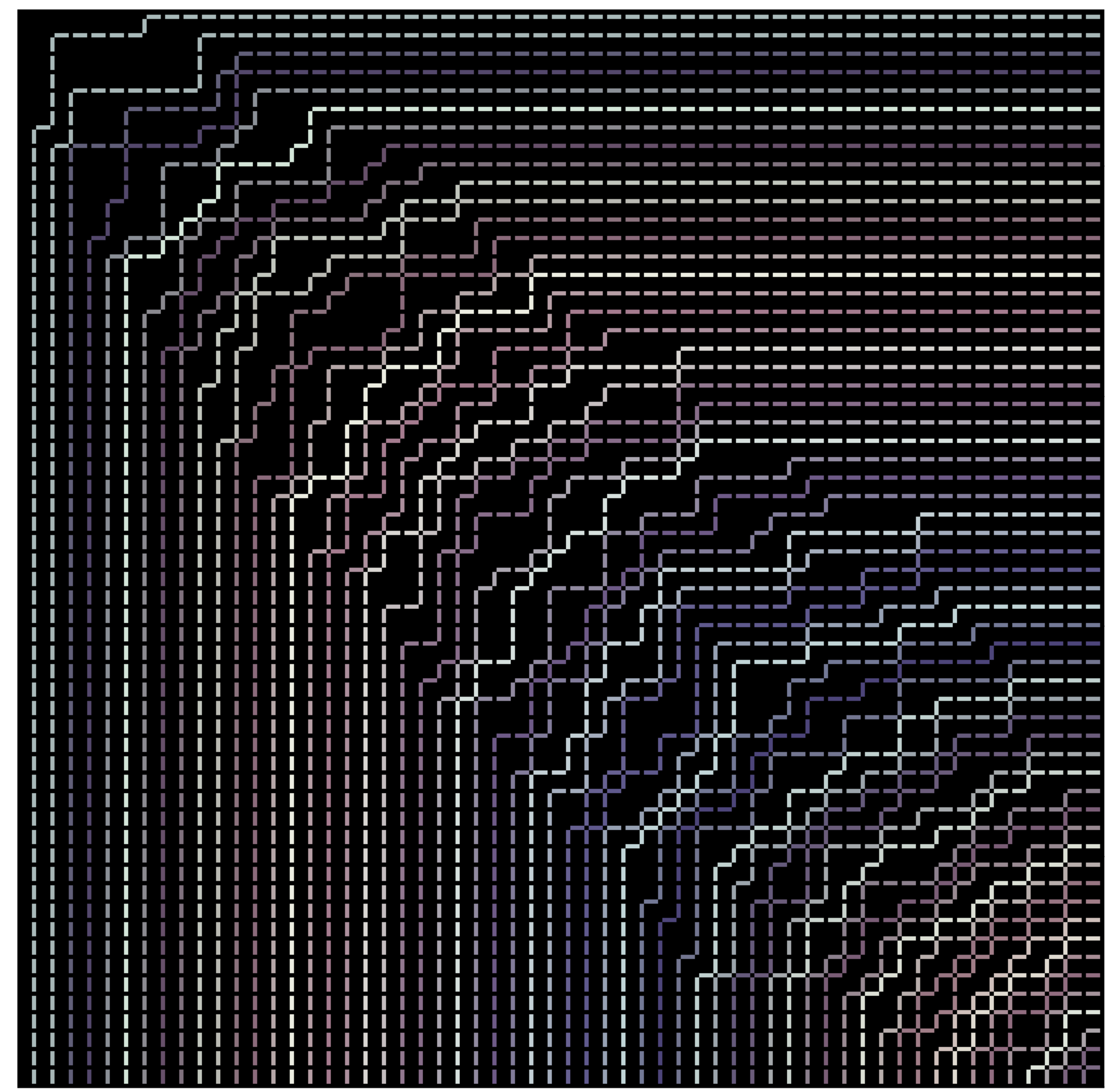
Stochastic six vertex model

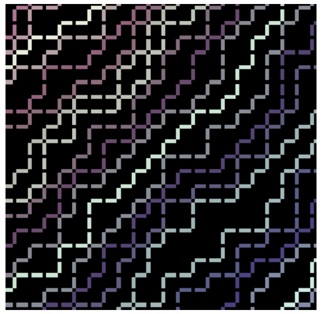
[Gwa-Spohn 1992, Borodin-Corwin-Gorin 2014]

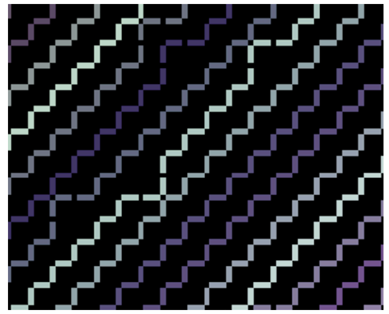


$$a_1 = a_2 = 1,\ b_i + c_i = 1$$

- The system is a **Markov process** (= stochastic interacting particle system)
- Partition functions are products, not determinants
- This and many other stochastic vertex models are exactly solvable to the point of asymptotics



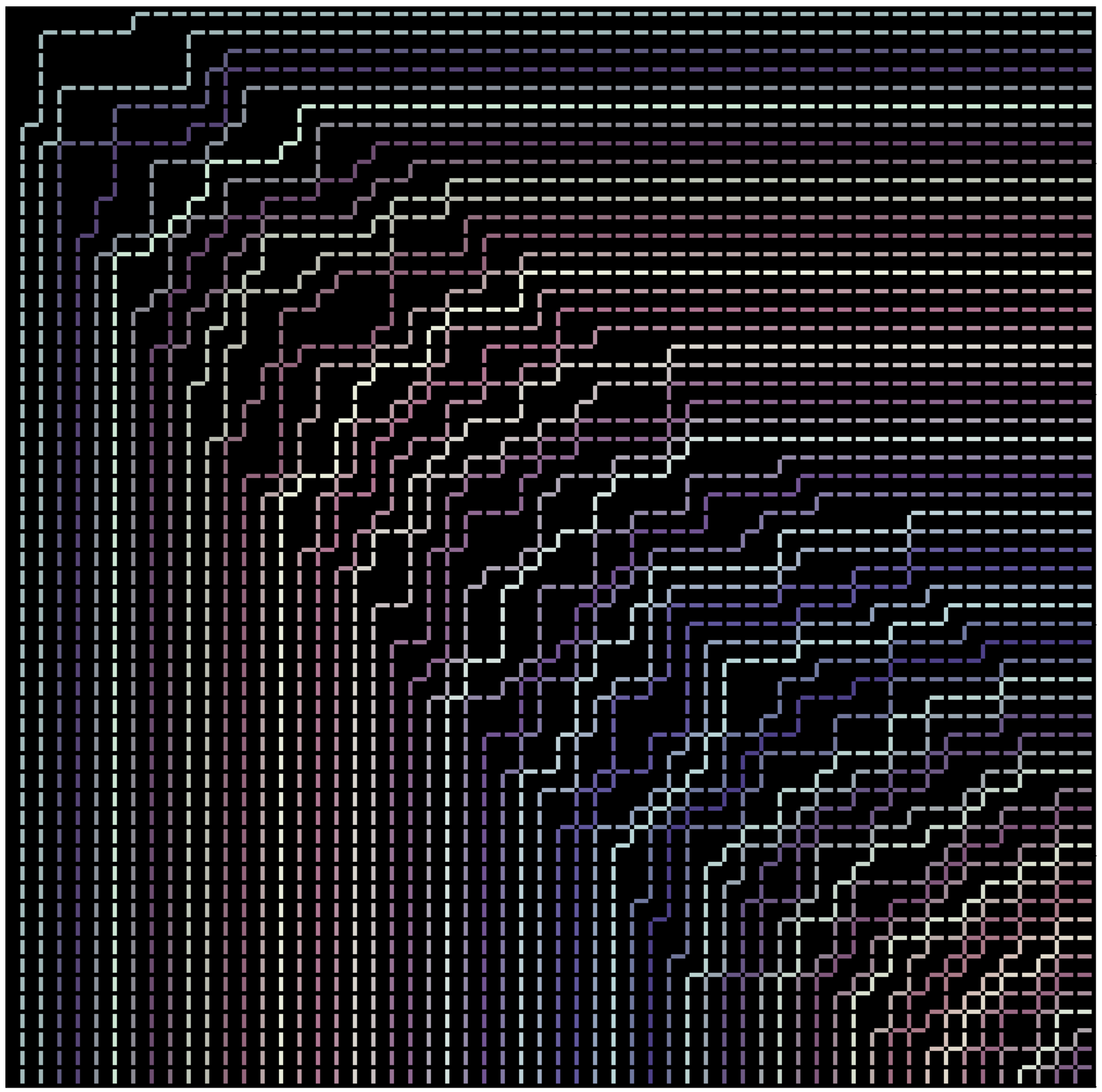




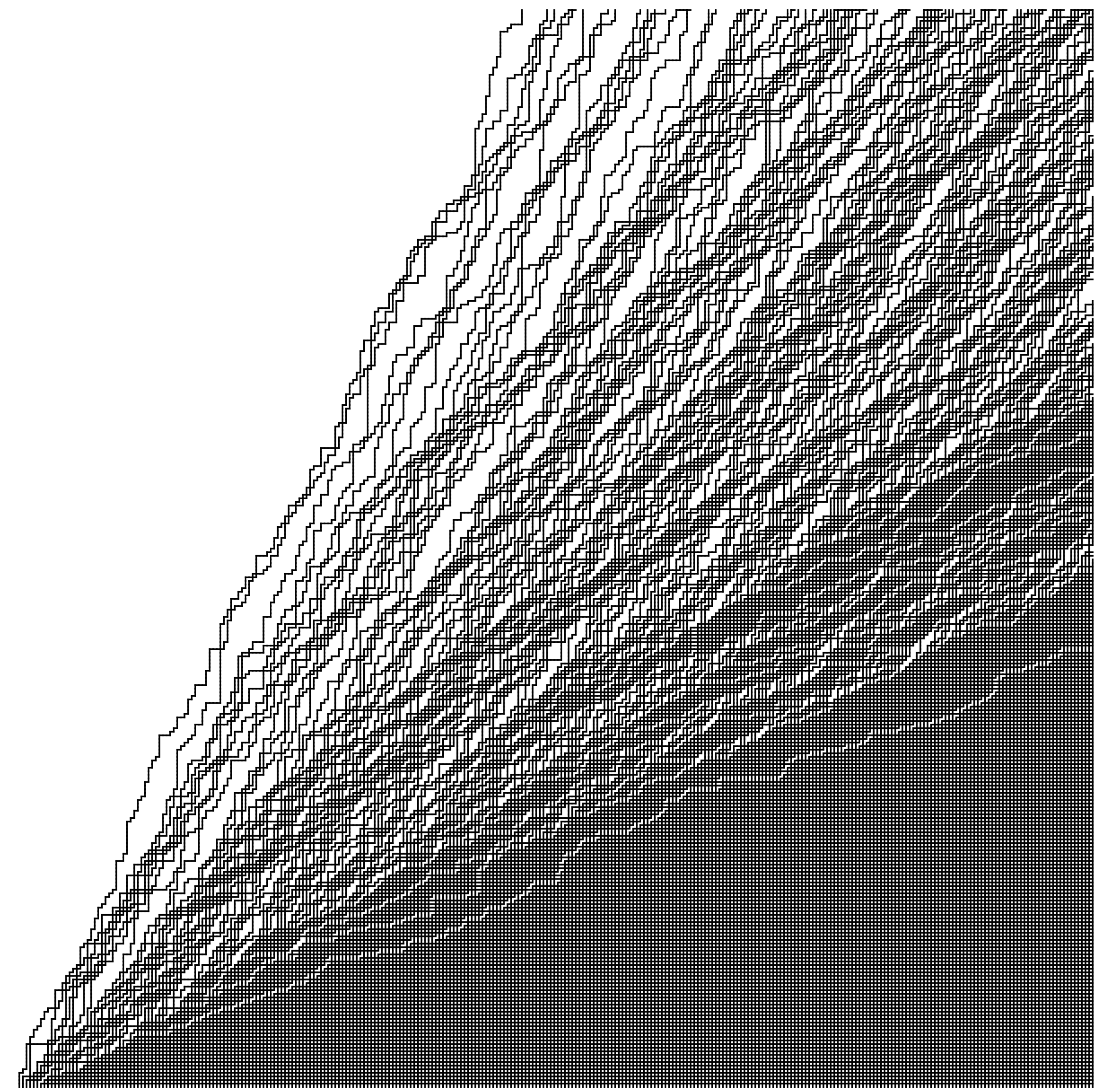
(richer behavior)

$a_1 = a_2 = 1$, $b_i + c_i = 1$

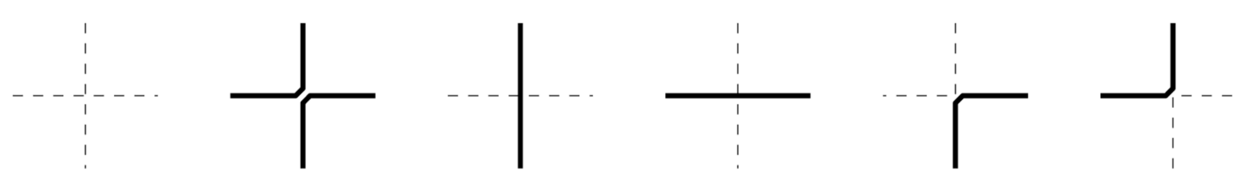
(easier to analyze)



$a_1 = a_2 = b_2 = c_1 = c_2 = 1$, $b_1 = 3$,
domain wall boundary conditions



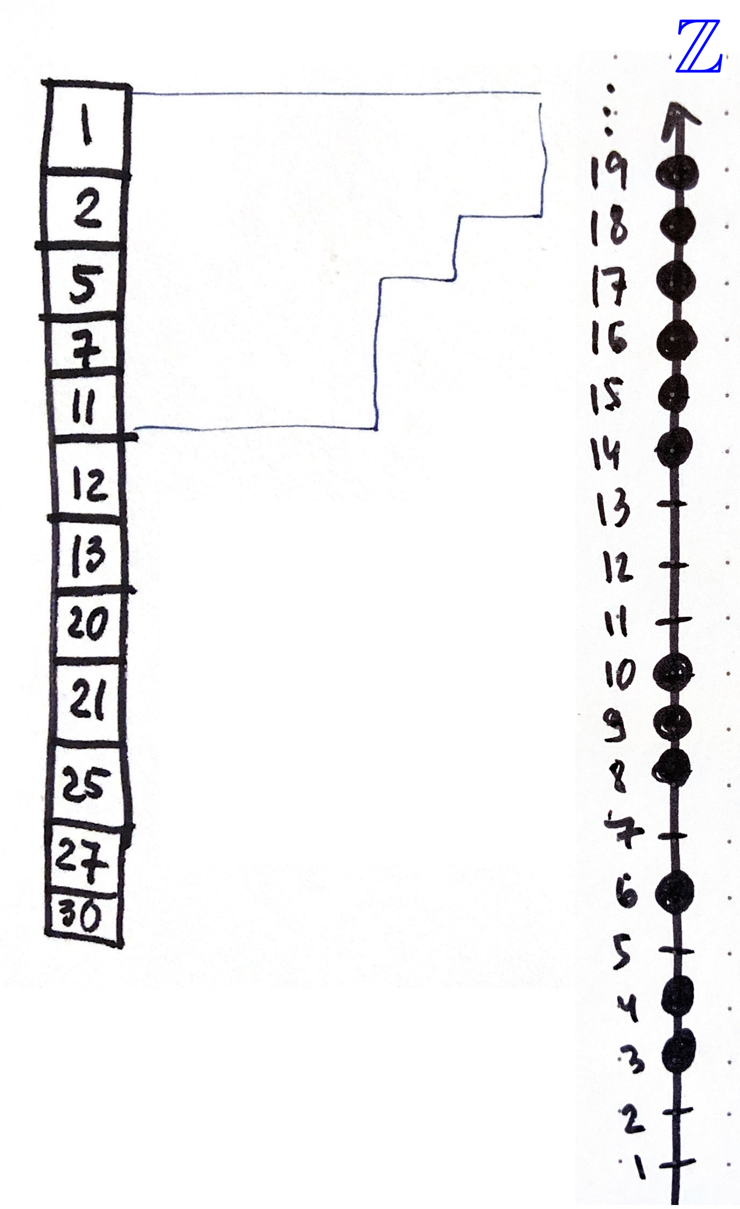
stochastic six vertex model in a quadrant



Marginally Markovian 1d projection
in the Hall-Littlewood Robinson-Schensted insertion

Keep only parameters $\alpha_1, \alpha_2, \ldots$, and let $\beta_j = \gamma = 0$
The Young tableau is semistandard

Defects in the first column of the tableau = particles on $\mathbb{Z}$

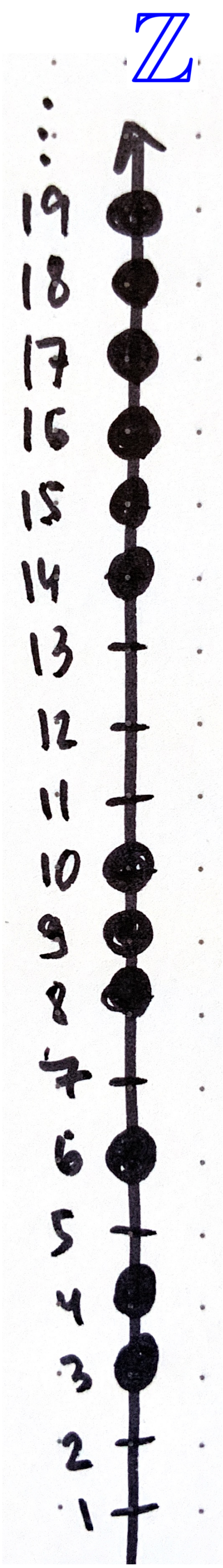


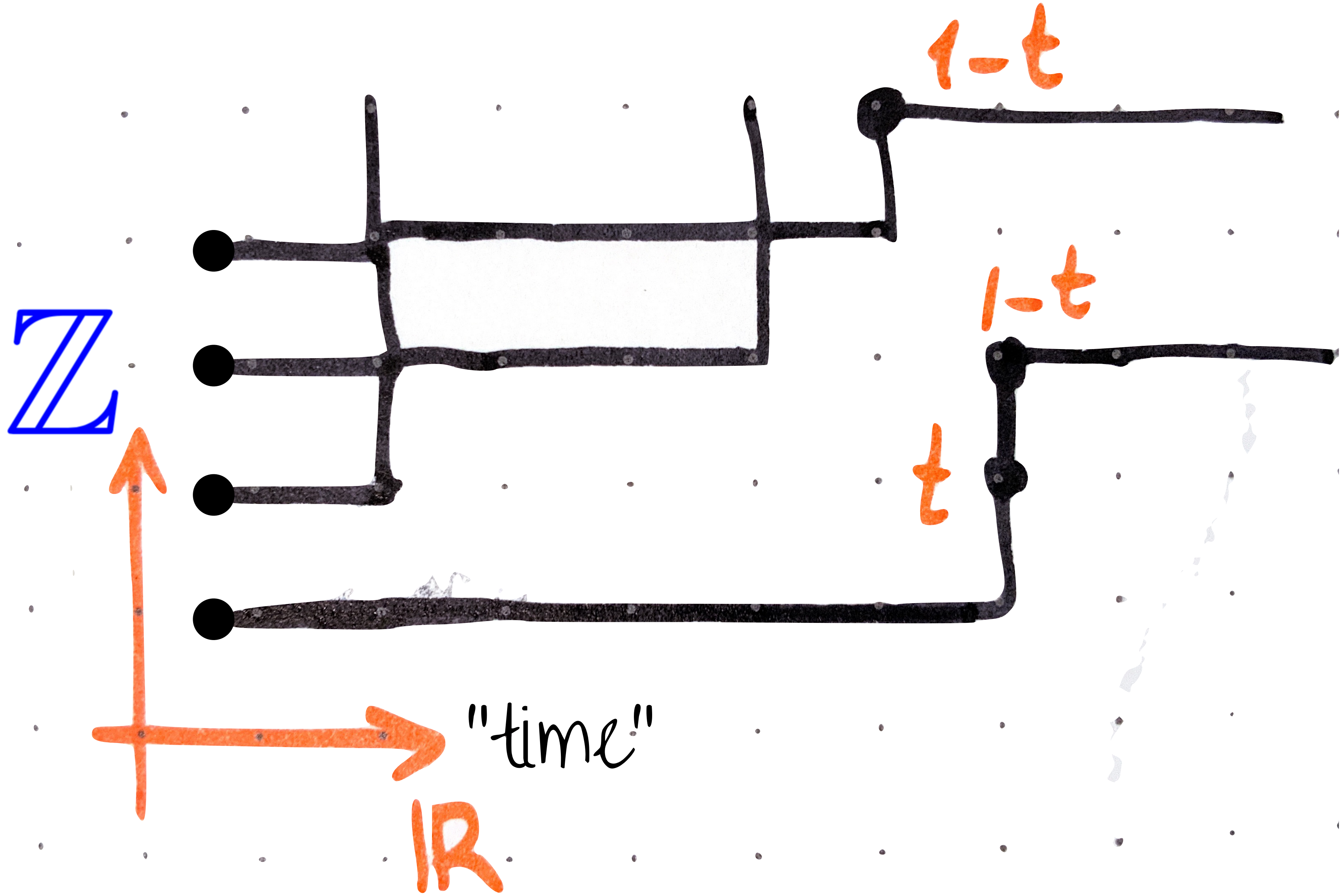
- RSK-insert $k \Leftrightarrow$ clock rings at site k (rate α_k)
- if there is a particle at k, it wakes up and jumps up by one
- if the destination is occupied, the next particle is pushed by one and wakes up, the pusher stops
- the active particle moves through the empty space with probability t per step; stops with probability $1 - t$

Called the **half-continuous stochastic six vertex model**

Asymptotics (homogeneous α) - **[Ghosal 2017]**

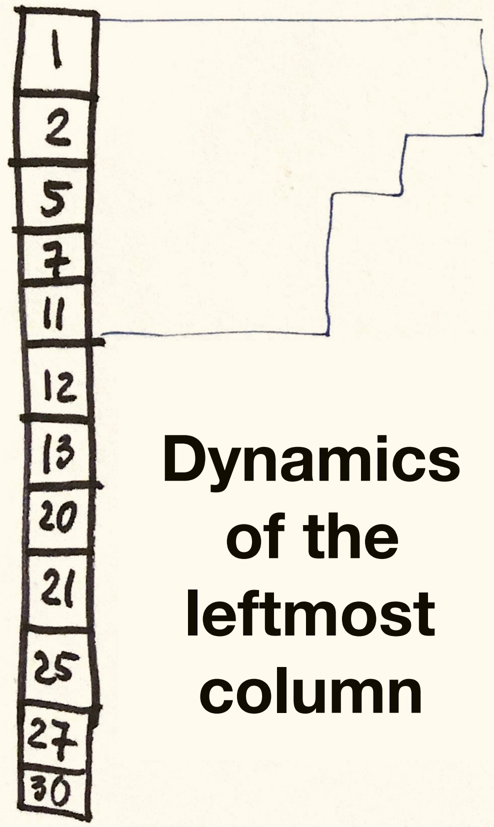
Get a continuous-time Markov chain on particle configurations. Plot trajectories of particles:





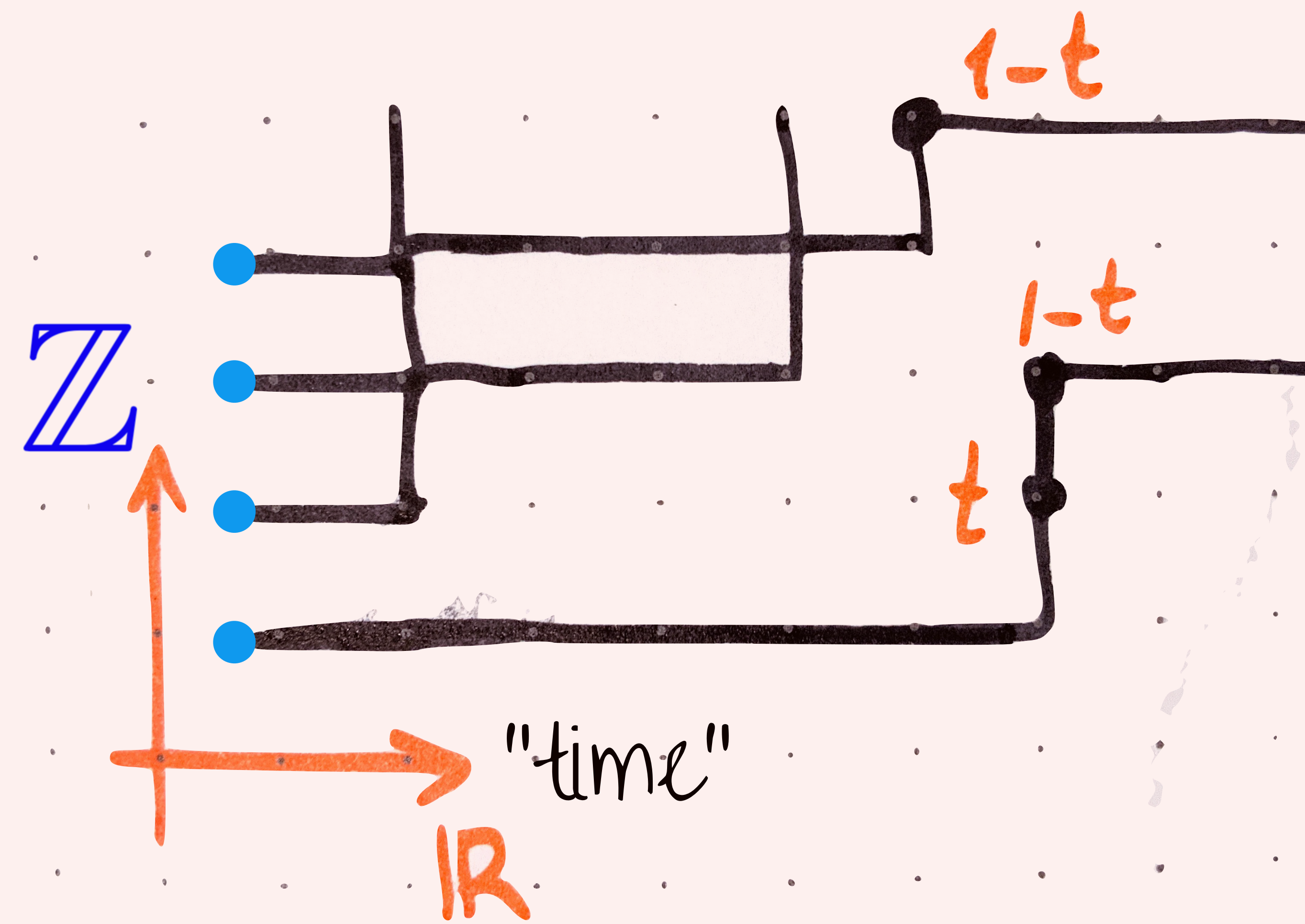
RSK type sampling of extreme central measure

$\beta_j = \gamma = 0$

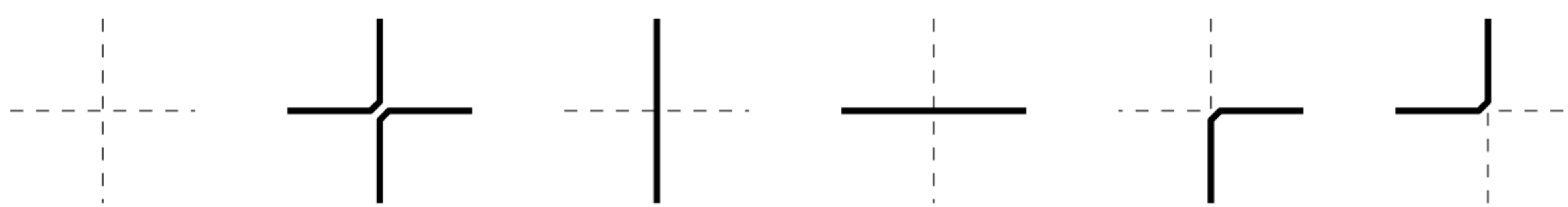


Dynamics of the leftmost column

Trajectories of particles:



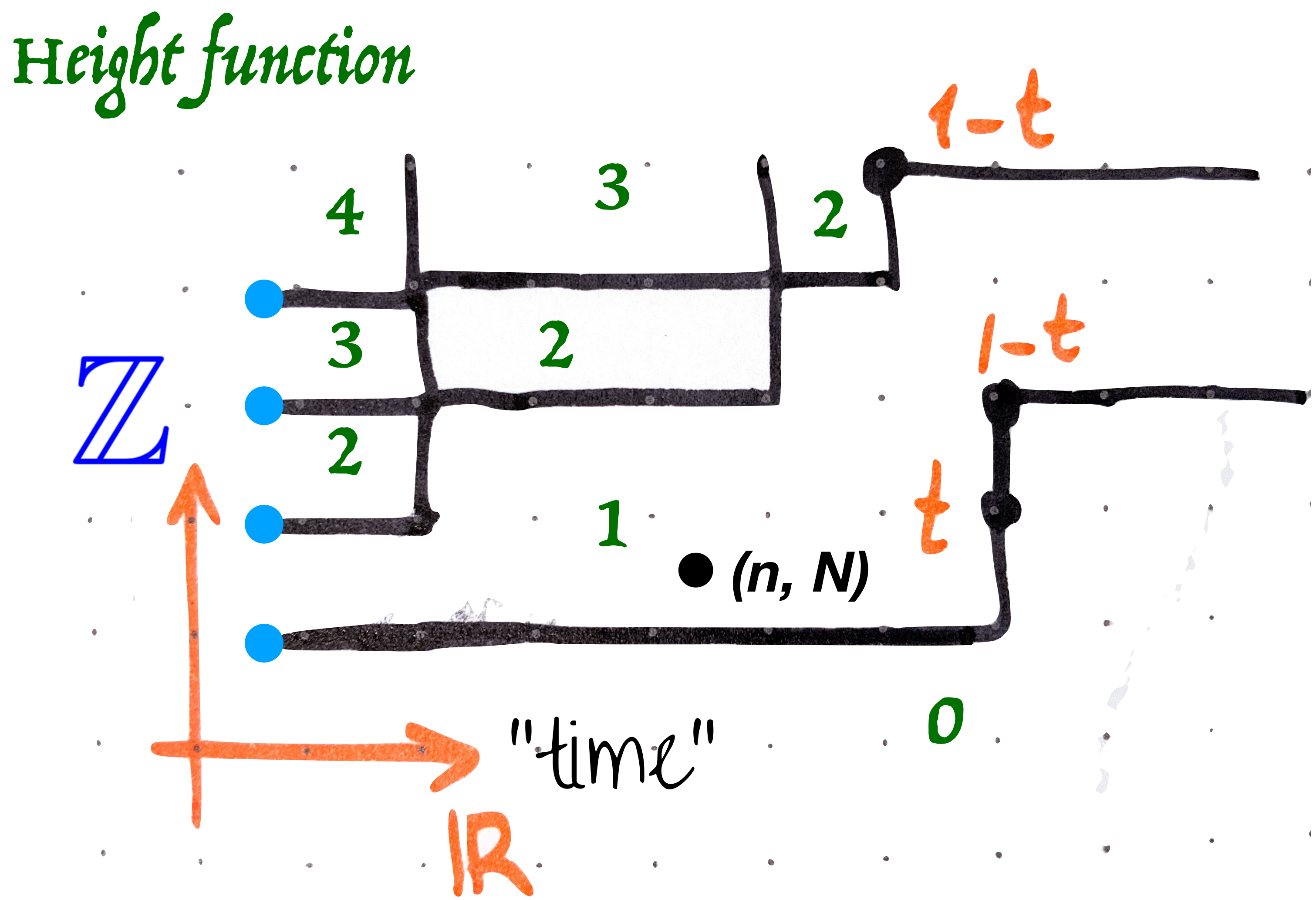
This is the same as...



$b_2 \to 1$, Poisson type limit in the horizontal direction

$b_1 = t$ fixed

Half-continuous stochastic six vertex model



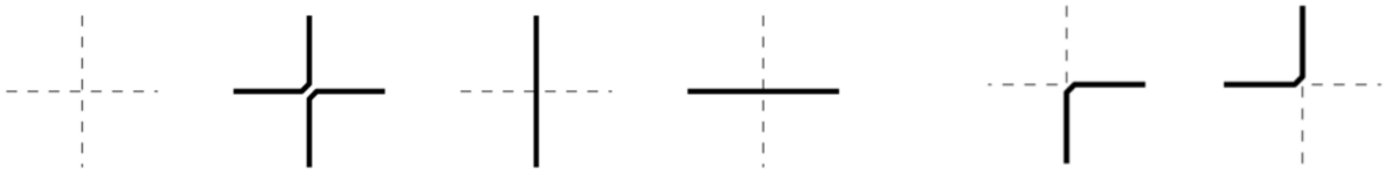
Theorem which follows from the constructions:

$N - \lambda_1'(n)$, where λ comes from $\omega = ((\alpha_1, \ldots, \alpha_N); 0; 0)$, is the height function of the half-continuous stochastic six vertex model at (n, N), where n is the number of (independent) jumps occurred

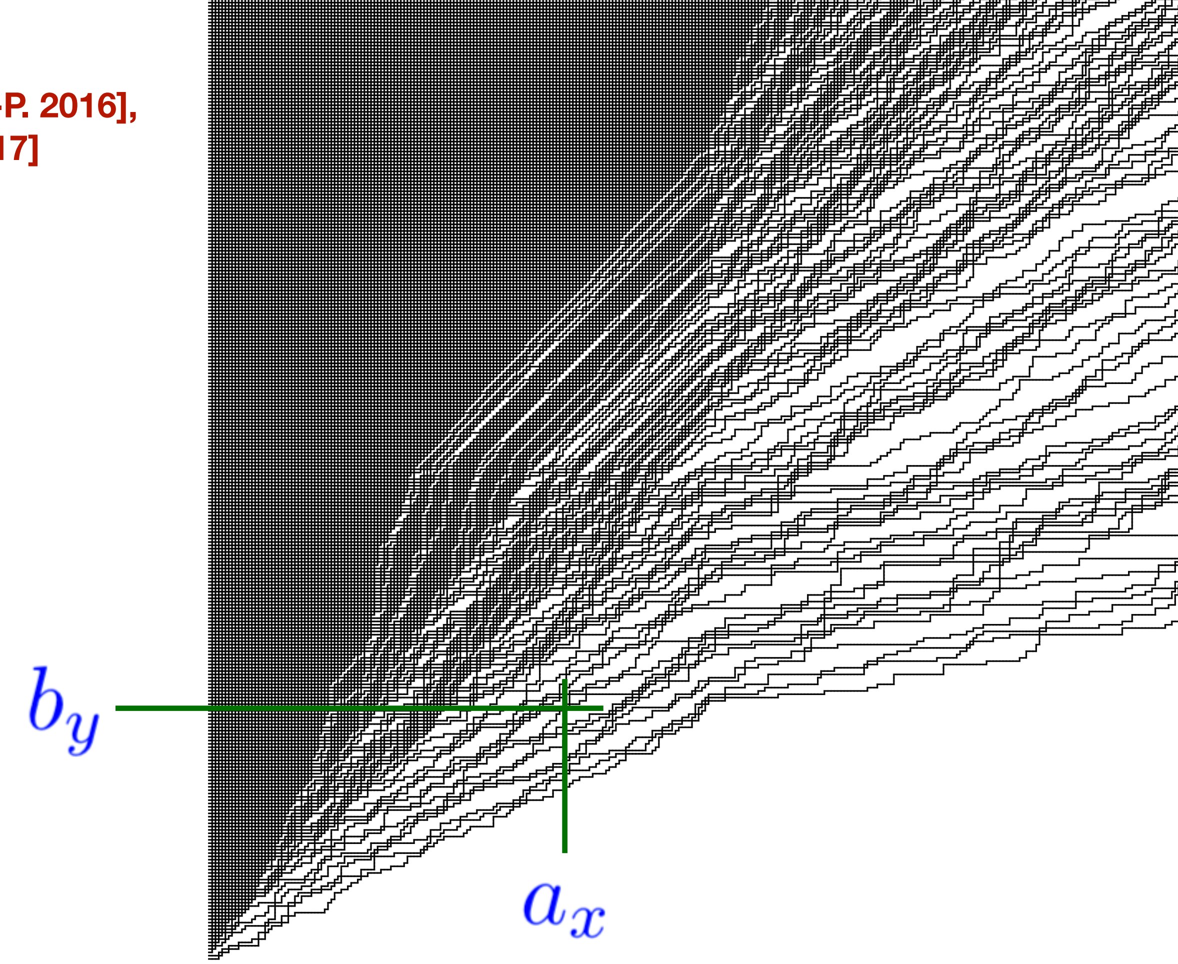
The distribution of $\lambda(n)$ is expressed through the Hall-Littlewood symmetric polynomials, which allows to write down explicit distributional formulas for $\lambda_1'(n)$, and obtain asymptotics

Summary for the fully discrete, inhomogeneous stochastic six vertex model

[Borodin-Corwin-Gorin 2014], [Borodin-P. 2016], [Bufetov-Matveev 2017], [Bufetov-P. 2017]



$1 \qquad 1 \qquad \frac{t(1-a_x b_y)}{1-ta_x b_y} \qquad \frac{1-a_x b_y}{1-ta_x b_y} \qquad \frac{1-t}{1-ta_x b_y} \qquad \frac{(1-t)a_x b_y}{1-ta_x b_y}$



Theorem

The height function at (x,y) is distributed as $y-\lambda'_1$, where λ has the Hall-Littlewood distribution $\propto P_\lambda(a_1,\ldots,a_x)Q_\lambda(b_1,\ldots,b_y)$

Theorem (t-moment formula) $\forall \ell = 1, 2, \ldots$

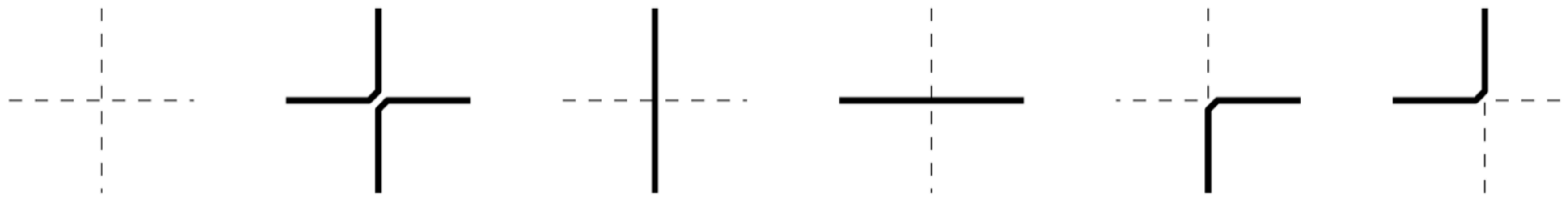
$$\mathbb{E}t^{\ell \cdot h(x,y)} = t^{\frac{\ell(\ell-1)}{2}} \oint \ldots \oint \prod_{1\le i<j\le \ell} \frac{w_i - w_j}{w_i - tw_j} \prod_{i=1}^{\ell} \left(\frac{dw_i}{2\pi \mathbf{i} w_i} \prod_{r=1}^{x} \frac{a_r - w_i}{a_r - tw_i} \prod_{r=1}^{y} \frac{tw_i - b_r}{w_i - b_r} \right)$$

Contours are around $\{b_i\}$ and 0 (in a certain order)

Has many proofs...

- A la [Tracy-Widom 2007+] for ASEP based on *coordinate Bethe Ansatz*
- Yang-Baxter equation and Cauchy identities via *q-correlations*
- *Randomized Robinson-Schensted-Knuth* plus Macdonald difference operators for Hall-Littlewood polynomials
- *Randomization of the Yang-Baxter equation* + HL polynomials

Asymptotics in the homogeneous stochastic six vertex model



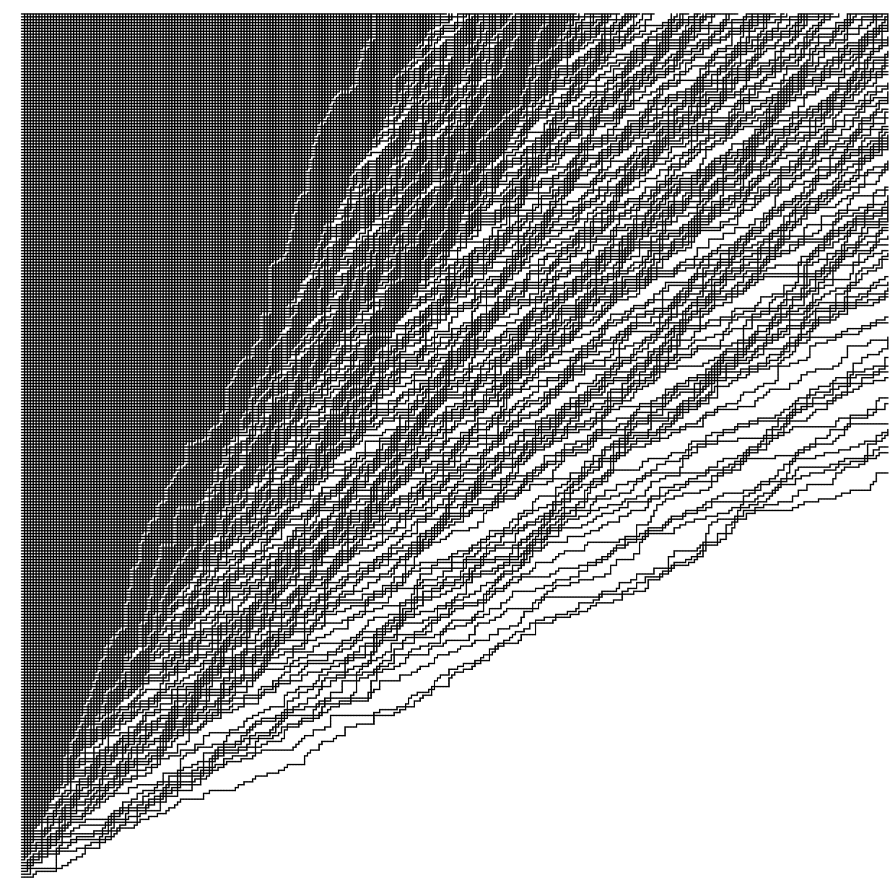
1 $\quad 1$ $\quad b_1$ $\quad b_2$ $\quad 1-b_1$ $\quad 1-b_2$

Height function has a limit shape $\frac{1}{L}h(Lx, Ly) \to \mathcal{H}(x,y)$

The nontrivial part of the limit shape is $\mathcal{H}(x,y) = \dfrac{\left(\sqrt{x(1-b_1)} - \sqrt{y(1-b_2)}\right)^2}{b_2 - b_1}$

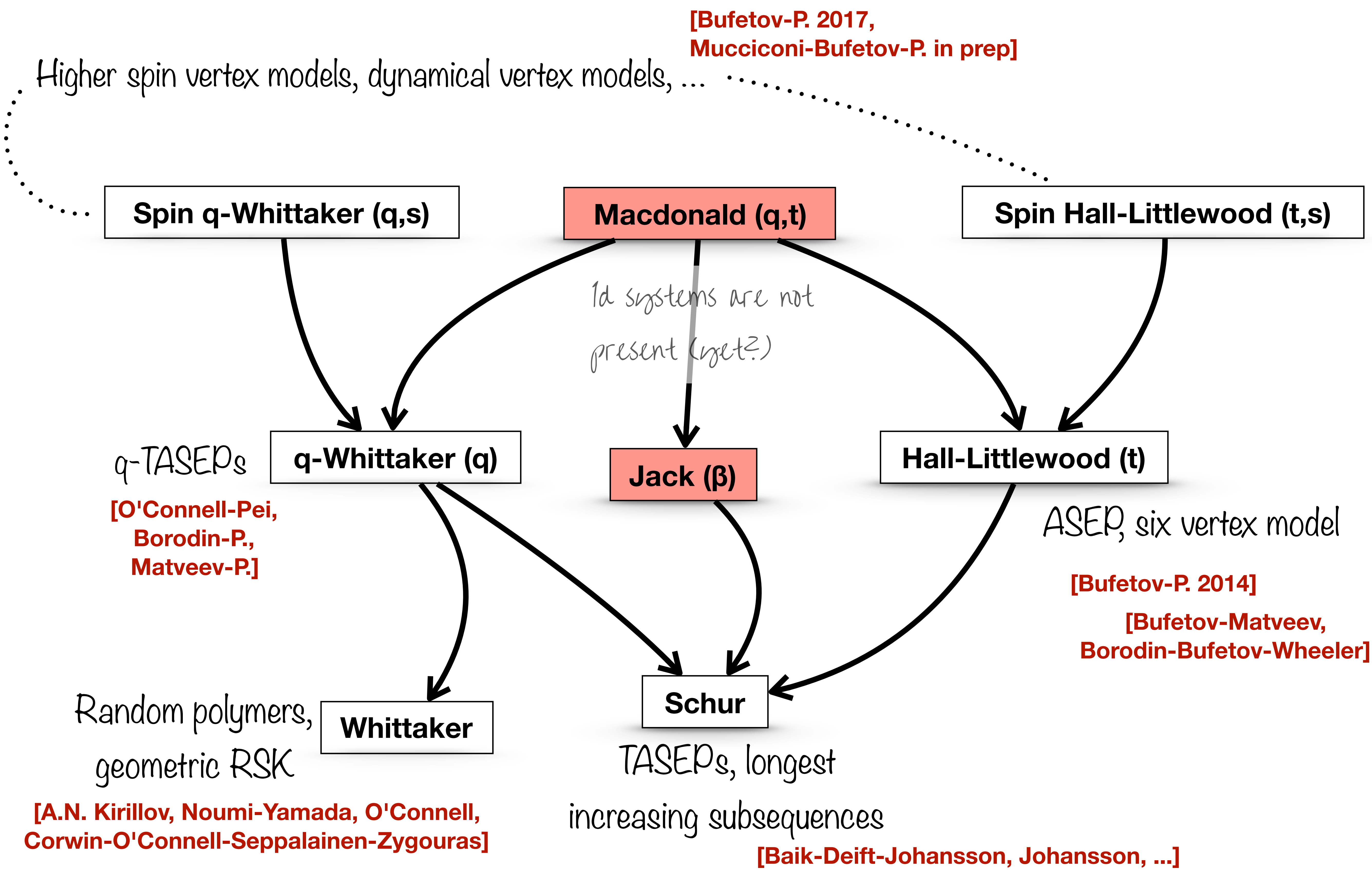
Fluctuations are governed by the GUE Tracy–Widom distribution (originated about 25 years ago in random matrix theory)

$$\lim_{L\to+\infty} \mathbb{P}\left(\frac{h(Lx,Ly) - L\mathcal{H}(x,y)}{\sigma_{x,y}L^{1/3}} \geq -s\right) = F_{GUE}(s)$$



- Higher spin versions + spin Hall-Littlewood polynomials
- Multilayer systems
- Degenerates to ASEP
- Limits to Kardar-Parisi-Zhang equation
- There is also a stochastic telegraph equation
- …

Conclusion: It is worthwhile to connect particle systems to random partitions associated with symmetric functions…



Thank you!