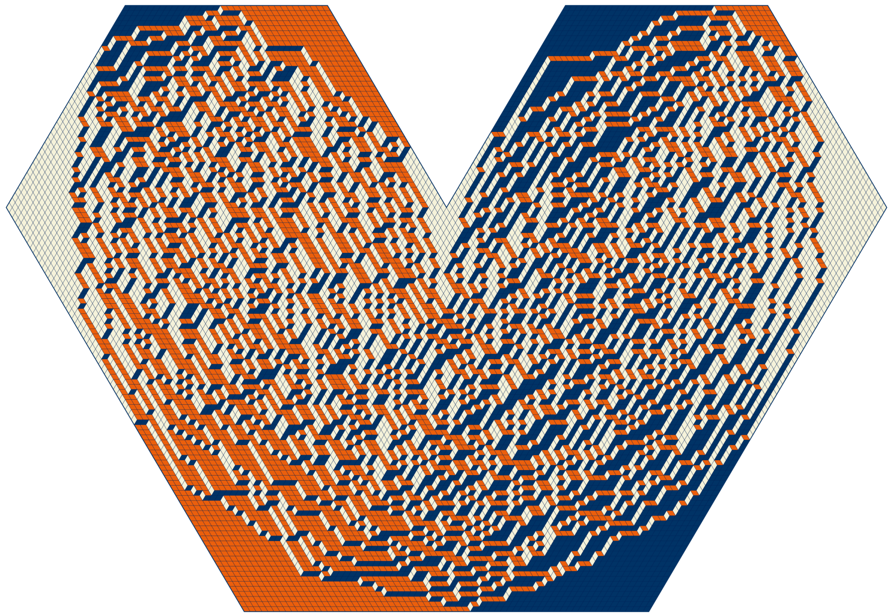
Gibbs measures, arctic curves, and random interfaces

Leonid Petrov
University of Virginia

December 3, 2018

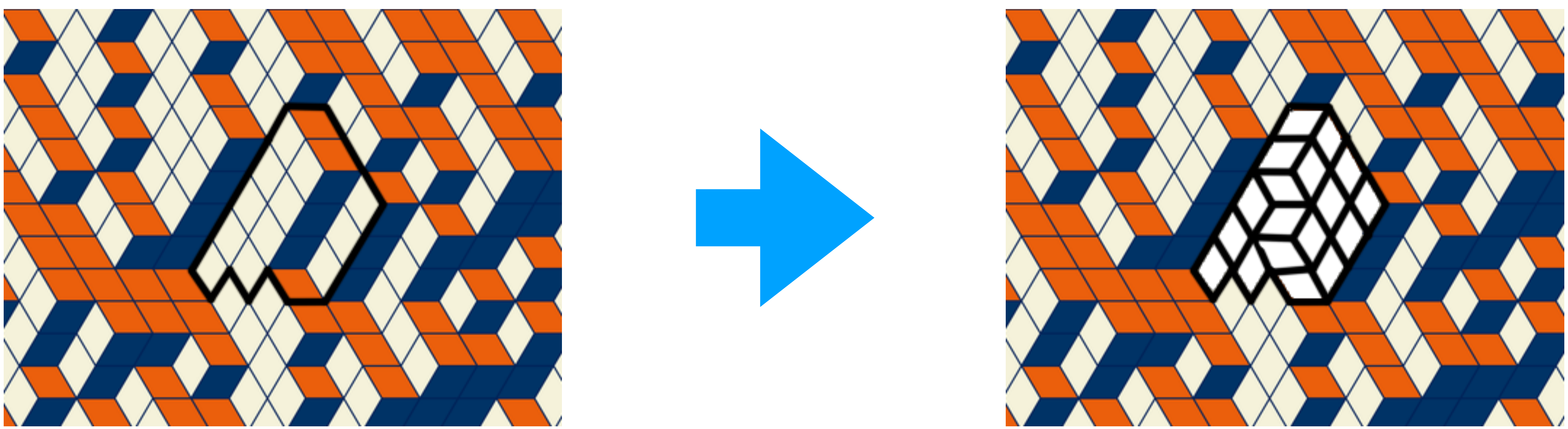


lpetrov.cc/Gibbs2018/

Gibbs Measures

Goal. Give an overview of universal asymptotic phenomena in 1 and 2 dimensions around Gibbs measures and dynamics on them

Definition (Dobrushin 1968, Lanford and Ruelle 1969)**.** Take a probability measure $\mathbb{P}$ on configurations (of particles / lozenges / dominoes / spins) in a space Λ.
$\mathbb{P}$ is called *Gibbs* if for any finite subset $A \subset \Lambda$, conditioned on the configuration outside of A, the distribution inside A depends only on the boundary values.



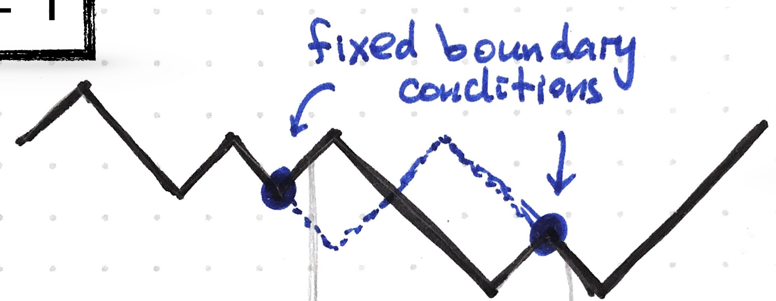
Resampling the configuration in a finite volume with fixed boundary and a prescribed distribution inside does not change the overall distribution $\mathbb{P}$

(the conditional distribution under the resampling does not have to be uniform)

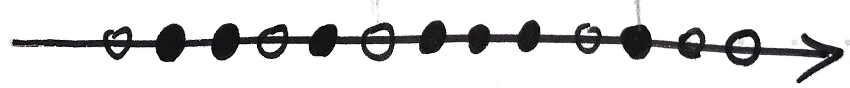
Examples

d = 1

Exchangeable sequences



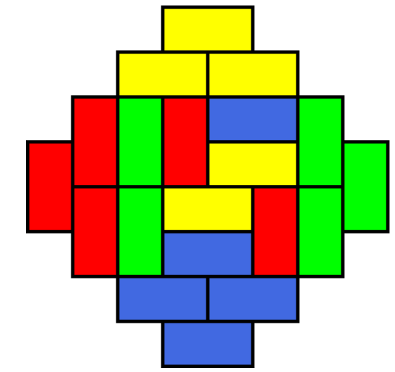
(de Finetti 1931) - classification of translation invariant ergodic Gibbs measures



d = 2

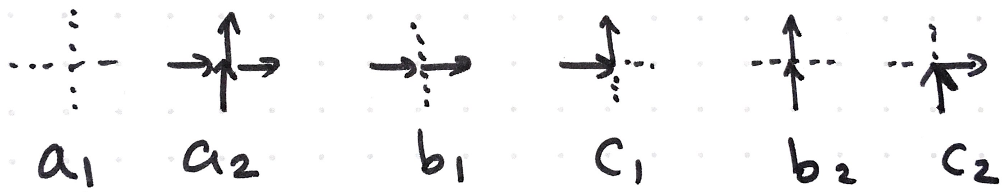
Lozenge tilings

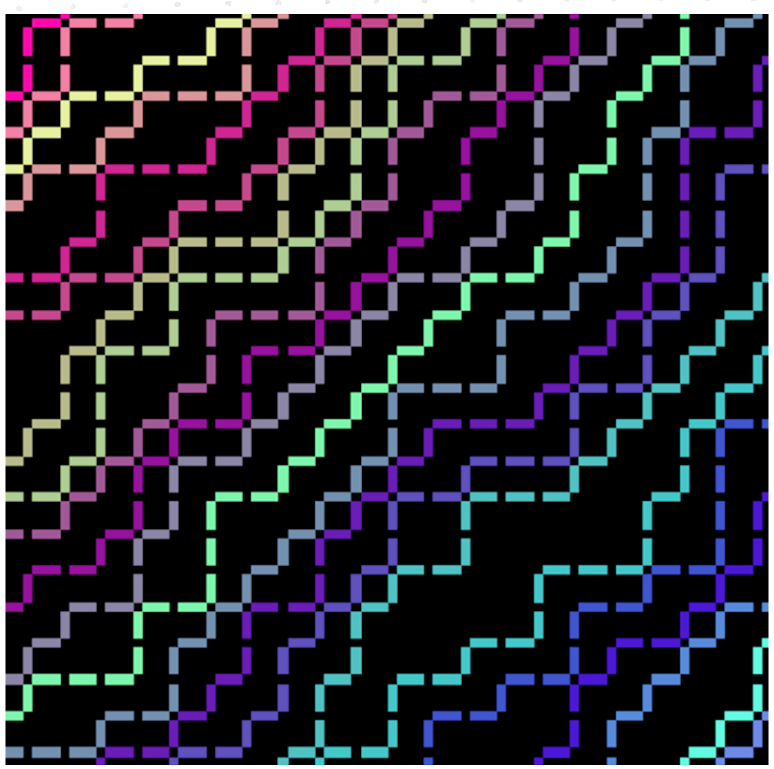
Domino tilings

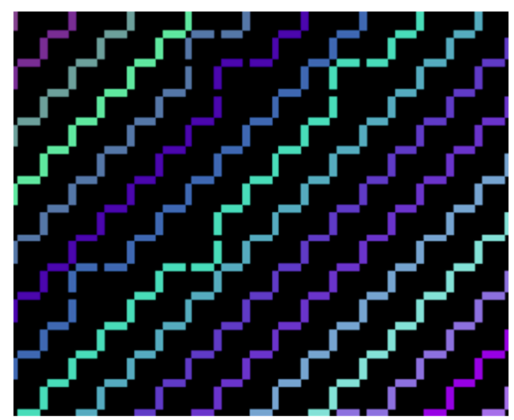


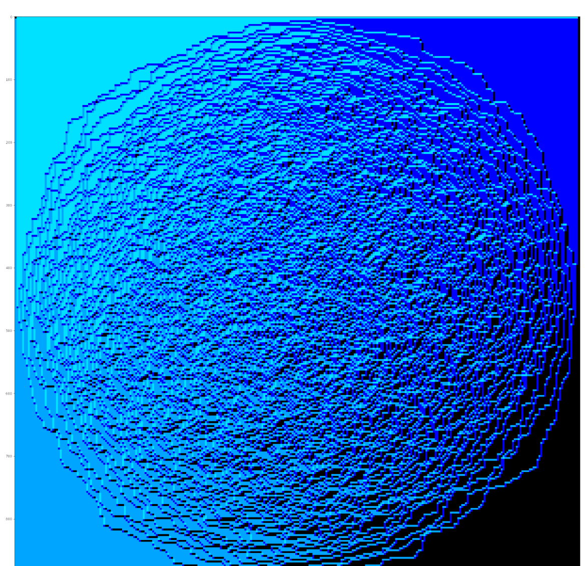
Classification of translation invariant ergodic Gibbs measures
— Sheffield 2005, also Kenyon, Okounkov, and Sheffield 2006

Six vertex model / spin systems





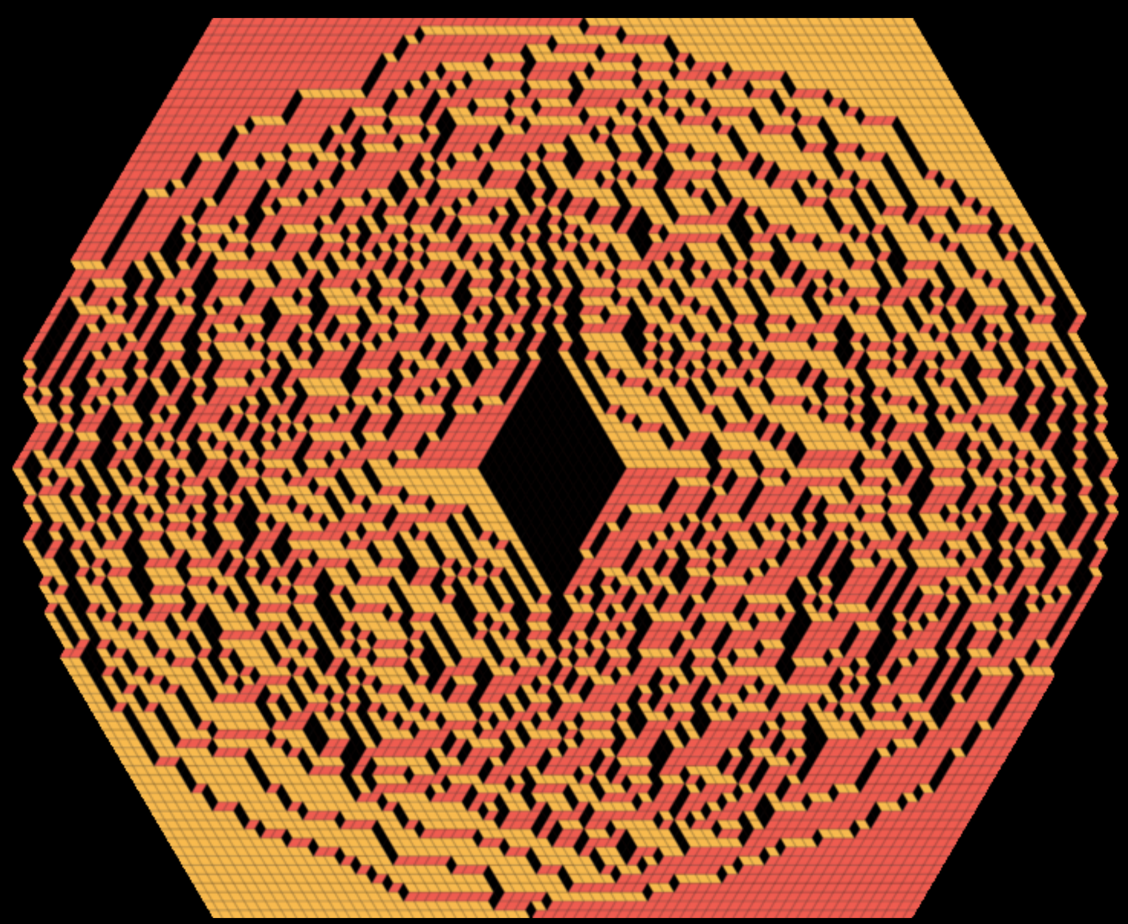




Alternating sign matrices

Ising Model

Uniformly Random Lozenge Tilings

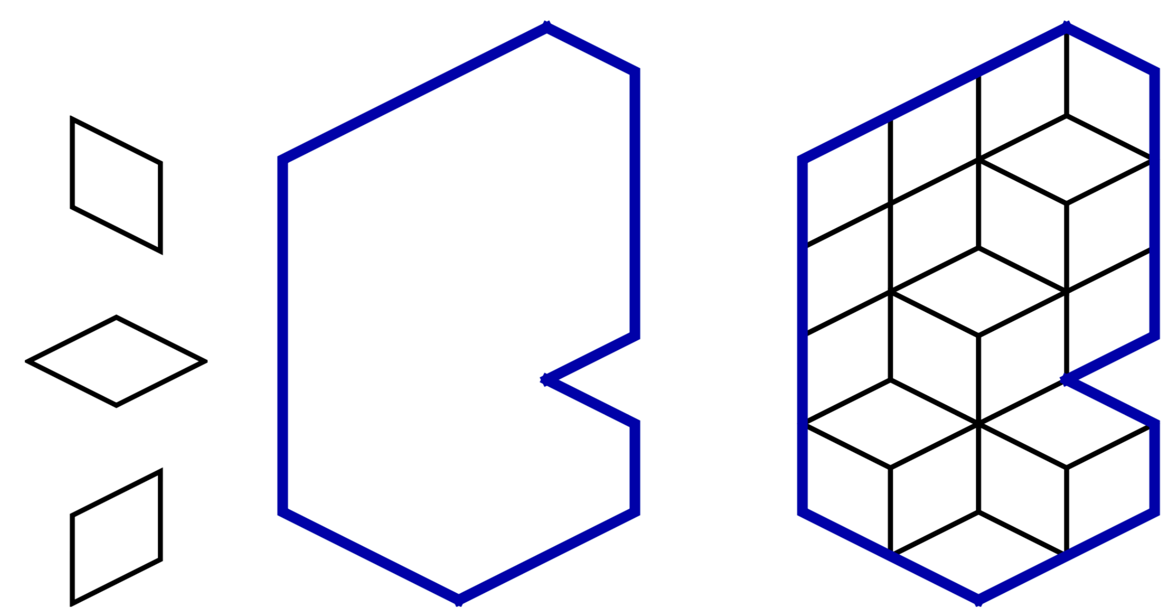


Random Lozenge Tilings

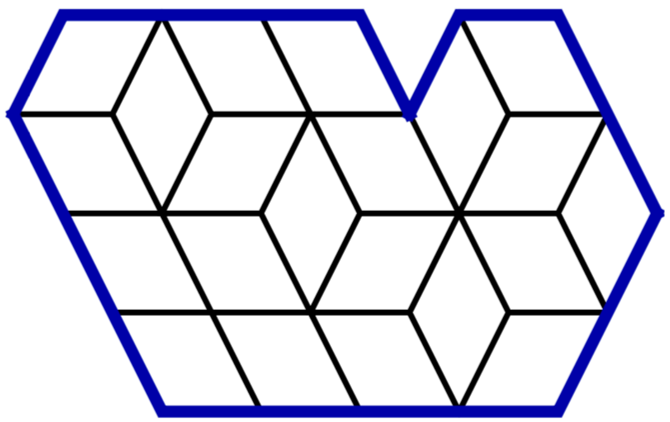
- Focus on Gibbs measures on lozenge tilings with **uniform** conditional distributions
- Their asymptotic behavior is well-understood, in infinite or finite (growing) domains:
  - **Limit Shape and Arctic curves**
  - **Global fluctuation behavior**
  - **Various local limits**
- Connections to Combinatorics, Representation Theory, Geometry, Statistical Mechanics, Random Matrix Theory, interacting particle systems in the Kardar-Parisi-Zhang universality class, Conformal Field Theory (Imaginary Geometry / SLE), ...
- Many interesting open questions (generalized weights or shapes, higher genera, …)

Uniformly random tilings = Gibbs measures in finite (growing) volumes

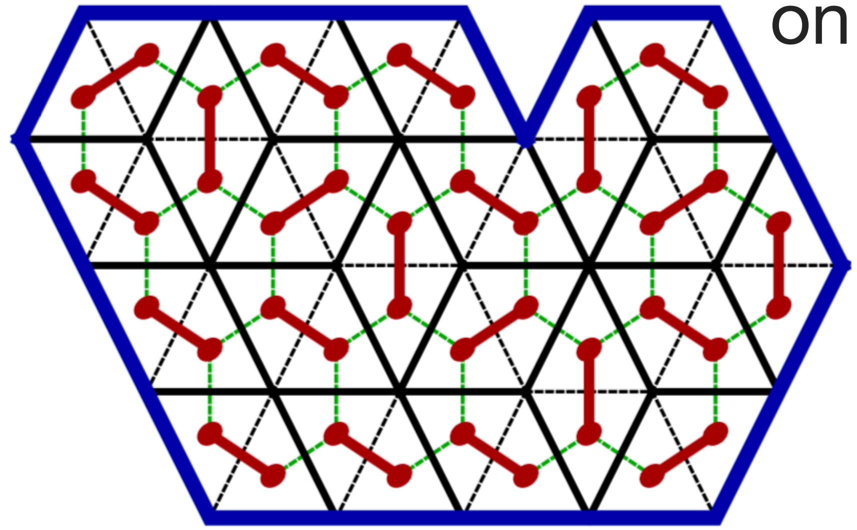
[Adler, Ahn, Beresticky, Borodin, Bufetov, Chhita, Cohn, Duse, Ferrari, Gorin, Guionnet, Johansson, Kenyon, Larsen, Laslier, Li, Metcalfe, Nordenstam, Novak, Okounkov, Panova, Petrov, Propp, Rains, Reshetikhin, Ray, Sheffield, Toninelli, van Moerbeke, Young]



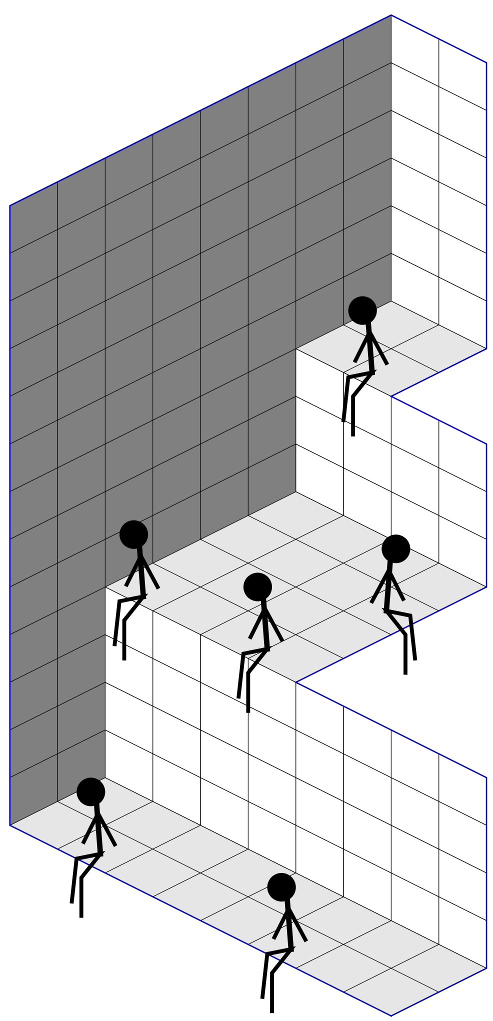
Lozenge tilings of polygons



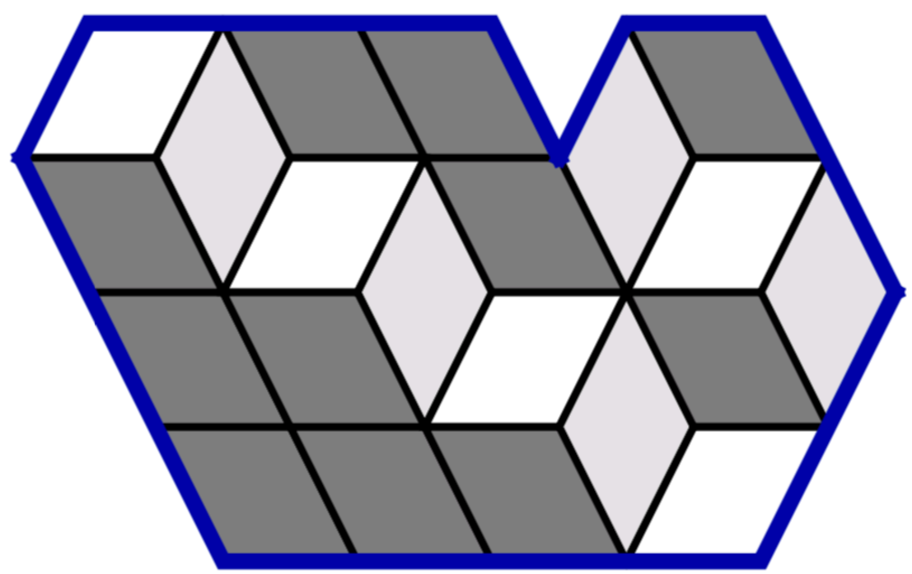
Dimer configuration (perfect matching) on the dual hexagonal graph

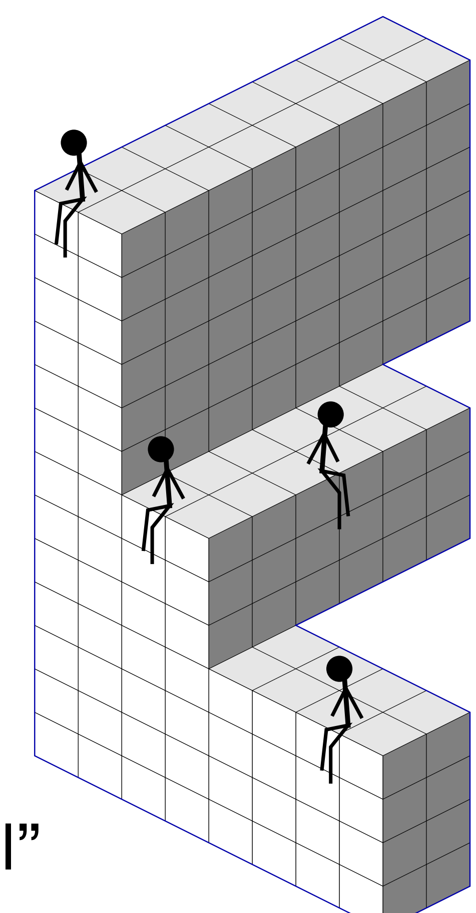


3d shape (2d interface; 3d Young diagram; plane partition)



“empty”

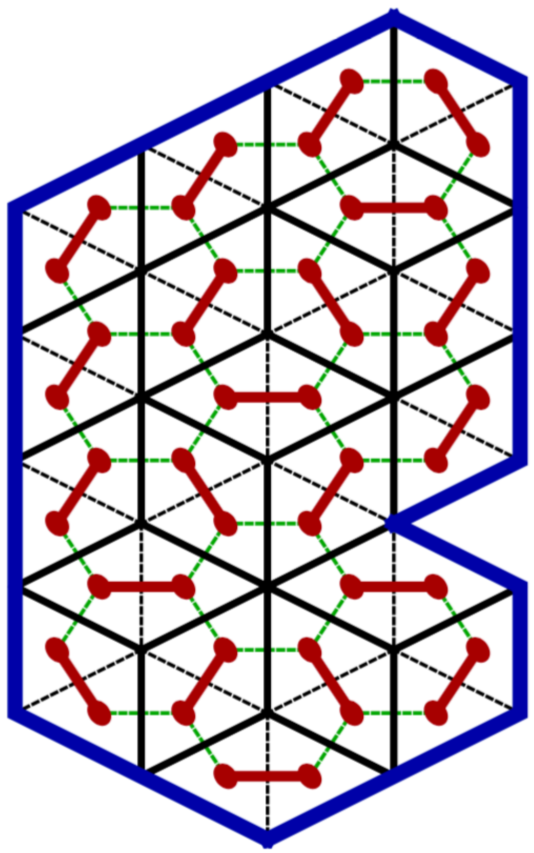




“full”

Solving lozenge tilings - determinantal method

Theorem 1 (Temperley–Fisher, Kasteleyn, 1960s). *The total number of coverings of a hexagonal graph is (the absolute value of) the determinant of the incidence matrix* $K(u,v)$



$$Prob(\text{dimers occupy } (u_1, v_1), \dots, (u_\ell, v_\ell))$$

$$= \left| \frac{\det[K]_{\text{graph without } (u_1,v_1),\dots,(u_\ell,v_\ell)}}{\det[K]_{\text{all graph}}} \right|$$

$$= \left| \det[K^{-1}(u_i, v_j)]_{i,j=1}^{\ell} \right|$$

[Johansson, Nordenstam,
Baik-Kriecherbauer-McLaughlin-Miller, Borodin,
Gorin, Rains, Adler-Johansson-van Moerbeke]

[P. 2012]

For trapezoidal polygons (trapezoids with cuts on one side): $K^{-1}(u,v)$ can be written in a double contour integral form

$$K^{-1}(u;v) \sim \text{additional summand} + \oint\oint f(w,z) \frac{e^{N[S(w;u)-S(z;v)]}}{w-z} dwdz$$

Asymptotic analysis via steepest descent / stationary phase method

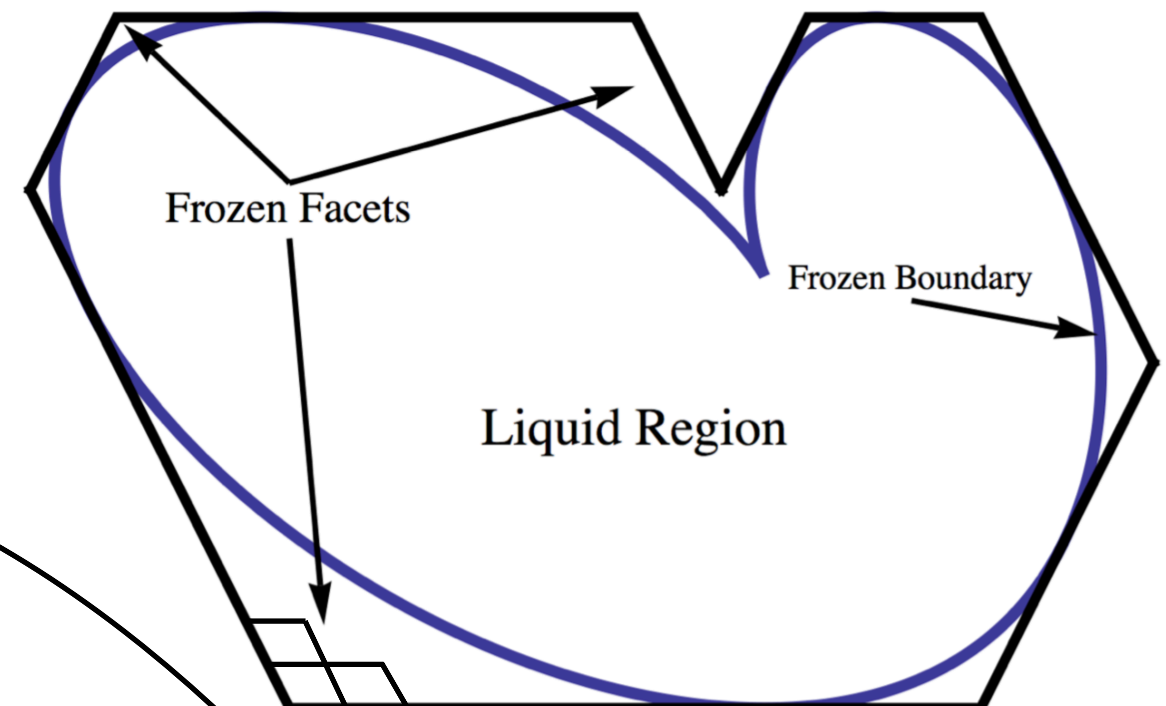
S - explicit function; u, v - physical locations inside the polygon

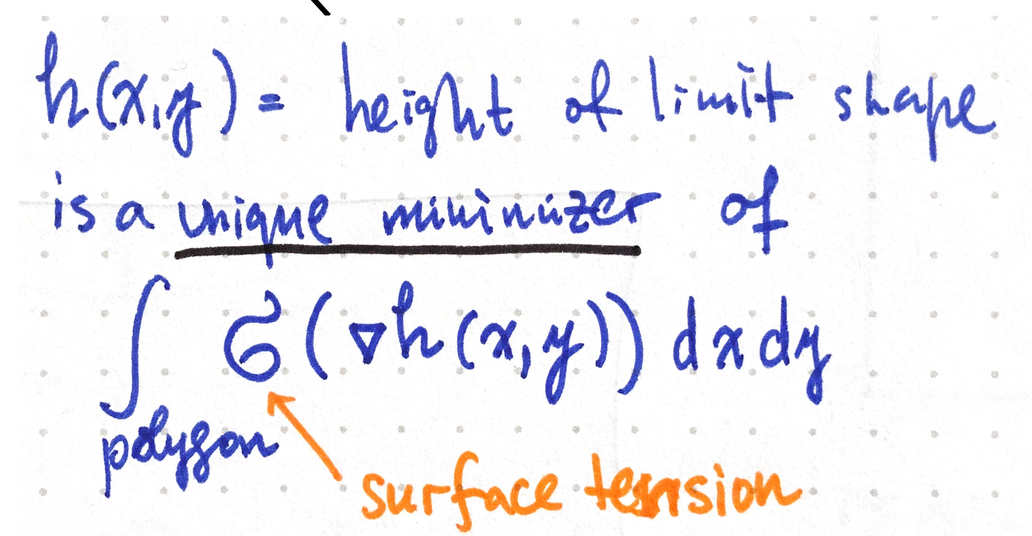
Limit shape and frozen boundary

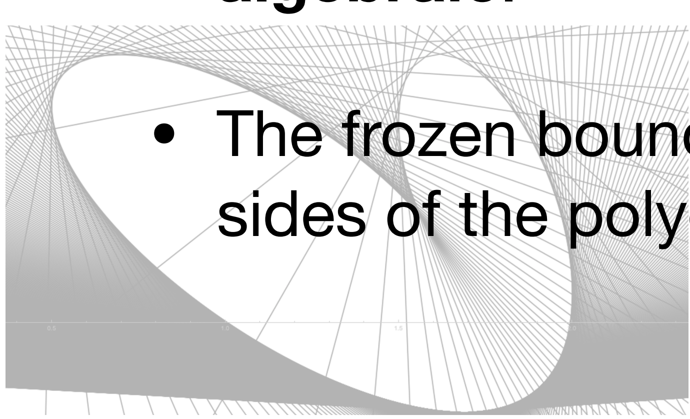
Cohn, Larsen, and Propp 1998, Cohn, Kenyon, and Propp 2001, Kenyon and Okounkov 2007

[P. 2012] - most explicit formulas for limit shapes / frozen boundaries in trapezoidal case

- (Law of Large Numbers) Fix a polygon and *scale the lattice mesh to zero*. Random 3D stepped surfaces concentrate around a **deterministic limit shape** surface which is a solution to a **variational problem**
- The limit shape develops **frozen facets**
- There is a connected **liquid (disordered) region** where all three types of lozenges are present
- The limit shape surface and the separating frozen boundary curve are **algebraic.**
- The frozen boundary is **tangent** to all sides of the polygon



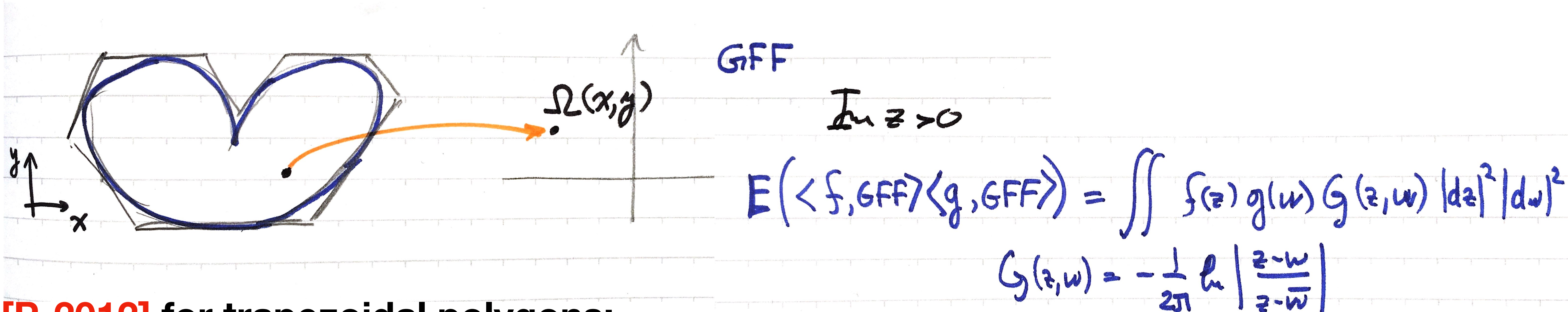




σ is the Legendre dual $(f^\vee(p^*) = \sup_p(\langle p, p^*\rangle - f(p)))$ of the Ronkin function of $z+w=1$,

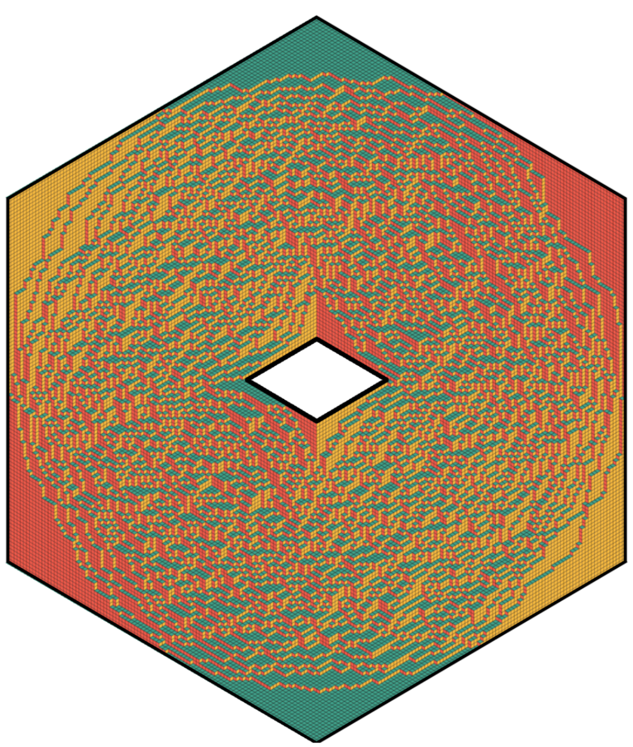
$$R(x,y) = \frac{1}{(2\pi i)^2}\int\int_{|z|=e^x,\ |w|=e^y} \log|z+w-1|\frac{dz}{z}\frac{dw}{w}$$

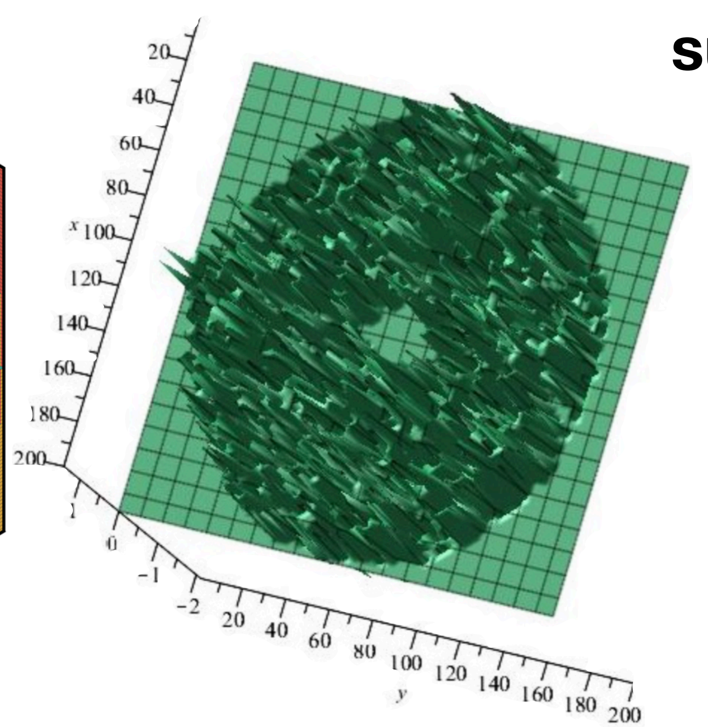
Global Gaussian fluctuations (“bulk” asymptotics)



[P. 2012] for trapezoidal polygons:

$$h_N([xN],[yN]) - \mathbb{E}(h_N([xN],[yN])) \to \mathsf{GFF}(\Omega^{-1}(x,y))$$





Tiling of a hexagon with a hole and fluctuations (complex structure is not in a simply connected domain as above)

GFF fluctuations are common in 2d random surfaces (including tilings, random matrices, representation-theoretic models)

Methods

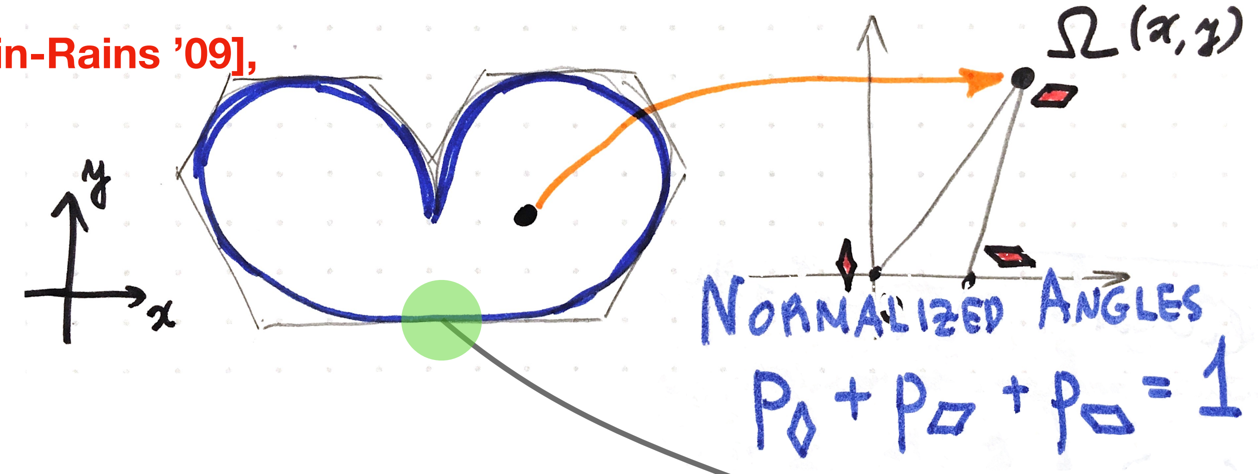
- Determinantal **[Kenyon, Borodin-Ferrari, Duits, Kuan, P., ...]**
- Random matrix computations **[Borodin, ...]**
- Methods of moments **[Borodin, Bufetov, Gorin, Knizel, ...]**
- Nekrasov (Schwinger-Dyson / loop) equations), e.g. for not simply connected **[Borodin, Borot, Dimitrov, Gorin, Guionnet, Knizel]**
- Connections to SLE **[Beresticky, Laslier, Ray]**

Local limits I

Locally inside the liquid region ("bulk")

[Baik-Kriecherbauer-McLaughlin-Miller '07], [Gorin '07], [Borodin-Gorin-Rains '09], [P. '12], [Duse–Johansson–Metcalfe '15], [Gorin '16]

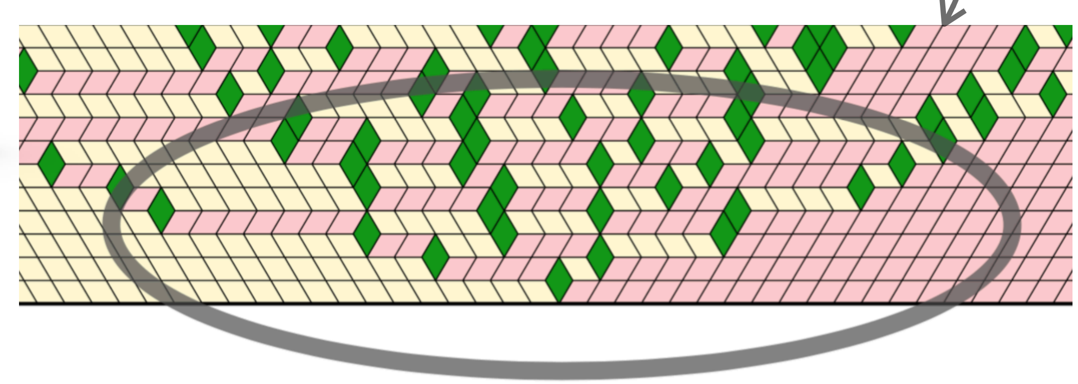
Locally around every point (x,y) in the liquid region, the *lattice* configuration converges to the **translation invariant ergodic Gibbs measure** of the corresponding slope



$$K(x,y) = \frac{\sin(\varphi(x-y))}{\pi(x-y)}, \qquad x,y\in\mathbb{Z}$$

For every given slope, such Gibbs measure is unique [Sheffield 2003]. Its 1d slice is the **discrete sine process** (determinantal)

At the bottom boundary of the trapezoidal domain



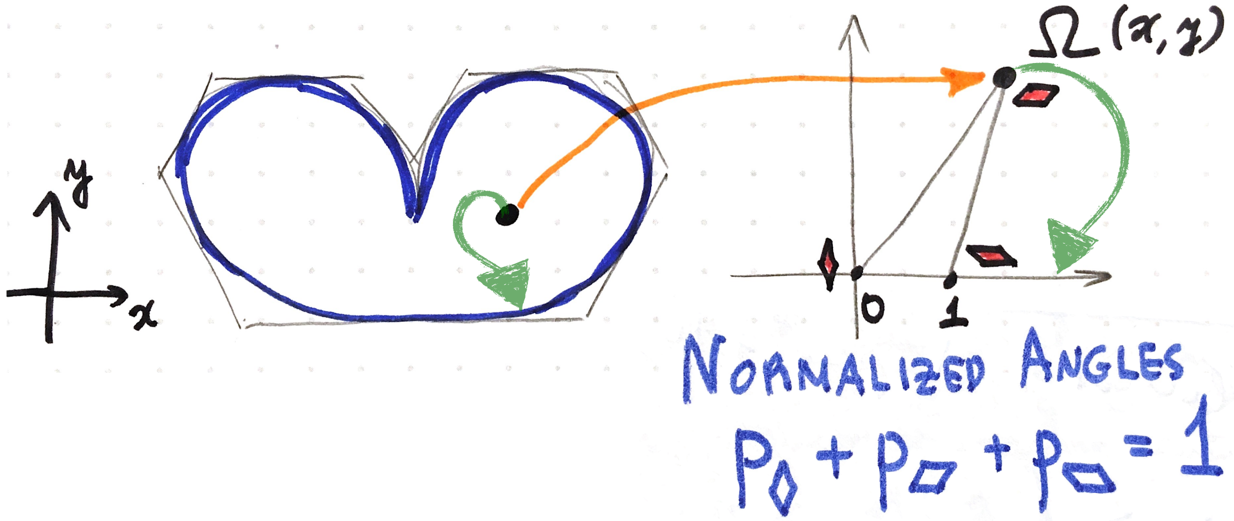
- Classification of "irreducible" characters of $U(\infty)$ [Edrei-Schoenberg et al 1950s; Voiculescu 1976; Vershik-Kerov 1980s; Okounkov-Olshanski 1990s; Borodin-Olshanski and P. 2011-12]
- Random matrix behavior [Gorin-Panova and Novak 2013+, Mkrtchyan-P. 2017]

Local limits II

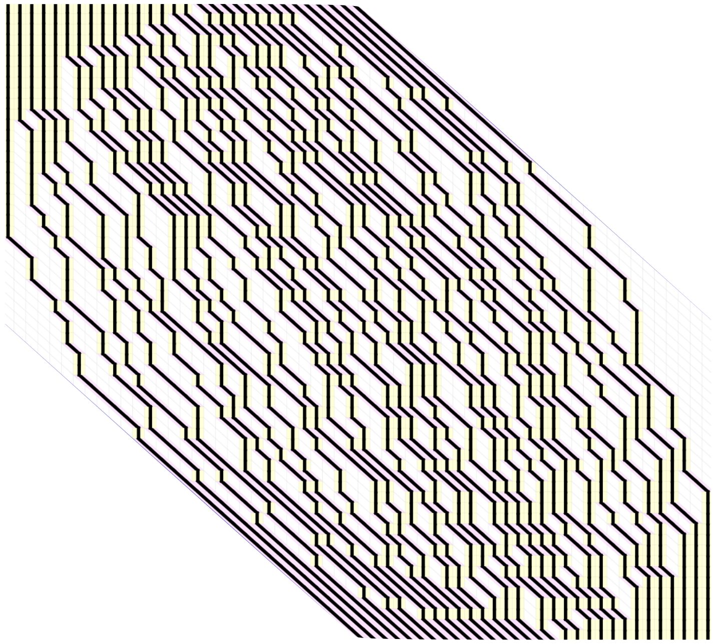
At the edge of the liquid region

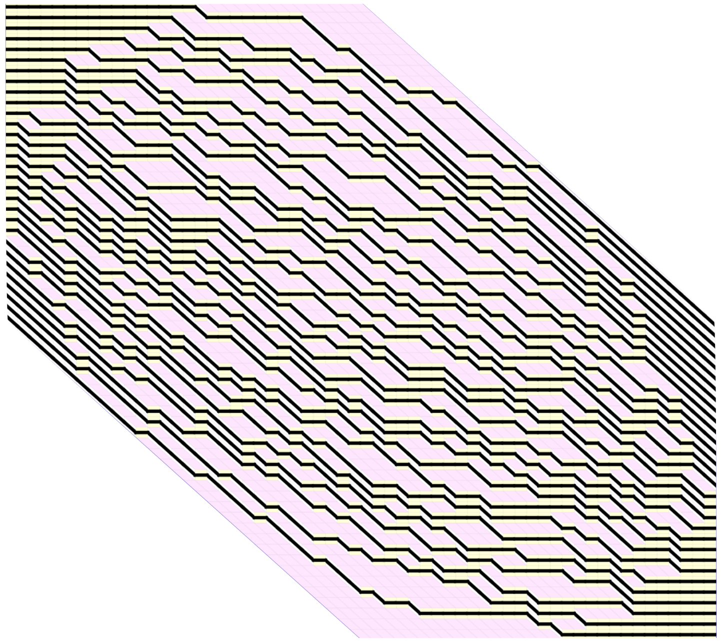
[P. ‘12], [Duse–Johansson–Metcalfe ‘15]

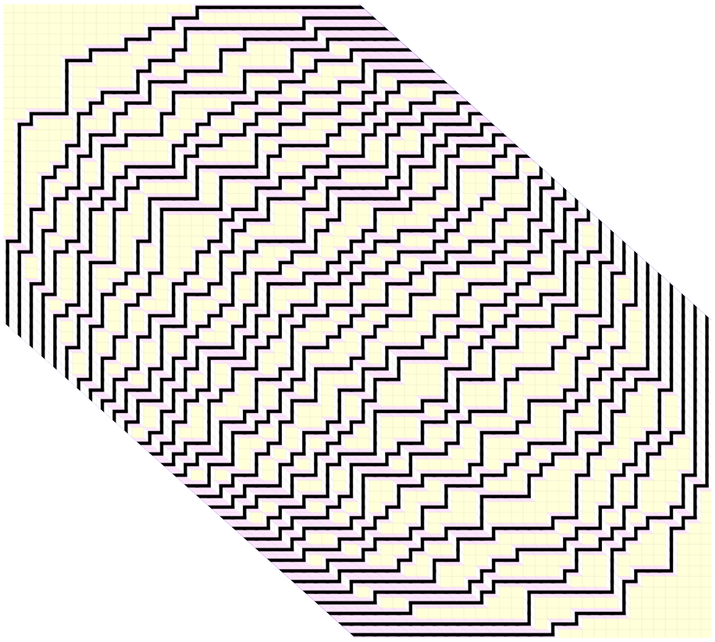
At *typical* points on the arctic curve: fluctuations $\sim N^{2/3}$ in tangent and $\sim N^{1/3}$ in normal direction. Converge to the Airy$_2$ line ensemble



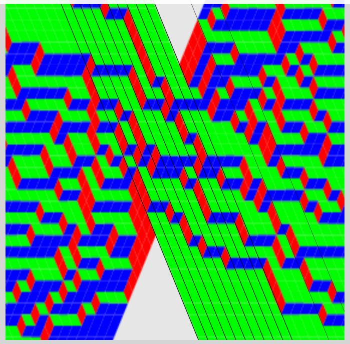
Airy$_2$ ensemble [Corwin and Hammond 2014] **(Brownian Gibbs property)**
Marginals — Tracy-Widom distributions [Tracy and Widom 1994]







There are other local limits:
Pearcey; symmetric Airy; tacnode
[Adler-Johansson-van Moerbeke 2015+]



Summary and future challenges

Bulk effects

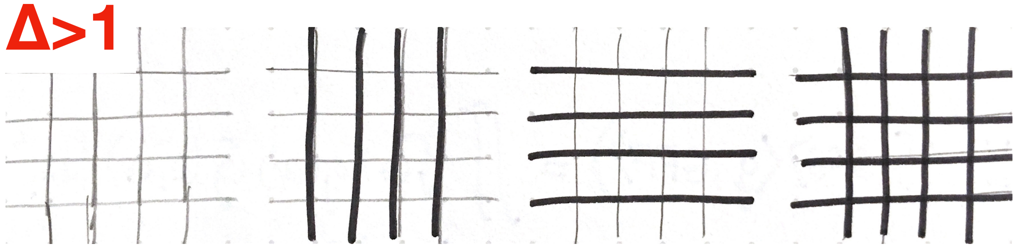
- GFF / discrete sine are **universal** in many models: Beta random matrices; Noncolliding random walks; Macdonald-distributed random tilings, Shapes with holes, …

- Easier to prove in certain Gibbs lozenge ensembles in **infinite regions** (plane partitions / 3D Young diagrams; Schur processes,…) [Okounkov-Reshetikhin, Borodin, Kuan, Ferrari ~2003-2008] prior to works in finite (growing) regions

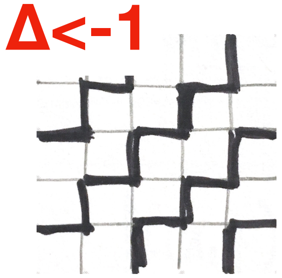
- **Six vertex models, periodic tilings**, … develop more complex local Gibbs phases (“antiferromagnet”,“gas”)

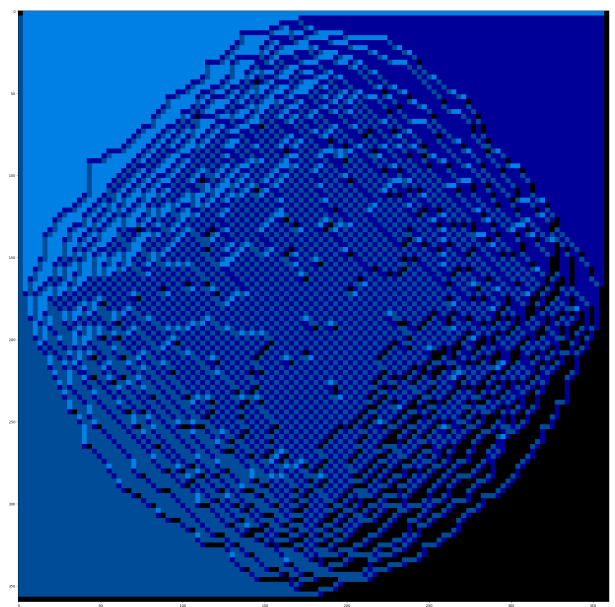
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

Δ>1

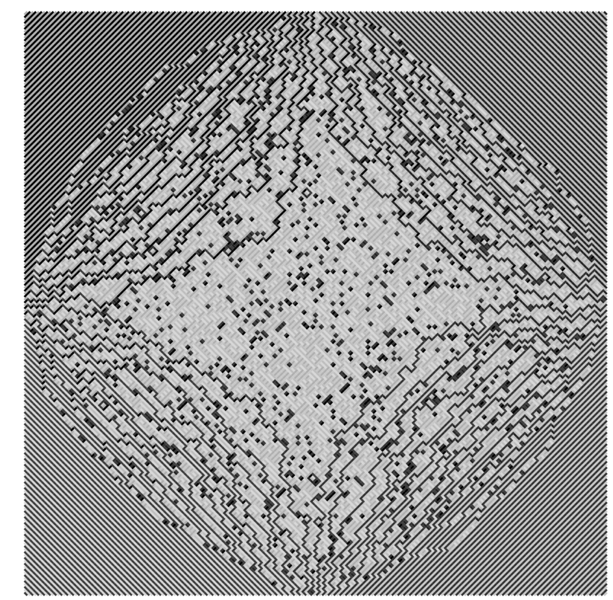


Δ<-1





Six vertex model, DWBC, Δ < -1

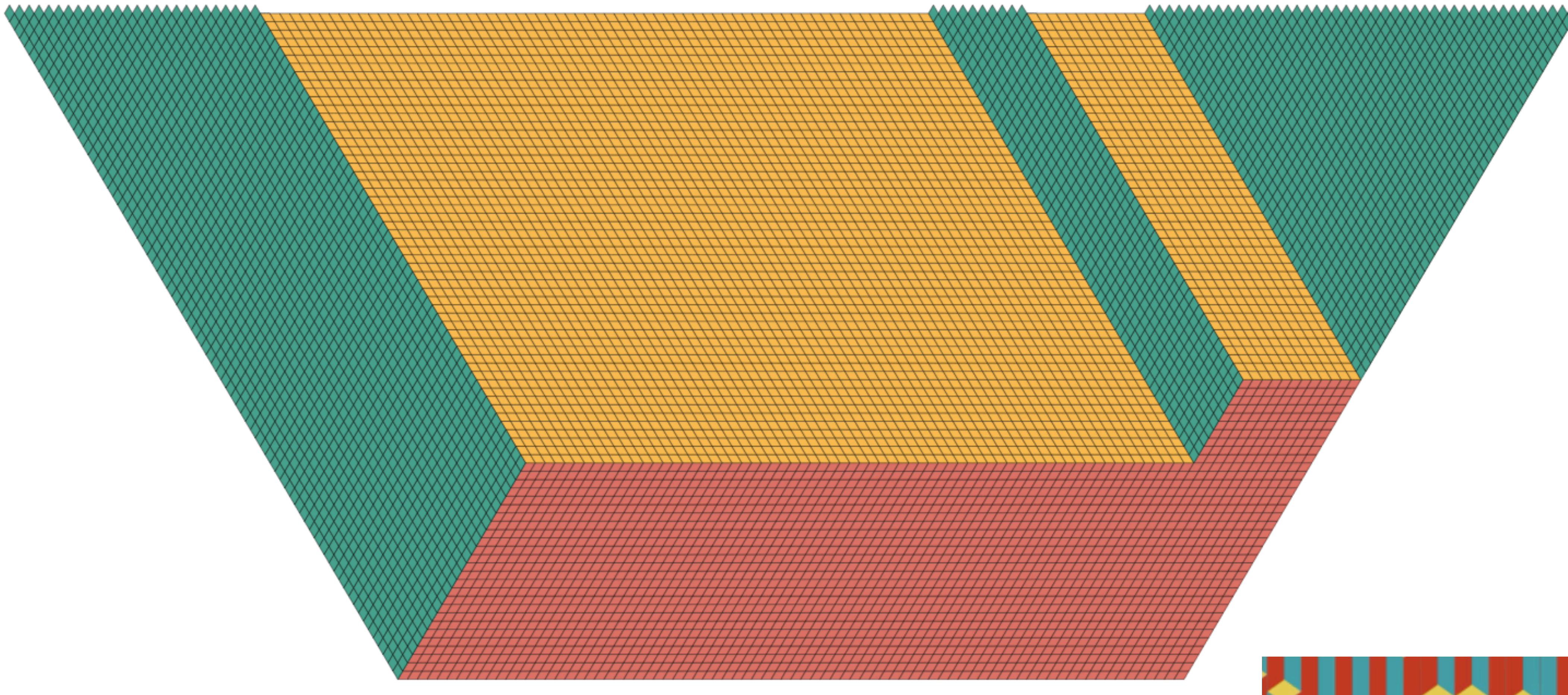


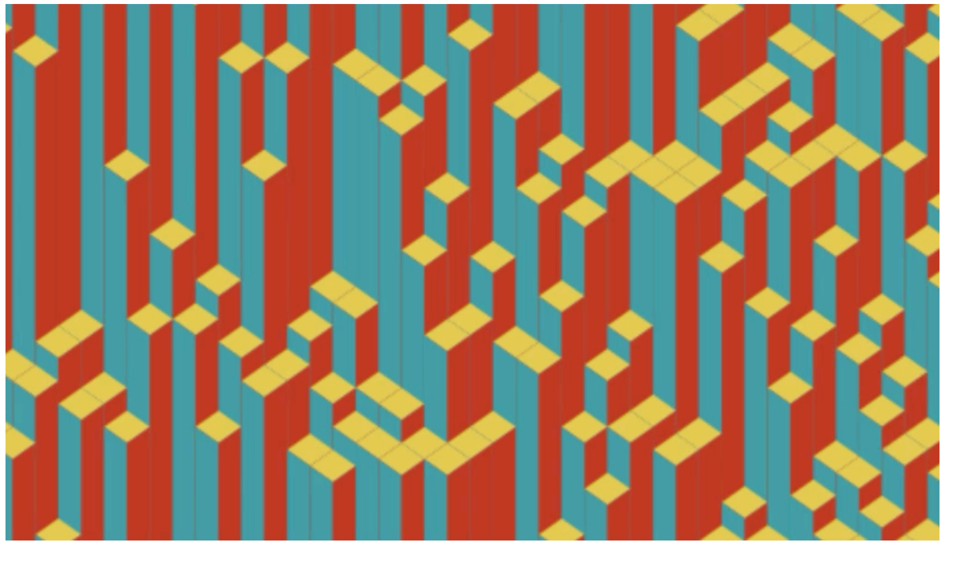
[Beffara, Chhita, Johansson, Young 2015+], doubly periodic domino tilings

Edge effects

- Airy line ensemble appears at an **interface** between two phases
- Fluctuation exponents 1/3, 2/3 follow (1+1)d **Kardar-Parisi-Zhang universality** (dynamics of random interfaces)
- Rich interplay between **Gibbs measures** and **Markov dynamics** in 1 or 2 space dimensions

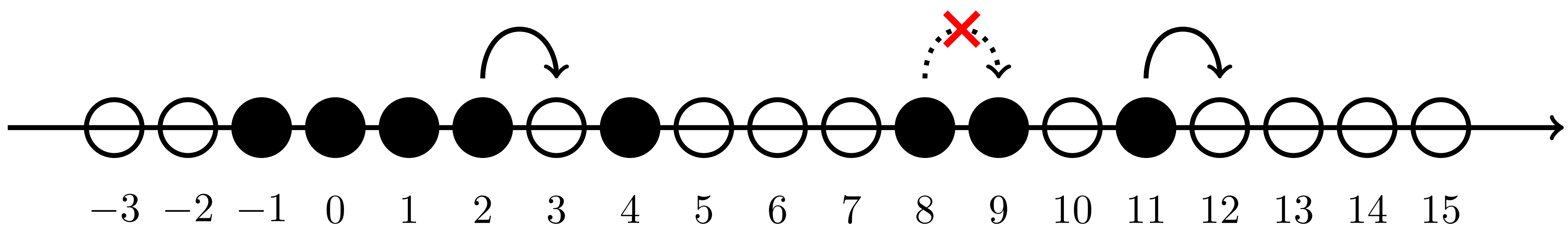
Random Interfaces and Markov Dynamics





Totally Asymmetric Simple Exclusion Process

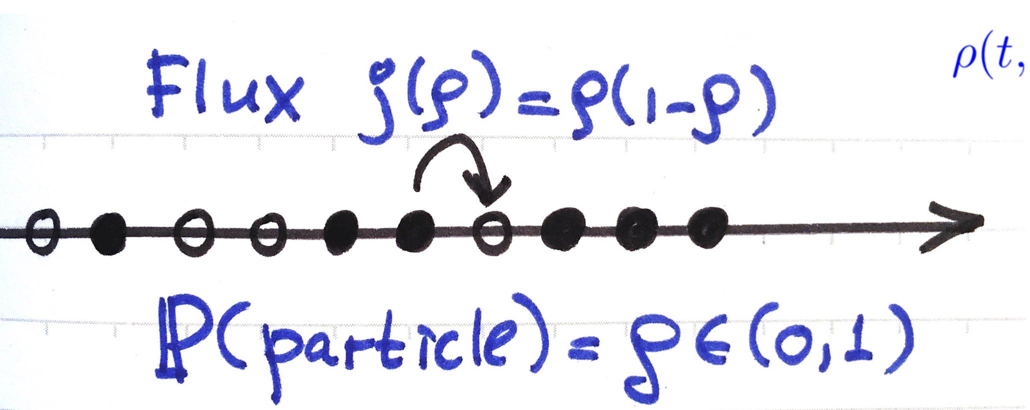
TASEP - dynamics on 1d Gibbs measures



Each particle has an exponential clock with rate 1: $\mathbb{P}(\text{wait} > s) = e^{-s}$, $s > 0$, clocks are independent for each particle.

When the clock rings, the particle jumps to the right by one if the destination is not occupied.

[Liggett 1976] All nontrivial translation invariant ergodic measures which are stationary under the TASEP dynamics are the *Bernoulli product measures*



$\rho(t,x)$ — limiting density as $\frac{time}{space} \sim \text{const}$

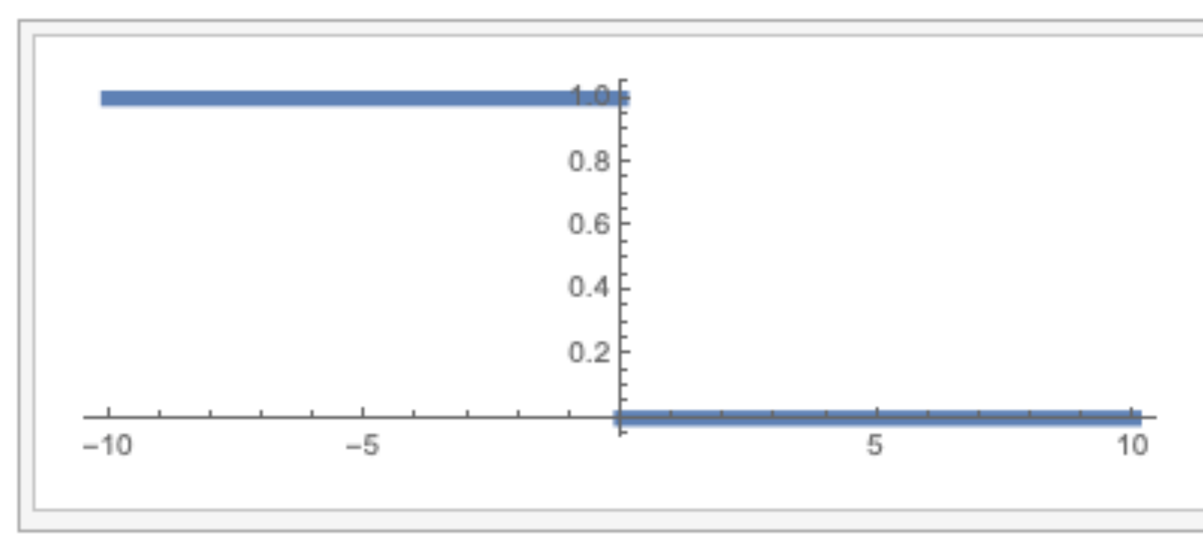
$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\Big(\rho(t,x)(1-\rho(t,x))\Big) = 0$$

$\rho(0,x) = \rho_0(x)$

(+variational representation)

Step initial condition: particles occupy $-1, -2, -3, \ldots$

$\rho_0(x) = \mathbf{1}_{x\le 0}$, and $\rho(t,x) = \frac{1}{2}(1 - x/t)$, $|x| < t$.

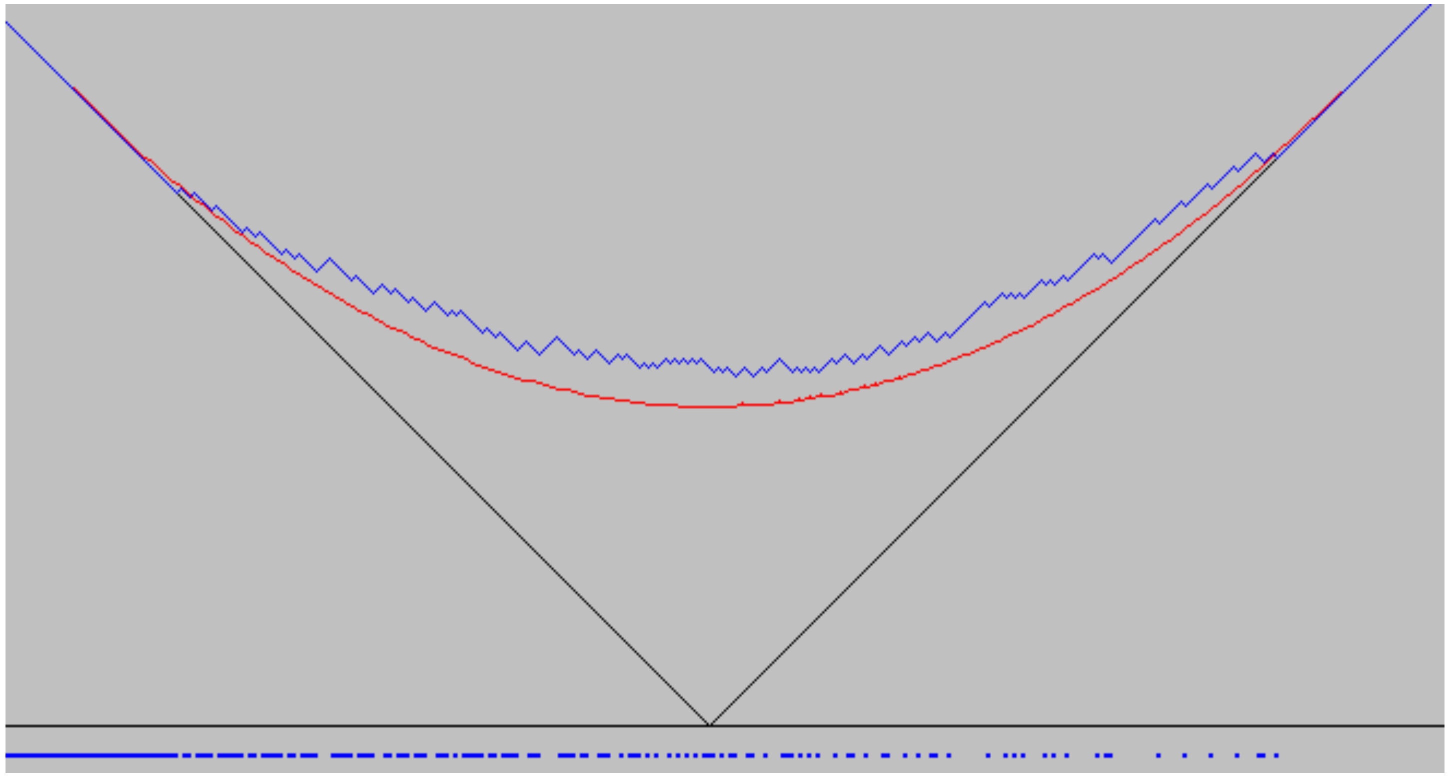


TASEP fluctuations

Theorem ([Johansson, 2000])
Start TASEP from the **step initial configuration** $x_i(0) = -i$, $i = 1, 2, \ldots$.
Let $h(t, x)$ be the height of the interface over x at time t. Then

$$\lim_{L\to+\infty} \mathbb{P}\left(\frac{h(\tau L, \chi L) - L\mathfrak{h}(\tau, \chi)}{c_{\tau,\chi} L^{\mathbf{1/3}}} \geq -s \right) = F_{GUE}(s),$$

where F_{GUE} is the **GUE** (Gaussian Unitary Ensemble) **Tracy–Widom distribution** originated in random matrix theory **[Tracy and Widom, 1993]**



Step IC is not Gibbs, but local lattice distributions for $t > 0$ are Gibbs

Limit shape **[Rost, 1981]**
Simulation **[Ferrari, 2008]**

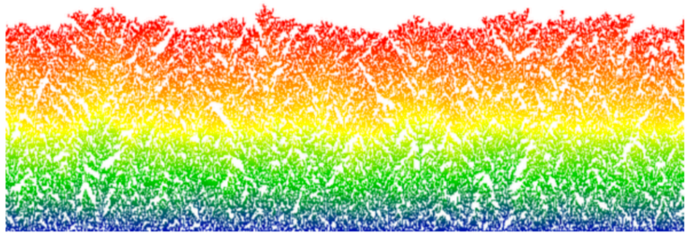
KPZ equation

KPZ equation **[Kardar, Parisi, and Zhang, 1986]** — a stochastic PDE model for randomly growing interface $h(t,x)$, $t>0$, $x \in \mathbb{R}$:

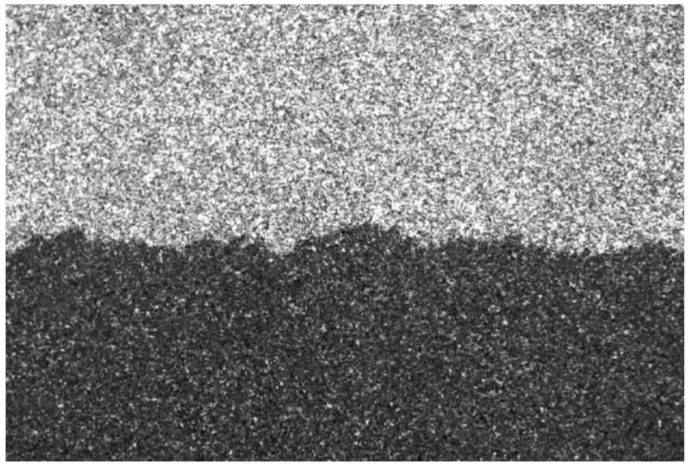
$$\frac{\partial h(t,x)}{\partial t} = \frac{\partial^2 h(t,x)}{\partial x^2} + \left(\frac{\partial h(t,x)}{\partial x}\right)^2 + \eta(t,x), \quad \mathbb{E}\eta(t,x)\eta(t',x') = \delta(t-t')\delta(x-x')$$

(the time evolution of the interface is governed by the smoothing and the slope-dependent growth terms, plus random noise)

- Existence and uniqueness of solutions **[Hairer, 2014]**, etc.
- Approximation of solutions of the KPZ equation by discrete-space interacting particle systems such as weakly ASEP **[Bertini and Giacomin, 1997]**, etc.
- Exact distributions and limits (e.g. $t \to +\infty$) of $h(t,x)$ for specific and (conjecturally) general initial data $h(0,x)$ **[Amir, Corwin, and Quastel, 2011, Matetski, Quastel, and Remenik, 2017]**, etc.



surface growth model



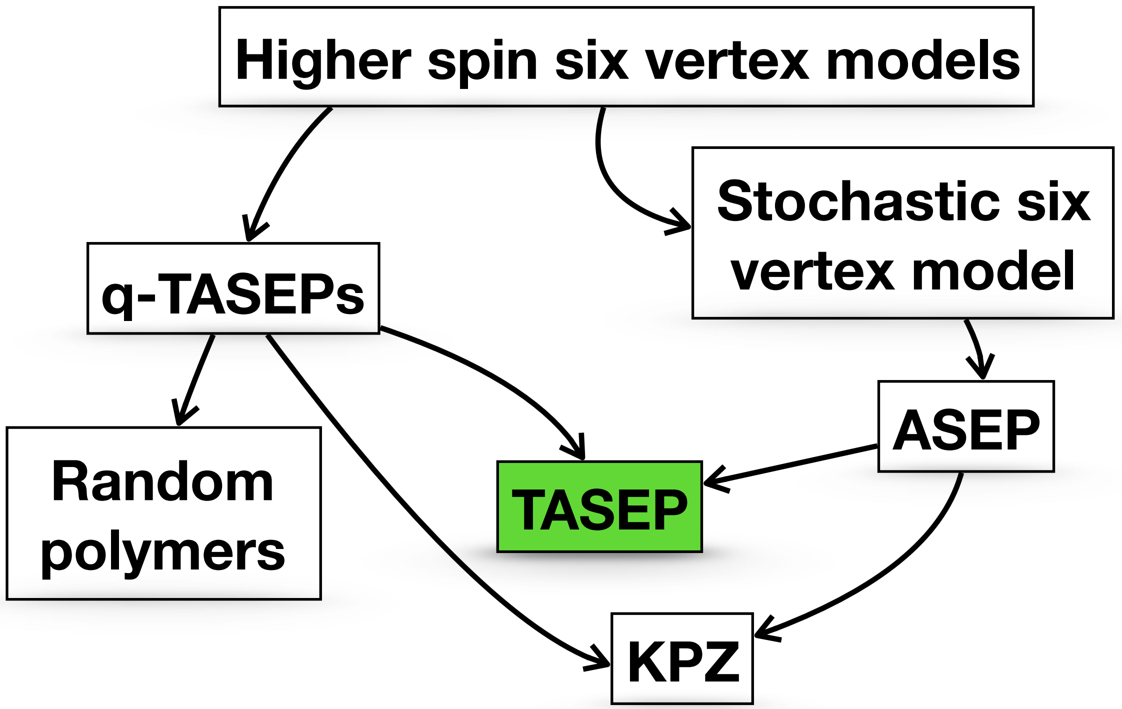
liquid crystal experiment
[Takeuchi and Sano, 2010]

TASEP and KPZ

KPZ universality principle / conjecture: models in KPZ class (including the KPZ equation) *at large times and scales* behave as TASEP at *large times and scales*

Starting from Johansson's theorem, there is a very good understanding of TASEP asymptotics:

- multipoint distributions
- particle-dependent speeds
- other initial conditions, including general
- extensions to other models such as ASEP



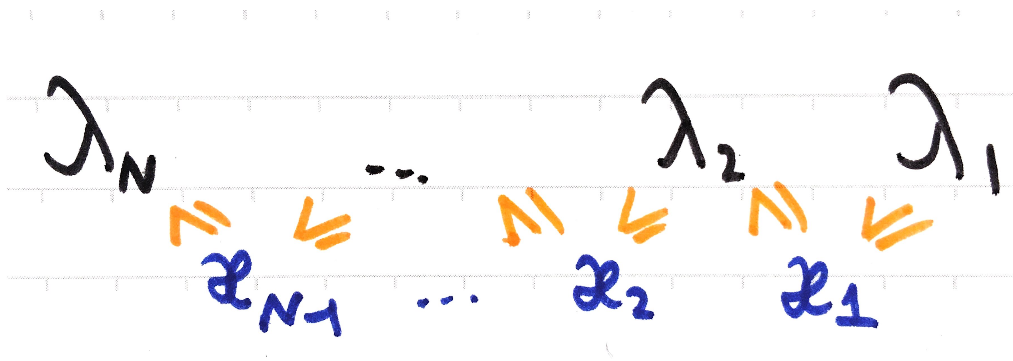
[Okounkov, 2001, Its, Tracy, and Widom, 2001, Gravner, Tracy, and Widom, 2002, Prähofer and Spohn, 2002, Borodin, Ferrari, Prähofer, and Sasamoto, 2007, Matetski, Quastel, and Remenik, 2017, Borodin, Ferrari, and Sasamoto, 2009, Duits, 2013, Tracy and Widom, 2009]

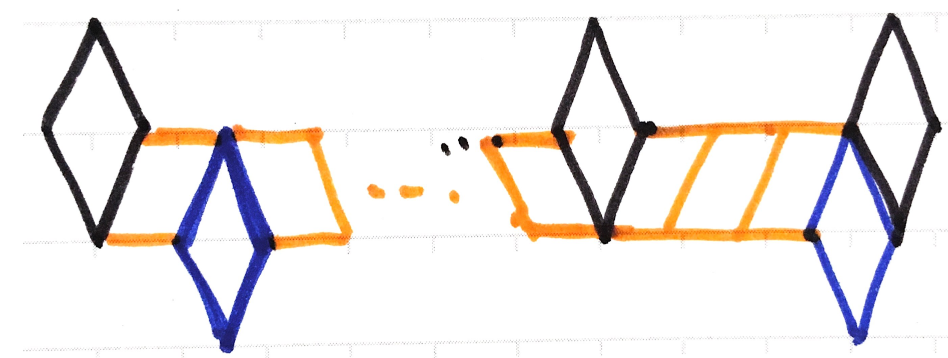
How to get fluctuation results for discrete KPZ particle systems?
Most common (and so far almost unique) method - **match them to 2d Gibbs systems**

Definition. Fields of Young diagrams [Bufetov-P. 2017]

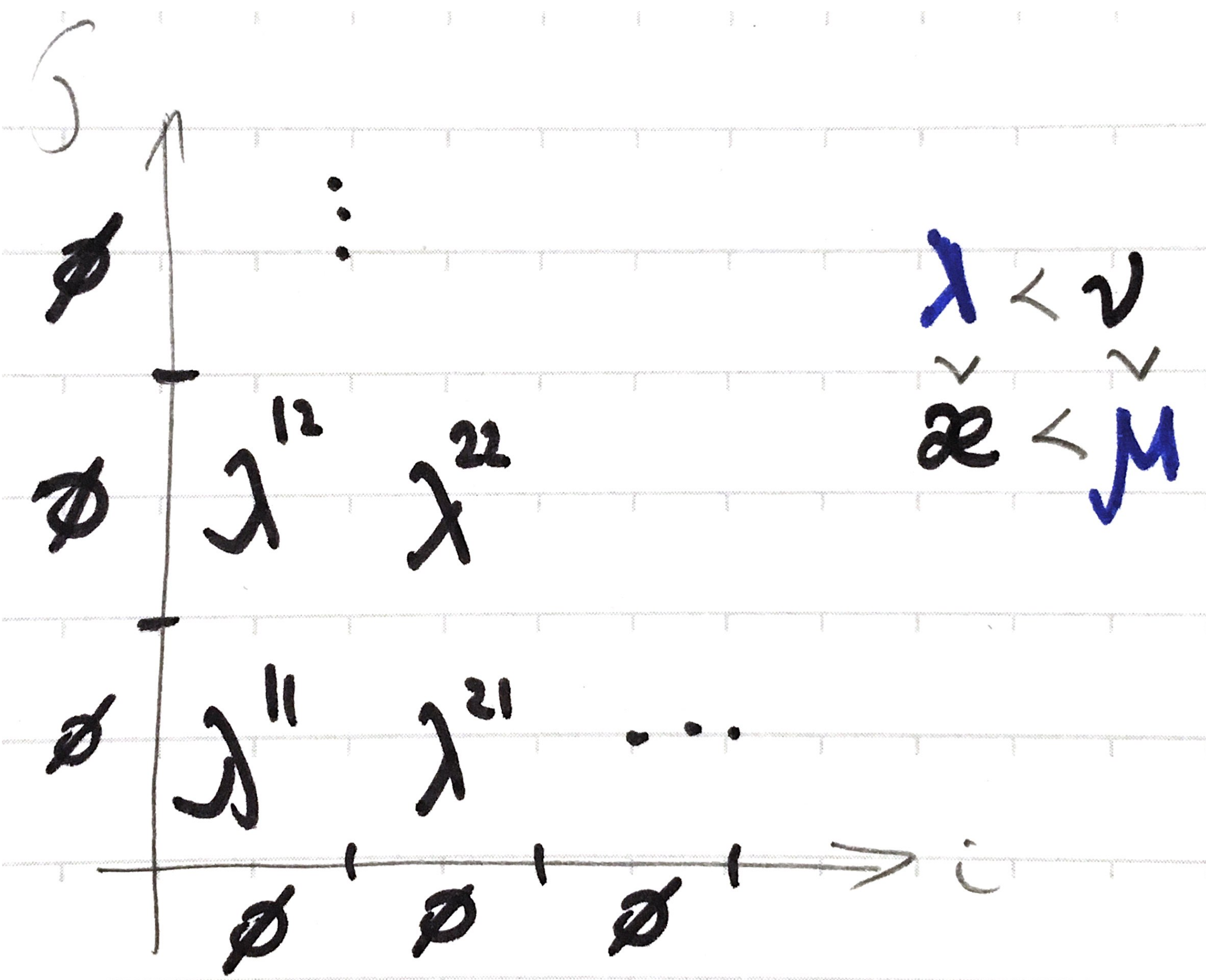
Young diagrams (partitions) $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$, $\lambda_i \in \mathbb{Z}$

Interlacing

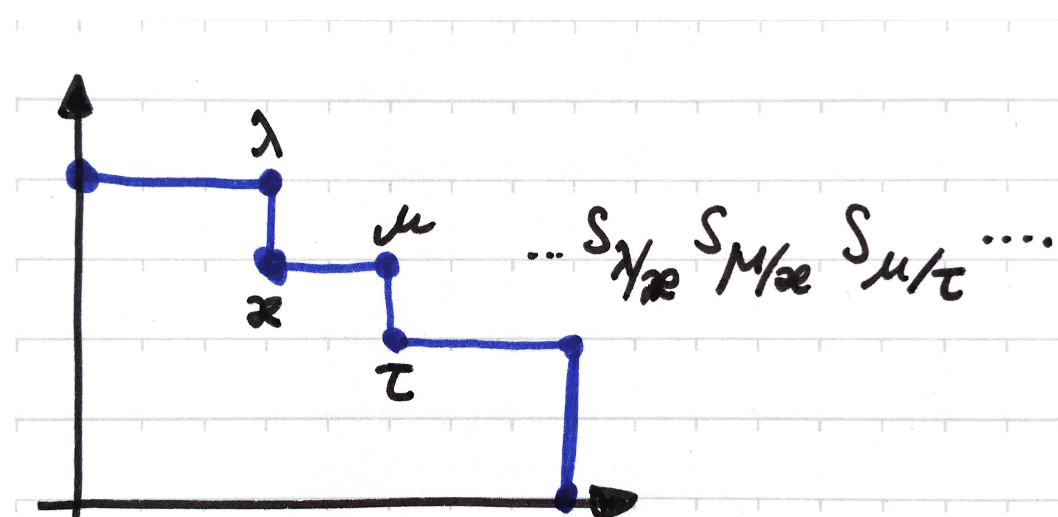




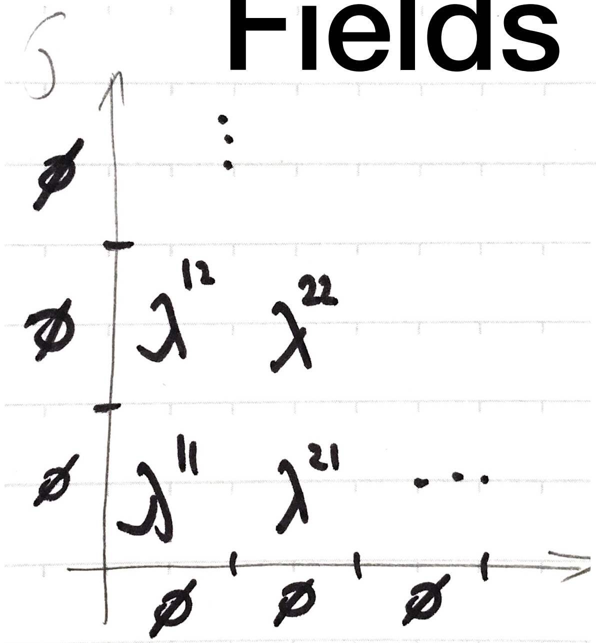
Random field of Young diagrams



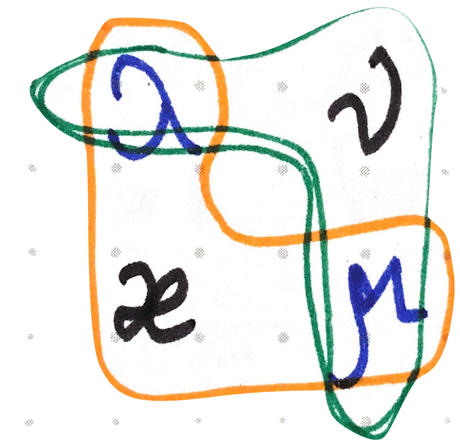
- Interlace along both directions
- Empty boundary conditions
- $\{\lambda^{(i,J)}\}_{i=1}^{\infty}$, $\{\lambda^{(I,j)}\}_{j=1}^{\infty}$ are Gibbs distributed (as lozenge tilings of the half plane)
- Distributions along down-right (= up-left) paths are *Schur processes*



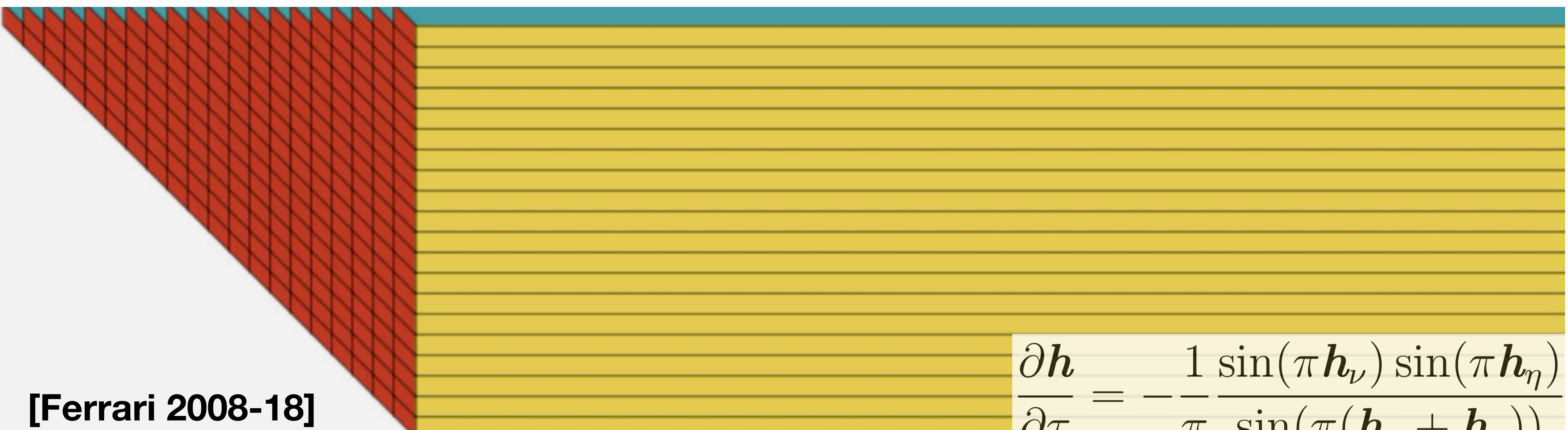
Fields of Young diagrams and dynamics



- Taking i or j as (discrete) time, get **Markov processes** on Gibbs measures on 2d lozenge tilings
- Does not define the distribution uniquely: at each elementary square, know only marginal distributions of $\varkappa$ and ν given λ, μ — but not the joint distribution of $(\varkappa, \nu)$



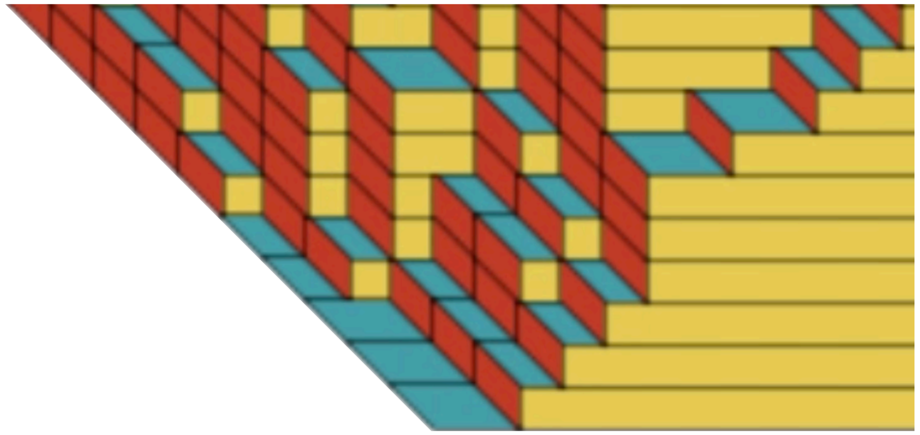
- Scalar field marginals of $\{\lambda^{(i,j)}\}$, when they are *Markovian*, are interacting particle systems in 1 space dimension
- TASEP arises as $x_j(t) = \lambda_j^{(t,j)} - j$ (edge of the Gibbs measure), in a continuous rescaling $i \to t$ **[Borodin-Ferrari 2008]**



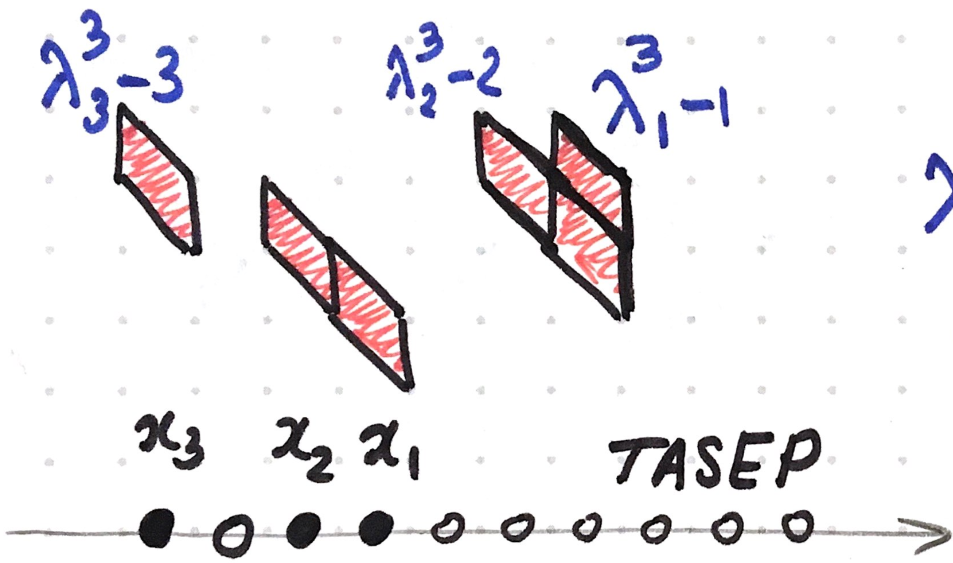
$$\frac{\partial \boldsymbol{h}}{\partial \tau} = -\frac{1}{\pi} \frac{\sin(\pi \boldsymbol{h}_\nu)\sin(\pi \boldsymbol{h}_\eta)}{\sin(\pi(\boldsymbol{h}_\nu + \boldsymbol{h}_\eta))}$$

http://wt.iam.uni-bonn.de/ferrari/research/jsanimationakpz/

How to solve TASEP with step IC



Each particle has an exponential clock and jumps to the right by one if can. Lower particles have priority, higher particles get pushed.



$$\lambda^{(i)}(t) \sim \underline{\text{Schur measure}}$$

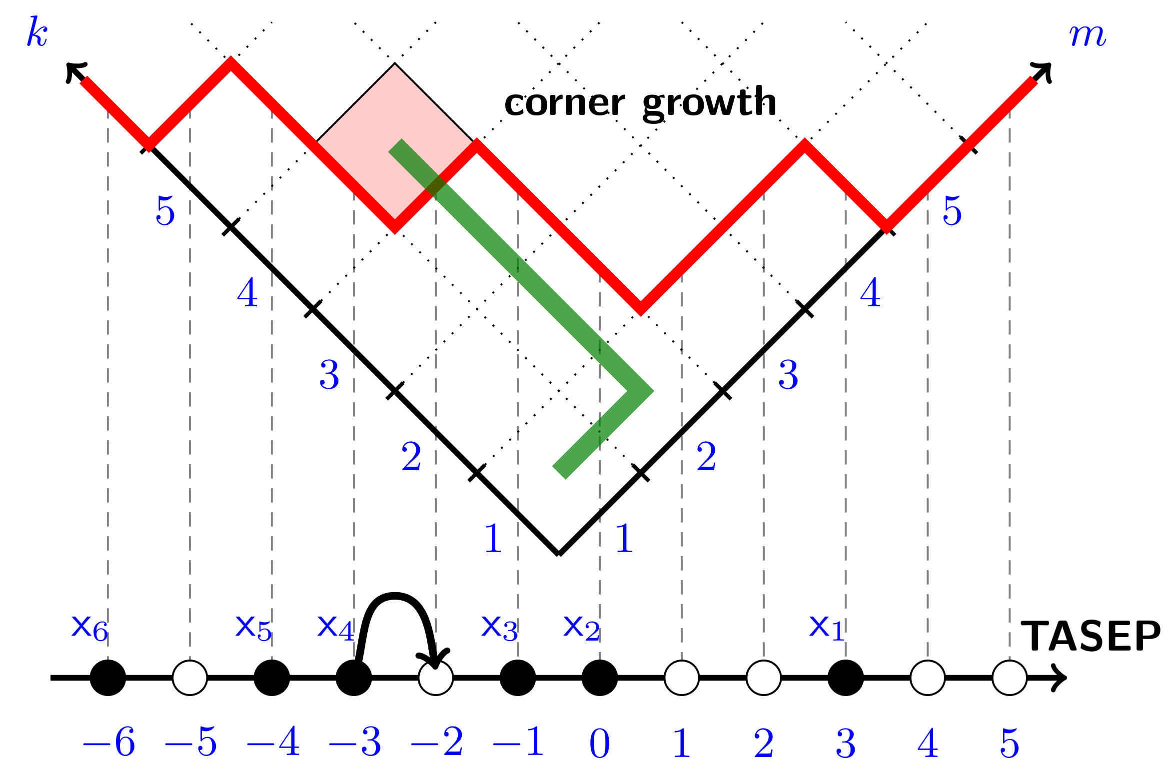
$$\propto s_{\lambda^{(i)}}(\underbrace{1,\dots,1}_{i})\, s_{\lambda^{(i)}}(\rho/t)$$

$$x_N(t) \stackrel{d}{=} \lambda^{(N)}_N(t) - N$$

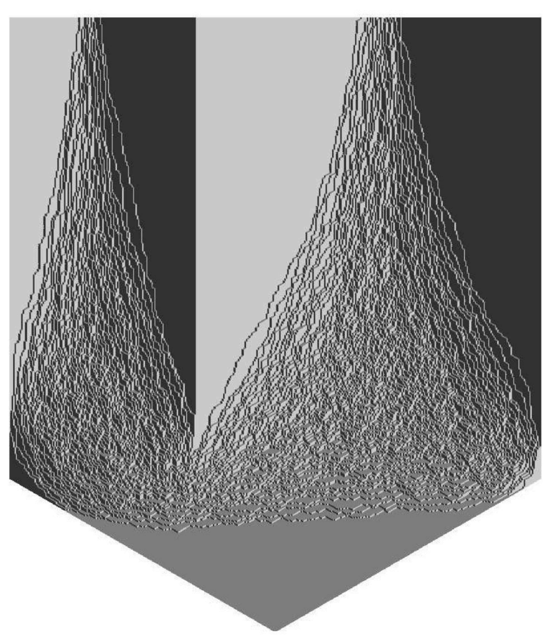
Schur measures are determinantal (Okounkov 2001), so $P(x_N(t) > x) = \det(\mathbf{1} - K)$

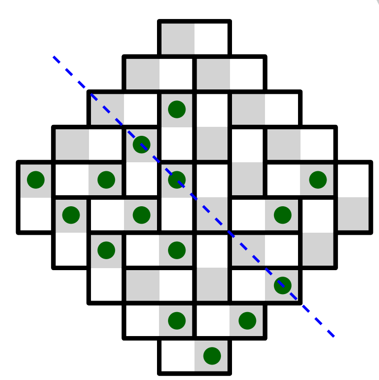
$$\underset{u,v\in\mathbb{Z}}{K(u,v)} = \frac{1}{(2\pi i)^2}\oint\oint \frac{dz\,dw}{z-w}\,\frac{w^v}{z^{u+1}}\,e^{t(z-w)}\left(\frac{1-z^{-1}}{1-w^{-1}}\right)^N$$

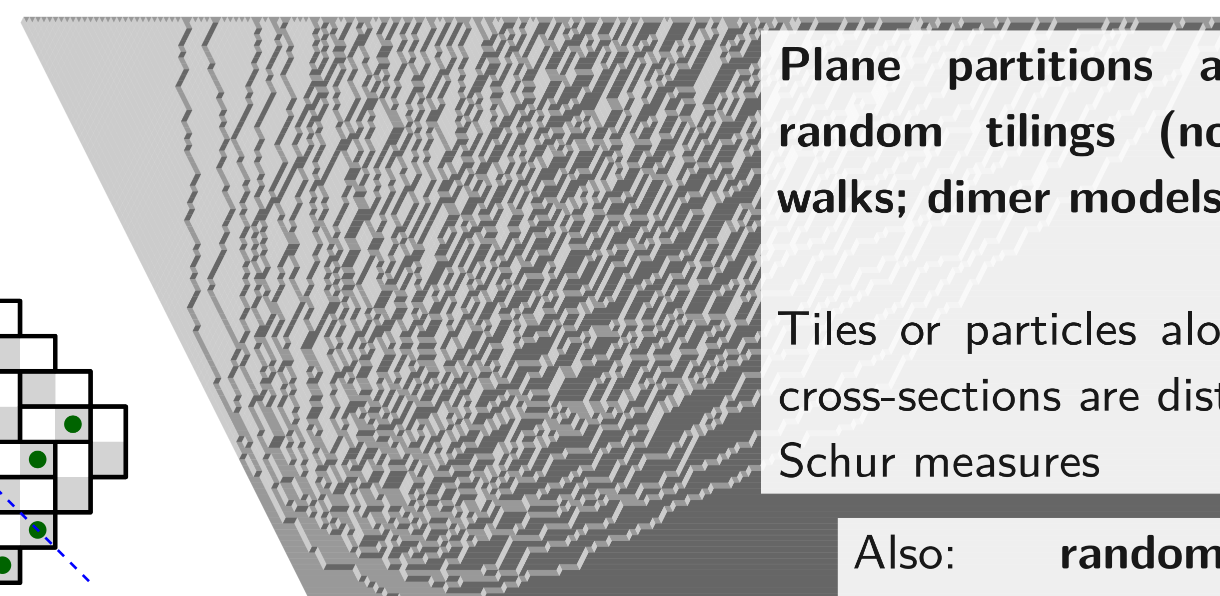
(A sample of) models solvable by Schur measures



- **Homogeneous** or **particle-inhomogeneous** TASEP on $\mathbb{Z}$
- directed last passage percolation
- corner growth
- longest increasing subsequences
- tandem queues







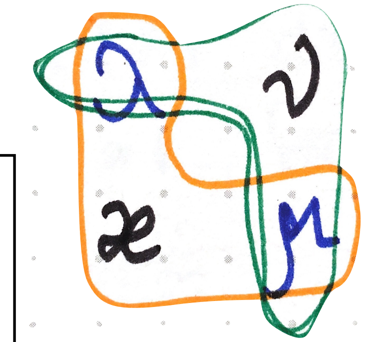
Plane partitions and other random tilings (noncolliding walks; dimer models; etc.)

Tiles or particles along certain cross-sections are distributed as Schur measures

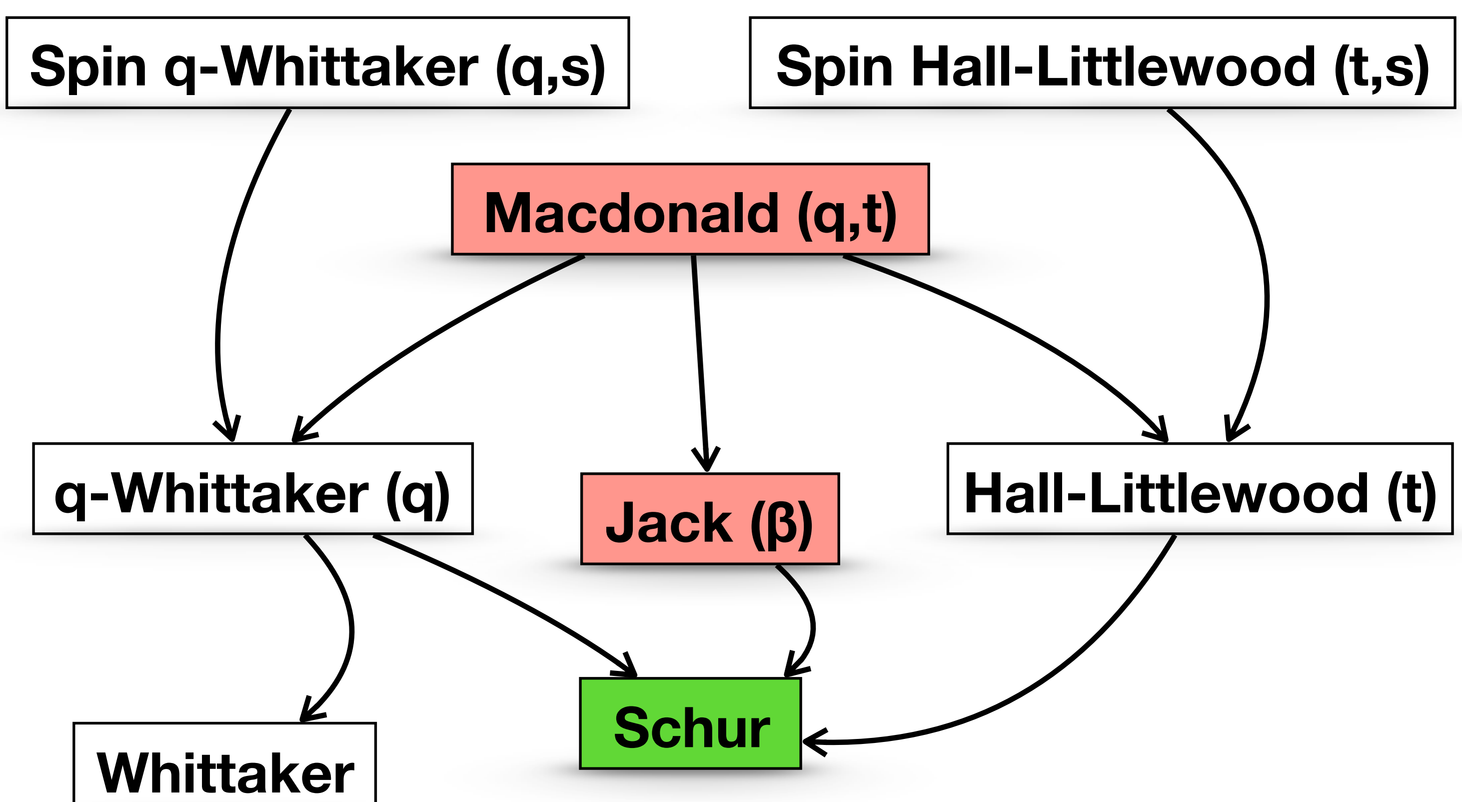
Also: **random matrix type models, z-measures, polynuclear growth, . . .**

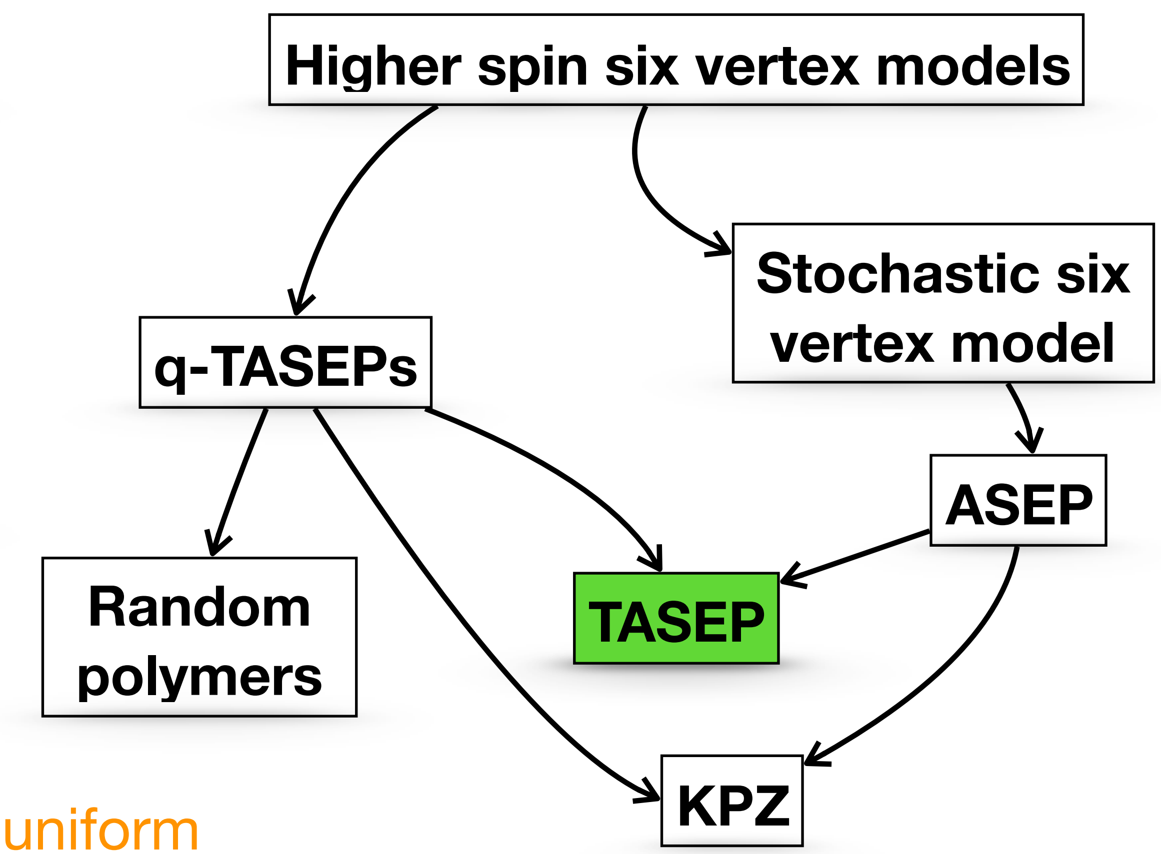
lozenge tilings pictures: **[Okounkov and Reshetikhin, 2003, Borodin and Ferrari, 2014]**

Fields for various symmetric functions



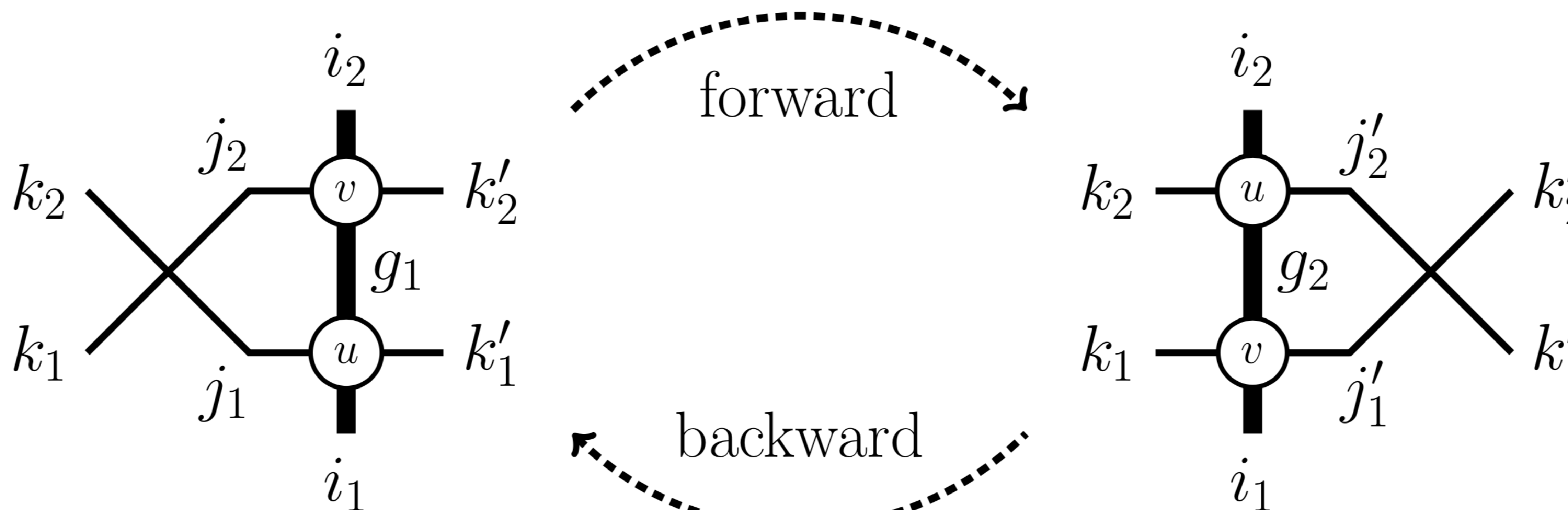
- **Robinson-Schensted-Knuth correspondence (TASEP)** [Robinson, Schensted, Knuth, Vershik-Kerov, O'Connell, Biane-Bougerol-O'Connell, Chhaibi, Baik–Deift–Johansson, Johansson]
- **Geometric RSK (polymers)** [Kirillov, Noumi-Yamada, Corwin, O'Connell, Seppalainen, Zygouras]
- **q-randomized RSK (q-TASEPs)** [O'Connell–Pei, Borodin-P., Matveev-P.]
- **Hall-Littlewood RSK (ASEP, six vertex model)** [Bufetov-P. 2014, Borodin-Bufetov-Wheeler, Bufetov-Matveev]
- **Push-block dynamics (TASEP, q-TASEP)** [Borodin–Ferrari based on Diaconis–Fill]
- **Yang-Baxter fields for sHL / sqW (higher spin six vertex models)** [Bufetov-P. 2017, Bufetov-Mucciconi-P. 2019]



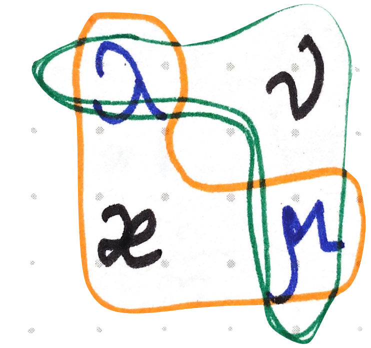


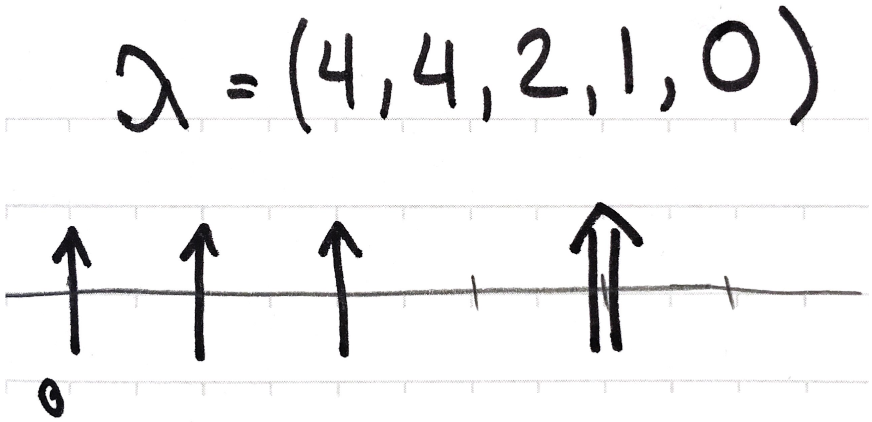
Note. Beyond Schur, Gibbs is not locally uniform

From Yang-Baxter equation to Markov dynamics

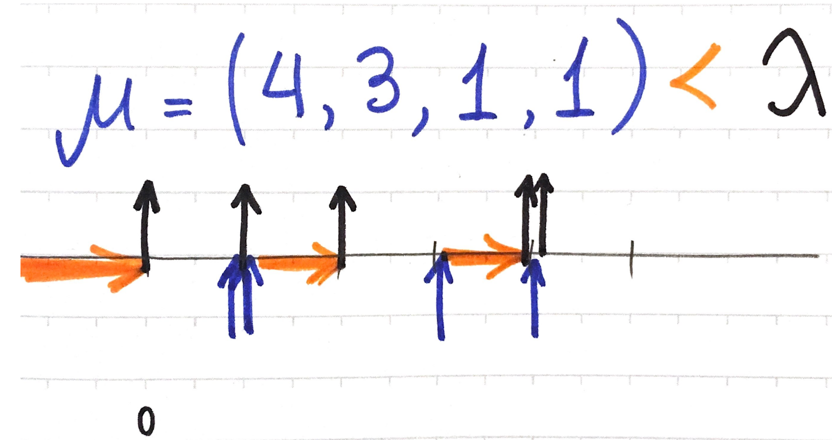


1. Represent the *quadruple step* as a **lattice spin configuration**, with vertex weights satisfying the Yang-Baxter equation
2. **“Bijectivise”** the Yang-Baxter equation to get Markov steps (somewhat like **domino shuffling**)

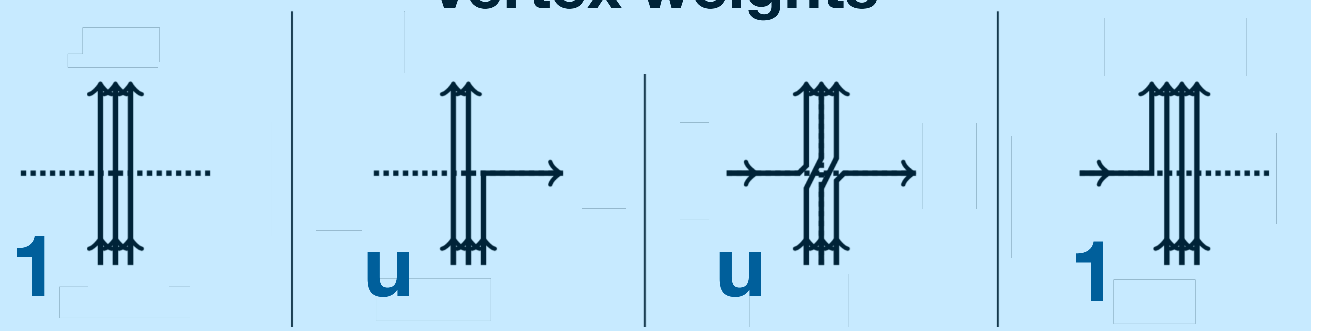




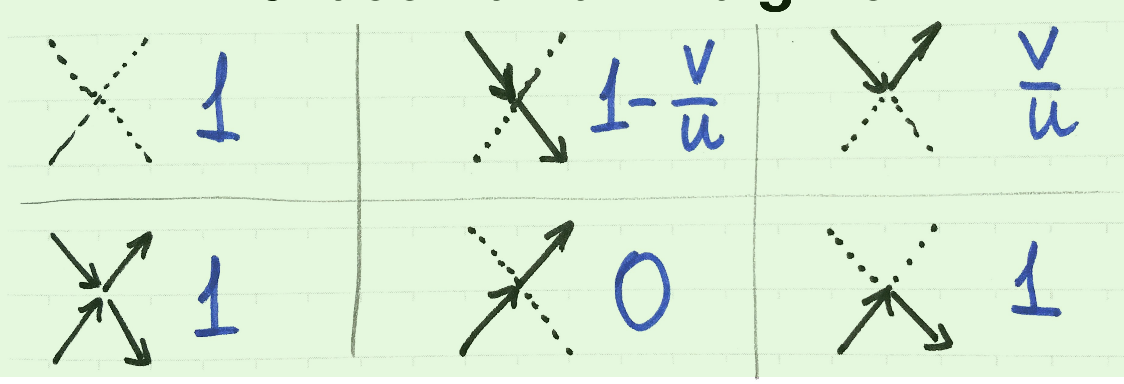
$$s_{\lambda/\mu}(u) = u^{|\lambda|-|\mu|} = u^2$$



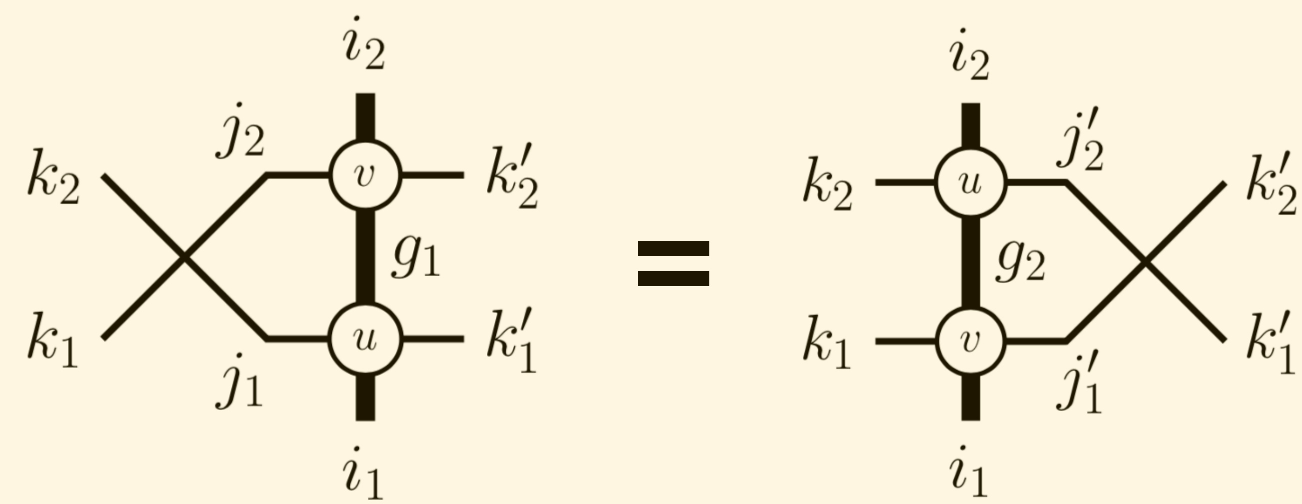
Vertex weights



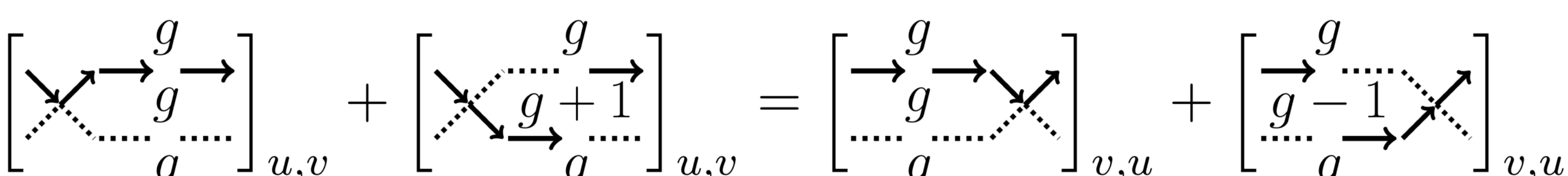
Cross vertex weights



$U_q(\widehat{\mathfrak{sl}_2})$ Yang-Baxter equation

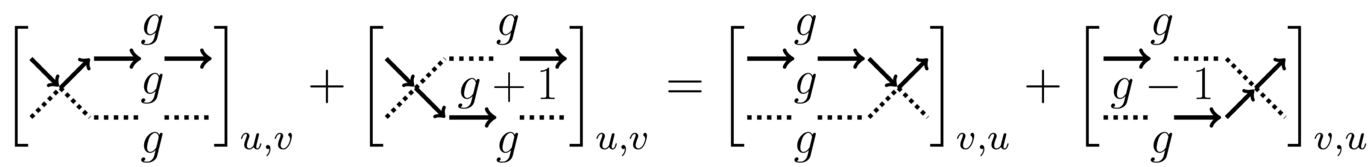


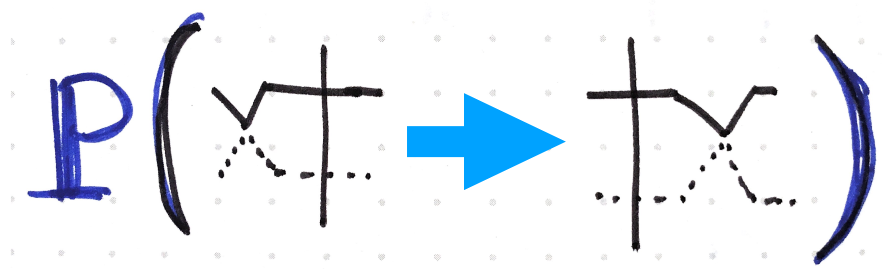
Example:



$$\frac{v}{u} \cdot v + \left(1 - \frac{v}{u}\right) \cdot v = \frac{v}{u} \cdot u + 0$$

Bijectivisation = coupling of terms in the Yang-Baxter equation





Example: 1 + 3 = 2 + 2

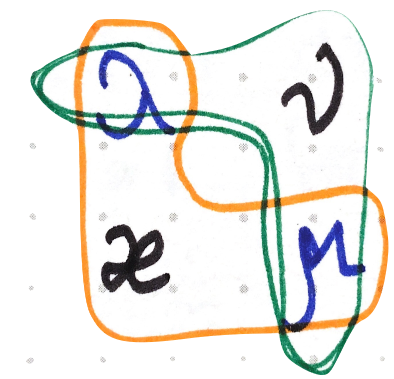
	2	2
1	1	0
3	1	2

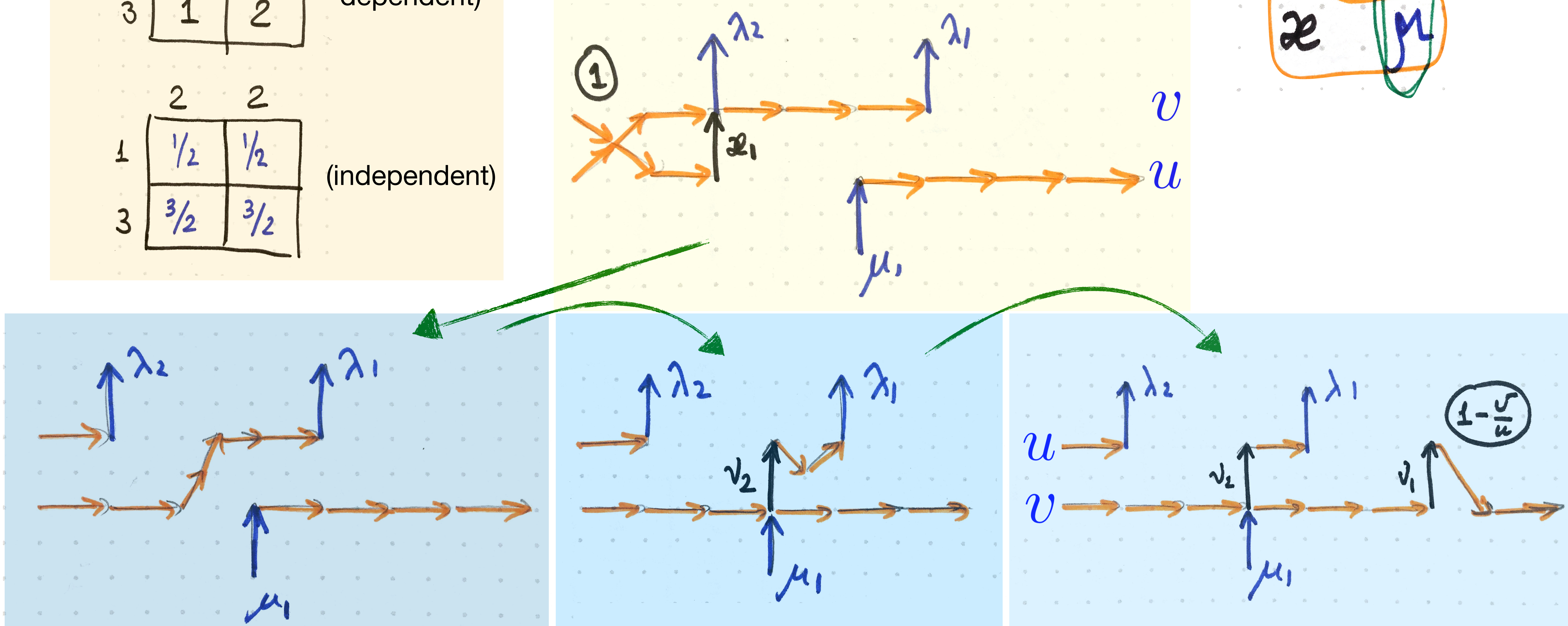
(maximally dependent)

	2	2
1	1/2	1/2
3	3/2	3/2

(independent)

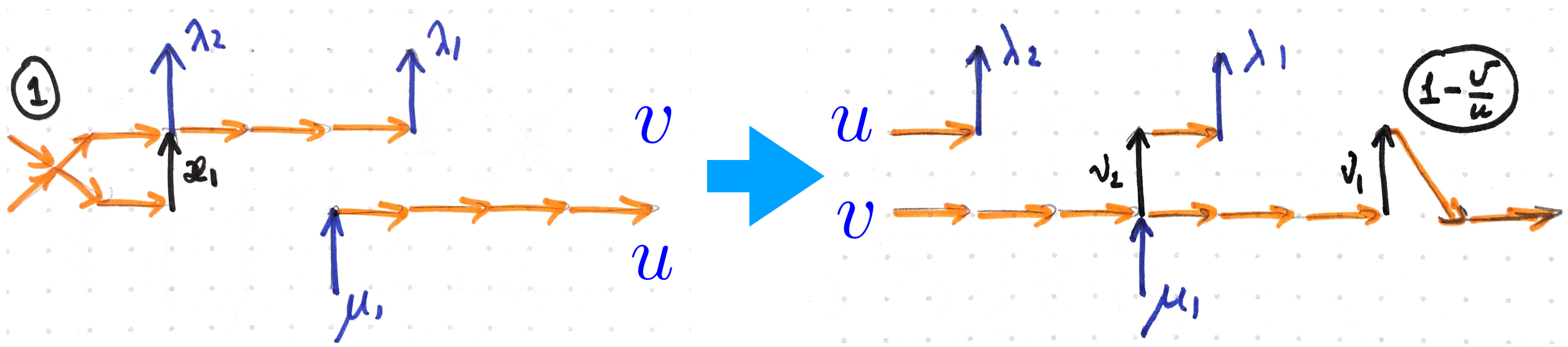
Quadruple step $\varkappa \to \nu$





$$1 \cdot s_{\lambda/\varkappa}(v)\, s_{\mu/\varkappa}(u^{-1}) \cdot \mathbb{P}(\varkappa \to \nu) = \left(1-\frac{v}{u}\right) \cdot s_{\nu/\lambda}(u^{-1})\, s_{\nu/\mu}(v) \cdot \hat{\mathbb{P}}(\nu \to \varkappa)$$

$$1 \cdot s_{\lambda/\varkappa}(v)\, s_{\mu/\varkappa}(u^{-1}) \cdot \mathbb{P}(\varkappa \to \nu) = \left(1 - \frac{v}{u}\right) \cdot s_{\nu/\lambda}(u^{-1})\, s_{\nu/\mu}(v) \cdot \hat{\mathbb{P}}(\nu \to \varkappa)$$

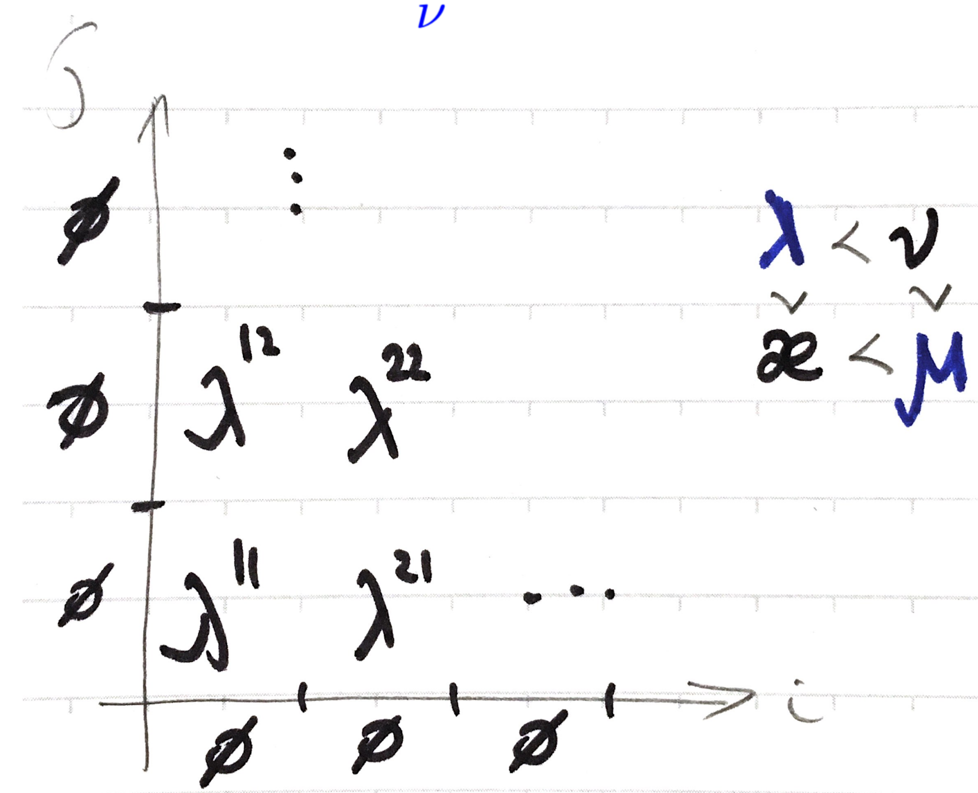


Skew Cauchy identity (new proof)

$$\frac{1}{1-v/u} \sum_{\varkappa} s_{\lambda/\varkappa}(v)\, s_{\mu/\varkappa}(u^{-1}) = \sum_{\nu} s_{\nu/\lambda}(u^{-1})\, s_{\nu/\mu}(v)$$

Constructed a Yang-Baxter field

- Has a Markovian scalar field marginal: **number of arrows in the leftmost column** (=> at the edge of the Gibbs configuration)
- The edge dynamics is a version of **TASEP** with discrete time and pushing. It converges to the usual TASEP
- Systematic view of **most known matchings** between KPZ particle systems and 2d Gibbs measures. Naturally includes **inhomogeneity**
- Associated to Schur polynomials, but **extends** to many other families. Brings new stochastic models



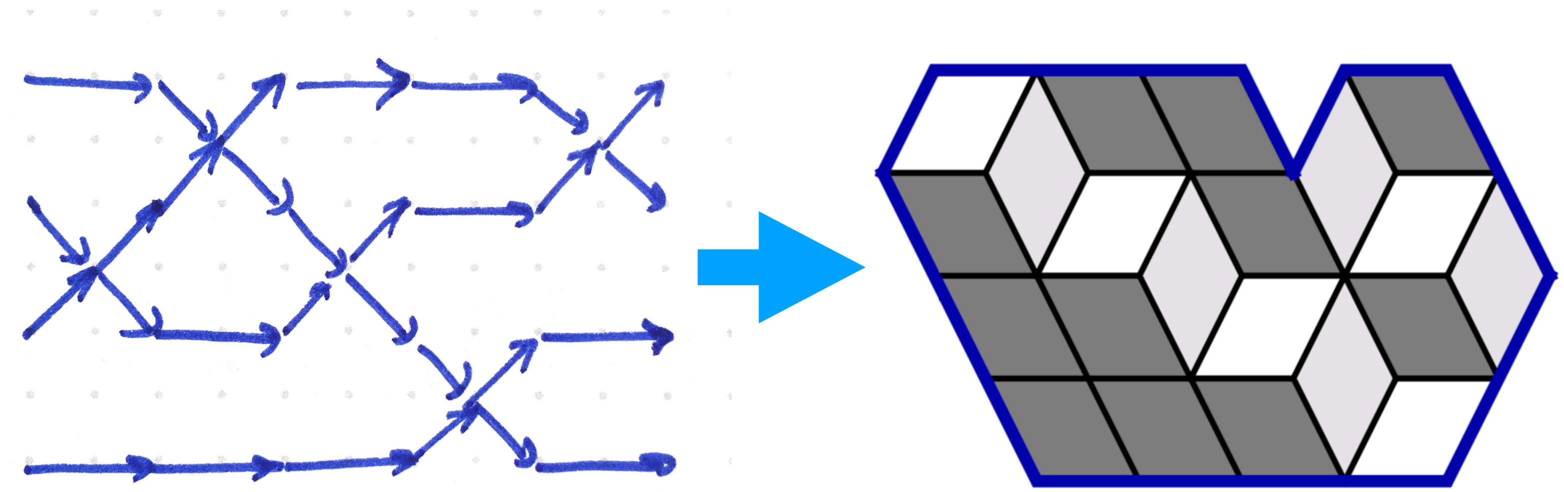
Yang-Baxter bijectivisation also applies to Gibbs measures in a finite volume [P., Saenz 2019]

- New Markov dynamics preserving distribution of **alternating sign matrices / six vertex model with domain wall boundary conditions** (refined with spectral parameters)

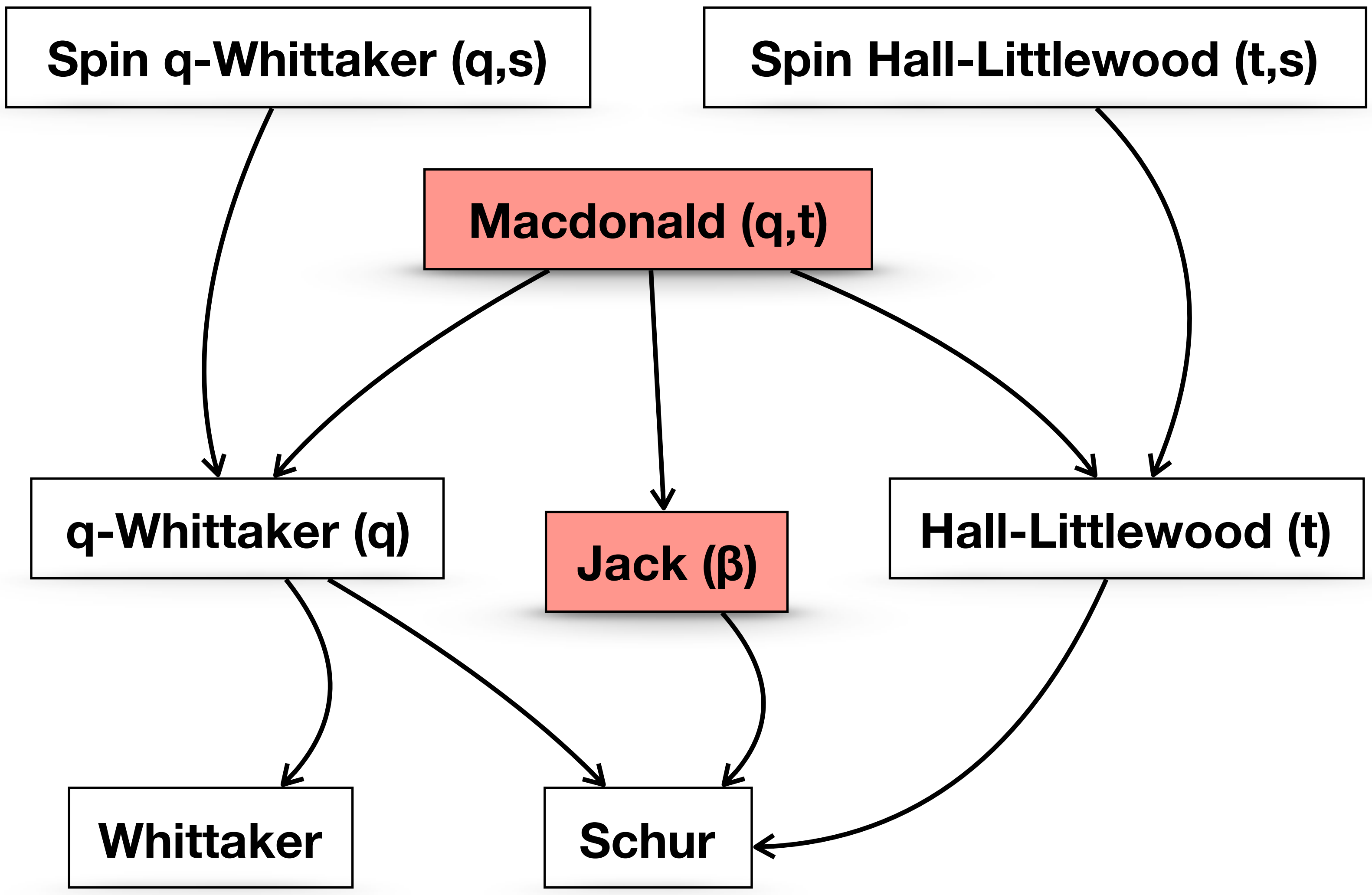
- Does not typically have Markovian 1d **marginals**, but dependence may be controllable

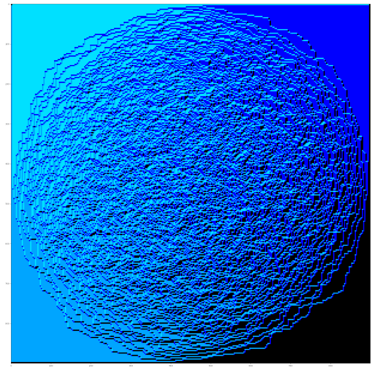
- In the bulk / continuous time, converges to **irreversible Markov dynamics** on 2d Gibbs measures (TASEP is the example in 1d)

- Applies systematically to known examples [Borodin, Bufetov, Corwin, Toninelli 2015+] and potentially to **any lattice models** possessing the Yang-Baxter equation



Interchanges spectral parameters in a Gibbs tiling





Summary and further development

- Gibbs measures are **rich** and **intriguing**. Six vertex model in finite domains is still largely open
- Used the Yang-Baxter equation in a unified way to **match** Schur-type Gibbs ensembles to 1d particle systems
- Applying these ideas in a finite volume to build new **Markov dynamics** preserving 2d Gibbs measures. Include known cases in a systematic way

- Can use Yang-Baxter equation or related duality to compute **observables** in non-determinantal models beyond Schur

  [Corwin-P. 2015], [Borodin-P. 2016]

- Elliptic and $U_q(\widehat{\mathfrak{sl}_n})$ (multispecies) **generalizations**

  [Aggarwal, Borodin, Wheeler 2016-18]

- Bethe ansatz **spectral theories** for stochastic particle systems of six vertex type, out of equilibrium **[Borodin, Corwin, P., Sasamoto 2012-14]**

- New integrable particle systems in **inhomogeneous space** solvable by Schur measures via interesting couplings beyond YB fields. New phase transitions

  [Orr-P. 2016], [Borodin-P. 2017], [Knizel-P.-Saenz 2018]

Thank you!

lpetrov.cc/Gibbs2018/