Spectral theory for interacting particle systems

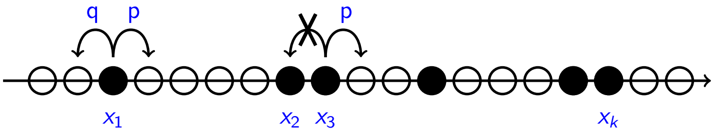
Leonid Petrov

University of Virginia

Outline

1. ASEP and its KPZ limit
2. Tracy-Widom's solution to the ASEP
3. q-Hahn zero-range process and its spectral theory
4. Degenerations

Asymmetric Simple Exclusion Process



Particles at locations $\vec{x} = (x_1 < x_2 < \ldots < x_k)\colon x_i \in \mathbb{Z}$ evolve in continuous time.

Each particle has an independent exponential clock with mean 1. When the clock rings, the particle attempts to jump to the right with probability $0 < \mathsf{p} < \frac{1}{2}$, and to the left with probability $\mathsf{q} = 1 - \mathsf{p}$. Jumping to occupied sites is prohibited.

$$\tau := \mathsf{p}/\mathsf{q} < 1.$$

Asymmetric Simple Exclusion Process

ASEP is the first *non-determinantal* model shown to belong to the **Kardar-Parisi-Zhang (KPZ) universality class**.

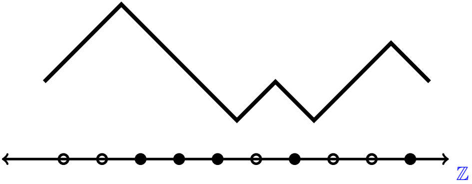
Start from the *step initial condition* $x_i(0) = i$, $i = 1, 2, 3, \ldots$. Let $N_0(t)$ be the number of particles to the left of the origin at time $t > 0$.

Theorem [Tracy-Widom '07+]

$$\lim_{t \to +\infty} \mathbb{P}\left(\frac{N_0(t/(\mathrm{q}-\mathrm{p})) - t/4}{2^{-1/3} t^{1/3}} \geq -s \right) = F_{GUE}(s),$$

where $F_{GUE}(s)$ is the Tracy-Widom GUE distribution.

ASEP and the KPZ equation



Under the *weak asymmetry scaling* $\tau = 1 - \sqrt{\epsilon}$, the ASEP interface (slope -1 over a particle, slope $+1$ over a hole)

converges [Bertini-Giacomin '97], [Amir-Corwin-Quastel '10] to the solution $h(t,x)$ of the **KPZ equation**

$$\partial_t h = -\frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + \xi \qquad (\xi \text{ — space-time white noise})$$

with *narrow wedge initial data* (corresponding to step initial data in ASEP).
($h = -\log Z$, where $Z(t,x)$ solves **SHE**; also [Hairer '11])

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TW's solution. Step 1

Apply the coordinate Bethe ansatz to find eigenfunctions of the ASEP generator

$$(\mathcal{H}_\tau^{\mathrm{ASEP}} f)(\vec{x}) = \sum_i \mathrm{p}(f(\vec{x}_i^{+}) - f(\vec{x})) + \sum_j \mathrm{q}(f(\vec{x}_j^{-}) - f(\vec{x}))$$

Here $\vec{x}_i^{+}$ is the configuration obtained by moving particle x_i one step to the right, and similarly for the configuration $\vec{x}_j^{-}$. The sums above are taken over allowed i and j.

Inspired by Bethe ansatz for Heisenberg XXZ spin chain (whose Hamiltonian is conjugate to $\mathcal{H}_\tau^{\mathrm{ASEP}}$: but with complex τ) [Bethe '31], [Yang-Yang '66];
also see [Gwa–Spohn '92], [Schutz '97], [Rakos–Schutz '05].

TW's solution. Step 1

The action of the generator $\mathcal{H}_\tau^{\mathrm{ASEP}}$ can be simplified. Let $k = 2$, then if x_1 and x_2 are not immediate neighbors,

$$\begin{aligned}\mathcal{H}_\tau^{\mathrm{ASEP}} f(x_1, x_2) =& \mathrm{p}(f(x_1+1, x_2) - f(x_1, x_2)) \\ &+ \mathrm{q}(f(x_1-1, x_2) - f(x_1, x_2)) \\ &+ \mathrm{p}(f(x_1, x_2+1) - f(x_1, x_2)) \\ &+ \mathrm{q}(f(x_1, x_2-1) - f(x_1, x_2)).\end{aligned}$$

If x_1 and x_2 are neighbors ($x_2 = x_1 + 1$), then

$$\begin{aligned}\mathcal{H}_\tau^{\mathrm{ASEP}} f(x_1, x_2) =& \mathrm{p}(f(x_1, x_2+1) - f(x_1, x_2)) \\ &+ \mathrm{q}(f(x_1-1, x_2) - f(x_1, x_2))\end{aligned}$$

If it turns out that for our specific f the discrepancy between the two expressions is 0 (for $x_2 = x_1 + 1$), then it *does not matter* which formula to use.

TW's solution. Step 1

In general, represent the action of $\mathcal{H}_\tau^{\mathrm{ASEP}}$ as a free generator (of independent particles)

$$(\mathcal{L}_\tau^{\mathrm{ASEP}} u)(\vec{x}) := \sum_{i=1}^{k} [\nabla_\tau^{\mathrm{ASEP}}]_i u(\vec{x})$$

$$(\nabla_\tau^{\mathrm{ASEP}} u)(y) := \mathsf{p}u(y+1) + \mathsf{q}u(y-1) - u(y), \quad y \in \mathbb{Z},$$

subject to $k-1$ *two-body boundary conditions*:

$$\Big(\mathsf{p}u(\vec{x}_i^{+}) + \mathsf{q}u(\vec{x}_{i+1}^{-}) - u(\vec{x})\Big)\Big|_{\vec{x}\in\mathbb{Z}^k:\, x_{i+1}=x_i+1} = 0.$$

That is, $\mathcal{H}_\tau^{\mathrm{ASEP}} u = \mathcal{L}_\tau^{\mathrm{ASEP}} u$ for $u(\vec{x})$ satisfying boundary conditions.

The ASEP is integrable in the Bethe sense i.e. no higher boundary conditions are needed.

TW's solution. Step 1

To find eigenfunctions of the *true* ASEP generator $\mathcal{H}_\tau^{\mathrm{ASEP}}$:

- Eigenfunctions of the *free* generator $\mathcal{L}_\tau^{\mathrm{ASEP}}$ are simply powers $\prod_{j=1}^{k} \left(\frac{1+z_j}{1+z_j/\tau} \right)^{-x_j}$, where $z_i \in \mathbb{C}$. Their eigenvalues are symmetric in z_i:

$$-\frac{(1-\tau)^2}{1+\tau} \sum_{j=1}^{k} \frac{1}{(1+z_j)(1+\tau/z_j)}$$

- Combine these eigenfunctions so that they satisfy the $k-1$ boundary conditions. These combinations will be eigenfunctions of $\mathcal{H}_\tau^{\mathrm{ASEP}}$.

TW's solution. Step 1

This approach gives the following eigenfunctions of $\mathcal{H}_\tau^{\mathrm{ASEP}}$:

Bethe ansatz eigenfunctions of the ASEP

$$\Psi_{\vec{z}}^{\mathrm{ASEP}}(\vec{x}) := \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(B)} - \tau z_{\sigma(A)}}{z_{\sigma(B)} - z_{\sigma(A)}} \prod_{j=1}^{k} \left(\frac{1 + z_{\sigma(j)}}{1 + z_{\sigma(j)}/\tau} \right)^{-x_j}.$$

Change of variables $\xi_i = (1 + z_i)/(1 + z_i/\tau)$, up to constant:

$$\Phi_{\vec{\xi}}^{\mathrm{ASEP}}(\vec{x}) := \sum_{\sigma\in S(k)} \mathrm{sgn}(\sigma) \prod_{1\le B<A\le k} \mathcal{S}_{\mathrm{ASEP}}(\xi_{\sigma(A)}, \xi_{\sigma(B)}) \prod_{j=1}^{k} \xi_{\sigma(j)}^{-x_j},$$

where $\mathcal{S}_{\mathrm{ASEP}}(\xi_1, \xi_2) := \tau - (1+\tau)\xi_1 + \xi_1\xi_2$

Note: these are "algebraic" eigenfunctions. For instance, they are not compactly supported.

TW's solution. Step 2

Use the eigenfunctions $\Psi_{\vec{z}}^{\mathrm{ASEP}}$ to write down the **transition probabilities of the ASEP**:

$P_t(\vec{x} \to \vec{y}) :=$ probability that the (k-particle) ASEP is at state $\vec{y}$ at time t given that it started from state $\vec{x}$ at time 0.

These probabilities solve the *master equation* (the same as *forward Kolmogorov equation*). This is an ODE with the right-hand side essentially $\mathcal{H}_{\tau}^{\mathrm{ASEP}}$. It is diagonalized in the $\Psi_{\vec{z}}^{\mathrm{ASEP}}$'s:

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(A) project the initial conditions $P_0(\vec{x} \to \vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}$ (as a function of the spatial variables $\vec{y}$) onto the eigenfunctions

(B) evolve the eigenfunctions: multiply by $\exp(\mathbf{ev}_{\mathrm{ASEP}} \cdot t)$

(C) reconstruct the solution $P_t(\vec{x} \to \vec{y})$ from the (evolved) eigenfunctions

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Steps (A) and (C) involve taking *direct* and *inverse* **Fourier-like transforms**, respectively, plus a *Plancherel-type isomorphism theorem* (discussed later).

TW's solution. Step 2

ASEP transition function [Tracy-Widom '07]

$$P_t(\vec{x} \to \vec{y}) = \sum_{\sigma \in S(k)} \oint \dots \oint \mathsf{A}_\sigma(\vec{\xi}) e^{t \cdot \mathbf{ev}_{\mathrm{ASEP}}(\vec{\xi})} \prod_{j=1}^{k} \xi_{\sigma(j)}^{-x_{\sigma(j)}+y_j-1} \frac{d\xi_j}{2\pi\mathbf{i}}$$

Integrals are over small circles around 0,

$\mathbf{ev}_{\mathrm{ASEP}}(\vec{\xi}) := \sum_{j=1}^{k} (\mathsf{p}\xi_j^{-1} + \mathsf{q}\xi_j - 1)$

$$\mathsf{A}_\sigma(\vec{\xi}) := \prod_{A<B:\ \sigma(A)>\sigma(B)} S_{\sigma(A)\sigma(B)},$$

$$S_{\alpha\beta} := -\frac{\mathsf{p} + \mathsf{q}\xi_\alpha\xi_\beta - \xi_\alpha}{\mathsf{p} + \mathsf{q}\xi_\alpha\xi_\beta - \xi_\beta} = -\frac{\mathcal{S}_{\mathrm{ASEP}}(\xi_\alpha, \xi_\beta)}{\mathcal{S}_{\mathrm{ASEP}}(\xi_\beta, \xi_\alpha)}.$$

TW's solution. Step 3

Use the transition function to get Fredholm determinantal expressions for one-point distributions $\mathbb{P}(x_m(t) \leq x)$ of the ASEP. This is done [Tracy-Widom '08, '09] for

1. *Step initial condition* $x_i(0) = i$, $i = 1, 2, \ldots$.
2. *Step-Bernoulli initial condition*: nothing to the left of the origin, and particles to the right of the origin independently at each site with some fixed probability.

In both cases, the passage from $P_t(\vec{x} \to \vec{y})$ to Fredholm determinants is very nontrivial and involves certain **symmetrization identities**.

TW's solution. Step 3

Symmetrization identity for the step initial condition [TW '07]:

$$\sum_{\sigma \in S(k)} \sigma \left(\prod_{i<j} \frac{\mathcal{S}_{\mathrm{ASEP}}(\xi_i, \xi_j)}{\xi_j - \xi_i} \frac{\xi_2 \xi_3^2 \dots \xi_k^{k-1}}{(1 - \xi_1 \xi_2 \dots \xi_k)(1 - \xi_2 \dots \xi_k) \dots (1 - \xi_k)} \right)$$

$$= \frac{\tau^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{k} (1 - \xi_j)}.$$

TW's solution. Step 3

Symmetrization identity for the step initial condition [TW '07]:

$$\sum_{\sigma \in S(k)} \sigma \left(\prod_{i<j} \frac{\mathcal{S}_{\mathrm{ASEP}}(\xi_i, \xi_j)}{\xi_j - \xi_i} \frac{\xi_2 \xi_3^2 \dots \xi_k^{k-1}}{(1 - \xi_1 \xi_2 \dots \xi_k)(1 - \xi_2 \dots \xi_k) \dots (1 - \xi_k)} \right)$$

$$= \frac{\tau^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{k}(1 - \xi_j)}.$$

Start with

$$P_t(\vec{x} \to \vec{y}) = \sum_{\sigma \in S(k)} \oint \dots \oint \mathrm{A}_\sigma(\vec{\xi}) e^{t \cdot \mathbf{ev}_{\mathrm{ASEP}}(\vec{\xi})} \prod_{j=1}^{k} \xi_{\sigma(j)}^{-x_{\sigma(j)} + y_j - 1} \frac{d\xi_j}{2\pi \mathbf{i}},$$

sum $\prod_{j=1}^{k} \xi_{\sigma(j)}^{-x_{\sigma(j)}+y_j-1}$ over $y < y_2 < \dots < y_k$ with y fixed, then apply symmetrization identity $\Rightarrow$ get a tractable expression for $\mathbb{P}(x_1(t) = y)$.

Further, can also write $\mathbb{P}(x_m(t) \le y)$ for any m as a **Fredholm determinant**.

TW's solution. Step 4

The Fredholm determinantal expression:

$$\mathbb{P}(x_m(t) \le y) = \int_{\text{large circle}} \frac{\det(\mathrm{Id} - \lambda \mathsf{q} K)}{\prod_{j=0}^{m-1}(1 - \lambda \tau^j)} \frac{d\lambda}{\lambda},$$

where K is the operator with kernel

$$K(\xi, \xi') := \frac{\xi^y e^{(\mathsf{p}\xi^{-1} + \mathsf{q}\xi - 1)t}}{\mathsf{p} + \mathsf{q}\xi\xi' - \xi}.$$

It "remains" to study asymptotics of this Fredholm determinant: very nontrivial transformations, then use steepest descent analysis.

TW's solution. Overview

1. **Diagonalize the generator** using Bethe ansatz eigenfunctions
2. Use Fourier-like transforms and Plancherel isomorphism theorems to get **transition probabilities**
3. Use nontrivial symmetrization identities to get **Fredholm determinantal expressions** for one-point functions
4. Study **asymptotics** of Fredholm determinants

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A new key: **focus on eigenfunctions** and study their properties such as Fourier-like transforms and Plancherel isomorphism theorems. This gives all results of steps 1–3.

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“Most general” (eigen?)functions

$$\Psi_{\vec{z}}(\vec{n}) := \sum_{\sigma \in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - \boxed{q} z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{\boxed{a} z_{\sigma(j)} + \boxed{b}}{\boxed{c} z_{\sigma(j)} + \boxed{d}} \right)^{-n_j},$$

$\vec{n} = (n_1 \geq n_2 \geq \ldots \geq n_k)$, $n_j \in \mathbb{Z}$
(note the difference in ordering: **now weakly decreasing**).

- For $a = c = d = 1$, $c = 1/q$, and $x_j = n_{k+1-j}$, these are eigenfunctions of the ASEP

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(note the difference in ordering: **now weakly decreasing**).

- For $a = c = d = 1$, $c = 1/q$, and $x_j = n_{k+1-j}$, these are eigenfunctions of the ASEP
- For $b = c = 0$, these are essentially the **Hall-Littlewood symmetric polynomials**

“Most general” (eigen?)functions

$$\Psi_{\vec{z}}(\vec{n}) := \sum_{\sigma \in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - \boxed{q} z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{\boxed{a} z_{\sigma(j)} + \boxed{b}}{\boxed{c} z_{\sigma(j)} + \boxed{d}} \right)^{-n_j},$$

$\vec{n} = (n_1 \geq n_2 \geq \ldots \geq n_k)$, $n_j \in \mathbb{Z}$

- Can rescale z_j's $\Longrightarrow$ in the linear fractional expression $\dfrac{az+b}{cz+d}$, only 2 parameters remain.

  Reparametrize as $\boxed{\dfrac{\theta - z}{1 - \nu z}}$.

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- Can rescale z_j's $\Longrightarrow$ in the linear fractional expression $\frac{az+b}{cz+d}$, only 2 parameters remain.

  Reparametrize as $\boxed{\frac{\theta - z}{1 - \nu z}}$.
- Up to an overall simple factor $\theta^{-n_1-\ldots-n_k}$, only the parameter ν remains.

q-Hahn eigenfunctions

$$\Psi^{\ell}_{\vec{z}}(\vec{n}) := \sum_{\sigma \in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1 - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}} \right)^{-n_j}$$

- Introduced in [Povolotsky '13] as eigenfunctions of a certain discrete-time stochastic particle system (q-**Hahn zero-range process**)

- Also appeared in the work of [Takeyama '14] in the context of a deformation of the affine Hecke algebra of type GL (essentially, deform the commutative part in a *linear fractional way*): leads to a stochastic particle system which is a continuous-time limit of the q-Hahn ZRP

- Spectral theory: [Borodin–Corwin–P.–Sasamoto '13–'14]

q-Hahn ZRP

q-Hahn jumping distribution

Let $0 < q < 1$ and $0 \le \nu \le \mu < 1$,

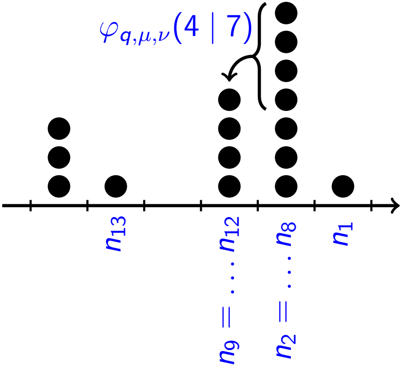
$$\varphi_{q,\mu,\nu}(j \mid m) = \mu^j \frac{(\nu/\mu; q)_j (\mu; q)_{m-j}}{(\nu; q)_m} \frac{(q; q)_m}{(q; q)_j (q; q)_{m-j}},$$

where $(a; q)_n := \prod_{j=1}^{n} (1 - aq^{j-1})$.
(sums to one; is orthogonality weight for the classical q-Hahn orthogonal polynomials)

q-Hahn ZRP

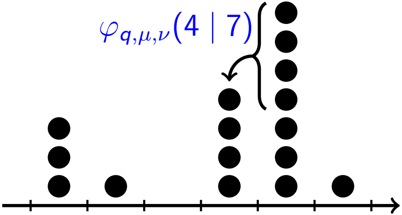
q-Hahn jumping distribution

$$\varphi_{q,\mu,\nu}(j \mid m) = \mu^j \frac{(\nu/\mu; q)_j (\mu; q)_{m-j}}{(\nu; q)_m} \frac{(q; q)_m}{(q; q)_j (q; q)_{m-j}}$$



q-Hahn zero range discrete-time Markov process on $\vec{n} = (n_1 \geq \ldots \geq n_k)$: in parallel, move j (out of m) particles in each column to the right with probability $\varphi_{q,\mu,\nu}(j \mid m)$

q-Hahn ZRP



Coordinate Bethe ansatz for q-Hahn ZRP [Povolotsky '13]

The functions

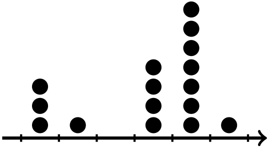
$$\Psi^{\ell}_{\vec{z}}(\vec{n}) = \sum_{\sigma \in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1 - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}} \right)^{-n_j}$$

are eigenfunctions of the q-Hahn ZRP's transition operator $\mathcal{H}_{q,\mu,\nu}$ with eigenvalues

$$\prod_{j=1}^{k} \frac{1 - \boxed{\mu} z_j}{1 - \nu z_j}$$

PT invariance

The q-Hahn ZRP operator $\mathcal{H}_{q,\mu,\nu}$ is not self-adjoint. This property is replaced by **PT invariance** (invariance under joint space and time reflection).



Let $\mathfrak{m}_{q,\nu}(\vec{n})$ be the product of $\dfrac{(\nu;q)_{c_i}}{(q;q)_{c_i}}$ over "clusters" of $\vec{n}$.

($\sim$ invariant product measure)

PT invariance

The operator $\mathfrak{m}_{q,\nu}^{-1}\mathcal{H}_{q,\mu,\nu}^{transpose}\mathfrak{m}_{q,\nu}$ coincides with the space reflection of $\mathcal{H}_{q,\mu,\nu}$

Left and right q-Hahn eigenfunctions

Left $\Psi^{\ell}_{\vec{z}}(\vec{n})$

$$\sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{1 - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}}\right)^{-n_j}$$

Right $\Psi^{r}_{\vec{z}}(\vec{n})$

$$(-1)^k (1-q)^k \mathfrak{m}_{q,\nu}(\vec{n}) \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(B)} - q z_{\sigma(A)}}{z_{\sigma(B)} - z_{\sigma(A)}} \prod_{j=1}^{k} \left(\frac{1 - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}}\right)^{n_j}$$

We have

$$\mathcal{H}_{q,\mu,\nu} \Psi^{\ell}_{\vec{z}} = \prod_{j=1}^{k} \frac{1 - \mu z_j}{1 - \nu z_j} \Psi^{\ell}_{\vec{z}}, \qquad \Psi^{r}_{\vec{z}} \mathcal{H}_{q,\mu,\nu} = \prod_{j=1}^{k} \frac{1 - \mu z_j}{1 - \nu z_j} \Psi^{r}_{\vec{z}}.$$

Direct transform

Two spaces of functions

1. Space of compactly supported functions of the spatial variables $\vec{n}$
2. Space of symmetric Laurent polynomials in $\frac{1-z_j}{1-\nu z_j}$

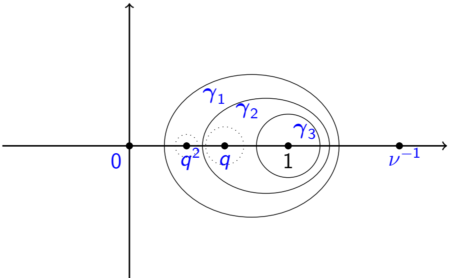
Direct transform of $f(\vec{n})$

$$(\mathcal{F}^{q,\nu} f)(\vec{z}) := \sum_{\vec{n}} f(\vec{n}) \Psi^r_{\vec{z}}(\vec{n}) =: \langle f, \Psi^r_{\vec{z}} \rangle_{\vec{n}}.$$

Inverse transform

Inverse transform of $G(\vec{z})$

$$(\mathcal{J}^{q,\nu}G)(\vec{n}) := \oint_{\gamma_1}\frac{dz_1}{2\pi\mathbf{i}}\cdots\oint_{\gamma_k}\frac{dz_k}{2\pi\mathbf{i}}\prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}$$
$$\times\prod_{j=1}^{k}\frac{1}{(1-z_j)(1-\nu z_j)}\left(\frac{1-z_j}{1-\nu z_j}\right)^{-n_j}G(\vec{z}).$$



Can write the inverse transform as $(\mathcal{J}^{q,\nu}G)(\vec{n}) = \left\langle G, \Psi^{\ell}(\vec{n})\right\rangle_{\vec{z}}$

Plancherel isomorphism theorems

Theorem [Borodin–Corwin–P.–Sasamoto '14]

(1) The composition $\mathcal{J}^{q,\nu}\mathcal{F}^{q,\nu}$ is an identity operator on the space of compactly supported functions in $\vec{n}$.

(2) The composition $\mathcal{F}^{q,\nu}\mathcal{J}^{q,\nu}$ is an identity operator on the space of symmetric Laurent polynomials in $\frac{1-z_j}{1-\nu z_j}$.

Proof of (1) is a direct combinatorial argument: show that $\big(\mathcal{J}^{q,\nu}\Psi^r_\bullet(\vec{x})\big)(\vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}$. In fact, contributions form each permutation σ in $\Psi^r_{\vec{z}}$ vanishes individually if $\vec{x} \neq \vec{y}$.

Proof of (2) relies on the presence of a commuting family of operators $\mathcal{H}_{q,\mu,\nu}$ (depending on additional parameter μ) diagonalized in the same eigenfunctions. This gives many relations that we use.

Biorthogonality

Spatial biorthogonality

$\left\langle \Psi^{\ell}_{\bullet}(\vec{x}), \Psi^{r}_{\bullet}(\vec{y}) \right\rangle_{\vec{z}} = \mathbf{1}_{\vec{x}=\vec{y}}.$

Follows immediately from the first Plancherel theorem.

Also this implies completeness of the Bethe ansatz for the q-Hahn ZRP.

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Follows immediately from the first Plancherel theorem.

Also this implies completeness of the Bethe ansatz for the q-Hahn ZRP.

Spectral biorthogonality

In a certain weak sense ($\mathbf{V}(\vec{z})$ is the Vandermonde),

$$\sum_{\vec{n}} \Psi^{r}_{\vec{z}}(\vec{n})\Psi^{\ell}_{\vec{w}}(\vec{n})\mathbf{V}(\vec{z})\mathbf{V}(\vec{w})$$

$$= (-1)^{\frac{k(k-1)}{2}} \prod_{j=1}^{k}(1-z_j)(1-\nu z_j) \prod_{A\neq B}(z_A - q z_B) \det[\delta_{z_i,w_j}]_{i,j=1}^{k}.$$

This statement in fact implies the second Plancherel theorem.

q-Hahn symmetrization identity

The second Plancherel theorem applied to a specific function $G(\vec{z})$ implies (after a change of variables)

$$\sum_{\sigma\in S(k)} \prod_{B<A} \frac{\mathcal{S}_{q,\nu}(\xi_{\sigma(B)}, \xi_{\sigma(A)})}{\xi_{\sigma(A)} - \xi_{\sigma(B)}} \Bigg(\sum_{n_1\geq\ldots\geq n_k\geq 0} \mathfrak{m}_{q,\nu}(\vec{n}) \prod_{j=1}^{k} \xi_{\sigma(j)}^{n_j} \Bigg)$$

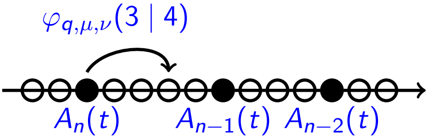
$$= \left(\frac{1-\nu}{1-q}\right)^k \left(\frac{q(1-\nu)}{1-q\nu}\right)^{\frac{k(k-1)}{2}} \prod_{j=1}^{k} \frac{1}{1-\xi_j}.$$

Here $\mathfrak{m}_{q,\nu}(\vec{n})$ is the product of $\frac{(\nu;q)_{c_i}}{(q;q)_{c_i}}$ over "clusters" of $\vec{n}$,

and

$$\mathcal{S}_{q,\nu}(\xi_1,\xi_2) := \frac{1-q}{1-q\nu} + \frac{q-\nu}{1-q\nu}\xi_2 + \frac{\nu(1-q)}{1-q\nu}\xi_1\xi_2 - \xi_1.$$

Duality with the q-Hahn TASEP



q-Hahn TASEP: in parallel, each paricle jumps by j steps with probability $\varphi_{q,\mu,\nu}(j \mid \text{next gap})$.

Duality [Corwin '14]

The q-Hahn TASEP is dual to the q-Hahn ZRP.
Let $H(\vec{A}, \vec{n}) := \prod_{j=1}^{k} q^{A_{n_j}+n_j}$. Then

$$\mathbb{E}^{TASEP}_{\vec{A}_0} H(\vec{A}(t), \vec{n}_0) = \mathbb{E}^{ZRP}_{\vec{n}_0} H(\vec{A}_0, \vec{n}(t)).$$

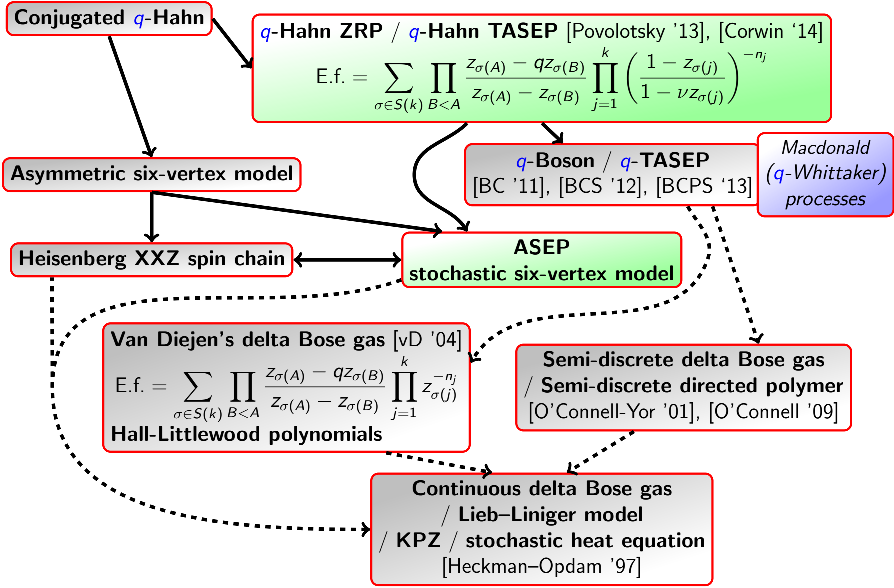
Duality plus spectral theory for ZRP allows to write moment formulas for q-Hahn TASEP with **arbitrary initial data** (these moments solve backward Kolmogorov equations for q-Hahn ZRP).

Overview of q-Hahn results

1. Applying coordinate Bethe ansatz to the q-Hahn ZRP, get **explicit eigenfunctions**
2. There are direct and inverse **Fourier-like transforms**, they are mutual inverses. In particular, this proves **completeness** of the Bethe ansatz.
3. The eigenfunctions satisfy certain **biorthogonality relations**.
4. Allows to solve forward and backward Kolmogorov equations for the q-Hahn ZRP with arbitrary initial data.
    - The forward equations give **transition probabilities** for the q-Hahn ZRP
    - By **duality**, the backward equations give **moment formulas** for the q-Hahn TASEP with arbitrary initial data
5. Also gives **symmetrization identities**.

Outline

Hierarchy of eigenfunctions

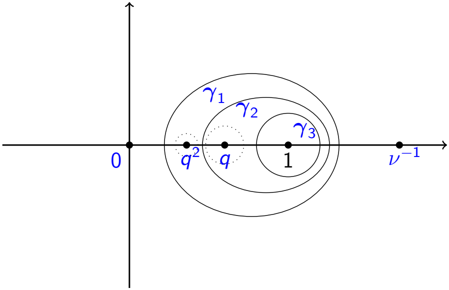


Degeneration to ASEP

If $q = 1/\nu = \tau$, the q-Hahn eigenfunctions become the ASEP eigenfunctions.

Note that in this case $\mathfrak{m}_{q,\nu}(\vec{n}) = 0$ unless all n_j's are **distinct**.

This is responsible for the weak/strict ordering in q-Hahn ZRP and ASEP, respectively.



Nested integration contours no longer work because $\nu^{-1} = q$.

With some work, all these contours collapse to a single small contour around 1

Degeneration to ASEP

There are direct and inverse Fourier-like transforms for the ASEP, and they are mutual inverses. In particular, this proves completeness of the Bethe ansatz for the ASEP.

The first Plancherel theorem implies the **Tracy-Widom's formula for the transition function** of the ASEP.

Note

An independent proof of the first Plancherel theorem $\big(\mathcal{F}^{\mathrm{ASEP}}\Psi_{\bullet}^{r,\mathrm{ASEP}}(\vec{x})\big)(\vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}$ (given in [Tracy-Widom '07]) is more involved combinatorially, because individual contributions of permutations σ in $\Psi_{\vec{z}}^{r,\mathrm{ASEP}}$ do not vanish if $\vec{x} \neq \vec{y}$ (after setting $q = \nu^{-1}$).

Degeneration to ASEP

The q-Hahn symmetrization identity

$$\sum_{\sigma\in S(k)} \prod_{B<A} \frac{\mathcal{S}_{q,\nu}(\xi_{\sigma(B)}, \xi_{\sigma(A)})}{\xi_{\sigma(A)} - \xi_{\sigma(B)}} \left(\sum_{n_1\geq\ldots\geq n_k\geq 0} \mathfrak{m}_{q,\nu}(\vec{n}) \prod_{j=1}^{k} \xi_{\sigma(j)}^{n_j} \right)$$

$$= \left(\frac{1-\nu}{1-q}\right)^k \left(\frac{q(1-\nu)}{1-q\nu}\right)^{\frac{k(k-1)}{2}} \prod_{j=1}^{k} \frac{1}{1-\xi_j}.$$

immediately implies the ASEP one. When $q = \nu^{-1}$, the sum over $\vec{n}$ above simplifies to the rational function from the Tracy-Widom's identity.

There are also more complicated identities corresponding to step-Bernoulli initial data in q-Hahn ZRP and ASEP, respectively.

Degeneration to ASEP

Finally, ASEP is self-dual (at the level of moments). So the spectral theory for the ASEP allows to write down certain moment formulas for arbitrary initial data.

These moment formulas were obtained for the step and step-Bernoulli initial data in [Borodin–Corwin–Sasamoto '12].

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At the level of processes, neither q-Hahn ZRP nor q-Hahn TASEP seem to converge to the ASEP.

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Note

ASEP eigenfunctions are the same as those of a certain stochastic six-vertex model. Symmetrization identities were used to prove KPZ-type results in [Borodin–Corwin–Gorin '14].

q-Boson / q-TASEP

Setting $\nu = 0$ in the q-Hahn eigenfunctions leads to the eigenfunctions of the stochastic q-Boson particle system introduced in [Sasamoto–Wadati '98].

The corresponding spectral theory was developed in [Borodin–Corwin–P.–Sasamoto '13].

This leads to moment formulas for the q-TASEP with arbitrary initial data (our original motivation).

q-TASEP was studied extensively in connection with the Macdonald processes, and was shown to belong to the KPZ universality class [Borodin–Corwin '11+], [Ferrari–Veto '13+].

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Our hierarchy of eigenfunctions thus unifies (at the spectral theory level) two discrete-space regularizations of the KPZ equation, namely, q-TASEP and ASEP.

Conjugated q-Hahn eigenfunctions

$$\Psi^{\ell;\boldsymbol{\theta}}_{\vec{z}}(\vec{n}) := \sum_{\sigma\in S(k)} \prod_{B<A} \frac{z_{\sigma(A)} - qz_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{k} \left(\frac{\boldsymbol{\theta} - z_{\sigma(j)}}{1 - \nu z_{\sigma(j)}} \right)^{-n_j}$$

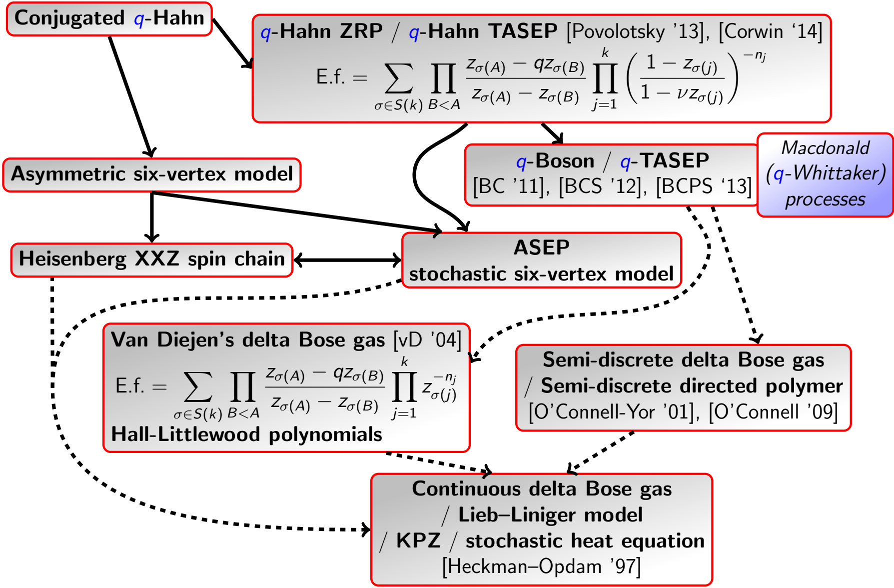
All $\boldsymbol{\theta} = 1$ results carry over to this case.

Eigenfunctions depend on $(\boldsymbol{\theta}, q, \nu)$, and for $\boldsymbol{\theta} \neq 1$ correspond to **non-stochastic** Hamiltonians.

Setting $\nu = 1/(q\boldsymbol{\theta})$, we arrive at eigenfunctions of the transfer matrix of the (non-stochastic) six-vertex model. Further this case degenerates to the Heisenberg XXZ spin chain.

Spectral theory for the latter was constructed (in a different form) by Thomas, Babbitt, and Gutkin (1977, 1990, 2000).

Hierarchy of eigenfunctions



Some (of the many) remaining questions

1. - At the q-Boson / q-TASEP level there are Macdonald processes [Borodin, Corwin et al., '11+].
   - The q-TASEP dynamics in 1+1 dimensions admits a (2+1)-dimensional lifting: dynamics on interlacing arrays (e.g., see [Borodin–P. '13]).

   How to lift these structures to the q-Hahn level?

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   How to lift these structures to the q-Hahn level?

2. At the six-vertex model level there is algebraic Bethe ansatz. Can it be lifted?

3. Having formulas for arbitrary initial data, can we derive asymptotics from them?